

The Geometry, Topology, and Dynamics of Consciousness: Four Preliminary Formal Hypotheses

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Abstract

The present paper develops a preliminary taxonomy of four mathematical orientations for investigating consciousness through relational and dynamical representations. The Variation Hypothesis locates the candidate connection to consciousness in the organization of transformations; the Trajectory Hypothesis considers selected paths, flows, or relationally ordered histories; the Generativity Hypothesis examines the coupled organization of change, structure, constraint, and difference, including possible transformations of the representational conditions themselves; and the Repertoire Hypothesis considers structured families of admissible histories. The programme uses manifolds, smooth and nonsmooth dynamics, CW-complexes and two-complexes, algebraic actions, and representation morphisms as alternative mathematical settings whose domains and interpretive commitments require separate specification. It distinguishes modal admissibility from temporal succession and treats parameterizations, causal order, and the historical status of observations as modeling choices. A previously developed semigroup metamodel supplies a minimal comparative vocabulary while leaving the appearance of new distinctions, evolving constraints, and the realization of subjective experience unresolved. Formal correspondences between representations may preserve specified structures while losing others; generative grammar serves only as a secondary illustrative instance. The four hypotheses are proposed as potentially overlapping explanatory orientations, not as established accounts of phenomenal consciousness. The paper's objective is to identify their mathematical commitments, discriminating examples, empirical interfaces, and explanatory limitations while maintaining ontological and epistemic restraint.

Keywords: consciousness; dynamical systems; differential geometry; algebraic topology; CW-complexes; two-complexes; relational generativity; variation; trajectory; repertoire.

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Research status, ethical scope, and representation notice

Status. This is an exploratory theoretical discussion. The four hypotheses are provisional distinctions among possible mathematical research programmes. The manuscript does not report experiments, validate a consciousness measure, or establish that a specified field, path, cellular complex, or repertoire has subjective experience.

Ontological and temporal restraint. Geometric state spaces, physical spacetime, epistemically reconstructed relational complexes, and mathematical parameter spaces are distinguished. The manuscript makes no assertion of an ultimate physical substrate of consciousness. It treats observer-relative records, modal admissibility, parameter choices, event order, and physical time as different objects requiring appropriate local assumptions. Expressions of futurity or historical succession, where needed, are understood relative to specified models or evidence.

Originality and attribution. The programme critically inherits standard methods from mathematical physics, topology, algebra, and consciousness science. The proposed classification is a research organization for examining mathematical objects and their explanatory interfaces. It does not claim invention of fields, neural manifolds, spin-foam models, semigroup actions, homology, or generative modeling. Existing related theories are credited in the manuscript according to a source-verification ledger.

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1 Introduction

Consciousness is examined across physical, neuroscientific, computational, and philosophical descriptions whose explanatory objects do not necessarily coincide. A reportable experience, the activity of a neural population, a dynamical law used to reconstruct that activity, and the range of processes permitted by that law can each become an object of inquiry. The distinction matters for a geometrically or dynamically formulated theory: a field of transformations, a selected path, the co-organization of changing structures and constraints, and a family of admissible paths possess different mathematical identities and preservation conditions. The present paper develops four preliminary hypotheses by separating these objects and investigating their possible relation to consciousness. Its purpose is a comparative formal inquiry into what each description assumes, expresses, and leaves unresolved. The hypotheses do not, by their mathematical form alone, establish the presence, character, or physical realization of subjective experience.

The need for this distinction arises in part from the plurality of current consciousness theories. Higher-order, global-workspace, recurrent and predictive-processing, and integrated-information approaches emphasize different explanatory constructs and make different empirical commitments; their precise comparability is itself a subject of methodological debate [36]. A preregistered adversarial comparison of global neuronal workspace theory and integrated information theory reported findings that challenged important predictions of both programmes, illustrating the value of explicit, theory-specific tests [12]. The present investigation does not propose these four formal hypotheses as replacements for that literature. Its question is prior to the selection of any one consciousness mechanism: when a theory employs a mathematical model of generation or change, *which aspect of that model is being related to consciousness*, and under what observational, physical, and philosophical assumptions?

Geometry and topology provide a useful setting for asking this question because they make different structural commitments visible. A smooth manifold supports local coordinates, tangent spaces, vector fields, and differentiable flows once appropriate regularity conditions are supplied [27]. A CW-complex is assembled by attaching cells, while a two-complex restricts attention to cells of dimension at most two; incidence, attachment, and topological invariants can be studied without assuming a global smooth-manifold structure [19]. Smooth manifolds may admit cellular decompositions, but a general two-complex need not be a manifold, and the mathematical relation between a particular state manifold and a particular relational two-complex must be constructed rather than inferred from terminology. Some models may also require variable carrier spaces, changing admissibility relations, or descriptions of structure-changing processes. In such cases, fixed-manifold flows and cellular attachments offer distinct, partially overlapping analytical resources.

Empirical research provides reasons to take these resources seriously while preserving the distinction between a dynamical description and an account of conscious experience. Neural population analysis has used state-space and dynamical-system models to examine the organization of neural computation across multiple tasks [40]. Topological methods have also been applied to structured neural correlations, including clique-complex constructions designed to recover aspects of intrinsic geometric organization [17]. These studies demonstrate mathematical and empirical uses of geometrical or topological representations of neural data. Their findings do not, by themselves, identify a topological invariant or a trajectory property sufficient for phenomenal consciousness. A model of measured neural activity is also different from physical spacetime, and a geometric state space introduced for data analysis should not silently acquire the ontological status of the system it represents.

The formal inquiry builds upon an author-developed metamodel for ontologically heterogeneous dynamics [22]. Its highest abstract layer identifies *Change*, *Structure*, *Constraint*, and *Difference* as four roles within a semigroup-action construction. The carrier is a set S without stipulated metric, order, or smooth structure; mechanisms belong to a semigroup M , with an action α and a compatible equivalence relation $\sim$ on S . The derived collapse relation is recalled in Equation 1, where $\Delta(m)$ records pairs of

states distinguishable under $\sim$ whose images under m are congruent.

$$\alpha : M \times S \longrightarrow S, \quad \Delta(m) = \{(x, y) \in S \times S \mid x \not\sim y, \quad \alpha(m, x) \sim \alpha(m, y)\}. \quad (1)$$

The action is required to respect the semigroup composition and the congruence. The metamodel derives precise consequences for quotient-level collapse under compositions of those mechanisms. It also explicitly leaves generativity, changes of the carrier, and endogenous revision of the congruence beyond its initial axioms. These limitations are productive for the current inquiry: a mathematical account of consciousness should state whether it uses only the fixed metamodel or proposes additional objects with which structures and constraints themselves can vary. In particular, the *Difference* primitive in Equation 1 is a specific relation of lost distinguishability; it is not a synonym for arbitrary perturbation, diversity, or continuous variation.

The first orientation, termed the *Variation Hypothesis*, considers whether consciousness may be associated with specified properties of the organization of change mechanisms or admissible transformations. In a differentiable specialization, a vector field is one possible mathematical object for that investigation. Under a suitable state manifold X , a selected parameter domain I , and regularity conditions sufficient for a solution, the familiar dynamical relation can be represented as Equation 2.

$$\frac{d\gamma}{d\lambda}(\lambda) = f(\gamma(\lambda)), \quad \gamma : I \longrightarrow X. \quad (2)$$

Within Equation 2, the vector field f and the path γ are distinguishable data: the field specifies a local prescription, while the chosen path satisfies it on the domain I . The symbol λ is a mathematical parameter and carries no automatic interpretation as fundamental physical time. The word *variation* is retained as a working label for transformational organization; it does not identify f with a variational-calculus perturbation δx , nor does it presume that possibility occupies a temporal stage awaiting actualization. At the more abstract level of Equation 1, the relevant objects may instead be semigroup actions, transition relations, or families of partial transformations.

The second orientation, the *Trajectory Hypothesis*, considers whether consciousness may be associated with selected paths, flows, or relationally organized histories. A path can be equipped with its image, parameterization, direction, labels, boundary conditions, and provenance, and the inclusion of each structure affects its equivalence relations. A trajectory is consequently more than an informal sequence of observations when its mathematical comparison requires incidence, orientation, or observational context. A cellular path in a two-complex provides a different representation from a smooth solution of Equation 2: cellular adjacency alone supplies neither causal precedence nor a clock, and a directed or temporally interpreted history requires specified additional relations. The hypothesis invites formal discrimination between a generator and selected paths satisfying it; the existence of a path itself is not taken as evidence of conscious experience.

The third orientation, the *Generativity Hypothesis*, places the possible connection to consciousness in the coupled organization of the four metamodel roles. The proposed object is richer than a collection of mechanisms and trajectories juxtaposed without interaction. It concerns how structures constrain transformations, how transformation mechanisms affect represented distinctions, and how the structures and constraints relevant to subsequent comparisons might themselves differ across related configurations. Because the supplied metamodel has a fixed carrier and fixed congruence, such a hypothesis requires explicit extensions when transformation of these components is studied. A diagram of represented systems, structure-respecting maps, and context-indexed constraints is one candidate mathematical setting. The existence of a self-modifying or coupled generative system is common across many domains; associating such organization with subjective experience would require additional, nontrivial explanatory principles. The hypothesis therefore examines a potential explanatory locus, not a sufficiency criterion.

The fourth orientation, the *Repertoire Hypothesis*, concerns the structured family of admissible trajectories or configurations under a declared model. Such a family could be presented as a set of solutions, a topological or measurable path space, a relation of reachability, or a repertoire furnished with weights,

Table 1: Four preliminary orientations and the mathematical objects they select for investigation.

Hypothesis	Primary orientation and illustrative mathematical objects
Variation	Organization of transformations: semigroup actions, vector fields, transition relations, and admissible local changes
Trajectory	Selected relational histories: smooth paths, cellular paths with declared directional structure, and decorated flows
Generativity	Coupled organization of change, structure, constraint, and difference across related represented systems
Repertoire	Structured families of admissible histories: solution sets, path spaces, and spaces of permitted configurations

boundary data, and labels. These choices determine which distinctions the repertoire retains. A bare set of outputs may conceal the generating process and the internal organization of paths that produce them; a repertoire of decorated histories may preserve further information. The Repertoire Hypothesis therefore cannot be reduced without qualification to the claim that the number or size of possible outcomes determines consciousness. It examines whether appropriately specified structures of possibility bear any explanatorily relevant properties, with empirical and phenomenological bridging assumptions stated separately.

Table 1 summarizes the four provisional orientations. The table specifies their initial objects of investigation without presuming that the hypotheses are mutually exclusive or exhaustive. The third orientation can incorporate transformation and trajectory data, while a repertoire may be derived from a generator after the admissibility conditions are fixed. Conversely, two generators may give rise to the same extensionally described repertoire while differing in internal organization. Equivalences therefore depend on the mathematical structures retained and on the property under comparison, and a separate study of equivalence conditions is a natural extension of the present programme.

These differences should be considered alongside a more general question about the meaning of a mathematical model of consciousness. A conscious experience might be associated with properties of neural dynamics, embodied physiological processes, environmental interactions, or physical relations not adequately described by the selected measurement apparatus. The degree to which particular neural processes are constitutive, necessary, causally sufficient, or merely correlated is a theory-specific empirical and philosophical matter. The present paper therefore adopts *ontological suspension*: it does not stipulate a unique physical substrate or a uniquely correct geometrical space for consciousness. It also does not infer a remote physical coupling mechanism from the mathematical availability of nonlocal or quantum descriptions. Empirical proposals involving such mechanisms require independently testable physical models and evidential support. The paper instead specifies the representational choices that any proposed substrate model would have to make in order to participate in the comparison.

The treatment of temporality requires similar care. In an ordinary account of a trajectory, expressions such as *before*, *after*, *realized*, and *possible* often import an implicit ordering. Within a formal model, the order of a path parameter, the composition of transformations, a causal partial order, a laboratory clock, a represented historical record, and the ontology of time are distinguishable objects. A relation of admissibility describes which configurations satisfy stated constraints; it does not identify the admitted configurations with events located after a privileged present. An embedding of one complex into another likewise gives an incidence-preserving inclusion, while its reading as historical continuation requires an independent interpretation. Similarly, noninvertibility of a map may establish loss of chosen quotient distinctions without establishing thermodynamic irreversibility or a universal arrow of time. This *temporal ontological restraint* is part of the methodology rather than a linguistic qualification added after construction.

The programme also faces a representation problem. Smooth differential equations, discrete transitions, cellular complexes, and rewriting or generative-grammar models may each express forms of change, composition, constraint, and difference once their domain-specific semantics are provided. A

correspondence between such models must be specified by maps or relations between their carriers, mechanisms, outputs, and observables. A semiconjugacy can preserve an evolution equation while collapsing state distinctions; a cellular map may preserve incidence while forgetting labels; two descriptions may agree on a selected family of paths while differing on unobserved transformations. The term *equivalent* is therefore incomplete until its preservation predicates are given. Generative grammar provides an instructive secondary comparison between a generator, a derivation, and a generated language, but the central analytical resources of this paper remain theoretical physics, geometry, topology, and algebraic dynamics. No universal or lossless translation is presumed.

A geometric and topological approach should also clarify the limits of its own invariants. Homeomorphic or homotopy-equivalent spaces may fail to preserve metric distances, parameterized flow rates, physically interpreted boundary conditions, or experiential contents. Two CW-complexes can possess the same homology groups while having different attaching maps and fundamental groups; a discrete Morse reduction can preserve homology while discarding historical labels or measurement provenance unless additional preservation requirements are imposed. These distinctions matter for each hypothesis: under Variation, the equivalence of generators may exceed topological equivalence of their carriers; under Trajectory, the image of a path may conceal its parameterization; under Generativity, a quotient may identify transformations that have different structural consequences; under Repertoire, equality of admissible-output sets may conceal heterogeneous derivation structures. Each comparison requires a specified invariant and a stated epistemic or physical interpretation.

The empirical interface remains unresolved and must be specified as part of any eventual evaluation of the hypotheses. A mathematical property such as recurrence, dynamical complexity, homological persistence, or the richness of a path space may occur in systems for which the attribution of consciousness is unsupported. Formally defining a property and observing its neural correlate do not establish phenomenal sufficiency. The inquiry consequently distinguishes a representation theorem, a generative model, a neural-data analysis, a proposed psychophysical bridge principle, and evidence concerning conscious experience. Different hypotheses may lead to different experimental discriminators only after observation operators, boundary conditions, counterfactual comparisons, and the specific phenomenon of consciousness under consideration are made explicit. The paper seeks such discriminators rather than claiming to have measured them.

The contribution of the proposed study is principally organizational and conditional. First, it articulates four formal orientations to the consciousness problem in terms of transformations, selected paths, coupled generative organizations, and structured repertoires. Second, it identifies the geometrical and topological data required to interpret these objects in smooth, cellular, and heterogeneous representations. Third, it establishes a programme of comparison using algebraic actions, morphisms, preserved distinctions, nonunique reconstruction, and synthetic counterexamples, with an explicit place for representation-dependent loss. Fourth, it articulates the limits introduced by observer dependence, temporal interpretation, incomplete access to modal structure, and the absence of an established bridge from formal dynamics to subjective experience. The taxonomy is preliminary, and the novelty or independent explanatory usefulness of each orientation requires a literature-level and mathematical assessment. The present introduction makes no claim that the four hypotheses are collectively exhaustive or experimentally validated.

The manuscript proceeds through a staged investigation. Section 2 reviews scientific and mathematical antecedents; Section 3 establishes ontological and epistemic commitments, including the status of temporality. Section 4 presents the minimal metamodel and its limitations. Section 5 develops manifolds, CW-complexes, two-complexes, fields, partial evolution relations, and representation interfaces. Section 6 constructs the comparative taxonomy. Sections 7–10 examine the Variation, Trajectory, Generativity, and Repertoire Hypotheses individually. Section 11 develops their geometric and topological interpretations, followed by the algebraic and categorical analysis in Section 12 and typed representation correspondences in Section 13. Section 14 offers illustrative constructions and counterexamples. Sections 15–17 investigate model reconstruction, physical and empirical interpretation, and explanatory limitations; Section 18 develops the wider discussion; and Section 19 concludes. Technical definitions,

cellular constructions, computations, proofs, and reproducibility materials are reserved for the appendices. Each stage is intended to support bounded mathematical or interpretive claims whose assumptions remain open to revision.

2 Scientific and Mathematical Background

This section establishes the scientific and mathematical context in which the four proposed consciousness hypotheses are to be interpreted. Its function is comparative. First, it locates the problem of conscious experience in contemporary neuroscientific debates and identifies the distinct evidential roles of neural correlates, dynamical models, and explanatory principles. Second, it examines the resources and limitations of geometric and topological approaches, with particular attention to manifolds, CW-complexes, and two-complexes. Third, it relates these approaches to the author’s minimal metamodel of change, structure, constraint, and difference. Finally, it states the modeling commitments through which these resources can be used without treating a mathematical representation as an ontology of consciousness. Formal definitions and hypothesis-specific developments are reserved for Sections 3–6, whose later drafts will build upon this background.

2.1 Physical and Mathematical Approaches to Consciousness

Explanatory targets and scientific evidence. Consciousness research investigates a heterogeneous range of phenomena: the occurrence of subjective experience, its particular phenomenal contents, distinctions between conscious and unconscious processing, the availability of content for report or action, and changes in behavioral or clinical responsiveness. The scientific question varies according to which of these phenomena is specified as the explanandum. A mathematically successful reconstruction of neural activity could support an account of behavioral computation while leaving open why the modeled process is accompanied by a particular experience. Conversely, an experimental difference between conscious and unconscious conditions could constrain competing neural theories without establishing a unique ontology of subjective experience. The distinction between neural correlates of consciousness, neural prerequisites, and consequences of conscious processing is consequently important. Koch and colleagues review these issues and distinguish full, content-specific, and background contributions in the study of neural correlates [23].

Consciousness theories also differ in the kind of explanatory relation they propose. Higher-order approaches commonly emphasize representational relations among mental states; global-workspace accounts emphasize access and broad availability of information; recurrent and predictive-processing approaches give special roles to particular interactions and inferential organizations; and integrated information theory places substantial weight on intrinsic causal organization [36]. These descriptions identify research programmes and do not exhaust their internal variants. In particular, integrated information theory advances a substantive proposal connecting phenomenological axioms to physical cause–effect structure [38]. The four mathematical hypotheses of the present paper occupy a different level of classification: they ask whether a candidate consciousness-relevant property is assigned to a transformation mechanism, a selected relational history, the coupling of generative structures, or an admissible repertoire. A global-workspace model or an integrated-information model could be examined under more than one of these orientations depending on its formalization. The taxonomy therefore crosses, rather than replaces, existing neuroscientific taxonomies.

The importance of separating a theory from a selected mathematical implementation is illustrated by recent adversarial testing. A preregistered comparison of integrated information theory and global neuronal workspace theory examined divergent predictions using multiple neural recording modalities. Its results aligned with some predictions while challenging important predictions of both approaches [12]. Such an outcome is not an endorsement of any one of the four orientations proposed here. It illustrates the need to identify the specific observable, experimental contrast, and auxiliary assumptions through which a formal construct acquires empirical content. A field, a homology group, or a path space can

become scientifically informative only after the proposed link to a discriminating observation has been formulated.

Dynamical models of neural computation. A dynamical approach often represents collective neural activity as a configuration in a state space. Its coordinates may denote measured firing rates, latent factors inferred from data, or additional variables of a mechanistic model. Reviews of neural population dynamics describe applications to movement, timing, decision-making, and working memory [40]. Studies of neural manifolds in motor control likewise investigate low-dimensional organization within high-dimensional population activity [16]. These results make geometry and dynamical systems effective empirical methods, while the experimental tasks studied in the cited work do not establish a geometric law of consciousness. The term *neural manifold* may refer to an inferred low-dimensional family or approximation; the smooth-manifold assumptions required for differential geometry must be separately justified for the particular data and model.

One common representation of measured activity separates a latent system configuration from the observation procedure. For a model carrier X , an indexed family of configurations $x_\lambda \in X$, and a real vector space Y of measurements (with $\varepsilon_\lambda \in Y$), an observational model may take the form of Equation 3.

$$y_\lambda = h_c(x_\lambda) + \varepsilon_\lambda, \quad h_c : X \longrightarrow Y. \quad (3)$$

Here c indexes relevant recording or interpretive conditions, h_c is the measurement map, and ε_λ denotes a measurement or modeling residual. Equation 3 is a schematic observation model, not a claim that neural measurements possess a unique latent representation. In particular, different latent configurations may yield the same observations, while a change of measurement apparatus may change which distinctions are accessible. The parameter λ indexes the observations under a declared protocol; it need not be interpreted as an ontologically privileged time coordinate.

A further distinction concerns the generator and its reconstructed activity. Fixed points and local linearizations have been used to analyze trained recurrent neural-network dynamics [37]. These methods identify dynamical structure that may organize observed computational behavior, yet their mathematical success leaves open the relation between implemented computation and experience. In general, a finite set of measured paths under Equation 3 constrains a candidate generator only relative to a selected model class and noise assumptions. The generator need not be uniquely recoverable, and observed agreement does not imply agreement on counterfactual or unobserved configurations.

Physical and generative explanations. Theories of perception and action also employ generative models in substantive but distinct senses. The free-energy principle, for example, investigates perception, action, and learning using probabilistic generative models and variational formulations [15]. This supplies a relevant instance of a physical and mathematical modeling programme involving dynamics, constraints, and inferential transformations. It does not establish an equivalence between its internal generative models and the four hypotheses under investigation. The word *generative* may designate a probabilistic data model, a mechanism with admissible transitions, a rewriting procedure, a variational dynamics, or a process that alters the structures of its own representations. Those meanings share formal interfaces only when their carriers, transformations, compositions, constraints, and observation maps have been explicitly specified.

The physical interpretation of a neural dynamical model deserves particular care. A state manifold for neural recordings is mathematically distinguishable from the physical spacetime in which experimental apparatus and biological processes are situated. It is also distinguishable from a cellular complex used to represent a relational history. The existence of a useful manifold of neural states establishes the effectiveness of a representation under its empirical conditions; it does not show that conscious experience occupies that manifold as its unique ontological substrate. Likewise, the availability of physical field theories, including theories with nonlocal correlations, supplies neither an identified mechanism of consciousness nor evidence for such a mechanism in the absence of a specified physical model and cor-

responding tests. The present paper leaves these questions open while treating neuroscientific findings as constraints on admissible interpretations.

2.2 Geometric and Topological Dynamics

Manifolds and laws of variation. Differential geometry supplies a language for transformations that are locally differentiable. Let X be a smooth manifold and f a smooth vector field on X . The relation between f and a selected integral curve γ is expressed by Equation 4.

$$\frac{d\gamma}{d\lambda}(\lambda) = f(\gamma(\lambda)), \quad \gamma : I \longrightarrow X. \quad (4)$$

In Equation 4, f assigns tangent vectors to configurations, while γ is one path satisfying the local differential prescription. Under regularity conditions, a local flow may be constructed; completeness and a global group action require additional assumptions [27]. A field is therefore not identical to any one integral curve, and an integral curve is not equivalent to the collection of admissible integral curves. These distinctions motivate, respectively, the Variation, Trajectory, and Repertoire orientations. The Generativity orientation asks a further question about coupling among transformations, constraints, distinctions, and representational structures, including cases in which the space carrying those objects is itself subject to alteration.

A fixed tangent bundle TX already assumes a specified smooth manifold. In some models it is useful to allow the underlying space to depend on an index or context, to consider singularities, or to describe changes that are not naturally encoded as smooth motion inside one manifold. A family $\{X_a\}_{a \in A}$ of spaces and structure-preserving maps between selected members may then be more appropriate than a single globally fixed X . The index set A need not carry a total temporal order. A relation on indices can be supplied to describe admissibility, experimental comparison, causal structure, or a chosen continuation convention. Differential equations on a fixed manifold and constructions among varying spaces consequently address different model classes; their relationship is a mathematical problem rather than a terminological identity.

CW-complexes and two-dimensional cellular structure. A CW-complex organizes topological data by attaching cells of different dimensions. For a two-dimensional CW-complex K , its zero-, one-, and two-dimensional skeleta are related by attaching maps of the form given in Equation 5.

$$K^{(0)} \subseteq K^{(1)} \subseteq K^{(2)} = K, \quad K^{(2)} = K^{(1)} \cup_{\{\varphi_j\}} \coprod_j D_j^2, \quad \varphi_j : S_j^1 \longrightarrow K^{(1)}. \quad (5)$$

The construction in Equation 5 records the attachment of two-cells through maps from their boundary circles; it does not impose a metric or smooth local coordinates on the resulting complex. In the relational interpretation adopted in the author's future-creation work, vertices may represent *events*, edges may represent *couplings*, and faces may represent higher-order primitives whose semantics require domain-specific specification. An individual, a cognitive system, or another entity is not thereby identified with a primitive vertex. Such an entity may require an extended, historically constituted configuration of cells and relations. Moreover, an oriented one-cell does not, by its orientation alone, assert physical temporal precedence, and the existence of a face does not establish a causal interaction unless its semantics supply that meaning.

The CW-complex framework is more general than a simplicial complex. A two-cell may be attached along a loop whose topology and incidence encode information that would be obscured by a particular triangulation if attaching data or provenance were discarded. Its geometric realization is a topological space, but a general two-complex need not be a two-dimensional manifold: edges may meet in nonmanifold configurations, attachments may identify regions, and the local neighborhood of a vertex need not resemble an open disk. Standard references on cellular complexes and homotopy supply the mathematical foundations for these distinctions [19]. The difference matters in consciousness research whenever a

Table 2: Mathematical representations and the data required for their intended interpretation.

Representation	Mathematically available structures	Further commitments requiring specification
Smooth manifold with field	Tangent spaces, local flows, differential invariants	State variables, differentiability, parameter semantics, physical realization
CW- or two-complex	Cells, incidence, attaching maps, homology and homotopy	Meaning of cells, dynamics, ordering, provenance, observational correspondence
Structured path space	Families of paths, admissibility, optional topology or measure	Which paths count, boundary conditions, observability, equivalence of parameterizations

relational structure is selected for its incidence or attachment properties, while a manifold is selected for its differentiability and local state-space geometry. Choosing one representation does not automatically justify the structures required by the other.

The relevance of higher-order structure is not restricted to a speculative ontology. Higher-order network research investigates systems in which interactions involving groups cannot be represented faithfully by pairwise edges alone; hypergraphs, simplicial complexes, and related constructions support different kinds of higher-order dynamics [3]. Such work provides mathematical motivation for examining cell- and face-level representations. However, a higher-order interaction is not in general synonymous with a two-dimensional CW face, and a social or neuronal interaction cannot be identified with a quantum spin foam solely on the basis of a similar combinatorial vocabulary. A specific model must state what its cells mean and whether its dynamics are defined by actions, amplitudes, transition operators, constraints, or some other domain-appropriate construction.

Topological invariants and their scope. For a selected two-complex K , the cellular chain groups and boundary maps allow calculation of homology. In dimension one, the relevant group is written in Equation 6.

$$H_1(K; \mathbb{Z}) = \ker(\partial_1 : C_1(K; \mathbb{Z}) \rightarrow C_0(K; \mathbb{Z})) / \text{im}(\partial_2 : C_2(K; \mathbb{Z}) \rightarrow C_1(K; \mathbb{Z})). \quad (6)$$

Equation 6 describes a topological invariant of the chosen complex. It is useful for detecting certain cycles and attachment effects, but it does not determine metric distances, detailed attaching-map homotopy classes, parameterized dynamical laws, or phenomenal contents. Distinct complexes may have isomorphic homology while differing in their fundamental groups or other invariants. An inference about consciousness based on homology therefore requires a declared bridge from a topological property to the specified conscious phenomenon. The presence of a nontrivial cycle by itself supplies no such bridge.

Topological methods have been used to examine neural correlations. Giusti and colleagues constructed clique-based topological summaries and studied their relationship to geometric organization in neural correlation data [17]. A clique complex induced by a correlation graph is a data-derived object; the presence of a simplex in that complex should not be equated with a physical multiway interaction or with an episode of consciousness. The observation operator, graph-construction rule, and thresholds affect the resulting topology. Such examples help identify a general principle: geometry and topology can expose patterns in measured relations, while the physical realization and experiential interpretation of those patterns depend on further models.

Table 2 contrasts three representative mathematical settings. The listed commitments are conditional; several settings can be used together when appropriate mappings are provided.

Paths and spaces of admissible processes. A path in a topological space is a continuous map from a specified parameter interval into that space. A dynamical solution is a path subject to an additional evolution law. A cellular history may instead be modeled through a sequence of cells, a directed incidence relation, or a morphism into a path or history category. These objects require different preservation

criteria. In particular, equal path images need not preserve orientation, multiplicity of traversal, parameterization, or physical labels. A repertoire of admissible processes may also be given extra structure, including compact-open or other topologies when technically appropriate, probability measures under explicit stochastic hypotheses, or boundary data derived from a physical problem.

The distinction between a prescribed generator and its repertoires can be sharpened through a schematic observation. Let $X = \mathbb{R}^2$ and consider the vector fields in Equation 7.

$$f_1(x, y) = (1, 0), \quad f_2(x, y) = (1, y), \quad \gamma(\lambda) = (\lambda, 0). \quad (7)$$

On the stated path, both fields in Equation 7 satisfy Equation 4. Off the line $y = 0$, however, their vertical components differ. A selected path thus underdetermines the generator, even in this elementary smooth setting. The example does not include measurement noise and therefore establishes a logical distinction independently of statistical estimation problems. Its mathematical role is to caution against treating an observed trajectory as a complete account of generative variation or as evidence that two generative repertoires coincide. Section 14 will develop corresponding examples in cellular and heterogeneous representations.

2.3 Historical Generativity and Relational Representations

The abstract roles of change, structure, constraint, and difference. The author's prior metamodel, *A Metamodel for Ontologically Heterogeneous Social Dynamics, Part I: The Highest Abstract Layer*, begins with a bare carrier S , a semigroup of mechanisms M , and an equivalence relation $\sim$ respected by the action [22]. The construction separates change by action, structure by composition, constraint by congruence, and difference by the collapse of distinctions. The relevant algebraic conditions and derived collapse relation are collected in Equation 8.

$$\begin{aligned} \alpha(mn, x) &= \alpha(m, \alpha(n, x)), \\ x \sim y &\implies \alpha(m, x) \sim \alpha(m, y), \\ \Delta(m) &= \{(x, y) \in S^2 : x \not\sim y, \alpha(m, x) \sim \alpha(m, y)\}. \end{aligned} \quad (8)$$

Under the assumptions in Equation 8, a distinction lost modulo $\sim$ by one transformation remains collapsed under further composition by mechanisms respecting that congruence. The author's metamodel derives this result, together with an induced-quotient injectivity criterion and a monotonicity property for Δ . These conclusions concern the specified algebraic system and the chosen congruence; they do not establish thermodynamic entropy production, universal temporal succession, or physical loss of information at every observational scale. This distinction is particularly important because a reversible microscopic model may induce a noninvertible coarse-grained description after an information-forgetting observation or quotient has been selected.

The fixed congruence in the metamodel determines what distinctions the formalism preserves or collapses. A differential-geometric variation, by contrast, may describe a tangent perturbation without identifying previously distinguishable states. Consequently the *Difference* primitive of Equation 8 must not be conflated with the word *variation* in the paper's first hypothesis. Their relationships can be investigated through additional maps, but the two concepts are mathematically different. The metamodel also explicitly postpones a formal account of novelty and endogenous changes of the congruence. Extending it to representations whose structures or distinguishing relations vary across models is therefore a research task, rather than a result already contained in its four axioms.

Relational histories and observer dependence. The historical constitution of a represented system can be studied through a collection of observations, records, and structural maps without assuming that the collection carries a universally valid ordering. A set of events may receive a causal partial order only when the model supplies and warrants such a relation; another inquiry may use laboratory measurement order, the inclusion of a subcomplex into a larger complex, or the compositional order of algebraic actions.

These orderings should be named separately because they preserve different information. A partial order may leave events incomparable; a parameterized curve supplies an interval order in its domain; and an embedding records attachment or inclusion while carrying no intrinsic assertion of temporal precedence. Neuroscientific discussions of internally constructed representations of space and time give additional reason to distinguish measured coordinates from neural representations of those coordinates [6].

One can represent an epistemic relation between two descriptions by maps $q_A : Z \rightarrow A$ and $q_B : Z \rightarrow B$ from a common underlying modeled configuration space Z , when such a Z is part of the stipulated model. A comparison between A and B can then ask which distinctions each map preserves. In models of heterogeneous reality there may be no warranted common carrier Z ; a span, correspondence, or family of context-dependent maps may be the appropriate replacement. Such objects permit analysis of representational translation without postulating that neural state coordinates, relational cells, and phenomenological reports inhabit one universally measured space. Their use does not resolve the independent epistemological question of how an observer comes to recognize and delimit a relevant structure.

Change of representation and preservation of distinctions. A dynamical correspondence between two semigroup-action descriptions can be specified by a carrier map $h : S_A \rightarrow S_B$ and a mechanism homomorphism $\phi : M_A \rightarrow M_B$. A common exact preservation condition is given by Equation 9.

$$h(\alpha_A(m, x)) = \alpha_B(\phi(m), h(x)). \quad (9)$$

Equation 9 is an intertwining or action-equivariance condition. If h is noninjective, it can preserve the specified transformation relation while identifying states that description A distinguishes. If a congruence is included, its preservation or reflection imposes additional conditions. The precise type and direction of the map also matter: an exact quotient factor, a simulation relation, an observational approximation, and an equivalence of categories express different strengths of correspondence. These distinctions will be developed in Sections 12 and 13. In particular, *translation cost* may refer to lost mathematical distinctions, lost empirically meaningful observables, diminished reconstruction capacity, or computational expense; these notions require separate definitions.

Geometric and cellular descriptions may consequently support complementary views of one represented process. A smooth path can sometimes be discretized into an incidence or transition structure, but the discretization may forget its metric, curvature, speed relative to a declared parameter, or local stability data. Conversely, endowing a complex with an embedding in a smooth manifold may introduce additional choices whose physical interpretation is neither unique nor guaranteed. A generative grammar can likewise be interpreted as a rewriting mechanism with derivation histories and an extensional language of outputs, provided its operational semantics have been defined. This supplies a secondary mathematical analogy: an internal generator, a chosen derivation, and an output repertoire are distinct, even when the same formal apparatus relates them. The central constructions in the present investigation remain geometric, topological, and physical; grammar provides one example of cross-representation comparison rather than the organizing foundation.

2.4 Scope of the Present Investigation

Four levels of inferential commitment. The scientific and mathematical background supports a hierarchy of increasingly substantive inferences. At the first level, a formal system is defined through specified carriers, fields, actions, cells, paths, or admissibility relations. At the second, mathematical consequences are proved relative to those definitions, such as invariance under a prescribed morphism or the nonuniqueness exhibited in Equation 7. At the third, the mathematical model is connected to observations, including the choice of measurement function, boundary conditions, uncertainty, and reproducible data analyses. At the fourth, an explanatory relation is advanced between identified physical or relational properties and a specified aspect of subjective experience. An inference established at one

Table 3: Primary modeling objects and additional evidential requirements of the four hypotheses.

Orientation	Formal object selected for comparison	Additional questions for consciousness research
Variation	Fields, transformation actions, admissible local changes	Which properties of transformations are measurable and discriminating?
Trajectory	Selected smooth, cellular, or relationally ordered paths	Which path structures and observation conventions matter?
Generativity	Coupled change, structure, constraint, difference, and their revisions	Which couplings are represented and which revisions are observable?
Repertoire	Structured families of configurations or admissible histories	Which boundary conditions, topologies, and sampling assumptions define the family?

level does not automatically discharge the requirements of the subsequent levels. A computational experiment may verify a proposition about a simulated generator, while a neural experiment may identify a correlate, and both may leave the phenomenal bridge principle unresolved.

Table 3 records the immediate obligations associated with each of the four hypotheses. It serves as a research map rather than a summary of empirical evidence in their favor.

Ontology, time, and epistemic reach. An independent physical theory can motivate a specific neural or non-neural model of consciousness, but the present taxonomy does not stipulate the truth of such a theory. It distinguishes the ontology of a physical process, the relational description through which it may be represented, and the manifested or reported phenomena through which it is investigated. Those levels may be related through proposed maps or explanatory principles, and the validity of any particular correspondence remains subject to argument. Similarly, a possible path is defined relative to a model and constraints; *possible* does not mean a future event waiting to occur. A selected path may be recognized from an observer-relative record without establishing a global division between past and future. The history language employed in the paper will therefore be indexed to declared mathematical or epistemic structures wherever the distinction affects an argument.

Tasks for the formal hypotheses. The four hypotheses must be assessed through their mathematical and empirical commitments, including cases in which different orientations coincide on selected observations but diverge on their generative or topological organization. The distinction between a transformation and its path, the distinction between a path and a repertoire, and the distinction between a fixed system and a coupled transformation of its structures have each been illustrated in this section. Sections 3–6 will convert these distinctions into explicit modeling conditions and taxonomy criteria. Sections 7–10 will investigate their separate implications, while Sections 11–14 will develop geometric, topological, algebraic, and representation-theoretic comparisons. The immediate result of the present background is a disciplined research setting: mathematical objects and their preservation properties can be specified with precision, empirical relevance requires independent modeling and measurement, and the constitution of subjective experience remains an explanatory problem to which the proposed taxonomy is addressed.

3 Ontological and Epistemic Foundations

The four consciousness hypotheses require a specification of what their mathematical objects represent, what observations can support claims about them, and which assumptions are introduced when their structures are interpreted as histories or as possible transformations. This section establishes those conditions before formal dynamical models are selected. Section 3.1 separates the conditional ontology of a modeled process from epistemic and manifestational descriptions. Section 3.2 introduces observer-relative

representations, observation-induced equivalence, and structural identifiability. Section 3.3 examines modality, indexing, and relational orders without assigning fundamental status to a privileged present or a global temporal direction. Section 3.4 develops the conditions under which identity or invariant properties can be transferred between representations. The constructions are methodological proposals for a comparative mathematical investigation. They supply neither an empirical mechanism of consciousness nor a proof that conscious experience is constituted by a field, path, complex, or repertoire.

3.1 Ontological Suspension and Physical Interpretation

A conditional physical target. A theory of consciousness may seek an association, realization relation, constitutive explanation, or identity between conscious phenomena and a class of physical processes. These explanatory commitments differ. An experimentally established dependence of reportability on a neural intervention supports a claim about the intervention and the measured phenomenon; a claim that a particular physical configuration constitutes phenomenal experience requires additional interpretation and argument. The study of neural correlates provides powerful empirical constraints on candidate descriptions, while the transition from correlation to explanation remains a distinct task [23, 36]. The present paper permits physical candidates of differing extent, including neural, bodily, environmentally coupled, or other processes for which a separately specified scientific theory supplies appropriate evidence. Mentioning a candidate physical process does not assign it a demonstrated role in the constitution of experience.

A topological or geometrical representation introduces its own conditions. A manifold supplies local charts and transition functions; a vector field assigns tangent vectors under differentiability assumptions; a two-complex supplies cells and attaching information. None of these conditions, considered alone, identifies the constituents of consciousness. The proposed framework therefore treats the physical substrate as an open explanatory variable while allowing individual examples to assume particular carriers. Such suspension is an investigative policy: it permits several theories to be compared through explicit mathematical consequences without asserting that their common notation identifies one ultimate reality. Philosophical discussions of structural realism distinguish epistemic commitment to representable structure from the stronger ontic thesis that relational structure constitutes the underlying world; the positions are not interchangeable [25]. The investigation developed here adopts neither position as a consequence of the mathematical constructions.

Three registers of reality. In the broader generative-relational programme, a useful conceptual distinction concerns *ontological reality*, *relational reality*, and *manifestational reality*. Their status in the present paper is explanatory rather than that of three independently established physical layers. Ontological reality denotes the open question of what obtains and how a candidate process is constituted. Relational reality denotes the historically and contextually constituted relations through which configurations, interactions, and possible transformations become effective or mutually significant. Manifestational reality denotes the phenomena accessible through specific modalities of experience, observation, expression, or interpretation. For consciousness research, an observed neural measurement, a reported experience, and a postulated physical mechanism may be related by a model without thereby becoming identical descriptions of one mathematical object.

One may stipulate, for a particular physical hypothesis, an underlying carrier Z , an epistemic representation $r_c : Z \rightarrow X_c$, and an observation map $o_c : X_c \rightarrow Y_c$. The index c specifies the observational context, modeling conventions, and admissible records. The resulting composite model is shown in Equation 10.

$$Z \xrightarrow{r_c} X_c \xrightarrow{o_c} Y_c, \quad O_c = o_c \circ r_c. \quad (10)$$

Equation 10 is conditional on the supplied Z and maps. It is neither a mandatory decomposition of the three reality registers nor a claim that an observer has access to Z . Some models may warrant only partial maps, relations, or a span of descriptions; some may leave the constitution of Z wholly unspecified. The epistemic state X_c may likewise depend on historically developed instruments, conceptual distinctions,

and capacities of the observer. When Z is a spacetime in a particular physical theory, X_c may be a reduced dynamical state manifold or a cellular representation; these mathematical carriers need not themselves be physical spacetimes.

Relations to subjective experience. The term *consciousness* may refer to occurrence of experience, specified phenomenal qualities, access to information, report, or discriminable behavioral capacities. A candidate physical interpretation must identify the targeted property. In the present paper a hypothetical bridge relation may be represented as $\mathcal{B}_c \subseteq X_c \times \text{Phen}_c$, where Phen_c is a stipulated domain of phenomenological descriptions or reports relative to context c . The relation is introduced only to mark an explanatory obligation. Choosing Phen_c does not establish that descriptions exhaust experience; constructing $\mathcal{B}_c$ does not establish phenomenal sufficiency. The hypotheses of Variation, Trajectory, Generativity, and Repertoire select different mathematical inputs for possible bridge relations, while leaving their validity open.

3.2 Epistemic Representation and Observability

The observer-relative data interface. Scientific observations are obtained under protocols with determinate sensitivities and interpretive constraints. A state-space reconstruction from neural population recordings, for example, depends upon channel selection, preprocessing, dimensionality reduction, and assumptions connecting measurements to modeled configurations. The empirical utility of neural population dynamics is well supported for several computational tasks, while interpretation of an inferred state coordinate as a conscious property requires additional evidence [40, 16]. The choice of a manifold, CW-complex, or relational configuration space may alter which distinctions can be reconstructed. Observer dependence here means dependence of the representation and its evidential warrant on an observational arrangement. It does not entail that the recorded physical process is created by observation or that different measurements are equally adequate.

Let X_c be the configuration carrier of a model and let $O_c : X_c \rightarrow Y_c$ be a deterministic observation map for fixed protocol c . Equation 11 defines an observational equivalence relation.

$$x \equiv_{O_c} x' \iff O_c(x) = O_c(x'), \quad x, x' \in X_c. \quad (11)$$

Equality in Equation 11 is equality of records under the selected model, rather than identity of the configurations represented by those records. The relation is reflexive, symmetric, and transitive. Its classes describe the distinctions that the observation map is unable to resolve. For noisy observations an alternative definition may compare the conditional data distributions $P_c(\cdot | x)$ and $P_c(\cdot | x')$; approximate similarity instead requires a justified divergence, threshold, or test procedure. The deterministic case provides a transparent basis for the initial analysis.

For models compared through experiments, let $\mathcal{M}_c$ be a candidate family and $\mathcal{U}_c$ a specified collection of admissible experimental conditions or interventions. Suppose $\mathcal{L}_c(M, u)$ is the predicted data law under model M and condition u . We define *experimental equivalence relative to that design* in Equation 12.

$$M \equiv_{\mathcal{U}_c} M' \iff \mathcal{L}_c(M, u) = \mathcal{L}_c(M', u) \quad \text{for each } u \in \mathcal{U}_c. \quad (12)$$

Equation 12 is explicitly design-relative. Equality under a finite set of experimental conditions may coexist with different predictions under other conditions. Even equality of selected observable distributions does not entail equality of unmeasured internal mechanisms or equivalence of subjective experience. Increasing the range of $\mathcal{U}_c$ may refine the model classes, although the ability to do so depends on available interventions and their ethical and physical admissibility.

Structural properties under observational abstraction. The question of what an observation permits one to reconstruct can be stated independently of the physical nature of the carrier. Let $I : X_c \rightarrow V$ denote a proposed structural property, with codomain V chosen solely for this analysis. A property is

identifiable through O_c when its value can be assigned from the observed record without distinguishing states that the observation map identifies. Proposition 3.1 gives the exact condition.

Proposition 3.1 (Observation-relative identifiability). *Let $O : X \rightarrow Y$ and $I : X \rightarrow V$ be maps of sets. There exists a map $\tilde{I} : O(X) \rightarrow V$ satisfying $I = \tilde{I} \circ O$, with the codomain of O restricted to its image, if and only if I is constant on every fiber of O .*

Proof. If $I = \tilde{I} \circ O$, then $O(x) = O(x')$ implies $I(x) = \tilde{I}(O(x)) = \tilde{I}(O(x')) = I(x')$. Conversely, if I is constant on each fiber, define $\tilde{I}(y) = I(x)$ for any x with $O(x) = y$. Constancy makes this definition independent of the representative and gives the claimed factorization. $\square$

The proposition is a standard set-theoretic factorization observation, used here to delimit what may be inferred from a representation. It is neither a new theory of measurement nor a proof that a purported consciousness invariant is empirically identifiable. An invariant under a dynamical symmetry, a Betti number of a complex, and a norm assigned to a physical field may behave differently under a particular observation map. Each requires its own factorization argument. If a candidate consciousness-relevant property fails this condition, claims about its uniqueness cannot follow from that observation channel alone.

Epistemic histories and revision. Representations may themselves be constituted through an accumulation and reinterpretation of records. Let c and c' denote different epistemic contexts. A revision may replace X_c with $X_{c'}$, modify the observation relation, change admissible evidence, or change which distinctions are available. A map $q_{c,c'} : X_c \rightarrow X_{c'}$ is one possible representation of a revision, provided it is warranted by the underlying descriptions. Such a map need not be injective, surjective, invertible, or available at all. Moreover, indexing c and c' does not, by itself, encode their temporal succession. A research setting may separately impose a revision relation on contexts and must then specify how that relation is justified. The earlier metamodel's fixed congruence gives a well-defined account of distinctions collapsed under prescribed mechanisms [22]; allowing observation relations and congruences to vary is a further formal problem requiring compatible comparisons between contexts.

3.3 Modality, Temporality, and Relational Ordering

Model-relative possibility. A repertoire of admissible histories is defined relative to a formal system and its conditions. Admissibility is a modal property of that system; it does not designate a process whose occurrence is temporally postponed. Likewise, a selected trajectory may be studied through a mathematical parameter that need not represent fundamental physical time. To keep these commitments visible, let J be an index set equipped, when the application requires one, with a relation $\prec_J$, let X be a carrier, and let $\mathcal{A}_c$ be a predicate on candidate maps $\gamma : J \rightarrow X$. Equation 13 defines the model-relative repertoire.

$$\mathcal{H}_c(X, J, \mathcal{A}_c) = \{\gamma : J \rightarrow X \mid \mathcal{A}_c(\gamma)\}. \quad (13)$$

The predicate $\mathcal{A}_c$ may express a differential equation, a cellular attachment condition, a transition relation, conservation requirements, empirical boundary information, or a combination of constraints. Each possible member of Equation 13 is admissible *relative to* that predicate. A choice or observation of a member requires additional data; it cannot be inferred from membership alone. A complete history set may also be unattainable epistemically, in which case the representation supplies only a subset, outer bound, sampling family, or approximation.

Distinct notions of order. The following distinctions are central to the physical interpretation of all four hypotheses. A causal order records influence or precedence within a specific causal model. An index order organizes mathematical arguments of a curve or diagram. A laboratory ordering describes a recording protocol. A compositional order specifies how operations combine. A cellular embedding records inclusion and attachment of complexes. These structures may be related through maps when a model warrants the relation, while none is automatically identical to an ontological past–present–future

division. In particular, $K \hookrightarrow L$ expresses a cellular inclusion; the embedding alone carries no temporal orientation. Similarly, a one-parameter semigroup can represent a directed operation family whose mathematical direction does not, without additional assumptions, establish a universal physical arrow of time.

When J is a real interval, a model may introduce an orientation and a differential operator with respect to a parameter λ . If X is a smooth manifold and f a smooth vector field, an integral curve satisfies Equation 4. That equation acquires a physical interpretation only after the parameter, relevant measurements, and association with the modeled system have been specified. In quantum, relativistic, or coarse-grained contexts, global ordering assumptions may require additional justification. The present investigation accordingly uses the expressions *trajectory*, *history*, and *continuation* in a model-relative sense. The words *earlier*, *later*, and *future* are employed only when a declared ordering and an observer-relative or physical interpretation are available.

Trajectories through changing carriers. A fixed state manifold is a substantive modeling choice. For a representation whose carrier varies over an indexing base J , a family of spaces X_j may be assembled, when possible, into a projection $\pi : E \rightarrow J$ with fiber $\pi^{-1}(j) = X_j$. A selected configuration across that family is then a section satisfying Equation 14.

$$\gamma : J \longrightarrow E, \quad \pi \circ \gamma = \text{id}_J. \quad (14)$$

Equation 14 describes a path-like selection through model-indexed carriers without identifying their points in a single universal state space. A fiber-bundle interpretation requires locally trivial structure; general evolving cellular complexes may be organized only as families, diagrams, or more general maps, particularly when homeomorphism type changes. A theory of transformation across fibers then requires comparison maps or relations and admissibility conditions. Thus, the existence of a family or projection alone supplies neither a dynamical law nor a physically fundamental time axis. The mathematical issue is especially relevant to the Generativity and Repertoire Hypotheses because the space of candidate configurations and its constraints may be among the objects represented as variable.

Irreversibility at different descriptive levels. The author's metamodel proves preservation of collapsed distinctions under further composition when mechanisms respect a fixed congruence [22]. The relevant loss is loss of distinguishability in the quotient specified by that congruence. Such algebraic irreversibility differs from microscopic reversibility, thermodynamic irreversibility, a directed experimental protocol, and the subjective phenomenology of temporal passage. A reversible physical model can admit a noninvertible quotient description when observation forgets distinctions. Conversely, changing the congruence may change which distinctions can be made at the epistemic level, requiring care before claiming restoration or permanent destruction. The scope of each irreversibility claim must therefore be stated along with its carrier, transformations, and preserved relations.

3.4 Representation-Dependent Identity and Invariance

Several meanings of sameness. Questions of equivalence arise whenever one physical or relational hypothesis receives multiple mathematical descriptions. Equality of observed records, bijective correspondence of state carriers, topological homeomorphism, homotopy equivalence, equivalence of dynamical actions, and equivalence of modal repertoires impose different conditions. A general CW-complex may share homology groups with a manifold while possessing different local structure or fundamental group; likewise, two vector fields may agree on a selected integral curve while assigning different dynamics elsewhere. Topological and geometric invariants help locate such distinctions but do not independently determine subjective experience [19, 27]. The three reality registers introduced in Section 3.1 add an interpretive caution: ontological identity, relationally effective structural equivalence, and manifestational equivalence under an observer are different claims. A criterion adopted for one register has to be justified separately if it is applied to another.

Congruence-aware dynamical correspondences. Consider two action models $(S_A, M_A, \alpha_A, \sim_A)$ and $(S_B, M_B, \alpha_B, \sim_B)$. A pair (h, ϕ) consisting of a carrier map $h : S_A \rightarrow S_B$ and a semigroup homomorphism $\phi : M_A \rightarrow M_B$ is equivariant if it satisfies Equation 9. To preserve distinguishability in both directions, one may additionally demand Equation 15.

$$x \sim_A x' \iff h(x) \sim_B h(x') \quad (x, x' \in S_A). \quad (15)$$

Equation 15 includes both preservation and reflection of the congruence on the image of h . If h is not surjective, the relation says nothing about configurations outside that image. If (h, ϕ) comprises isomorphisms and respects the actions, the induced correspondence preserves the metamodel's collapse relation as expressed by Equation 16.

$$\Delta_B(\phi(m)) = (h \times h)[\Delta_A(m)]. \quad (16)$$

Equation 16 follows directly by substitution in the definition of Δ ; its reverse inclusion uses surjectivity of h and the inverse correspondences. A merely equivariant noninjective factor map can identify distinctions that its source retains, so the displayed equality is not a general property of representation translations. Nor does preservation of the metamodel collapse operator establish equivalence of consciousness: it concerns the selected algebraic notion of distinguishability, whose physical and phenomenological meaning is separately specified.

An epistemic preservation criterion. The factorization criterion in Proposition 3.1 and the correspondence condition in Equation 16 address complementary questions. The first asks whether a proposed property can be recovered from a particular observation. The second asks whether an explicitly chosen kind of structural equivalence preserves a derived property. Both require specifying the property of interest. Thus, a topological invariant of a trajectory space, an observation-based discriminability class, or a geometric feature of a vector field can be carried into a second representation only under a map that preserves the relevant construction. An equivalence of outputs from two generative models, for example, may fail to preserve their derivation multiplicities, hidden states, admissible interventions, or constraints. In physics, equally useful coordinate representations may be related by transformations that preserve specified observables; whether their apparently different degrees of freedom are physically distinct is an interpretive question to be settled within the relevant theory.

Consequences for the consciousness taxonomy. The four hypotheses should be formulated as proposed connections between conscious phenomena and explicitly declared classes of mathematical properties. For Variation, the candidate may concern generators, transformations, or admissible local variations. For Trajectory, it may concern the geometry or topology of selected paths and their relational ordering. For Generativity, the candidate may concern the coupled organization of changes, constraints, structures, and distinctions, including model-relative revisions. For Repertoire, it may concern an admissible-history space with specified topology, measure, or other structure. The possibility that an invariant of one class is determined by another should be investigated through formal comparisons; their ontological equivalence does not follow from a shared notation. Sections 4–6 provide the common metamodel, representation types, and taxonomy; Sections 7–14 develop separate constructions and equivalence tests.

The immediate outcome of the present section is a limited but operational foundation. Candidate consciousness properties are related to data through context-specified maps or relations. Modal repertoires and realized selections are distinguished without assigning either a fundamental temporal location. Mathematical identities are assessed according to their explicitly preserved structures, and physical or phenomenological identities remain separate explanatory questions. Within those conditions, geometric, topological, and algebraic constructions can yield precise consequences while retaining the possibility that the appropriate substrate, level of description, and explanatory bridge remain unsettled.

4 A Minimal Metamodel for Generative Dynamics

The four proposed consciousness hypotheses require a common language in which a mechanism of change, a selected history, a coupled generative organization, and a family of admissible histories can be distinguished without imposing a common metric or differential structure upon their carriers. The author's earlier metamodel of ontologically heterogeneous social dynamics supplies a deliberately minimal starting point [22]. Its formal results concern actions on a carrier with a fixed relation of distinguishability. The present section reconstructs that formal layer, gives proofs of its relevant consequences, and identifies the additional assumptions required before the construction can be used in physics or consciousness research. Sections 4.1–4.4 treat the four original primitives of Change, Structure, Constraint, and Difference; Section 4.5 then examines model-relative irreversibility, observation, and the limits of this abstraction. The metamodel is used here as an algebraic comparison device. No claim that these axioms characterize physical consciousness follows from its mathematical consequences.

4.1 Change and Transformation Mechanisms

Carrier and action. The highest abstract layer begins with a set S of configurations and a set M of transformation mechanisms. At this layer, the elements of S receive no metric, order, vector-space structure, chosen coordinates, or privileged zero. The original definition also includes an equivalence relation on S , which is made explicit in Section 4.3. A transformation mechanism acts through the map in Equation 17.

$$\alpha : M \times S \longrightarrow S, \quad (m, x) \longmapsto \alpha(m, x). \quad (17)$$

Equation 17 supplies the *Change* primitive of the source metamodel. It specifies that a modeled mechanism assigns a configuration to each input configuration; neither the physical nature of the mechanism nor an interpretation of the configuration is fixed by the notation. Different domains may interpret S as neural configurations, discrete relational states, admissible cellular assignments, or other modeled configurations, provided the claimed action is well defined on that carrier. The same symbols should not be interpreted as establishing an equivalence among these domain-specific carriers.

The choice of an endomap is already a substantive limit. An action on a fixed carrier represents changes among elements of S ; it does not, without additional machinery, represent the emergence of a new carrier or of a configuration lying outside S . In particular, assigning every admissible configuration to a sufficiently encompassing set at the outset is mathematically possible in many examples, but may hide the epistemic question of how that set is constituted and revised. The status of admissible configurations remains a model-relative assumption.

Change and differential equations. For a smooth manifold X and a sufficiently regular vector field $f \in \Gamma(TX)$, an integral curve has the familiar form of Equation 18.

$$\frac{d\gamma}{d\lambda}(\lambda) = f(\gamma(\lambda)), \quad \gamma : J \longrightarrow X. \quad (18)$$

The vector field in Equation 18 is a specialization of a transformation-generating description, subject to manifold, differentiability, existence, and domain conditions. Under appropriate well-posedness conditions, it gives local flow maps that may instantiate actions on a suitable carrier. The original map α does not require such a generator, or even a continuously indexed family of transformations. Conversely, a differential equation is not automatically an instance of the metamodel until its mechanism set, composition, carrier, and selected congruence have been specified. The index λ acquires an interpretation as a physical time coordinate only through separately stated assumptions; its appearance in a derivative is a property of the mathematical representation.

For the Variation Hypothesis, the candidate consciousness-relevant object might be a property of α , a subset of its mechanisms, or a differential operator such as f in a justified smooth instantiation. This proposal concerns the *organization of possible transformations*; its distinction from a selected solution

γ is mathematical and does not assert a temporal priority of possibility over realization. The existence of such a transformation description is shared by many systems for which consciousness has not been established, making any sufficiency claim a further and highly demanding problem.

4.2 Structure and Composition

Associativity of mechanisms. The *Structure* primitive in the source metamodel assigns an associative product to the mechanisms, making M a semigroup whose product is respected by the action. These assumptions appear in Equation 19.

$$\begin{aligned} (m_1 m_2) m_3 &= m_1 (m_2 m_3), \\ \alpha(m_1 m_2, x) &= \alpha(m_1, \alpha(m_2, x)). \end{aligned} \tag{19}$$

By convention, $m_1 m_2$ in Equation 19 acts as m_2 followed by m_1 . A semigroup need not be supplied with an identity or inverses, although particular semigroups may possess them. The earlier manuscript deliberately does not *assume* either structure. Even when M happens to possess a group identity e , the composition equation alone does not force $\alpha(e, x) = x$; this additional *unitality* axiom is required when recovering states by inverse mechanisms. Associativity licenses algebraic regrouping of composite mechanisms. It does not by itself provide a partial order on events, a metric duration, an orientation of physical spacetime, or a global distinction between past and future. A model seeking those interpretations must establish how compositional structure corresponds to the relevant empirical or physical relations.

Composition is especially useful for comparing descriptions of generative processes. When an action is defined, a selected composite transformation may be expressed by one element of M , whereas the interpretation of its internal stages can require an additional representation of factorizations or provenance. The equality of two composite actions on the carrier need not determine whether they have the same factorization or physical implementation. Even faithful knowledge of an input-output function may consequently leave the constituent process underdetermined.

Quotient-level dynamics. An equivalence relation preserved by the action, introduced below, permits the mechanisms to act on the quotient set $S/\sim$. The quotient records classes of configurations under specified distinctions; its mathematical existence does not establish that its classes are physically elementary or phenomenologically identical. A quotient may be computationally helpful precisely because it forgets differences that another account regards as relevant. The relations between transformations and such quotient dynamics will be made precise in Section 4.3.

4.3 Constraints and Congruence

Preservation of identified configurations. The metamodel equips S with an equivalence relation $\sim$ and requires it to be preserved by each mechanism. Equation 20 gives this action congruence condition.

$$x \sim y \implies \alpha(m, x) \sim \alpha(m, y) \quad (m \in M). \tag{20}$$

Under Equation 20, a mechanism respects the distinctions *already identified as equivalent* by the model. The source manuscript interprets $\sim$ as expressing what mechanisms are obliged to preserve, and assigns this modeling choice considerable normative significance [22]. Formally, the condition ensures preservation of equivalence; it permits distinct equivalence classes to become equivalent after transformation. Determining which observed or ethical distinctions the relation encodes is a separate act of interpretation. Thus, the relation is part of the representation's declared structure, not a mathematically discovered universal norm.

The induced action on the quotient is the map in Equation 21.

$$\bar{\alpha}_m : S/\sim \longrightarrow S/\sim, \quad \bar{\alpha}_m([x]) = [\alpha(m, x)]. \tag{21}$$

The congruence condition in Equation 20 makes Equation 21 well defined: choosing a different representative of the same class gives the same transformed class. The quotient action retains the semigroup composition law. Neither construction requires metric geometry, a probability distribution, or a continuum parameter. The minimality is valuable for heterogeneous representations, though it also limits the empirical predictions that can be extracted from the model alone.

Observation-relative congruences. A measurement or interpretation map $O_c : S \rightarrow Y_c$ defines an observational equivalence whenever configurations producing the same output are declared equivalent. Equation 22 specifies this relation.

$$x \sim_c y \iff O_c(x) = O_c(y). \quad (22)$$

The observational equivalence of Equation 22 is a congruence for the chosen action only if the action preserves its classes. That property cannot be inferred solely from the existence of an observation map. If the property holds, the observation factors the modeled action through an induced transformation of its output image. If it fails, an output at one configuration does not uniquely determine the output after transformation, which is often precisely the information hidden by a coarse-grained observation. This observation-dependent construction parallels the epistemic factorization criterion in Proposition 3.1, while placing an additional requirement on the evolution of observation classes.

For consciousness research, the interpretation of $\sim_c$ depends upon O_c . Report, behavior, physiological recording, and phenomenological description induce different potential equivalence classes. Equal measured outputs provide a mathematical equivalence under the declared protocol. They do not, without an additional explanatory bridge, constitute a proof of equal conscious experiences.

4.4 Difference and the Collapse of Distinctions

The derived relation. The *Difference* primitive of the source metamodel is a relation on pairs of configurations. For each $m \in M$, its collapse relation is defined by Equation 23.

$$\Delta(m) = \{(x, y) \in S \times S : x \not\sim y \wedge \alpha(m, x) \sim \alpha(m, y)\}. \quad (23)$$

An ordered pair belongs to Equation 23 when the configurations are distinguishable under the stipulated congruence and their images under the selected mechanism are equivalent. The definition identifies a *lost discrimination* in the quotient representation; it specifies neither a distance between states nor a magnitude of material loss. Since Δ is defined from α and $\sim$, it adds a named diagnostic construction without adding an independent structural degree of freedom to the given action model. The source manuscript presents Difference as its fourth axiom, with the derived nature of the operator stated explicitly [22].

Proposition 4.1 (Persistence of collapsed distinctions). *For a semigroup action satisfying Equation 20, if $(x, y) \in \Delta(m)$, then $\alpha(nm, x) \sim \alpha(nm, y)$ for every $n \in M$.*

Proof. The definition of $\Delta(m)$ gives $\alpha(m, x) \sim \alpha(m, y)$. Applying Equation 20 with n preserves that equivalence. The action law in Equation 19 identifies the two resulting configurations with $\alpha(nm, x)$ and $\alpha(nm, y)$. $\square$

Proposition 4.1 establishes persistence of a collapsed discrimination under further *compositions of the specified action*. It does not assert that the physical configurations have become identical. In particular, the conclusion is relative to one carrier, a fixed congruence, and the modeled composite mechanisms. Interpreting the algebraic composition as chronological succession would introduce an additional assumption.

Proposition 4.2 (Injectivity criterion). *For any $m \in M$, the relation $\Delta(m)$ is empty if and only if the induced quotient map $\bar{\alpha}_m$ in Equation 21 is injective.*

Table 4: Iterated transformations on a three-configuration carrier, with equality as the congruence. The semigroup index is an algebraic iteration count without an imposed physical time interpretation.

Configuration	x	$\alpha(1, x) = q(x)$	$\alpha(2, x) = q^2(x)$
a	a	a	a
b	b	a	a
c	c	b	a

Proof. The induced map fails to be injective precisely when two distinct quotient classes $[x] \neq [y]$ have equal images $[\alpha(m, x)] = [\alpha(m, y)]$. By the definition of the quotient, these conditions are equivalent to $x \not\sim y$ and $\alpha(m, x) \sim \alpha(m, y)$. Together they are equivalent to $(x, y) \in \Delta(m)$. $\square$

Proposition 4.2 characterizes loss of the chosen discriminations without a numerical information measure. An injective transformation on $S/\sim$ can change configurations substantially while leaving $\Delta(m)$ empty. The absence of collapse therefore cannot be equated with absence of change or of a modeled history.

Proposition 4.3 (Accumulation under composition). *For every $m, n \in M$, the collapse relations satisfy Equation 24.*

$$\Delta(m) \subseteq \Delta(nm). \quad (24)$$

Proof. Membership in $\Delta(m)$ fixes the initial nonequivalence $x \not\sim y$. Proposition 4.1 establishes equivalence of the images under nm . These two conditions give membership in $\Delta(nm)$. $\square$

Equation 24 defines monotonicity relative to a right-divisibility or composition relation among mechanisms. It does not require numerical monotonicity, increasing thermodynamic entropy, or a fundamental ordering of physical events.

Proposition 4.4 (Unital group actions preserve quotient distinctions). *Let M be a group with identity e , and suppose its action on S satisfies the semigroup composition law together with $\alpha(e, x) = x$ for every $x \in S$. If the action preserves $\sim$, then $\Delta(m) = \emptyset$ for every $m \in M$.*

Proof. Suppose $\alpha(m, x) \sim \alpha(m, y)$. Applying the inverse group element m^{-1} preserves equivalence, and the composition law and stipulated unitality give $\alpha(m^{-1}, \alpha(m, x)) = \alpha(e, x) = x$ and similarly for y . Thus $x \sim y$, and there are no pairs satisfying Equation 23. $\square$

The unitality qualification is essential under the manuscript's minimal semigroup-action definition. For a counterexample without it, let $M = \{e\}$ be the one-element group, take $S = \{0, 1\}$ with equality as its congruence, and set $\alpha(e, x) = 0$ for both x . The composition law and congruence condition hold, but $\Delta(e) = \{(0, 1), (1, 0)\}$; the element e does not act as the identity map. Proposition 4.4 applies to the defined collapse operator. It does not assert that a reversible physical model lacks trajectories, histories, or other forms of dynamics. Its role is to distinguish the persistence of *quotient-level information loss* from more general concepts of physical or phenomenal irreversibility. Under the fixed-congruence assumptions, genuine quotient collapse requires a noninjective induced action. Invertibility of the induced action rules such collapse out.

A finite semigroup example. To make the definitions concrete, let $S = \{a, b, c\}$, take equality as $\sim$, and define an endomap q by $q(a) = a$, $q(b) = a$, and $q(c) = b$. Let $M = (\mathbb{N}_{>0}, +)$ act through positive iterates of q . The example is summarized in Table 4. Table 4 exhibits two different collapse patterns. The first iterate identifies a and b while keeping c distinguishable from them; the second identifies all three configurations. The latter collapse relation strictly contains the former. This example has no state-space metric and supports the results of Propositions 4.1–4.3 directly. It remains an algebraic example; its configurations are not identified with conscious states, and its three elements carry no phenomenological interpretation.

Observational formulation of loss. An observation map offers another expression for the collapse relation. If the equivalence is induced by O_c as in Equation 22 and satisfies the action congruence condition, a pair belongs to $\Delta(m)$ precisely when its observed outputs are different under the selected protocol and the outputs of its transformed configurations coincide. The resulting statement is given by Equation 25.

$$\Delta_c(m) = \{(x, y) : O_c(x) \neq O_c(y), O_c(\alpha(m, x)) = O_c(\alpha(m, y))\}. \quad (25)$$

Equation 25 makes the dependence of reported loss on observational resolution explicit. Choosing another justified map $O_{c'}$ may generate a different collapse relation. It would then be necessary to specify a correspondence between the two observational contexts before asserting that a distinction was recovered, destroyed, or preserved across representations. Such comparisons are outside the fixed-congruence axioms.

4.5 Scope of the Abstract Metamodel

Formal achievements and what they leave open. The preceding propositions are consequences of the source metamodel, reconstructed in the present manuscript from the author's supplied paper. They establish statements about composition, induced quotient maps, and the loss of selected distinctions. The axioms do not provide a measure of variation, a smooth field, a topological invariant, a causally oriented physical spacetime, a probability law, or an account of conscious experience. Nor do they specify a physically privileged equivalence relation. Each of these objects requires additional assumptions about the modeled domain and the evidence through which its structures may be warranted.

The source metamodel explicitly identifies two further limits relevant to the planned consciousness taxonomy. First, its action on a fixed carrier does not express the emergence of an element whose type lies outside that carrier. Second, its equivalence relation is stipulated at the outset, even when the capacity to make distinctions is itself historically constituted [22]. These limits are particularly consequential for the Generativity Hypothesis. A model may need to represent changes in transformation rules, admissibility conditions, carriers, or distinguishing relations, possibly without a common set of states. A diagram of partially related models, a dependent family of carriers, or a category of typed dynamical systems may offer alternative extension strategies. The suitability of these strategies must be investigated rather than assumed.

Representations with variable structure. As a preliminary formal schema, consider a family of action models indexed by a set or category C . A possible indexed object is written in Equation 26.

$$\mathfrak{D}_c = (S_c, M_c, \alpha_c, \sim_c), \quad c \in C. \quad (26)$$

Equation 26 supplies a notation for variability among model structures, not an evolution law relating them. A map or relation between $\mathfrak{D}_c$ and $\mathfrak{D}_d$ requires additional compatibility conditions among their carriers, mechanisms, and congruences. A specified direction in C might describe inferential refinement, computational abstraction, a physical process, or another relation. The mathematical index carries no intrinsic declaration of temporal succession. If the relation between contexts is partial, the indexed family need not possess global maps that identify all configurations across contexts.

In the language of geometry and topology, a particular S_c may be represented by points of a manifold, states supported on a CW-complex, or configurations of a two-complex. The geometric realization of a CW-complex, its homology, and the cellular incidence structure are different levels of description. Continuous maps between geometric realizations may fail to preserve cell labels, relational provenance, or the dynamical action. Conversely, abstract action isomorphisms may preserve quotient-level collapse while carrying no meaningful spatial or topological interpretation. The representational interface must therefore specify the invariant or observation of interest. Section 5 develops these carriers explicitly, and Sections 11–14 examine their structural comparisons and limitations.

Four hypotheses and four distinct formal commitments. The metamodel supports the proposed classification by identifying questions that attach to different mathematical objects. The Variation Hypothesis directs attention to properties of the transformation organization, including field-like specializations. The Trajectory Hypothesis directs attention to selected paths or histories whose admissibility is defined relative to a model; a transformation action alone does not specify such a path without a selection and an indexing or ordering structure. The Generativity Hypothesis directs attention to coupled changes in the transformations, constraints, structures, and discriminations themselves; the fixed-layer metamodel motivates that question but does not solve it. The Repertoire Hypothesis directs attention to structured families of admissible paths or histories; constructing such a family requires a choice of admissibility predicate and often topology, measure, or boundary information. These are investigative orientations whose mutual implications, dependence on representation, and consciousness relevance remain open.

Interpretive boundary for the consciousness problem. A transformation action and its quotient properties can be reproduced in purely symbolic automata, engineered controllers, and abstract cellular models. An isolated instance of algebraic collapse or noninvertibility consequently offers no sufficient criterion for subjective experience. The empirical question requires operational definitions of the conscious phenomena under investigation and warranted associations with candidate physical or relational mechanisms. The ontological question requires additional argument about the constitution of experience. The mathematical question concerns the invariant properties, inferential limitations, and translation behavior of the proposed representations. Keeping these inquiries distinct permits the four hypotheses to be examined for substantive differences without assigning consciousness to every system admitting a semigroup action.

The resulting minimal foundation is therefore a collection of precise structural resources: an action that represents change, an associative law that specifies composition, a congruence that determines model-relative identifications, and a derived collapse relation that records selected distinctions rendered indiscernible. Their formal consequences are exact under their premises, and their interpretive force depends upon the declared carrier, observational conditions, and preservation criteria. This bounded starting point allows Sections 5 and 6 to introduce geometric and topological models and to construct a taxonomy whose hypotheses remain distinguishable without presupposing a unique physical substrate or a fundamental past–present–future partition.

5 Mathematical Representations of Relational Dynamics

The minimal algebraic metamodel of Section 4 supplies an action, a composition rule, a congruence, and a derived notion of collapsed distinctions. Its carrier has no stipulated metric, topology, differentiable structure, or cell decomposition. The present section develops several mathematically distinct ways to enrich such a carrier for the study of transformation mechanisms, selected histories, and families of admissible configurations. The purpose is to establish a controlled comparison among smooth manifolds, cellular spaces, differential prescriptions, and spaces of paths, while making explicit the commitments introduced by each representation. The argument proceeds from smooth state spaces to CW-complexes and relational two-complexes, then to equations and transition relations, to trajectory repertoires, and finally to mappings between these descriptions. Each representation supports different kinds of inference. None supplies, merely through its mathematical construction, an identification of its states or invariants with subjective experience.

5.1 Manifolds and Geometric State Spaces

Smooth carriers and local structure. A smooth d -manifold X is locally modeled on open subsets of $\mathbb{R}^d$ through an atlas with smooth transition maps. Its points may be physical configurations, parameterized neural population states, collective variables, or abstract positions in a theoretical state space. These possibilities have different evidential and ontological commitments. In particular, a manifold describing

statistical coordinates inferred from recordings is distinct from physical spacetime. The smooth structure licenses tangent vectors, derivatives, and differential forms, but does not itself select a metric, a causal orientation, or a physical temporal parameter [27].

The tangent bundle TX assembles the tangent spaces T_xX at all points $x \in X$. A smooth vector field is a section of this bundle. Equation 27 records its local mathematical type.

$$f \in \Gamma(TX), \quad f(x) \in T_xX \quad (x \in X). \quad (27)$$

A Riemannian metric g would additionally provide an inner product on each tangent space and make lengths and angles available; a symplectic form would furnish a different structure with different dynamical implications. Neither is implied by Equation 27. A change of coordinates alters the components of f according to the tangent-map rule without replacing the geometric field itself. This coordinate independence makes differential geometry attractive for comparing models with different measured or latent coordinates. It does not establish that a coordinate-free description is empirically unique: the state manifold, the embedding of observations, and the modeled variables still require justification.

Subspaces, constraints, and observability. Physical constraints may be represented by an embedded submanifold $Q \hookrightarrow X$, by a distribution of admissible tangent directions, or by an equation defining a constraint surface. For an ordinary vector field to generate curves remaining in an embedded constraint manifold, a relevant local condition is tangency along Q . Equation 28 states that condition.

$$f(x) \in T_xQ \quad \text{for each } x \in Q. \quad (28)$$

Under the usual regularity and local existence–uniqueness assumptions, the condition in Equation 28 makes Q invariant under the corresponding local solutions. A singular constraint set may not admit tangent spaces of constant dimension; stratified or nonsmooth formulations may then be needed. Constraints can also act upon which configurations are experimentally accessible without being intrinsic equations of motion. These differences matter when relating a generative model to an observer’s restricted sample of neural dynamics.

A point in a modeled neural state manifold may collect the readings of an instrument or coefficients of a latent dynamical reconstruction. The coordinate assignment is itself a modeling operation. If $O : X \rightarrow Y$ is an observation map, the same observation may be compatible with several points of X , and a lower-dimensional observed manifold need not inherit a closed dynamical law. The question of whether a closed law survives projection is addressed formally in Section 5.5. For the proposed consciousness taxonomy, this is relevant because a transformation-level property inferred from observed trajectories may fail to be identifiable, even when those trajectories admit a compact geometric visualization.

Geometry of the carrier and geometry of its transformations. A state manifold and the spaces of maps acting upon it are distinct mathematical objects. For example, a family of diffeomorphisms of X may have a group structure, while its selected subfamily of admissible transformations may form only a semigroup. An integral curve may be represented by a map from an interval into X ; its image may have much lower dimension than X . The topology of X , the geometry of a particular curve, and the algebra of mechanisms that act on X therefore require separate definitions. In physical modeling, these structures may be linked by differential equations, constitutive principles, or observation protocols. The Variation, Trajectory, and Repertoire orientations are intended to isolate different parts of this linked construction, and the Generativity orientation investigates the conditions under which the linking structures themselves can vary.

5.2 CW-Complexes, Two-Complexes, and Cellular Realizations

Cells and attaching maps. A CW-complex builds a topological space by adjoining disks of various dimensions through continuous maps from their boundaries. For a two-dimensional CW-complex K , the

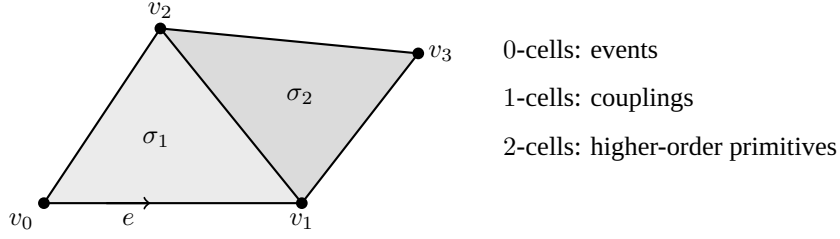


Figure 1: One simplicial example of a relational two-complex. Two faces share an edge, and all labels express a proposed model interpretation. A general CW two-complex admits broader attaching maps; no manifold or physical-time interpretation is inferred from the drawing.

successive skeleta are a set of vertices, a graph-like one-skeleton, and attached two-cells. Its construction is described in Equation 29.

$$K^{(0)} \subseteq K^{(1)} \subseteq K^{(2)} = K, \quad K^{(2)} = K^{(1)} \cup_{\{\varphi_a\}} \coprod_{a \in A_2} D_a^2, \quad \varphi_a : S_a^1 \longrightarrow K^{(1)}. \quad (29)$$

The attachment notation in Equation 29 designates the relevant quotient construction: boundary points of each disk are identified with their images under the corresponding attaching map. Different attaching maps on the same one-skeleton may yield different topological spaces. The dimensions label cell types; they do not impose an ordering of physical events or imply that a two-cell is a surface embedded in an ambient physical spacetime. Cellular homotopy and homology are standard tools for distinguishing such constructions [19].

A finite simplicial complex of dimension two gives a special case with triangular faces and combinatorially restricted attachments. The CW category also admits non-simplicial cells, loops formed from a single one-cell, and attaching maps that traverse the one-skeleton more than once. A general two-complex can have branching, nonmanifold neighborhoods, and singular incidence configurations. Its geometric realization $|K|$ is a topological space; a smooth structure on $|K|$ requires additional hypotheses and may fail to exist globally. Accordingly, a two-complex used to represent relational history and a manifold used to represent dynamical state variables need not be interconvertible by a structure-preserving isomorphism.

Relational interpretation and the status of faces. Within the present GR-inspired illustrative interpretation, vertices represent events, oriented one-cells represent couplings, and two-cells represent higher-order primitives whose physical or relational semantics are supplied by the particular model. Events are not stipulated to be individual people, neurons, subjects, or self-contained entities. A represented agent or observer may instead be associated with a historically constituted subcomplex or another relational organization; this proposal alone does not explain how such an organization realizes consciousness. Figure 1 provides a schematic simplicial instance of a two-complex and its kinds of cells.

The diagram in Figure 1 does not prescribe the semantics of a face. A two-cell may encode a compositional coupling, a constraint, a higher-order relational primitive, or another model-defined feature. This caution matters when borrowing terminology from spin-foam or higher-order network theories: a shared two-complex syntax establishes a possible structural comparison, not an identity among physical mechanisms, neural processes, or quantum amplitudes. Higher-order network research likewise distinguishes several mathematical devices for multiway interaction [3]. The object represented by a face must be justified separately in any empirical instantiation.

Cellular chains and attachment-sensitive invariants. For an oriented finite CW two-complex, the cellular chain groups $C_q(K; \mathbb{Z})$ are free abelian groups on the q -cells. The cellular differentials and

homology are given in Equation 30.

$$C_2(K; \mathbb{Z}) \xrightarrow{\partial_2} C_1(K; \mathbb{Z}) \xrightarrow{\partial_1} C_0(K; \mathbb{Z}), \quad H_1(K; \mathbb{Z}) = \frac{\ker \partial_1}{\text{im } \partial_2}. \quad (30)$$

The map ∂_2 records signed cellular degrees associated with face attachments. If K has one vertex, one loop one-cell, and one two-cell attached along a map of degree n , then $\partial_1 = 0$ and ∂_2 is multiplication by n . For a nonzero integer n , the resulting computation is stated in Equation 31.

$$C_2 \cong \mathbb{Z} \xrightarrow{\times n} C_1 \cong \mathbb{Z} \xrightarrow{0} C_0 \cong \mathbb{Z}, \quad H_1(K_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}, \quad H_2(K_n; \mathbb{Z}) = 0. \quad (31)$$

For $n = 1$, the first homology is trivial; for $n = 2$, it is $\mathbb{Z}/2\mathbb{Z}$. The calculation in Equation 31 shows how distinct face attachments alter a topological invariant while leaving the counts of cells unchanged. Homology does not, in general, recover the original attaching maps, the names or provenance of the cells, or the chosen dynamics. Two complexes can have isomorphic homology groups while differing in their fundamental groups or other homotopical structure. A model of conscious experience based on a homology statistic would therefore require an independent argument explaining which distinctions remain relevant after this coarse reduction.

Cellular change and relational provenance. A map of cellular complexes can preserve some attachments while identifying cells or merging distinct historical structures. Continuous maps induce homomorphisms on homology, while a cellular embedding may preserve a specified subcomplex together with its incidence relations. Such a map need not preserve cell labels, the domain-specific meaning of a face, or the relation between a structural event and its observational record. Conversely, the choice to regard one complex as a subcomplex of another does not by itself make the first a physical past or the second a physically later state. Topology-changing constructions such as adjoining cells, making quotients, or forming pushouts can be given interpretations of historical continuation only when an additional ordering, causal structure, or epistemic protocol is specified. These observations motivate the use of typed relational maps, provenance data, and explicit interpretation conditions in later sections of the paper.

5.3 Vector Fields, Differential Equations, and Evolution Relations

Evolution prescriptions and their domains. A smooth vector field supplies a local prescription for an integral curve, given a chosen parameter domain. An autonomous ordinary differential equation can be written as Equation 32.

$$\frac{d\gamma}{d\lambda} = f(\gamma(\lambda)), \quad \gamma : J \longrightarrow X, \quad J \subseteq \mathbb{R}. \quad (32)$$

For a suitable regular field, Equation 32 has unique local solutions with specified initial configurations. A complete vector field generates a globally defined one-parameter group of diffeomorphisms; models involving partial domains or one-sided admissibility can yield local flows or semiflows [27]. The relation between λ and physical time, and the extent to which solutions describe realized physical processes, are additional modeling commitments. In particular, the existence of a reversible group action does not imply the absence of histories in any general sense; it only precludes collapse of a fixed congruence under the conditions established by the source metamodel.

Set-valued and relational prescriptions. A deterministic field is one possible specialization. A differential inclusion may assign a collection of admissible tangent vectors to a configuration, as written in Equation 33.

$$\frac{d\gamma}{d\lambda}(\lambda) \in F(\gamma(\lambda)), \quad F(x) \subseteq T_x X. \quad (33)$$

The solution concept for Equation 33 depends on regularity and measurability conditions that have not been imposed here. The set $F(x)$ expresses a modeled space of admissible local changes; it is not a

collection of physically future events awaiting realization. On a bare carrier S , a transition relation $R \subseteq S \times S$ is another mathematically economical prescription. It can encode allowed pairs without a smooth structure. A specified collection of composable transitions may in turn form paths in a directed graph or a category of transformations, although those structures need explicit domains and composition laws.

Field equations and admissible sections. Partial differential equations support a different organization. A field u can be modeled as a section of a bundle $E \rightarrow B$, with derivatives described through an appropriate jet bundle. An equation of finite differential order can be represented schematically in Equation 34.

$$u \in \Gamma(E), \quad \mathcal{E}(j^k u) = 0, \quad \mathcal{E} : J^k E \longrightarrow W. \quad (34)$$

Here $j^k u$ denotes the k -jet of a local section, and W denotes a suitable target bundle over the base where the zero condition is defined. The formulation in Equation 34 needs further domain and regularity assumptions for analytical use. Boundary data, gauge choices, compatibility conditions, and solution spaces influence the resulting model. The base B can represent physical spacetime in a particular field theory, but it can also serve another geometrical role; its use does not establish an ultimate ontology of consciousness. A PDE's solution space and the transformations acting on that space can be represented through semigroups or categorical maps under appropriate conditions. A PDE, a state-space transition model, and a cellular rewriting rule are therefore potentially comparable at specified interfaces without being definitionally identical.

Changing geometric and cellular carriers. The fixed-carrier dynamics described so far cannot on its own represent a system in which the carrier or its topology varies. A preliminary extension assigns models to objects of an index category $\mathcal{C}$, as in Equation 35.

$$\mathcal{K} : \mathcal{C} \longrightarrow \mathbf{CW}_{\leq 2}, \quad c \longmapsto K_c, \quad (35)$$

where $\mathbf{CW}_{\leq 2}$ is a chosen category of CW-complexes of dimension at most two and suitable maps. Equation 35 is a possible representation of a diagram of cell complexes. It does not prescribe how a consciousness-bearing system would produce those complexes or whether morphisms in $\mathcal{C}$ signify causation, experimental refinement, historical attachment, or abstraction. A nontrivial cellular attachment can alter homology, whereas a homotopy equivalence preserves it. If objects have different carriers, comparison of their states and fields requires maps or correspondences between those carriers. The abstract change of an indexing object alone does not define a physical evolution.

A related smooth construction may involve a family of fibers X_c whose geometry depends on c , perhaps assembled into a bundle when the necessary local triviality conditions hold. Fiber bundles do not in general admit topology-changing fibers, since local triviality constrains their types; more flexible families, correspondences, or singular fibrations are needed for topology-changing cases. This is one point where manifold and two-complex methods make genuinely different commitments. The interpretation of a historically constituted geometry must also specify which relations between different fibers or complexes constitute meaningful identifications of configurations.

5.4 Transformations, Paths, and Structured Repertoires

Selected paths and their order structures. A path in a topological space X is a continuous map from a chosen interval, or more generally from a specified ordered or parameterized domain, into X . For the purposes of the proposed taxonomy, a *selected history* is a path or other relational configuration designated by evidence, a model, or a representational protocol. The word *history* describes that modeling role and does not declare an objective temporal boundary. The parameterization of a curve may encode duration, a laboratory sampling index, a causal surrogate, or a purely mathematical ordering. Identifying two paths that traverse the same geometric image in opposite directions requires an explicit choice about orientation and what is retained from the parameterization.

A model D may designate certain paths as admissible. For a fixed domain I and carrier X , a preliminary repertoire is the family in Equation 36.

$$\mathcal{H}_I(D) = \{\gamma \in \text{Map}(I, X) : \text{Adm}_D(\gamma)\}. \quad (36)$$

The predicate in Equation 36 may express differential equations, transition relations, boundary conditions, conservation laws, or another specified admissibility rule. Admissibility is therefore relative to D ; the repertoire does not automatically enumerate ontologically possible conscious experiences. Depending on I , X , and the regularity class, the ambient map space may be equipped with compact–open, uniform, differentiable, Sobolev, or other topologies. These choices distinguish notions of closeness among histories and affect possible geometrical analyses of the repertoire.

Homotopy of paths and the retention of history. Two paths with prescribed endpoints may be homotopic relative to those endpoints, even when their parameterized images and local dynamical organization differ. A homotopy forgets some distinctions relevant to a selected trajectory. For the circle, the space of continuous free loops admits a particularly transparent topological classification. The connected components satisfy Equation 37.

$$\pi_0(\text{Map}(S^1, S^1)) \cong [S^1, S^1] \cong \mathbb{Z}. \quad (37)$$

The integer in Equation 37 is the degree, or winding number, of a continuous loop. The classification applies to the full continuous mapping space with its usual compact–open topology; a specified vector field can select a much smaller class of admissible solutions. Winding classes reveal a robust topological distinction without characterizing the speed, local field, or experiential interpretation of any trajectory. This illustrates why a repertoire can carry meaningful topology independently of both a generator and any one path [19].

Four orientations within one mathematical example. Consider a vector field on the circle, written in angular coordinates as $f_+(\theta) = +1$, and another field $f_-(\theta) = -1$. Both define motions whose unparameterized complete orbits occupy the entire circle. Their directed integral curves and parameterized transformations differ. A record that retains only the occupied geometric orbit cannot distinguish the two models; a record that retains an oriented parameterized curve may distinguish them. The full admissible history families depend on the precise path domain, parameterization rule, and observational equivalence being used. The example illustrates four separable questions: which transformation prescriptions apply (Variation); which curve is selected (Trajectory); whether the prescription and its constraints or carrier change together (Generativity); and what family of curves or configurations is admitted (Repertoire).

An extension allowing the underlying topology to vary requires care about the word *whole*. A repertoire may be the whole family relative to one stipulated dynamical law and a fixed carrier, but a different model class or observational refinement may admit additional paths or alter the classifying space. No observer-independent exhaustive catalogue follows from Equation 36. An appropriate consciousness hypothesis would need to specify whether its candidate explanatory property concerns the set of histories, its topology, a probability structure on it, relations of counterfactual accessibility, or some other invariant.

5.5 Representation Bridges and Modeling Assumptions

Exact dynamical correspondence. A map between two represented systems may preserve their transformation laws while identifying configurations. Let (X, Φ) and (Y, Ψ) carry actions indexed by the same semigroup M . A continuous map $h : X \rightarrow Y$ is an intertwining map when Equation 38 holds.

$$h(\Phi_m(x)) = \Psi_m(h(x)) \quad (m \in M, x \in X). \quad (38)$$

The relation in Equation 38 can be regarded as a semiconjugacy or factor correspondence when the usual supplementary conditions, such as continuity and suitable surjectivity, are satisfied. A noninjective h

can satisfy it and still erase represented distinctions. For example, the constant map from S^1 to a point intertwines any continuous circle action with the unique action on a point. Yet the induced map on first homology is the zero map from $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ to $H_1(\{*\}; \mathbb{Z}) = 0$. Preservation of the algebraic action equation therefore does not establish preservation of spatial topology, winding information, or phenomenological meaning.

Projection of smooth generators. Whether a vector field can be translated to a lower-dimensional state representation has a sharper local criterion. The following standard projectability fact is stated here because it will later supply a controlled comparison between the Variation and Trajectory hypotheses.

Proposition 5.1 (Projectability of a vector field under a submersion). *Let $h : X \rightarrow Y$ be a smooth surjective submersion of smooth manifolds and let $f \in \Gamma(TX)$ be smooth. A smooth vector field $g \in \Gamma(TY)$ satisfying the relation in Equation 39 exists if and only if $dh_x(f(x))$ is constant on each fiber $h^{-1}(y)$ as a vector in $T_y Y$.*

$$dh_x(f(x)) = g(h(x)) \quad (x \in X). \quad (39)$$

Proof. If g exists, Equation 39 gives the required equality for any two points in one fiber. Conversely, if the projected tangent vector is constant on each fiber, define $g(y)$ to be that common vector. Smoothness follows locally from the existence of smooth sections of a surjective submersion: near a chosen y , compose f with a local section and apply the differential of h . The fiberwise condition makes the resulting local definitions agree on overlaps. $\square$

The proposition makes the cost of abstraction precise: even when a coarse variable $h(x)$ is well defined, it need not support an autonomous smooth generator. Consider $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $h(x, y) = x$, and $f(x, y) = (y, 1)$. Equation 40 exhibits the obstruction.

$$dh_{(x,y)}(f(x, y)) = y, \quad h^{-1}(x) = \{(x, y) : y \in \mathbb{R}\}. \quad (40)$$

The expression in Equation 40 varies along each fiber, so there is no autonomous $g(x)$ whose field exactly reproduces the projection of f on all configurations. A projected observed path $x(\lambda)$ may nevertheless exist and be fitted by an effective history-dependent or augmented-state model. This distinction separates preservation of selected trajectories from preservation of the generating law.

Cellular and geometric abstraction. The same discipline applies when comparing a smooth manifold with a CW model. A triangulation or cellular approximation, when available under justified hypotheses, may preserve selected topological properties while replacing smooth tangent data with incidence structures. A continuous map of CW realizations induces maps on homology, but the induced maps need not be isomorphisms. A cellular chain model may retain boundary matrices while omitting the full attaching homotopy classes; a quotient may identify previously distinguishable events; and a smooth discretization may preserve approximations to selected solutions while losing local derivatives or invariant measures. These are different notions of representational loss. Table 5 collects the main preservation questions required by subsequent comparisons. Table 5 underscores that *equivalent* has no unqualified mathematical meaning across heterogeneous representations. An action equivalence, a homotopy equivalence, a geometric isometry, a behavioral equivalence, and an observational equivalence preserve different properties. A functor between suitably defined categories can express the preservation of compositions and identities, while faithfulness or equivalence of categories expresses stronger properties under separate assumptions. These algebraic notions will be developed in Section 12. The present aim is to identify the carriers and interfaces that such a comparison would have to act upon.

Implications for the consciousness taxonomy. The manifold description facilitates questions about differentiable variation fields and local flows. The cellular description facilitates questions about relational attachments and topology-changing structures. A path-space description isolates selected histories

Table 5: Candidate representations and the structures whose preservation requires explicit verification. The listed losses describe possible failures, not universal consequences of each representation.

Representation or bridge	Explicitly modeled structure	Potentially omitted or altered information
Smooth state manifold	Local coordinates, tangent spaces, differentiable maps	Relational provenance, historical cell labels, measurement semantics
CW/two-complex	Cellular incidence, attachments, topological invariants	Smooth derivatives, metrics, domain-specific face semantics
Path or trajectory reconstruction	Selected ordered configurations and admissibility	Generator away from observed paths, counterfactual range
Homology reduction	Chain-level cycles, boundaries and quotient groups	Attaching homotopy, cell-level history, metric geometry
Semiconjugacy or factor	Intertwining of specified actions	States distinguished in the source, non-preserved invariants
Projected differential model	Coarse observables and, if projectable, closed field	Hidden-state dependence and fiberwise variations

from their ambient dynamical prescription, while a repertoire encodes the model-relative collection of admissible possibilities. The indexed-family description makes explicit a possible formal venue for the transformation of the carriers, constraints, and relations employed by a model. Consequently, assigning two systems the same observed trajectory, homology group, or action quotient may establish a restricted mathematical equivalence without establishing identity of their generative mechanisms or conscious experience.

These mathematical resources prepare the taxonomy in Section 6. Their common role is comparative rather than reductive. Physical and empirical work would have to identify plausible carriers, observations, regularity conditions, and discriminating predictions. Philosophical interpretation would have to address the relationship between the resulting structures and subjective experience. Neither task is discharged by the existence of a manifold, a two-complex, a differential equation, or a repertoire of solutions. The four hypotheses become useful to the extent that they isolate distinct candidate explanatory dependencies and enable formally controlled comparisons among them.

6 A Taxonomy of Four Consciousness Hypotheses

The present section establishes a comparative taxonomy for four preliminary hypotheses relating consciousness to different aspects of represented dynamical systems. Its objective is to specify the mathematical object emphasized by each hypothesis, identify the evidential and interpretive commitments needed to associate that object with experience, and delimit the comparisons that can be drawn prior to a physical instantiation. The classification uses the semigroup metamodel of Section 4 and the geometric, cellular, and dynamical representations developed in Section 5. The four hypotheses are introduced as investigative orientations; their detailed mathematical analyses occupy Sections 7–10. Relations among the hypotheses are considered after their individual objects have been distinguished.

A consciousness hypothesis must make at least two kinds of commitments. It selects a mathematical object, such as a field, a path, a family of heterogeneous dynamical structures, or an admissible-history space. It also proposes a relation between that object and an independently specified aspect of conscious phenomena. The first commitment can be studied through definitions, invariants, and formal examples. The second involves evidence, interpretation, and potentially unsettled questions concerning physical constitution. In particular, an algebraic, topological, or dynamical property is not by its mathematical existence a sufficient condition for subjective experience. The taxonomy helps locate candidate explanatory dependencies without assigning consciousness to generic mathematical systems.

Table 6: Four preliminary consciousness hypotheses classified by their principal mathematical objects and comparison criteria. The candidate dependencies are proposals requiring separate empirical and philosophical justification.

Hypothesis	Principal object	Candidate explanatory dependence	Relevant comparisons
Variation	Transformation prescription or local change organization	Organization of transformations and generative tendencies	Action intertwining; field correspondence; generator invariants
Trajectory	Selected path, flow segment, or relational history	Organization of a particular represented process	Path equivalence; ordering and parameter dependence; homotopy when defined
Generativity	Coupled family of change, structure, constraint, and difference	Relations among constituent capacities and their structural transformation	Diagrams of models; constraint revision; provenance and cross-model compatibility
Repertoire	Structured family of admissible histories or configurations	Organization of modeled possibilities taken collectively	Equivalence of history spaces; inclusion, accessibility, topology and measure

6.1 Criteria and Limits of the Classification

A classification based on dynamical systems can begin with the distinction among a transformation prescription, a selected path compatible with that prescription, the relational constitution and possible transformation of the prescription itself, and a family of admissible paths. Each object calls for different mathematical structures and preservation criteria. The same physical investigation may employ several objects simultaneously, and a proposed consciousness mechanism may concern more than one of them. Consequently, membership in the taxonomy is understood as explanatory emphasis rather than exclusive ontological membership.

For continuity with the metamodel, let $\mathfrak{D} = (S, M, \alpha, \sim)$ denote a fixed-carrier action model satisfying the conditions of Section 4. Additional choices, including a differentiable structure on S , an admissibility predicate, a representation map, and an observation protocol, are stipulated only where required. A selected path γ may be indexed by a structured domain I , while a family of heterogeneous models may be indexed by a set or category J . A schematic list of the respective objects appears in Equation 41.

$$\begin{aligned} V(\mathfrak{D}) &= (S, M, \alpha), & T(\mathfrak{D}; I, \gamma) &= (I, \gamma), \\ G(\mathbf{D}) &= (J, \{\mathfrak{D}_j\}_{j \in J}, \mathcal{R}_J), & R(\mathfrak{D}; c) &= (\mathcal{H}_{\mathfrak{D}, c}, \mathcal{Q}_{\mathfrak{D}, c}). \end{aligned} \quad (41)$$

The symbols in Equation 41 are *modeling roles*, not canonical functors or a complete common ontology. In V , the carrier is retained to type the transformation action. In T , admissibility of γ and the ordering or other structure of I remain additional assumptions. In G , $\mathcal{R}_J$ records whatever cross-model relations have been explicitly defined, without presupposing that every pair of models is comparable. In R , $\mathcal{H}_{\mathfrak{D}, c}$ denotes a model- and context-relative family of admissible histories, and $\mathcal{Q}_{\mathfrak{D}, c}$ denotes any further declared structure, such as a topology, a relation of accessibility, or a measure. The family of models used by G may possess changing carriers, constraints, or mechanisms; it is therefore richer than a single instance of the fixed-carrier metamodel.

The criteria organizing the four hypotheses are summarized in Table 6. The column describing candidate relevance to consciousness indicates a research question, not a demonstrated association. A hypothesis about consciousness must additionally specify which features of experience are at issue: reportability, wakefulness, perceptual content, phenomenal character, or another carefully distinguished target. Evidence associated with one target need not establish claims about another [36, 23].

Three restrictions apply across Table 6. First, a mathematically richer object does not automatically provide a better physical explanation. Second, the use of the word *generative* does not stipulate that consciousness constitutes a fundamental physical force or that a model of dynamics is an ontology of experience. Third, references to selected histories and possible continuations do not assume a universal

past–present–future partition. Selection, ordering, causal interpretation, and claims of realized occurrence are separately indexed to explicit models and evidential settings.

Explanatory strength and attribution. A proposed association between a mathematical object and consciousness could be interpreted as a correlation, an explanatory dependence, a realization condition, or an identity claim. These interpretations have different evidential burdens. Throughout the present taxonomy, the neutral expression *may be associated with* leaves their relationship open; subsequent formal propositions concern the mathematics of the candidate objects. The four headings consequently identify questions to investigate rather than conclusions about phenomenal sufficiency.

6.2 Variation: Transformational Organization

The Variation Hypothesis isolates the organization of change-producing mechanisms as a candidate object relevant to consciousness. It includes semigroup actions, transformation relations, and, when smooth structure is warranted, vector fields or families of admissible tangent directions. In the first case, a mechanism $m \in M$ acts according to $\alpha(m, \cdot)$; in the second, a vector field f assigns a tangent vector $f(x) \in T_x S$ to each state x of a smooth manifold. The object of interest is the transformation prescription in its declared domain, together with any structure needed to identify its invariants.

Hypothesis 6.1 (Variation). Specified properties of a system’s transformation organization may bear an explanatory relation to specified conscious phenomena, under an independently justified physical interpretation and observational protocol.

The statement is deliberately compatible with continuous and discrete models. In a smooth specialization, the local transformation prescription is represented by Equation 42.

$$f: S \longrightarrow TS, \quad \pi_S \circ f = \text{id}_S. \quad (42)$$

Equation 42 describes a section of the tangent bundle, provided S is a smooth manifold. A dynamical equation involving f introduces further assumptions about admissible parameterizations, solution concepts, and regularity. A vector field can specify local transformations even at configurations for which its assigned vector vanishes. This fact should be described as a property of the field, without treating a zero vector as change waiting to occur in a privileged future.

The word *variation* has established uses in the calculus of variations, where it may denote a perturbation δx of a configuration or path. The present hypothesis employs the broader sense of organization of transformations. A perturbation family, a vector field, and an observed derivative are distinguishable objects. Whether a variation-based physical model supplies an action functional, a constraint distribution, a stochastic transition rule, or another prescription must be stated at instantiation. Section 7 will examine these distinctions and their representation-dependent invariants.

A field-based consciousness proposal faces a challenge of specificity. Vector fields occur in many mathematical and physical systems, including idealizations for which no evidence of subjective experience is supplied. The hypothesis therefore requires some additional, discriminating property of the field or transformation organization, and a justified account of its connection to conscious phenomena. Neither smoothness nor the existence of a transformation law alone provides that property.

6.3 Trajectory: Selected Relational Histories

The Trajectory Hypothesis selects a path, orbit, relational history, or otherwise organized process as the candidate object of consciousness inquiry. Its relation to a generating model is expressed through an admissibility condition. The indexing domain may be a real interval, a discrete ordered set, a causal poset, or another structure suited to the intended description. A selected path is a mathematical object with a declared domain; whether the path corresponds to an experimentally recorded process depends on a separately specified reconstruction procedure.

Hypothesis 6.2 (Trajectory). Specified properties of a selected admissible trajectory or relational history may bear an explanatory relation to specified conscious phenomena, conditional on the meaning of selection, ordering, and observation in the model.

For a smooth example, let $I \subseteq \mathbb{R}$ be an interval and f a smooth vector field on a manifold S . The admissibility condition in Equation 43 is a chosen dynamical specialization.

$$\gamma: I \longrightarrow S, \quad \frac{d\gamma}{d\lambda}(\lambda) = f(\gamma(\lambda)). \quad (43)$$

The real parameter λ in Equation 43 specifies the differential equation and its solution domain. Interpreting λ as laboratory time, proper time, computational iteration, or an abstract ordering coordinate requires an additional premise. For changing carriers, a trajectory may instead be modeled as a compatible family of configurations in fibers of a bundle, a diagram of state spaces, or another indexed arrangement, subject to explicit gluing or transition conditions.

There are several levels of trajectory equivalence. Equality of images as subsets of S can ignore orientation and traversal. Equality up to reparameterization can preserve an ordered path while disregarding its particular parameter scale, but the admissible class of reparameterizations must be specified. Homotopy class can retain a topological distinction while forgetting substantial dynamical information. These comparisons provide mathematical tools for analyzing a trajectory hypothesis, and they are not interchangeable with claims of equivalence of conscious experience.

The Trajectory Hypothesis introduces a particular epistemic advantage and a particular inferential limit. Selected processes can sometimes be reconstructed from observational sequences, whereas a complete generating rule generally requires evidence extending beyond a single record. Yet the existence of an observational record alone does not identify the underlying trajectory without an observation and reconstruction model. Different transformation prescriptions may agree along one selected path while differing elsewhere. Thus, the inference from an identified trajectory to its generator is commonly underdetermined under restricted observations. Section 8 will investigate trajectory invariants and the limits of that reconstruction.

6.4 Generativity: Coupled Constitutive Structures

The Generativity Hypothesis takes the relational constitution of generation itself as the candidate explanatory object. It is informed by the four roles in the author's metamodel: Change, Structure, Constraint, and Difference [22]. Those roles are conceptually distinct: Change specifies an action, Structure constrains composition, Constraint specifies preserved equivalences, and Difference records distinctions collapsed under a mechanism. In the source metamodel the Difference operator is derived from the action and congruence; it is not an independent extra field or a synonym for unrestricted variability. The metamodel's fixed-carrier conditions do not themselves derive emergence of new carriers or endogenous revision of congruence. The present hypothesis investigates extensions in which constitutive relations among these roles can themselves be represented and, under additional assumptions, transformed.

Hypothesis 6.3 (Generativity). Specified properties of a coupled organization of change, compositional structure, constraints, and distinguishing relations, including compatible descriptions of their transformation, may bear an explanatory relation to specified conscious phenomena.

One preliminary extension uses an indexed collection of action models as written in Equation 44.

$$\mathbf{D} = (J, \{(S_j, M_j, \alpha_j, \sim_j)\}_{j \in J}, \mathcal{R}_J). \quad (44)$$

In Equation 44, J may have only set-theoretic structure, or it may carry explicitly declared morphisms; $\mathcal{R}_J$ represents proposed relationships across model instances. If S_j and S_k are different carriers, their elements require an explicitly constructed correspondence before statements about preservation or change can be made. If $\sim_j$ and $\sim_k$ differ, the quotient distinctions and collapse relations are model-dependent.

The ability to compare them is a separate problem, potentially involving spans, simulations, functors, or common observation spaces with stated limits. Merely indexing models does not supply an endogenous physical evolution or a semantics for the index.

The distinction between fixed-model action and change of the modeling conditions is central here. A trajectory under a fixed vector field can be described through successive points in one state manifold. A model in which a relational two-complex acquires an additional cell may require a map between distinct cellular carriers, together with attachment data and an account of which quantities are retained. A change in the modeled capacity to distinguish configurations can require revising the congruence rather than assigning another point in the original quotient. Treating these cases as equivalent without such mappings would conceal the problem the hypothesis seeks to investigate.

The Generativity Hypothesis can encompass relations between mechanisms and selected histories, but its defining emphasis lies in the *constitution and possible transformation of the conditions of generation*. In this sense it is broader than juxtaposing the Variation and Trajectory hypotheses. Its formal adequacy depends on identifying what counts as a constitutive component, which structural conditions bind the components, and what information a proposed cross-model transformation preserves. Section 9 will investigate families of manifolds, cellular structures, and typed dynamical models suitable for making those questions precise.

6.5 Repertoire: Structured Families of Admissible Histories

The Repertoire Hypothesis considers an admissible family of configurations, histories, or processes as a mathematical object in its own right. The family can be derived from a transformation model and a specified admissibility condition, and may carry geometric, topological, relational, or measure-theoretic structure. In contrast to a selected path, the focus lies on comparative and collective properties of the modeled possibilities. The claim that a repertoire is complete is always relative to the declared carrier, laws, admissibility assumptions, and epistemic context.

Hypothesis 6.4 (Repertoire). Specified properties of the structured repertoire of histories or configurations admissible under a declared model may bear an explanatory relation to specified conscious phenomena.

A minimal repertoire can be specified by an admissibility predicate, as in Equation 45.

$$\mathcal{H}_{\mathfrak{D},c} = \{\gamma \in \text{Map}(I, S) : \text{Adm}_{\mathfrak{D},c}(\gamma)\}. \quad (45)$$

The path domain I , carrier S , and context c in Equation 45 are fixed for the displayed instance. Where structures vary across an index, a repertoire can instead be assembled from a compatible class of sections, diagrams, or other typed histories. The symbol $\text{Map}(I, S)$ alone supplies no topology or probability distribution. Such structure may be added through explicit hypotheses and is potentially significant: two models can have the same set of admissible paths while differing in their probability laws, accessibility relations, or physical interpretation.

The distinction between generator and repertoire corresponds to the difference between an *intensional* description and an *extensional* family of admitted outcomes. A concrete illustrative instance occurs in generative grammar, where a grammar G is distinguished from its generated language $L(G)$. This analogy clarifies the comparison while occupying a secondary role in a physics-oriented inquiry. For a field model, the analogous objects are the transformation prescription and its specified class of solutions. For a two-complex model, a repertoire might consist of admissible cellular continuations with their attachment and incidence information. The construction must specify whether histories are identified up to cell isomorphism, homotopy, observational equivalence, or some other relation; each choice can change the object being classified.

A repertoire carries a distinctive epistemic burden. Observations of a finite selection of trajectories generally do not establish the totality of a model's admissible family, especially where the model itself is underdetermined. Its physically relevant content depends on the class of interventions, initial or

Table 7: Pairwise comparisons relevant to the four-hypothesis taxonomy. An equivalence claim requires explicit preservation conditions for the mathematical objects compared.

Comparison	Principal distinction	Required further specification
Variation / Trajectory	Prescription versus selected compatible path	Observation, solution concept, parameterization and reconstruction
Variation / Generativity	Fixed-model action versus constitutive coupling across models	Cross-model maps, changing constraints and structural provenance
Variation / Repertoire	Mechanism versus family of admissible processes	Boundary conditions, path equivalence, admissibility and weighting
Trajectory / Generativity	Selected history versus organization of generative conditions	How paths relate to changes in carriers, structures, and constraints
Trajectory / Repertoire	One selected history versus an indexed class of histories	Selection rules, modality, completeness and observation domain
Generativity / Repertoire	Constitutive organization versus modeled possibility family	Preservation of mechanisms, provenance, types and path-space structure

boundary configurations, and evidence used to constrain admissibility. Section 10 will compare path-space topology, reachability and accessibility structures, and representation-dependent restrictions on repertoire equivalence.

6.6 Overlap, Distinguishability, and Non-Exhaustiveness

The four hypotheses are neither an ontological partition nor a hierarchy of increasing explanatory adequacy. Their mathematical objects may be related through constructions with substantially different information-preservation properties. A generating model can define admissible histories; a trajectory can be a selected member of such a family; and a richer constitutive model can include mechanisms and records of selected histories among its components. These relations alone do not establish that the proposed consciousness-relevant properties coincide.

Comparisons among candidate objects. Variation and Trajectory are distinguishable because a law of transformation constrains a family of possible paths while a selected path ordinarily provides restricted information about that law. Variation and Repertoire distinguish a prescription from the admitted family; the relation between them depends on solution concepts, path identifications, and any further weighting of histories. Trajectory and Repertoire distinguish a selected member from a structured family; their difference may disappear under deliberately degenerate model assumptions, but such cases do not define the general taxonomy. Generativity permits relations among models whose constitutive structures themselves differ; it may incorporate the other three objects, although reducing an indexed model to one fixed state space can discard the provenance or type information needed to identify those relations.

Table 7 records the central distinctions and the corresponding requirements for any claimed equivalence or inference. It specifies research tasks rather than universal separation results.

Indistinguishability under restricted representation. The question of what can be inferred across these hypotheses can be stated in a modest form. Let $\mathcal{M}$ be a class of represented systems, let $O : \mathcal{M} \rightarrow \mathcal{E}$ record the evidence made available by a specified observation protocol, and let $P_i : \mathcal{M} \rightarrow \mathcal{Z}_i$ denote a candidate mathematical property for hypothesis i . Whenever P_i factors through O , observational agreement suffices for equality of that property. This fact is an instance of the observation-relative identifiability criterion proved in Section 3. The converse need not hold: mathematical equality of selected trajectories, or of a coarse repertoire, can coexist with different structures in the generating models. Such comparisons are meaningful only after their observation and equivalence relations have been declared.

Interfaces with geometry and physical interpretation. The taxonomy is compatible with different classes of mathematical carriers. A manifold may support a vector field and a path space; a CW-complex

or two-complex may encode attachments and higher-order relational structure; and a family of such carriers may model varying incidence or topology. Some comparisons preserve continuous geometric structure while losing cell provenance, and others preserve cellular homology while omitting local dynamics. The presence of an invariant in one representation does not assign it phenomenal significance. Experimental discrimination requires a well-defined target phenomenon, a physically motivated instantiation, and observations that distinguish the explanatory dependencies of competing hypotheses.

The classification is intentionally provisional and non-exhaustive. Phenomenal theories organized around integration, information, representation, embodiment, or other constructs can intersect several of its mathematical orientations, and their theoretical commitments cannot be inferred from a shared vocabulary alone. Likewise, a single physical system may admit several models of different mathematical kinds without their equivalence being established. The taxonomy is valuable insofar as it makes the explanatory target explicit, clarifies which structures are being compared, and identifies where additional mathematics or evidence is required. Sections 7–10 pursue the four hypotheses individually; Sections 11–14 develop their geometrical, topological, algebraic, and representational comparisons.

7 The Variation Hypothesis

The Variation Hypothesis concerns the organization of transformations attributed to a represented system. Its immediate mathematical object is a rule, relation, or field specifying how configurations can be transformed, together with the structures that give those transformations meaning. Section 6 distinguished this object from a selected trajectory, a coupled evolution of the constitutive conditions of a model, and the model’s full repertoire of admissible histories. The present section develops the first distinction through algebraic actions, differential geometry, constrained systems, and representation maps. It derives several elementary preservation and non-identifiability results, then states the additional physical and phenomenological requirements for a consciousness interpretation. The formal constructions are intended to clarify a hypothesis; they do not themselves identify a conscious physical substrate.

The term *variation* is adopted as a concise name for the explanatory focus of Hypothesis 6.1. In the present usage, it designates the *organization of transformations*, including their dependence on configuration and the domain in which they are defined. A variation in the sense of a perturbation of a path in the calculus of variations remains a related, more specialized object. A local vector assigned to a configuration, the rule assigning such vectors, and the derivative of a chosen curve occupy different mathematical roles. Their differentiation makes the Variation Hypothesis sufficiently precise for comparison with the Trajectory Hypothesis.

7.1 Transformation Actions and Generating Descriptions

A first formulation adopts the author’s semigroup-action metamodel from Section 4. Its carrier S specifies possible configurations in a declared representation; M contains mechanisms composed associatively; and the action $\alpha : M \times S \rightarrow S$ assigns the effect of each mechanism. Equation 17 gives the minimal action. Within that formalism, the Variation Hypothesis directs attention to properties of α and M relative to S and to the chosen congruence $\sim$. The semigroup operation gives a compositional organization of mechanisms, whereas $\sim$ determines which differences among configurations the description preserves.

Definition 7.1 (Transformation organization). For a fixed-carrier metamodel $\mathfrak{D} = (S, M, \alpha, \sim)$, a *transformation organization* is the typed action (M, S, α) together with declared structural conditions and any observational or physical interpretation supplied for the application. A *variation property* is a property of this organization, such as the existence of fixed configurations, distinguishability under the action, isotropy, accessibility, or preservation under an explicitly chosen class of action morphisms.

Definition 7.1 deliberately refrains from associating consciousness with every transformation organization. For instance, a three-state carrier with an idempotent map may satisfy the same algebraic axioms

as an abstraction of a more elaborate physical system. The presence of the action and the ability to compose it carry no independent evidence of phenomenal experience. A consciousness application therefore needs a further predicate, invariant, or measurable relation whose relevance to a defined conscious phenomenon is argued and empirically constrained.

One way to specify the appropriate sense of representation invariance is through an action-intertwining pair. Let $\mathfrak{D}_A$ and $\mathfrak{D}_B$ be action models and let $h : S_A \rightarrow S_B$ and $\phi : M_A \rightarrow M_B$ be maps. Equation 46 defines the commuting-action condition.

$$h(\alpha_A(m, x)) = \alpha_B(\phi(m), h(x)), \quad (m, x) \in M_A \times S_A. \quad (46)$$

If ϕ is a semigroup homomorphism, the pair respects composition of mechanisms as well as their action on configurations. A bijective pair with an inverse satisfying the same conditions supplies a strong algebraic equivalence. If h identifies distinct configurations, the relation may describe an abstraction with information loss. In either case, Equation 46 says nothing by itself about the experiential significance of the equivalence. The epistemic meaning of h must be supplied by the representation and observation protocols, as discussed in Section 3.

The difference operator Δ in Equation 23 provides one candidate diagnostic property. If a mechanism identifies configurations that the chosen congruence distinguished, then the action collapses those distinctions in the metamodel. The property is explicitly *relative to the congruence*; it is different from thermodynamic entropy production, geometrical contraction, or a direct measure of subjectivity. Further, the metamodel's composition is algebraic. Interpreting composition as a physical temporal sequence introduces assumptions not entailed by associativity alone.

7.2 Vector Fields and Infinitesimal Transformation Structure

Smooth manifolds offer a more geometrical realization of transformation organization. Let Q be a finite-dimensional smooth manifold, and let $X \in \Gamma(TQ)$ be a smooth vector field. At each $q \in Q$, $X(q)$ belongs to the tangent space $T_q Q$. A local integral curve γ is constrained by Equation 47.

$$\gamma : I \longrightarrow Q, \quad \frac{d\gamma}{d\lambda}(\lambda) = X(\gamma(\lambda)). \quad (47)$$

Equation 47 uses a chosen real parameter λ to express a differentiable solution. Neither the smooth structure of Q nor this parameter automatically identifies Q as physical spacetime or λ as fundamental time. Where a physical parameterization is available, it is an additional interpretation to be stated. In this section the field X , rather than an individual solution γ , is the central object of the Variation Hypothesis. Standard treatments of vector fields and smooth dynamical systems provide the mathematical background [27, 21].

A vector field can be described through its local expression $X = X^i(q)\partial/\partial q^i$ in a coordinate chart. Under a change of coordinates $q' = \psi(q)$, the components transform by the differential $d\psi$, so equality of coordinate arrays is generally an unsuitable invariant. A useful geometric comparison instead employs a diffeomorphism between state manifolds. Equation 48 expresses the condition that X_A and X_B are h -related.

$$dh_q(X_A(q)) = X_B(h(q)), \quad h : Q_A \longrightarrow Q_B. \quad (48)$$

If h is a diffeomorphism and Equation 48 holds, the fields represent the same differential transformation organization under the specified change of coordinates or model identification. When h is a surjective submersion, the condition instead describes a projected vector field, provided it can be satisfied consistently on each fiber. The necessary projectability issue was established in Proposition 5.1. A representation may therefore preserve an observed coordinate while losing the variables required to specify its autonomous dynamics.

The differential expression does not exhaust physically relevant variations. A family of admissible velocities may be represented by a set-valued assignment $F(q) \subseteq T_q Q$, producing a differential inclusion.

Equation 49 states a common solution condition.

$$\frac{d\gamma}{d\lambda}(\lambda) \in F(\gamma(\lambda)) \quad \text{for almost every } \lambda \in I. \quad (49)$$

A differential inclusion can describe constraint-dependent admissibility and controlled variation under suitable regularity assumptions. Existence and invariance results depend on properties of F and the state space; a set-valued prescription alone guarantees neither [2]. The collection $F(q)$ is a family of mathematically admitted directions, and its modal status should not be re-described as an inventory of events awaiting actualization. A selected solution and the full collection of solutions remain different objects even when the same formalism represents both.

7.3 Constraint Geometry, Perturbations, and Local Organization

Constraints affect what a transformation organization can express. For a smooth submanifold $C \subseteq Q$, a vector field tangent to C satisfies Equation 50.

$$X(q) \in T_q C, \quad q \in C. \quad (50)$$

Under regularity assumptions ensuring local existence and uniqueness, tangency is the familiar geometric condition supporting invariance of C under compatible integral curves. More general constraints can be represented by a distribution $\mathcal{D} \subseteq TQ$, a family of admissible sets, or an admissibility relation in a cellular model. These alternatives need separate specifications. In particular, a distribution may fail to be integrable, and a CW-complex has no canonical tangent bundle analogous to that of a smooth manifold [27, 1].

A geometric-control specialization makes the distinction between prescribed mechanisms and accessible paths especially visible. Let $X_1, \dots, X_r$ be smooth vector fields and $u = (u^1, \dots, u^r)$ an admissible control. Equation 51 specifies a controlled family of local directions.

$$\frac{dq}{d\lambda} = \sum_{a=1}^r u^a(\lambda) X_a(q), \quad u(\lambda) \in U. \quad (51)$$

The generating fields, restrictions on u , and the Lie brackets of the fields can carry information about accessibility that a single recorded trajectory does not retain. Bracket calculations and accessible-set results belong to geometric control theory and require conditions on regularity, admissible controls, and the chosen parameter domain [1]. A consciousness interpretation of such a control system would need an empirically meaningful account of what the controls represent. Equating formal controllability with volition, agency, or awareness would add assumptions not established by the mathematics.

The word *variation* also has a narrower role in variational mechanics. An action functional is constructed from a Lagrangian and a parametrized family of curves. Equation 52 gives a schematic finite-dimensional example.

$$\mathcal{A}[\gamma] = \int_{\lambda_0}^{\lambda_1} L(\gamma(\lambda), \dot{\gamma}(\lambda), \lambda) d\lambda, \quad \delta \mathcal{A}[\gamma] = 0. \quad (52)$$

Stationarity is taken over a specified class of variations, often with boundary conditions. With a sufficiently regular Lagrangian, the Euler–Lagrange equations express local consequences of this condition. The stationarity relation in Equation 52 is one possible *construction of a dynamical prescription*; it is not the definition of the Variation Hypothesis. A model may be based on nonvariational dynamics, stochastic transitions, or discrete actions and still be investigated under the hypothesis. Conversely, the fact that a physical system admits a variational principle does not by itself establish any connection with conscious experience.

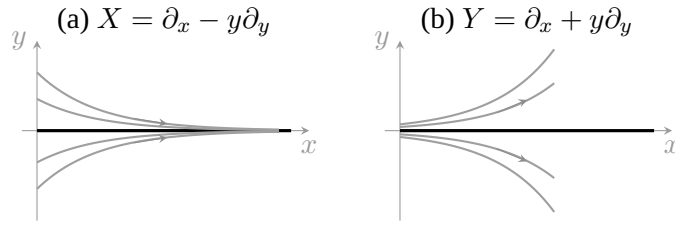


Figure 2: Two smooth transformation prescriptions sharing the integral curve $y = 0$ while differing in transverse evolution relative to the chosen increasing parameter. Gray curves represent selected neighboring solutions; axes indicate the modeled state coordinates, not physical space.

7.4 Representation Equivalence and Trajectory Underdetermination

Two formal comparisons clarify what a variation-based claim would need to preserve. The first examines the relation between the field and the parameterization of its integral curves. The second exhibits identical selected trajectories generated by systems with different behavior around those trajectories. Both depend on an explicitly chosen smooth state manifold and an ordered real parameter; they make no claim concerning the fundamental ontology of time.

Proposition 7.2 (Positive rescaling preserves unparameterized integral curves). *Let Q be a smooth manifold, X a smooth vector field, and $a : Q \rightarrow (0, \infty)$ a smooth function. Write $Y = aX$. Every local integral curve of X admits an orientation-preserving local reparameterization that is an integral curve of Y , and conversely. Thus X and Y have the same local unparameterized oriented integral curves and the same zero set, although their parameterized flows may differ.*

Proof. Let γ satisfy $\gamma'(\lambda) = X(\gamma(\lambda))$. Solve locally for a strictly increasing $\lambda = \lambda(s)$ satisfying $\lambda'(s) = a(\gamma(\lambda(s)))$, which is possible since a is smooth and positive. The chain rule yields $(\gamma \circ \lambda)'(s) = a(\gamma(\lambda(s)))X(\gamma(\lambda(s))) = Y(\gamma(\lambda(s)))$. Exchanging the fields and using a^{-1} gives the converse. Since $a(q) > 0$, $Y(q) = 0$ precisely where $X(q) = 0$. $\square$

Proposition 7.2 is a standard consequence of reparameterization of integral curves. It shows that equality of unparameterized path images with their orientations leaves part of the transformation prescription unidentified. Any attempt to identify phenomenally relevant features with a particular parameterized rate therefore needs an account of what sets the physical or experimental parameter scale. Conversely, a hypothesis involving only orbit structure may intentionally disregard such rates; that choice changes which models count as equivalent.

Proposition 7.3 (A selected trajectory does not determine transverse dynamics). *On $Q = \mathbb{R}^2$ with coordinates (x, y) , let $X = \partial_x - y\partial_y$ and $Y = \partial_x + y\partial_y$. The curve $\gamma(\lambda) = (\lambda, 0)$ is an integral curve of both fields. Nevertheless, the transverse dependence of nearby integral curves differs: relative to increasing λ , deviations from $y = 0$ contract exponentially under X and expand exponentially under Y .*

Proof. For either field, the first coordinate satisfies $x' = 1$ and thus $x(\lambda) = x_0 + \lambda$ on a chosen parameter interval. For X , the second coordinate satisfies $y' = -y$, giving $y(\lambda) = y_0 e^{-\lambda}$. For Y , it satisfies $y' = y$, giving $y(\lambda) = y_0 e^{\lambda}$. Choosing $y_0 = 0$ produces the same γ in both systems, while $y_0 \neq 0$ yields the stated distinction. $\square$

Figure 2 illustrates Proposition 7.3 using a representative family of local path traces. The example demonstrates why a selected history does not uniquely determine the change mechanism. It also cautions against inferring a generator's structural stability from observations restricted to an invariant subset.

The selected-trajectory comparison also differs from an observational comparison. Suppose an observation function returns only x in the systems of Proposition 7.3. It reports the same derivative $x' = 1$ in both systems for all represented initial configurations. It therefore conceals the transverse difference even when many initial states are sampled. With a richer observation mapping, the models could be distinguished. The observation-relative identifiability criterion of Section 3 provides the general setting for such questions.

7.5 Topology, Cellular Models, and Geometric Limitations

The type of carrier influences which variation structures can be defined. On a smooth manifold, differential geometry supplies tangent bundles, vector fields, Lie brackets, and smooth coordinate transformations. Global topology can constrain whether such fields possess particular properties. The classical hairy-ball theorem, for example, prevents a continuous nowhere-vanishing tangent vector field on the two-sphere. Thus, certain smooth variation organizations depend on the topology of their carrier, even before a physical interpretation is chosen. This observation is mathematical; it is not a claim that spherical state spaces have a distinctive phenomenology.

On a CW-complex, cells are attached through specified maps and yield incidence and homology data, as developed in Section 5.2. A general two-complex need not be a manifold, and singular attachments may obstruct the construction of a globally smooth tangent field of the kind used in Equation 47. Variation can instead be represented by cellular self-maps, combinatorial transition relations, or explicitly specified piecewise-smooth rules with compatibility conditions on shared boundaries. Those choices preserve different information: a cellular endomorphism can encode action on cells and induced homology maps, whereas a smooth vector field can encode local derivative information. A homology-level map alone generally cannot reconstruct either the entire cellular action or all local dynamics.

One simple example uses a finite CW-complex K and a cellular map $f : K \rightarrow K$. Its induced homomorphisms on cellular homology provide a coarse family of descriptors. Equation 53 records the induced maps for each degree k .

$$f_{*,k} : H_k(K; \mathbb{Z}) \longrightarrow H_k(K; \mathbb{Z}). \quad (53)$$

Equation 53 is a familiar functorial construction, but distinct maps can have the same induced maps on homology. The description therefore keeps a subset of the transformation's topological information. Comparing such invariants with differential generators across discrete and continuous representations requires explicitly constructed bridges, a task developed further in Sections 11–13. The model's two-complex cells are assigned their meanings at instantiation; a vertex is not assumed to be an atomic conscious subject, and a cellular face is not assigned a universal physical interpretation.

Topology-changing representations require particular caution. If a model uses different carriers K_j and K_k , variation across them needs a morphism, span, cobordism-like description, or another stated correspondence. The fixed-carrier action α by itself does not generate a new cell or define an identification between inequivalent carriers. This limitation is an important reason for distinguishing the Variation Hypothesis from the broader Generativity Hypothesis in Section 6. It also respects the methodological restraint that indexing models by an ordered set does not, without interpretation, provide an ontological past or future.

7.6 Candidate Consciousness Correlates and Empirical Requirements

The Variation Hypothesis becomes a hypothesis *about consciousness* only when its transformation properties are connected to a defined phenomenon of experience. Possible targets include reported conscious content, experimentally assessed level of consciousness, or theoretically specified aspects of phenomenal organization. These targets may involve different observation and intervention protocols. Existing research on neural population dynamics and consciousness theories provides potential contexts for designing such protocols; methods established for motor control or recurrent neural networks do not thereby constitute evidence for any of the four consciousness hypotheses [40, 16, 37, 36, 23].

One candidate research design compares two physically motivated transformation models under a common observation protocol. The study would specify (i) the state representation and its reconstruction uncertainty, (ii) the variation property predicted to covary with a separately operationalized conscious phenomenon, (iii) the invariance class under which that property remains meaningful, and (iv) a competing model that preserves observed trajectories while changing the proposed generative mechanism. Proposition 7.3 provides a minimal mathematical template for the fourth component. Interventions and

Table 8: Requirements for evaluating a variation-based consciousness proposal. The requirements identify distinct validation tasks rather than a sufficient set of conditions for consciousness.

Task	Required specification	Illustrative failure mode
Mathematical typing	Carrier, transformation law, regularity, and constraints	A vector field assumed on a singular CW-complex
Representation invariance	Maps and observables that the comparison preserves	Coordinate-dependent scores reported as intrinsic
Model discrimination	Competing generators and informative interventions	One trajectory compatible with incompatible transverse dynamics
Physical instantiation	Substrate, parameters, and empirical measurement protocol	Formal controllability equated with agency
Phenomenal connection	Defined experience-related target and independent evidence	Generic dynamical structure treated as sufficient for awareness

observation protocols would then determine whether the competing prescriptions are distinguishable on the relevant domain.

The following criteria summarize the requirements for advancing the Variation Hypothesis beyond structural analogy. Table 8 distinguishes mathematical adequacy from physical and experiential adequacy; satisfaction of one column does not imply satisfaction of the others.

A deeper explanatory problem remains even when a physically grounded field or operator is reconstructed accurately. A mathematical model may organize potential transformations, identify invariants, and explain why certain patterns of activity are possible under its assumptions. A claim that the modeled organization is necessary for, sufficient for, identical with, or causally responsible for conscious experience involves additional theoretical and evidential commitments. Those commitments should be distinguished from a merely observed covariation. Contemporary adversarial comparisons of consciousness theories indicate why predictions and discriminating empirical designs matter; they do not adjudicate the present Variation Hypothesis [12].

Provisional conclusion. The Variation Hypothesis isolates change-generating prescriptions as its principal mathematical object. Actions and vector fields can be compared through intertwining morphisms, projectability, orbit equivalence, and other stated relations. Local field data, constraint organization, and topological restrictions can contain information absent from an observed selected history. Proposition 7.2 illustrates the distinction between a field and its unparameterized integral curves, while Proposition 7.3 makes the limits of trajectory-based reconstruction explicit. These conclusions concern mathematical representations. Their relevance to consciousness depends on independently justified instantiation, observation, and phenomenal interpretation. The Trajectory Hypothesis developed in Section 8 therefore addresses a related but distinct candidate explanatory object.

8 The Trajectory Hypothesis

The Trajectory Hypothesis considers whether properties of selected paths, flows, or relational histories can bear an explanatory relation to specified conscious phenomena. A trajectory is a mathematical object only after its carrier, domain of definition, admissibility conditions, and intended interpretation have been declared. The present section develops trajectory-based description on smooth manifolds and cellular spaces, examines the relations among parameterization, geometry, topology, and observation, and provides counterexamples to overly strong interpretations of trajectory equivalence. It also describes selected histories in families of changing spaces. These constructions specify what the hypothesis can investigate; they do not identify any particular path invariant with phenomenal experience. The contrast with the transformation-oriented account of Section 7 concerns the type of mathematical object being proposed as explanatorily relevant, rather than a presumed metaphysical partition of processes and their

generating conditions.

In the vocabulary of Section 6, a *selected* trajectory is an element singled out by a mathematical condition, a reconstruction procedure, or an observation protocol. Selection in this sense entails neither conscious choice nor a privileged ontological present. When a parameter is interpreted as physical time, that interpretation belongs to an explicit physical model and is subject to its own empirical and theoretical conditions. When an ordering is provided by cellular incidence, causal dependence, a parameter set, or an abstract relation, those orderings should likewise be distinguished. These restrictions preserve the temporal and ontological restraint developed in Section 3.

8.1 Path Objects, Selection, and Admissibility

Let Q be a topological space and $I = [0, 1]$ be an interval used solely as a mathematical parameter domain. The continuous mapping space $C(I, Q)$ has a compact–open topology. An admissibility model singles out a subset $\mathcal{H}_{\mathfrak{D}}(I, Q) \subseteq C(I, Q)$, or a more general class of maps, according to a declared dynamical or relational prescription. A particular trajectory is a chosen element of this subset. The distinction is recorded in Equation 54.

$$\mathcal{H}_{\mathfrak{D}}(I, Q) = \{\gamma : I \rightarrow Q \mid \text{Adm}_{\mathfrak{D}}(\gamma)\}, \quad \gamma_{\star} \in \mathcal{H}_{\mathfrak{D}}(I, Q). \quad (54)$$

The index $\mathfrak{D}$ identifies the represented model, including its constraints and any required regularity. Equation 54 distinguishes a selected path $\gamma_{\star}$ from the admissible family. The former is the immediate mathematical focus of the Trajectory Hypothesis; the latter is the focus of the Repertoire Hypothesis. Since even the definition of admissibility can depend on mechanisms or structural relations, the hypotheses share formal ingredients without identifying the same explanatory object.

For a smooth manifold Q and a vector field X , a differentiable admissible curve may satisfy the relation in Equation 55.

$$\frac{d\gamma}{d\lambda}(\lambda) = X(\gamma(\lambda)), \quad \lambda \in I. \quad (55)$$

Equation 55 is a special instantiation: it presupposes a common differentiable carrier and a parameter with respect to which derivatives are meaningful. A discontinuous physical model, differential inclusion, stochastic system, or topologically varying cellular model may require different admissibility predicates. In particular, for an inclusion $\gamma'(\lambda) \in F(\gamma(\lambda))$, a selected solution need not determine which allowable directions in F were available at configurations outside its image [2].

A trajectory also differs from its image $\gamma(I) \subseteq Q$. The image records which points are traversed while omitting information about repeated passages, their order on I , and the parametrization assigned to those passages. Two curves can have identical images and endpoints yet follow different ordered traversals. If the explanatory hypothesis depends on recurrences, transitions, or order-sensitive observables, the image alone is an insufficient representation. A point sequence produced by a recording instrument is another object again, as measurement and interpolation can identify or remove distinctions among candidate paths.

8.2 Parameterized Geometry and Trajectory Equivalence

The geometry of a selected curve depends on which structures are assigned to its carrier. On a Riemannian manifold (Q, g) , the length of a piecewise continuously differentiable path is defined by Equation 56.

$$\ell_g(\gamma) = \int_0^1 \sqrt{g_{\gamma(\lambda)}(\gamma'(\lambda), \gamma'(\lambda))} d\lambda. \quad (56)$$

Equation 56 acquires a numerical interpretation only after the metric g has been justified. A metric reconstructed from neural activity need not coincide with a spatial metric of an organism, and a state-space distance need not represent physical separation or an experienced difference. Curvature, acceleration, torsion, and related differential invariants introduce additional regularity and geometric assumptions.

They may be valuable descriptors of recorded or reconstructed processes, while their interpretation as consciousness-related quantities requires independent argument [27, 40].

One common equivalence considers curves related by an orientation-preserving change of parameter. Let $\rho : I \rightarrow I$ be a continuously differentiable, strictly increasing diffeomorphism fixing the two endpoints. Write $\tilde{\gamma} = \gamma \circ \rho$. The defining relation is given by Equation 57.

$$\gamma \simeq_{\text{rep}} \tilde{\gamma} \iff \tilde{\gamma} = \gamma \circ \rho \text{ for an admissible } \rho. \quad (57)$$

This equivalence preserves the oriented geometric traversal while setting aside its particular parameter scale. It is weaker than equality of parameterized curves and stronger than equality of images for many path classes. The class of permitted ρ is part of the model; permitting parameter reversal would yield a distinct quotient with different information-preservation properties.

Proposition 8.1 (Metric length under orientation-preserving reparameterization). *Let (Q, g) be a Riemannian manifold and $\gamma : I \rightarrow Q$ be piecewise continuously differentiable. If $\rho : I \rightarrow I$ is an orientation-preserving continuously differentiable diffeomorphism fixing the endpoints, then $\ell_g(\gamma \circ \rho) = \ell_g(\gamma)$.*

Proof. The chain rule yields $(\gamma \circ \rho)'(\lambda) = \gamma'(\rho(\lambda))\rho'(\lambda)$. Since $\rho'(\lambda) > 0$, the metric norm of this vector is $\|\gamma'(\rho(\lambda))\|_g \rho'(\lambda)$. Substitution $u = \rho(\lambda)$ in Equation 56 establishes the equality. $\square$

Proposition 8.1 expresses invariance of one specific geometric quantity; it does not show that all possible dynamical or physical observables are reparameterization invariant. For example, a measured rate of change relative to laboratory time, a prescribed control budget, or a dissipative constitutive law can depend on the parameterization selected for the physical model. If λ is interpreted as physical time, replacing it by an arbitrary ρ need not preserve physical solutions to the same time-dependent equation. The proposition consequently clarifies the mathematical conditions for quotienting trajectories without silently erasing physically relevant information.

For Euclidean paths of bounded variation, iterated-integral signatures provide a richer, order-sensitive family of features with useful reparameterization properties. Their study also illustrates why the choice of equivalence class and path regularity matters: signatures can identify certain classes of paths only modulo specified degeneracies, and topologies on unparameterized path spaces are nontrivial mathematical objects [10]. The signature apparatus is one possible analytical instrument; no claim of an established phenomenal correspondence follows from applying it to a recorded path.

8.3 Homotopy, Fundamental Groupoids, and Cellular Histories

Topology supplies another family of relations among paths. Two continuous paths $\gamma_0, \gamma_1 : I \rightarrow Q$ with common endpoints are homotopic relative to their endpoints if a continuous map $H : I \times I \rightarrow Q$ deforms one into the other while keeping both endpoints fixed. The corresponding equivalence classes form the morphisms of the fundamental groupoid $\Pi_1(Q)$, with objects the points of Q and composition induced by concatenation of paths. These are topological notions whose interpretation as physically permitted deformations depends on the model. Homotopy need not preserve the differential equation generating either path, and a homotopy parameter carries no automatically privileged physical-time meaning [19].

For a based loop $\gamma : I \rightarrow S^1$, the winding number distinguishes elements of $\pi_1(S^1) \cong \mathbb{Z}$. That distinction is lost when one records only the image of the loop: loops of several degrees can have the same image S^1 . Conversely, attaching a disk to the circle along a degree-one boundary map makes the original circle contractible in the enlarged space. The degree- n cellular construction in Section 5 gives a broader family of examples, for which the homology groups and the fundamental group depend on the attaching map. A trajectory's homotopy classification is therefore relative to its ambient space and its attaching data, as well as to its own map.

This observation is especially relevant to relational two-complexes. In the present program, vertices may represent events, edges represent couplings, and faces represent higher-order primitives whose domain-specific meaning requires specification. An edge path on the one-skeleton $K^{(1)}$ is a sequence

of composable oriented edges with matching incidences. The inclusion $K^{(1)} \hookrightarrow K$ can impose additional homotopy identifications through the attaching maps of two-cells. The cellular boundary operator constrains homology, while homotopy retains certain nonabelian distinctions invisible to ordinary first homology. Neither object determines a causal order among the represented events without added orientation or admissibility structure.

A simple finite example makes the last point explicit. Let K have one vertex v , two oriented loop edges a, b , and a two-cell attached along the word $aba^{-1}b^{-1}$. Its geometric realization has the homotopy type of a torus, with fundamental group $\mathbb{Z}^2$. Paths winding once around a and once around b carry independent homotopy information. If a trajectory model observes only a projection onto one circular coordinate, the winding in the other direction can be concealed. The next subsection gives a precise observational counterexample based on this structure.

The use of ordinary path homotopy need not be appropriate when a model distinguishes allowable direction of traversal. Directed algebraic topology studies spaces equipped with a chosen family of directed paths and develops homotopy-theoretic tools in which the direction constraint is meaningful [18]. A directed structure is additional data: an abstract CW-complex does not select physically meaningful directions by virtue of its cellular topology. This distinction is especially consequential when interpreting event relations or irreversible processes, since a chosen orientation on an edge and a demonstrated causal or temporal precedence are separate claims.

8.4 Observed Trajectories and Topological Nonidentifiability

A recording protocol rarely supplies a complete trajectory in the proposed underlying carrier. It typically produces an observed path through a representation map $h : Q \rightarrow Y$, possibly with noise or sampling. A candidate latent trajectory γ determines $h \circ \gamma$, while distinct latent curves can give the same observation. The observation-relative identifiability criterion in Section 3 applies directly: a trajectory property can be inferred uniquely from the observed path only if it is constant over all trajectories compatible with that observed record.

Proposition 8.2 (A projected trajectory can conceal a homotopy distinction). *Let $Q = S^1 \times S^1$, let $Y = S^1$, and let $h : Q \rightarrow Y$ be projection onto the first coordinate. There exist based loops $\gamma_0, \gamma_1 : I \rightarrow Q$ with identical projected paths $h \circ \gamma_0 = h \circ \gamma_1$ and different homotopy classes in Q .*

Proof. Define the two based loops by Equation 58.

$$\gamma_0(\lambda) = (e^{2\pi i \lambda}, 1), \quad \gamma_1(\lambda) = (e^{2\pi i \lambda}, e^{2\pi i \lambda}). \quad (58)$$

Projection gives $h \circ \gamma_0(\lambda) = h \circ \gamma_1(\lambda) = e^{2\pi i \lambda}$. The classes of γ_0 and γ_1 in $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ are $(1, 0)$ and $(1, 1)$, respectively, and so differ. $\square$

Figure 3 depicts the two lifted loops in a square fundamental domain and their common observed winding under the projection h .

Proposition 8.2 demonstrates a precise limit on what a partial trajectory observation can establish. Perfect observation of the first circular coordinate does not determine the homotopy class of the underlying path in the two-dimensional torus. The example does not require noise, finite sampling, or an ambiguous choice of parameter. It is a structural failure of identifiability induced by the observation map. A further projection to path images would discard additional ordering and winding distinctions.

Table 9 records several frequently conflated equivalence notions. Each row specifies what an equality statement says and which distinctions its quotient can lose. The table describes mathematical information preservation under chosen models, rather than a ranking of their suitability for consciousness research.

The counterexample also has a practical epistemic analogue. Geometric and topological properties of neural population data can be reconstructed and compared under suitable measurement assumptions. For example, Yoon and colleagues developed a method for identifying corresponding topological features across neural manifolds using activity-derived dissimilarities, with simulation and in vivo demonstrations [41]. Such work supports the usefulness of topology for investigating neural representations; it does

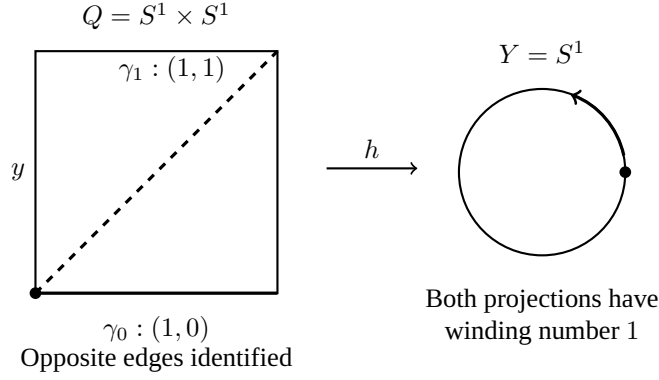


Figure 3: Two based loop classes on the torus with identical first-coordinate projections. The square is a fundamental domain with opposite edges identified; the solid horizontal and dashed diagonal lifts represent classes $(1, 0)$ and $(1, 1)$. Their projections onto the first S^1 factor agree. The drawing is a schematic of the topological construction, not a physical model of consciousness.

Table 9: Comparative notions of trajectory equivalence and information retained. Their physical relevance depends on the declared observation and admissibility structures.

Relation	Preserved structure	Potentially unrecorded distinctions
Parameterized equality	Assigned map from parameter domain to carrier	Latent distinctions eliminated by prior measurement
Oriented reparameterization	Ordered geometric traversal	Parameter rate and time-dependent observables
Image equality	Visited subset of carrier	Traversal order, multiplicity, winding
Endpoint-fixed homotopy	Homotopy class and fixed endpoints	Metric length, local field, detailed route
Observed-path equality	Path under a specified observation map	Unobserved dimensions and latent homotopy class

not establish that an inferred loop, manifold, or matching of topological features realizes subjective experience. Phenomenological interpretation remains an additional, empirically constrained step.

8.5 Relational Histories on Changing Carriers

A fixed carrier Q is convenient but may be inadequate for models whose relational structure changes. A family of manifolds, CW-complexes, or two-complexes can be organized over an index category $\mathcal{J}$. One approach assigns a carrier Q_j to each object $j \in \mathcal{J}$ and specified transition or compatibility relations to morphisms of $\mathcal{J}$. The index category may encode a modeling order, observational resolution, or a particular process organization. Its identification with a universal temporal order requires further argument.

Let $Q : \mathcal{J} \rightarrow \mathbf{Top}$ be a diagram of spaces. When the model instead supplies relations $R_{jk} \subseteq Q_j \times Q_k$ for selected comparable indices, a relational history can be specified as a compatible family according to Equation 59.

$$\text{Hist}_R(Q) = \left\{ (x_j)_{j \in \text{Ob } \mathcal{J}} : x_j \in Q_j, \quad (x_j, x_k) \in R_{jk} \text{ for every stipulated comparison } j \rightarrow k \right\}. \quad (59)$$

Equation 59 is a provisional relational definition: it requires consistency conditions on the specified R_{jk} if those relations are expected to compose. Such conditions should be explicit in a concrete model. When only a proper subset of indices is observed, the resulting record can be a partial compatible family, and its admissible extensions need not be unique. A category with branching or incomparable indices also avoids building a universal linear order into the definition.

For a bundle $p : E \rightarrow B$, another construction uses sections $\sigma : B \rightarrow E$ satisfying $p \circ \sigma = \text{id}_B$. Its fibers may carry different geometric descriptions even when the bundle has a fixed total space. For

topology-changing carriers, a global fiber-bundle description with locally trivial fixed fiber type may be inappropriate. Diagrams with cellular attachment maps, cobordisms under specified hypotheses, or stratified families may be more suitable. A change in attachment data is not equivalent to motion through a fixed smooth state manifold merely because both can be indexed by the same parameter set.

The trajectory perspective should retain the historical provenance of the representation. If an original cellular edge is collapsed under a quotient, an observed path on the quotient may remain available while the distinction between alternative edge histories is lost. If a new cell is added, homotopies available in the enlarged complex can identify paths that were distinct in the original complex. An account of continuity or identity of a modeled subject through these constructions would require additional criteria of correspondence among subcomplexes; the combinatorial transformations alone do not determine such identity.

8.6 Physical Interpretation and Phenomenological Requirements

The Trajectory Hypothesis proposes a candidate explanatory orientation: specified properties of selected, interpreted relational processes might be systematically associated with specified aspects of consciousness. A physical instantiation must identify a substrate or measurement domain, define the observed variables, justify the reconstruction of trajectories, and describe which interventions could discriminate competing interpretations. In a neural-population instantiation, a fitted low-dimensional trajectory may summarize measured activity while depending on data selection, preprocessing, dimensionality reduction, and observation conditions. The phenomenon to be explained may concern reportability, perceptual content, levels of wakefulness, or subjective experience more broadly; these targets impose different empirical requirements [23, 36, 40].

A trajectory invariant may be reproducible across observations without being sufficient for consciousness. Circular paths, recurrence, low-dimensional manifolds, and path signatures also arise in explicitly nonconscious physical and computational systems. To give a consciousness interpretation evidential content, a proposed invariant would need appropriate negative controls, comparisons across conscious and unconscious conditions, careful analysis of confounds, and evidence of robustness under legitimate changes of representation. Its dependence on a chosen metric, cell decomposition, observation map, and notion of trajectory equivalence should be tested rather than left implicit.

The mathematical analyses also delimit what a trajectory-based approach can claim about generative conditions. Proposition 8.2 proves that even an exact observed trajectory can fail to identify a latent homotopy class. Proposition 7.3 from Section 7 establishes a complementary limitation: one selected integral curve can be compatible with different transverse fields. Thus, a trajectory description can support a meaningful analysis of a selected process while leaving counterfactual transformations and the broader landscape of admissible histories underdetermined. The distinction motivates the Generativity and Repertoire Hypotheses examined in Sections 9 and 10, without requiring those hypotheses to be phenomenologically superior.

A further question concerns the status of a trajectory itself. Under some physical accounts, trajectories describe evolution relative to physical time; under other representations, they record a mathematical ordering or a reconstruction of relational changes. The use of expressions such as “already realized,” “still possible,” or “not yet actual” can import a privileged temporal standpoint and a modal ontology that the model has not justified. In the present paper, a selected trajectory is a mathematically distinguished member of a specified class, and the epistemic status of that selection is declared separately. This distinction keeps the paper open to heterogeneous physical and philosophical accounts of temporal structure.

The present section establishes a family of formal comparisons: equality of images, equivalence under parameter changes, homotopy, cellular attachment dependence, and observational indistinguishability. Each comparison preserves a different portion of trajectory information. A rigorous Trajectory Hypothesis must specify which portion is proposed to be consciousness-relevant, demonstrate its robustness under appropriate representations, and connect it to evidence about a defined conscious phenomenon.

Geometry and topology supply useful tools for stating these tasks. Their successful application to a mathematical history remains distinct from an explanation of subjective experience.

9 The Generativity Hypothesis

The Generativity Hypothesis investigates whether the coupled organization of change, structure, constraint, and difference bears an explanatory relation to specified conscious phenomena. Its distinguishing mathematical problem concerns relations among the conditions under which transformations are defined, including variations in the carrier, its available mechanisms, its relational organization, and its criteria of distinguishability. The minimal semigroup metamodel of Section 4 establishes results within a fixed carrier and congruence; the present section develops explicitly typed comparisons among such models. Smooth manifolds and cellular two-complexes supply complementary examples in which cross-model changes require additional data. The analysis proceeds from constitutive model families through comparison morphisms, congruence variation, and geometric and topological constructions to observation and physical interpretation. The formal results identify conditions for preserving information across represented systems; they neither assign consciousness to those systems nor settle the ontology of its physical realization.

The hypothesis concerns *constitutive generativity*: the structural relationships by which mechanisms, constraints, available differences, and realized configurations jointly delimit a modeled class of transformations. Such relationships can sometimes be represented within a fixed state space by enlarging the state description. They can also require comparing heterogeneous mathematical objects, such as manifolds of different dimensions or cellular complexes with distinct attachment structures. These representational choices are not interchangeable by definition. In particular, the possible transformation of a mathematical law is a property of a specified family of models, and its index does not establish an absolute order of past and future. The temporal and epistemic qualifications of Section 3 consequently remain part of the hypothesis's formal specification.

9.1 Constitutive Data and Indexed Models

An extended model specification. Let $\mathcal{I}$ be a small category indexing descriptions, contexts, or comparisons. For each object $i \in \mathcal{I}$, consider a model $\mathfrak{D}_i$ with the constituents displayed in Equation 60.

$$\mathfrak{D}_i = (S_i, M_i, \alpha_i, \sim_i; K_i, \mathcal{A}_i), \quad \alpha_i : M_i \times S_i \longrightarrow S_i. \quad (60)$$

Here S_i is an unstructured carrier at the metamodel's highest abstract level, M_i is a semigroup of transformations acting through α_i , and $\sim_i$ is an action-compatible equivalence relation. The optional object K_i supplies a geometrical, topological, or relational realization and need not be present in every instance. The symbol $\mathcal{A}_i$ groups additional domain-specific admissibility and interpretive assumptions, including possible constraints on paths, observations, or physical interpretations. A cellular K_i and the carrier S_i should be explicitly related at instantiation, for example when S_i consists of admissible labelings of cells of K_i . They are not assumed identical.

The original four primitives are preserved in Equation 60: Change is specified by α_i , Structure by associative composition in M_i , Constraint by the congruence $\sim_i$, and Difference by the derived collapse relation of Equation 23. Difference remains a derived object rather than an independently assigned state variable. In particular, the source metamodel contains no general dynamics of the congruence, no rule generating a carrier outside its specified set, and no universal interpretation of the states [22]. The additional entries in $\mathfrak{D}_i$ are a proposed modeling extension, not axioms inherited from that paper.

Different ways to index a family. An arrow $u : i \rightarrow j$ in $\mathcal{I}$ can designate a comparison of theoretical descriptions, a change in observation, a physically modeled intervention, or a relation of structural dependence. Its interpretation is part of $\mathcal{A}_i, \mathcal{A}_j$ and the chosen cross-model data. A category may contain

incomparable objects, noninvertible maps, and more than one arrow between the same pair of objects; none of these features uniquely defines physical temporal succession. Where an empirical application uses laboratory time, that parameter and its relation to $\mathcal{I}$ must be separately supplied. Where a collection of models represents alternative hypotheses rather than an unfolding process, interpreting its arrows as historical transitions would be unwarranted.

A family of descriptions need not admit a single encompassing state manifold on which every physical distinction is simultaneously meaningful. Embedding model identifiers and local states into a disjoint union produces one set, but does not supply canonical dynamics between its components or a common metric. The construct therefore retains information about model boundaries and the work required to compare them. This distinction is important whenever the proposed consciousness-relevant property concerns changes in the conditions of generation rather than only changes of a coordinate within a settled representation.

9.2 Morphisms of Generative Descriptions

Action-preserving comparisons. A particularly strict cross-model comparison consists of a semi-group homomorphism $\phi_u : M_i \rightarrow M_j$ together with a carrier map $h_u : S_i \rightarrow S_j$ satisfying Equation 61.

$$h_u(\alpha_i(m, x)) = \alpha_j(\phi_u(m), h_u(x)), \quad m \in M_i, x \in S_i. \quad (61)$$

Equation 61 is an intertwining condition. It specifies exactly which transformation structure the pair (h_u, ϕ_u) preserves. When the comparison also satisfies $x \sim_i y \Rightarrow h_u(x) \sim_j h_u(y)$, it preserves the selected congruence and induces a well-defined map between quotient carriers. Neither property establishes that h_u is injective, that every mechanism of M_j has a preimage, or that the compared physical systems are equivalent. More general comparisons may be partial, relational, approximate, or observation-based; the algebraic and categorical conditions for these cases are taken up in Section 13. Categorical models of interactive dynamical systems, including polynomial-functor approaches, provide a related body of mathematical techniques without making the present indexed specification canonical [32].

When transformations can be compared along composable arrows, the requirements $h_{v \circ u} = h_v \circ h_u$ and $\phi_{v \circ u} = \phi_v \circ \phi_u$ give a coherent diagram of model descriptions. The comparison then has a categorical interpretation in an appropriately defined category of actions. Even here, a functor between such diagram categories need not be faithful, full, or an equivalence. These distinctions matter because a translation can preserve the compositional shape of transformations while identifying previously distinguishable structures.

Proposition 9.1 (Preservation of modeled collapse under a reflecting morphism). *Suppose (h, ϕ) satisfies Equation 61 and, for every $x, y \in S_i$, the relation $x \sim_i y$ holds exactly when $h(x) \sim_j h(y)$. Then, for every $m \in M_i$, the following inclusion holds:*

$$(h \times h)[\Delta_i(m)] \subseteq \Delta_j(\phi(m)). \quad (62)$$

Proof. If $(x, y) \in \Delta_i(m)$, then $x \not\sim_i y$ while $\alpha_i(m, x) \sim_i \alpha_i(m, y)$. Reflection of the congruence gives $h(x) \not\sim_j h(y)$, and congruence preservation gives $h(\alpha_i(m, x)) \sim_j h(\alpha_i(m, y))$. Intertwining identifies these latter images with $\alpha_j(\phi(m), h(x))$ and $\alpha_j(\phi(m), h(y))$. Thus $(h(x), h(y)) \in \Delta_j(\phi(m))$, proving Equation 62. $\square$

Proposition 9.1 gives a sufficient condition for preservation of lost distinctions relative to the specified model congruences. The inclusion need not be an equality because S_j may possess additional configurations or pairs outside the image of h . The proposition also concerns a fixed comparison, rather than evidence that one conscious experience is preserved by a transformation. If the congruence is merely preserved and its reflection is omitted, even inclusion may fail. The example in Section 9.3 supplies an explicit failure of that stronger conclusion.

9.3 Changing Constraints and Distinguishability

The metamodel records a distinction as collapsed when a mechanism sends states lying in distinct $\sim_i$ -classes into a common class. This definition makes the chosen congruence epistemically and theoretically consequential. If two instances use different congruences, the relations Δ_i and Δ_j have different criteria of discrimination. Even when their underlying carrier sets and transformations coincide, a comparison of collapse sets requires an account of the two equivalence relations. The monotonicity result of Section 4 applies to composition of mechanisms with a fixed compatible congruence; it provides no general ordering between collapse sets calculated under different congruences.

A finite example isolates the issue. Take $S = \{a, b\}$, the singleton semigroup $M = \{m\}$ with $m^2 = m$, and the action $m(a) = m(b) = b$. Equip the same action separately with equality $\sim_{\text{disc}}$ and with the indiscrete equivalence $\sim_{\text{ind}}$, which identifies a and b . Both are congruences of this action. The derived collapse relations are compared in Equation 63.

$$\Delta_{\text{disc}}(m) = \{(a, b), (b, a)\}, \quad \Delta_{\text{ind}}(m) = \emptyset. \quad (63)$$

The comparison in Equation 63 changes the discrimination criterion while retaining the same transformation. Accordingly, the difference between these two collapse sets does not demonstrate that a physical transformation restored a previously destroyed distinction. It demonstrates that the meaning of destruction in this metamodel depends on the relational resolution at which states are distinguished. Taking the identity carrier map from the equality-congruence model to the indiscrete-congruence model preserves congruence in one direction but fails to reflect it, illustrating the qualification in Proposition 9.1.

Historically constituted congruences. For a family of models, it is possible to attach a congruence $\sim_i$ to each description and compare how its quotient $S_i / \sim_i$ is represented under cross-model maps. This permits investigation of changing conventions of classification, different perceptual discriminations, evolving scientific taxonomies, or revisions in a relational system's normative boundaries. Such interpretations require evidence appropriate to the domain. An epistemic change in the distinctions available to an observer is not automatically identical to a change in an ontological substrate. The relation between recognition, physical change, and experienced difference is particularly unsettled in consciousness research.

A more ambitious dynamics in which the congruence itself is generated by a lawful relational process remains open. Merely writing $\sim_i$ for each index does not determine its evolution or explain why particular distinctions become recognizable. A circularity can arise if generation is defined through distinction while distinction is, in turn, defined through the output of generation. The earlier metamodel explicitly identifies the evolution of $\sim$ as a central unresolved task [22]. The present construction makes comparison possible once congruences and maps are supplied, while leaving their constitution as an explanatory question.

9.4 Cellular Attachments and Topological Variation

A two-complex can represent higher-order attachment information absent from an unstructured state set. In a domain-specific relational model, zero-cells might denote events, one-cells couplings, and two-cells selected higher-order primitives. Those meanings remain interpretive choices requiring empirical or theoretical justification. The cellular formalism nevertheless permits exact calculations of how different attachment maps affect topology. The following example identifies a change in relational organization without requiring a globally fixed manifold or assigning temporal precedence to one topology.

Let $K_0 = S^1 \vee S^1$ be a CW-complex with one vertex and two oriented one-cells a, b . Consider two spaces obtained by attaching a two-cell to the same one-skeleton. In K_a , the attaching loop is a ; in $K_{[a,b]}$, it is the commutator loop $aba^{-1}b^{-1}$. Both admit canonical cellular inclusions from K_0 . Their

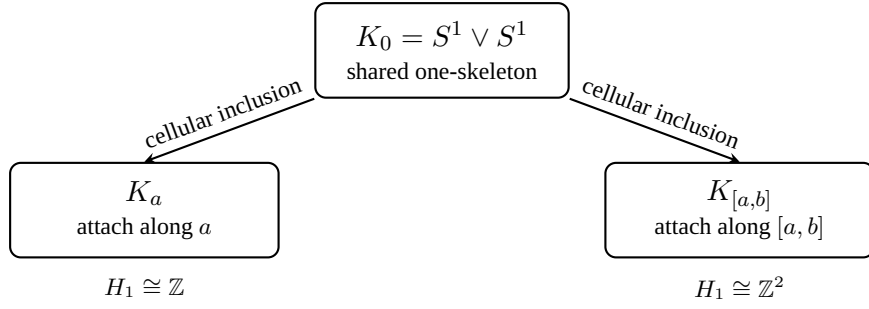


Figure 4: Two admissible cellular attachments sharing a one-skeleton. Arrows denote inclusions of CW-complexes, not an assumed physical direction of time. The attachment maps retain information that the common one-skeleton alone does not determine.

two-cellular boundary maps and first homology groups are shown in Equation 64.

$$\begin{aligned} \partial_2^{K_a}(1) &= (1, 0), & H_1(K_a; \mathbb{Z}) &\cong \mathbb{Z}, \\ \partial_2^{K_{[a,b]}}(1) &= (0, 0), & H_1(K_{[a,b]}; \mathbb{Z}) &\cong \mathbb{Z}^2. \end{aligned} \tag{64}$$

The first complex has one independent one-dimensional homology class and the second has two. The vanishing cellular boundary of the commutator attachment records its zero total exponent sums; the attachment is nevertheless homotopically consequential, giving $\pi_1(K_{[a,b]}) \cong \mathbb{Z}^2$ rather than the free group $\pi_1(K_0) \cong F_2$. For the standard CW structures in this construction, $K_{[a,b]}$ is homeomorphic to a torus. The calculations follow from cellular homology and the van Kampen presentation [19].

Figure 4 displays the two constructions as a span of inclusions, while Equation 64 identifies one invariant by which their results differ. These spaces provide distinct mathematical possibilities for attaching higher-order structure to a common cellular representation. A model that treats the one-skeleton as its complete state description cannot recover the two-cell attachment map without further data. Conversely, an observed homology group need not identify the attaching map uniquely; distinct cellular models can share homology while having different homotopy structure. The present example therefore establishes a relation between constitution and representational discrimination, not an association between the topology of a two-complex and conscious experience.

For a diagram of multiple cellular models, functorial homology assigns groups and induced homomorphisms to compatible cellular maps. Zigzag persistence offers an established framework for studying diagrams in which maps among spaces point in different directions [7]. Such methods could help compare structural records from changing neural or relational models. Their applicability depends on choosing justified complexes, maps, and data; a barcode alone does not identify the realized physical mechanism, much less its phenomenal interpretation.

9.5 Manifold Families and Changes in Geometric Type

Smooth differential geometry provides a complementary approach when the constitutive objects admit manifold structure. A smooth fiber bundle $\pi : E \rightarrow B$ with typical fiber F can organize a family of state manifolds over a parameter or context base B . Local triviality gives a precise meaning to comparing a neighborhood of fibers, while additional connections or transport rules are needed to identify particular points, vectors, or dynamical fields across fibers. A field defined on one fiber does not canonically extend to another solely because the fibers are diffeomorphic.

Proposition 9.2 (Topological constancy of the fibers of a locally trivial bundle). *Let $\pi : E \rightarrow B$ be a locally trivial topological fiber bundle over a nonempty base B , with typical fiber F . Then any two fibers E_b and $E_{b'}$ are homeomorphic to F . In the smooth-bundle setting with smooth local trivializations, the corresponding fibers are diffeomorphic.*

Table 10: Distinct commitments in applying the Generativity Hypothesis. Each inferential level requires the conditions specified in its middle column.

Investigation	Required specification	Limitation or relevant control
Metamodel comparison	Actions, congruences, carrier maps, and preservation laws	A homomorphism may identify important distinctions
Constitutive dynamics	Cross-model constraints, mechanism changes, and admissibility	Diagrammatic arrows have no automatic physical-time meaning
Geometric/topological change	Manifold or cellular construction, topology, and map semantics	Sampling and reconstruction may create apparent structure change
Observation	Instruments, observable mappings, intervention context	Equivalent data may arise from different generative structures
Consciousness relation	Defined phenomenal target, comparison conditions, competing models	Generic generativity does not establish experience

Proof. By local triviality, each point $b \in B$ has an open neighborhood U for which $\pi^{-1}(U)$ is (smoothly, where appropriate) identified with $U \times F$ by a fiber-preserving trivialization. Restriction of this trivialization to $\{b\} \times F$ gives a (diffeomorphic) homeomorphism between E_b and F . The same holds for any other point b' . Composition through F gives the claimed equivalence. $\square$

The proof of Proposition 9.2 is pointwise in the base, so no path-connectedness assumption is needed once the bundle is stipulated to have one fixed typical fiber. The result illustrates a limitation of a familiar geometric representation. A family claimed to change the homotopy type of its state-space fibers cannot be a locally trivial fiber bundle of one fixed fiber type under the proposition's assumptions. Describing such changes may call for singular fibers, stratified spaces, cobordisms, diagrams of CW-complexes, or other constructions. These alternatives carry different interpretive and mathematical obligations. The statement concerns the geometry of model carriers, and should not be interpreted as a theorem about the topology of physical spacetime or of neural consciousness.

A representational family can also change its coordinates, reconstruction dimension, or sampling regime while the physically underlying system remains unchanged in the respects under investigation. Therefore, detecting different estimated manifolds across two data sets does not, by itself, establish a change in the ontological geometry of a conscious subject. Distinguishing modeling revisions from physical structural changes requires stable empirical definitions and cross-representation tests. In the neural context, comparative topological analyses of population representations offer potentially relevant methods, while their relation to conscious phenomenology requires separate hypotheses and evidence [41].

9.6 Consciousness-Related Commitments and Model Discrimination

The distinctive explanatory commitment of the Generativity Hypothesis is that specified conscious phenomena may depend on the constitutive coupling among transformation mechanisms, constraints, relational structures, and available distinctions, possibly including transformations of these components under a stated model. The hypothesis is consequently broader than combining one selected trajectory with one vector field. It treats the organization of admissibility and distinguishability as potential explanatory variables, while recognizing that there are many nonconscious dynamical systems with such mathematical organization. A substantive consciousness hypothesis must identify additional properties, mechanisms, or boundary conditions and explain why those features should be connected to conscious experience.

Table 10 separates the mathematical question from its epistemic and phenomenal extensions. The distinction is important because a representation morphism may preserve collapse, a cellular attachment may change homology, and a manifold family may change geometric type, yet none of these formal facts alone identifies the presence, absence, or character of experience.

A practical investigation could therefore select a physically motivated system, specify alternative

generative descriptions, and compare their identifiable consequences under controlled interventions. If a claim concerns changes in the system's admissible transformations, the experimental design would need data beyond a single passive trajectory. If it concerns the topology of a reconstructed state space, it would need stability to choices of sampling, metric, thresholds, and reconstruction procedure. If it concerns a purported necessary or sufficient condition for consciousness, it would need a justified connection between the modeled condition and independently measured or reported phenomenal targets, with appropriate controls for arousal, task behavior, reportability, and other confounds. Such designs belong to the empirical development of the hypothesis; the current section supplies conceptual and formal distinctions for evaluating them.

Provisional assessment. The indexed-model construction clarifies one mathematical meaning of generativity as constitutive coupling across potentially heterogeneous representations. Proposition 9.1 gives a sufficient condition under which the collapse relation of a metamodel is preserved across a specified comparison. Equation 63 demonstrates that changing the chosen relation of distinguishability changes which collapses are recorded even for the same transformation. Equation 64 shows how distinct two-cell attachments alter topological invariants, and Proposition 9.2 identifies an assumption that precludes topology-changing fibers in ordinary bundles. These conclusions make the Generativity Hypothesis mathematically distinct from a selected-trajectory account. They remain statements about represented structures. Further work is required to define the physical and phenomenological quantities through which the hypothesis could be tested, compared with alternative explanations, or rejected. Section 10 will investigate the corresponding question at the level of structured families of admissible histories.

10 The Repertoire Hypothesis

The Repertoire Hypothesis investigates whether the organized family of admissible configurations or histories associated with a relational or dynamical description has explanatory significance for specified conscious phenomena. The hypothesis directs attention to a class of possibilities defined by a model, including its geometric and topological organization, rather than exclusively to its generating mechanism, one represented history, or the coupled constitution of its generative conditions. The mathematical challenge is to characterize this family without turning model-relative admissibility into an unexamined claim about ontological possibility. This section introduces typed repertoire objects, develops topology on path families, supplies a constrained-system construction, examines cellular and two-complex interpretations, and derives consequences for observational equivalence. Its final subsection identifies the commitments required to connect formal repertoires to phenomenology.

A repertoire may be defined over a fixed carrier, over a specified class of starting configurations or boundary conditions, or over a family of mathematical structures. These choices produce different objects even when some observed outcomes coincide. Accordingly, neither the word *possible* nor the indexing of histories carries an intrinsic temporal interpretation here. An admissible history is an element satisfying stipulated conditions. Its physical interpretation, including the interpretation of its parameter, requires additional assumptions of the kind discussed in Section 3.

10.1 Admissible History Families and Their Scope

A typed repertoire object. Let D denote a specified dynamical or relational description, let $I = [0, 1]$ be a compact parameter interval, and let S be a topological space. Write $\mathcal{P}(I, S) = C(I, S)$ for its space of continuous paths. An admissibility predicate $\mathcal{A}_D$ determines the repertoire in Equation 65.

$$\mathcal{R}(D; I, S, \mathcal{A}_D) = \{\gamma \in \mathcal{P}(I, S) : \mathcal{A}_D(\gamma)\}. \quad (65)$$

The predicate can encode a differential equation, differential inclusion, set of allowable couplings, directed-cellular rules, or boundary conditions. Equation 65 is schematic; a concrete instance must

supply path type, regularity, and the actual admissibility conditions. For a discrete carrier, the repertoire may consist of composable finite histories rather than continuous maps from I .

A deterministic initial-value problem with a fixed starting value and uniqueness of solutions produces a singleton path repertoire. Varying the initial set, control inputs, or model context generally changes its scope. To expose these dependencies, use the indexed repertoire of Equation 66.

$$\mathcal{R}_{D,B,U,c} = \{\gamma \in \mathcal{P}(I, S) : \gamma \text{ is admissible under } (D, B, U, c)\}. \quad (66)$$

Here B denotes starting or boundary data, U an admissible input class, and c a representational or experimental context. This decomposition distinguishes a change to the generative description from a change to the conditions under which it is investigated.

Relation to the other hypotheses. The Variation Hypothesis in Section 7 studies transformation mechanisms used to define admissibility. The Trajectory Hypothesis in Section 8 studies a selected member of a path family. The Generativity Hypothesis in Section 9 studies the coupled constitution and transformation of the conditions defining such families. The Repertoire Hypothesis investigates properties of the *family as a structured whole*. It may carry connectivity, compactness, a topology of neighborhoods, a measure, a stratification, or distinguished boundary conditions. None of these is inherent in an unstructured set of paths.

Two descriptions can yield the same set of represented outcomes while using different generators, controls, costs, or derivations. Equality of the sets $\mathcal{R}(D_1) = \mathcal{R}(D_2)$ is thus a relatively weak form of equivalence. Stronger comparisons must specify which additional structures should be preserved. This is crucial when comparing representations of proposed consciousness-related phenomena.

10.2 Topology and Geometry of Path Repertoires

Topology is part of the model. For a Hausdorff state space S , the compact-open topology supplies a standard topology on $C(I, S)$. If S is metric with metric d , this topology for compact I is induced by the uniform metric in Equation 67.

$$d_\infty(\gamma, \eta) = \sup_{\lambda \in I} d(\gamma(\lambda), \eta(\lambda)). \quad (67)$$

Equation 67 compares parameterized paths. Its interpretation differs when paths are identified under reparameterization, tree-like path equivalence, or other relations; quotient path-space topology requires its own construction [10]. A Riemannian state space permits further quantities such as path length or energy. A CW-complex supports path spaces without possessing a canonical smooth tangent bundle.

A repertoire inherits a subspace topology when it is presented as a subset of $C(I, S)$. One may investigate connectedness, compactness, contractibility, and partitions into components. A probability measure on the repertoire requires additional mathematical and empirical assumptions: membership in an admissible-history family alone assigns no unique probability to a path.

A compact constrained repertoire. Choose $S = \mathbb{R}$, $I = [0, 1]$, a fixed initial value $\gamma(0) = 0$, and the almost-everywhere differential inclusion $\dot{\gamma}(\lambda) \in [-1, 1]$. Equation 68 defines the associated repertoire.

$$\mathcal{R}_{\text{Lip}} = \{\gamma \in AC([0, 1], \mathbb{R}) : \gamma(0) = 0, \quad |\dot{\gamma}(\lambda)| \leq 1 \text{ for a.e. } \lambda\}. \quad (68)$$

Here AC denotes absolutely continuous functions. The derivative bound is an admissibility condition and gives no ontological temporal interpretation to λ . Differential inclusions and viability constraints provide a general mathematical context for this example [2].

Proposition 10.1 (Compactness of a bounded-slope repertoire). *Under the uniform metric of Equation 67, the family $\mathcal{R}_{\text{Lip}}$ of Equation 68 is compact. Its endpoint evaluation image under $e_1(\gamma) = \gamma(1)$ is $[-1, 1]$.*

Proof. Absolute continuity and the derivative bound imply $|\gamma(s) - \gamma(t)| \leq |s - t|$. Since $\gamma(0) = 0$, the family is uniformly bounded and equicontinuous. Arzelà–Ascoli gives relative compactness in $C([0, 1], \mathbb{R})$ [24]. A uniform limit retains the Lipschitz bound and initial value, and is absolutely continuous with the stipulated derivative bound almost everywhere. Hence the family is closed and compact. Every endpoint lies in $[-1, 1]$; conversely, $\gamma_a(\lambda) = a\lambda$ has endpoint a for each $a \in [-1, 1]$. $\square$

Proposition 10.1 establishes a property of a whole admissible-history family, rather than of a distinguished path. Endpoint equivalence is also weaker than path equivalence: the paths $\gamma_0(\lambda) = 0$ and $\gamma_1(\lambda) = \lambda(1 - \lambda)$ belong to the family and share endpoint 0, while their interiors differ. A record of endpoints alone cannot identify the repertoire of complete paths.

10.3 Control Geometry and Repertoire Boundaries

Families defined by constrained dynamics. For a smooth state manifold M , consider a controlled vector field $F : M \times U \rightarrow TM$ with $F(x, u) \in T_x M$, an admissible control class $\mathcal{U}$, and a starting set $B \subseteq M$. One repertoire is given in Equation 69.

$$\mathcal{R}(F, B, \mathcal{U}) = \left\{ \gamma : \begin{array}{l} \gamma(0) \in B, \quad \gamma \text{ absolutely continuous,} \\ \dot{\gamma}(\lambda) = F(\gamma(\lambda), u(\lambda)) \text{ a.e.,} \\ u \in \mathcal{U} \end{array} \right\}. \quad (69)$$

Equation 69 requires suitable regularity and existence assumptions for the stated controls. In geometric control, attainable configurations, allowable control directions, and boundary constraints can supply distinct invariants of the repertoire [1]. Eliminating the control variable in favor of a set-valued velocity map $\mathcal{F}(x) \subseteq T_x M$ can retain the family of state histories while losing which controls realized a particular velocity. Repertoire equality therefore need not imply equality of control mechanisms.

Endpoint evaluation e_1 maps the path repertoire to its reachable endpoint set. An endpoint set is only one projection of the full repertoire. A family of evaluations $e_\lambda : \gamma \mapsto \gamma(\lambda)$ supplies additional constraints, yet it need not capture correlations among parameter positions unless their joint relations are retained. The choice of parameter interval and allowable reparameterizations is material to this comparison. It does not automatically describe physical temporal succession.

An illustrative constraint envelope. Figure 5 displays selected paths of Proposition 10.1. The shaded cone indicates the pointwise bound $|\gamma(\lambda)| \leq \lambda$; the derivative bound of Equation 68 is an additional condition for membership. This distinction matters because many curves inside the cone violate the required Lipschitz condition. A diagram of reachable regions generally supplies less information than the represented family of admissible histories.

10.4 Cellular and Two-Complex Repertoires

Paths and homotopy classifications. For a CW-complex K , the geometric realization $|K|$ admits a continuous path space $C(I, |K|)$ without requiring a global smooth vector field. An admissible repertoire may be selected by endpoints, cell incidences, allowed edges, local transition rules, or additional constraints. Homotopy relative to endpoints gives the morphisms of the fundamental groupoid $\Pi_1(|K|)$. Attachment maps in a CW-complex therefore influence how path histories are classified [19]. An ordinary homotopy classification may identify paths that a directed dynamics distinguishes; the relevant directed-topological structure must then be specified separately [18].

Consider $K_\partial = S^1$ and the two-complex $K_{\text{fill}} = S^1 \cup_{\text{id}} D^2$, with inclusion $j : K_\partial \hookrightarrow K_{\text{fill}}$. Equation 70 gives its induced fundamental-group map.

$$j_* : \pi_1(K_\partial) \cong \mathbb{Z} \longrightarrow \pi_1(K_{\text{fill}}) \cong \{e\}, \quad j_*(n) = e. \quad (70)$$

Under Equation 70, loops with different winding numbers in the boundary circle become homotopic within the filled disk. The attachment changes the homotopy classification of paths while retaining the

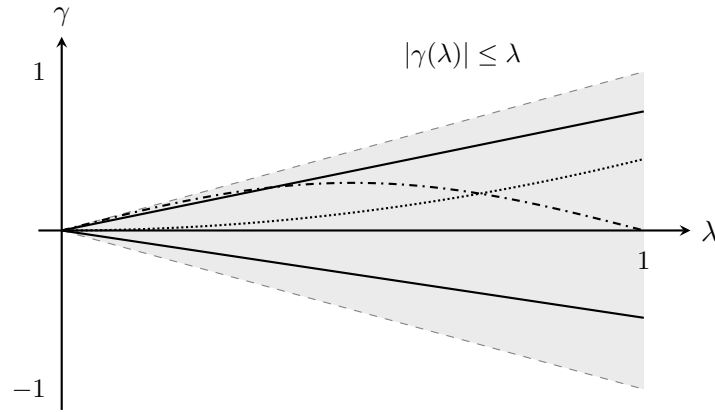


Figure 5: Selected admissible paths and the pointwise envelope for a compact constrained repertoire. The shaded region is a necessary pointwise bound, not a sufficient criterion for membership. The example makes no claim of correspondence to subjective experience.

boundary and its paths as a subspace. Whether paths may physically traverse the disk interior remains a separate matter of admissibility; topology alone supplies no dynamical law or consciousness interpretation.

The example separates two constructions: enlargement of an available path space and coarsening of an equivalence classification. A relational two-complex additionally carries an interpretation of cells that must be specified at the chosen model level. In our program vertices represent events, edges represent couplings, and faces carry model-dependent higher-order roles. A path through a geometric realization, a physically realized process, and a recorded history of emergent agents require explicit representation bridges. The existence of a 2-cell does not by itself determine the generative or causal relations of such agents.

Repertoires on heterogeneous cellular carriers. For a family $i \mapsto K_i$ indexed by a category $\mathcal{I}$, let $\mathcal{R}_i \subseteq C(I, |K_i|)$. A continuous map $q_{ij} : |K_i| \rightarrow |K_j|$ induces the ambient path-space map in Equation 71.

$$(q_{ij})_* : C(I, |K_i|) \longrightarrow C(I, |K_j|), \quad \gamma \longmapsto q_{ij} \circ \gamma. \quad (71)$$

Equation 71 restricts to a map $\mathcal{R}_i \rightarrow \mathcal{R}_j$ only when the relevant admissibility conditions are preserved. A cellular map can collapse distinct homotopy classes and may fail to preserve selected transition rules. Zigzag persistence gives one mathematical method for comparing invariants across non-monotone diagrams of spaces when the maps are explicitly supplied and justified [7]. Index objects may encode representations, measurement contexts, or other comparison conditions; the category $\mathcal{I}$ does not intrinsically encode past or future.

10.5 Repertoire Equivalence and Observational Projection

Observation maps on path families. Let $O : S \rightarrow Y$ be a continuous observation map. Composition induces $O_* : C(I, S) \rightarrow C(I, Y)$. Equation 72 defines the observed repertoire.

$$\mathcal{R}_O^{\text{obs}}(D) = O_*[\mathcal{R}(D)] = \{O \circ \gamma : \gamma \in \mathcal{R}(D)\}. \quad (72)$$

This representation presumes a pointwise observation map. Actual measurement procedures can depend on earlier samples, a measurement intervention, or a reconstruction algorithm; those cases require a history-dependent observation operator. Equality of the images in Equation 72 is an *observation-relative equivalence* of repertoires. It need not imply a correspondence of the underlying dynamical laws or of the full state histories.

Table 11: Candidate equivalence criteria for generative repertoires.

Criterion	Preserved information	Principal qualification
Outcome-set equality	Selected outcomes in one representation	Omits internal paths and generators
Observed-path equality	Complete observed path family under a fixed map	Can conflate different full-state families
Topological equivalence	Chosen topology of repertoire spaces, up to homeomorphism	Does not imply generator equivalence
Dynamical correspondence	Specified transformations or controls and their interplay	Requires a defined intertwining map
Measure-preserving equivalence	Selected measures and measurable structures	The measure requires justification
Phenomenological equivalence	Specified properties of subjective experience	Needs an additional explanatory and evidential bridge

Proposition 10.2 (Coincidence of observed repertoires under projection). *Let $S = \mathbb{T}^2$ with angular coordinates (θ, φ) and $O(\theta, \varphi) = \theta$. Let $\mathcal{R}_X$ and $\mathcal{R}_Y$ be the integral-curve repertoires over $I = [0, 1]$, with all starting states in $\mathbb{T}^2$, for $X = \partial_\theta$ and $Y = \partial_\theta + \partial_\varphi$. Then $\mathcal{R}_X \neq \mathcal{R}_Y$ as subsets of $C(I, \mathbb{T}^2)$, while $O_*[\mathcal{R}_X] = O_*[\mathcal{R}_Y]$.*

Proof. Every integral curve of X has the form $\gamma_{a,b}^X(\lambda) = (a + \lambda, b)$ modulo 2π , while every integral curve of Y has the form $\gamma_{a,b}^Y(\lambda) = (a + \lambda, b + \lambda)$. The path sets differ in their second-coordinate evolution. Both project onto precisely the paths $\eta_a(\lambda) = a + \lambda$ on $\mathbb{T}^1$, for $a \in \mathbb{T}^1$. Hence the projected repertoires coincide. $\square$

Proposition 10.2 extends the selected-path nonidentifiability example of Section 8. Even if the observed path *family* were completely determined, transverse dynamics could remain unspecified by the observation map. This is a mathematical factorization result, not evidence that the two systems have indistinguishable subjective experience. Access to an additional observable sensitive to the second angular coordinate would distinguish these particular repertoires.

Relations among candidate repertoire equivalences. A hierarchy of equivalence criteria must make preservation conditions explicit. Table 11 contrasts six candidates. They are not generally linearly ordered: topological and measure-theoretic equivalences, for example, compare different structures and need not imply one another.

A structural property P of a full repertoire is identifiable from its observed image only if P is constant on the classes of descriptions with equal images under the selected observation procedure. This applies the general identifiability principle of Section 3 to history families. It also frames a research problem for the separate investigation of equivalence structures across consciousness hypotheses: representation maps preserve some invariants, forget others, and do not automatically preserve phenomenal properties.

10.6 Modal Scope, Physical Interpretation, and Phenomenology

The admissibility boundary. The membership statement $\gamma \in \mathcal{R}_D$ is relative to D and its admissibility rules. It does not assert that γ is a physically possible future or an unrealized event awaiting actualization. The parameter interval supplies mathematical ordering and regularity conditions. The distinction between model-relative possibility, epistemically accessible history, and physical realization therefore remains a substantive assumption, rather than an implication of the notation.

For empirical reconstruction, the boundary of a neural-state repertoire depends on recording modalities, the chosen dimensionality-reduction procedure, available interventions, sampled conditions, and the observation horizon. Neural population manifolds and analyses of neural topology show the relevance of geometric and topological representations to neural science, but their reconstructed invariants do not

by themselves measure the presence or character of subjective experience [40, 41]. Mathematical enlargement of a repertoire by relaxing constraints should not automatically be interpreted as enlargement of consciousness.

Conditions for a consciousness-specific hypothesis. A substantive Repertoire Hypothesis needs a defined property of the structured path family and an explicit target phenomenon. Potential quantities include the topology of accessible path components, dimensions of admissible-history families, sensitivity to constraints, or invariance under justified representation maps. They are mathematical candidate correlates rather than already validated markers. A study must explain why a chosen property would have a relation to conscious experience, compare alternative physical interpretations, and discriminate it from nonconscious systems with similar mathematical organization. Contemporary theoretical debates and adversarial comparisons underscore the need to distinguish an internally coherent formal model from an empirically supported account of consciousness [36, 12].

The four hypotheses assign different inferential objects to related experimental data. Perturbation-response observations may inform transformation mechanisms; a represented episode may inform trajectory analysis; comparisons of changing constraints and relational organization may inform generativity; and experimentally constrained families of responses may inform repertoire analysis. A study can test several of these perspectives, provided the mappings from observations to their formal objects are specified. Their common use of geometry, topology, and dynamics does not entail their equivalence as explanatory theories.

Provisional assessment. The Repertoire Hypothesis identifies a structured family of admissible histories as a possible explanatory object. Proposition 10.1 establishes compactness of a particular constrained path repertoire, while Proposition 10.2 exhibits distinct full-state repertoires with identical observed images. Equation 70 establishes an attachment-sensitive change in path homotopy classification, while retaining the distinction between topological paths and dynamically admissible processes. These results are conditional mathematical facts. They neither demonstrate that repertoires constitute consciousness nor establish the physical completeness of a chosen repertoire. The comparative geometric, topological, and algebraic consequences of the four explanatory orientations are developed in Sections 11–13.

11 Geometric and Topological Interpretations

Geometry and topology provide several distinct kinds of information about dynamical and relational descriptions. Smooth structure specifies which local variations can be differentiated and composed; metric and differential-form data characterize selected curves; cellular attachments determine relations among events, couplings, and higher-dimensional primitives; and topologies on spaces of paths describe the organization of admissible families. The objective of this section is to examine how these constructions discriminate the Variation, Trajectory, Generativity, and Repertoire Hypotheses without identifying their explanatory objects. The argument develops explicit examples, propositions, and representation criteria across smooth manifolds and relational two-complexes. It also identifies information lost under selected abstractions and the additional assumptions required for a physical or consciousness-related interpretation.

The geometries investigated here are geometries of *mathematical representations*. A neural state manifold inferred from recordings, the geometric realization of a cellular complex, a physical spacetime, and a manifold parameterizing possible histories can be mathematically related while remaining distinct objects. A structural correspondence among them requires a typed map and a stated preservation condition. Moreover, an index assigned to models, a parameter along a curve, and an ordering of cellular operations have different roles. None supplies a privileged physical present or a universal order of realizations. These distinctions implement the epistemic and temporal commitments of Section 3 within the more detailed constructions below.

11.1 Differential Geometry of Change Mechanisms

Infinitesimal actions and generated directions. Let M be a smooth manifold and $X_1, \dots, X_r \in \mathfrak{X}(M)$ be smooth vector fields. Each field supplies a local one-parameter family of transformations through its local flow, wherever the flow exists. Their pointwise linear span determines the distribution in Equation 73.

$$\mathcal{D}_x = \text{span}_{\mathbb{R}}\{X_1(x), \dots, X_r(x)\} \subseteq T_x M. \quad (73)$$

The rank of $\mathcal{D}$ may vary with x . When it has constant rank, it defines a smooth subbundle of TM ; under the additional involutivity condition, Frobenius-type results identify local integral submanifolds [27]. A collection of allowed vector fields need not be involutive. Their Lie brackets can generate directions not present in the pointwise span of the original fields. Such bracket effects are fundamental in geometric control theory and depend on the permitted composition of flows and controls [1].

There are consequently several candidate geometric properties of a transformation description: the rank of its immediate direction distribution, the Lie algebra generated by its fields, the geometry of attainable sets, the behavior of transverse perturbations, and the compatibility of these structures with observation maps. The Variation Hypothesis may concern some of these properties, whereas a single selected path retains only a restricted record of their action. The Repertoire Hypothesis studies the family of admissible concatenations produced when the control and boundary rules are fixed. In the Generativity Hypothesis, the space of fields, the control restrictions, or the underlying carrier may themselves enter the constitutive data under comparison.

A bracket-generating construction. Consider $M = \mathbb{R}^2$, with coordinates (x, y) , and the vector fields defined in Equation 74.

$$X = \frac{\partial}{\partial x}, \quad Y = x \frac{\partial}{\partial y}, \quad [X, Y] = \frac{\partial}{\partial y}. \quad (74)$$

The pointwise span of X and Y has rank one on the line $x = 0$ and rank two at points where $x \neq 0$. However, their Lie bracket supplies the missing vertical direction on that line. Assume that positive and negative segments of both local flows are admitted and may be concatenated. Starting at $(0, 0)$, apply the X -flow with parameter a , the Y -flow with parameter b , and then the reverse X - and Y -flows. The resulting displacement is given in Equation 75.

$$(0, 0) \xrightarrow{X, a} (a, 0) \xrightarrow{Y, b} (a, ab) \xrightarrow{X, -a} (0, ab) \xrightarrow{Y, -b} (0, 0). \quad (75)$$

The notation in Equation 75 designates compositions of specifically admitted control segments; the parameters are mathematical control durations and acquire a physical temporal meaning only in an appropriate model. With $a \neq 0$, varying b produces arbitrary vertical displacement. Horizontal segments then provide access to arbitrary points of $\mathbb{R}^2$, for the present unrestricted control class. By comparison, the system generated solely by X reaches only the horizontal line through its starting state. Both systems admit the selected path $\gamma(\lambda) = (\lambda, 0)$, yet they differ in their transformation organization and attainable repertoires.

Proposition 11.1 (Selected-path coincidence with distinct generated reachability). *Let D_X admit finite concatenations of positive and negative flow segments of $X = \partial_x$, and let $D_{X,Y}$ admit the analogous segments of $X = \partial_x$ and $Y = x\partial_y$ on $\mathbb{R}^2$. From the starting configuration $(0, 0)$, the attainable endpoint sets are $\mathbb{R} \times \{0\}$ and $\mathbb{R}^2$, respectively, although both descriptions admit $\gamma(\lambda) = (\lambda, 0)$ for $0 \leq \lambda \leq 1$.*

Proof. The X -flow preserves the y coordinate, proving the first endpoint assertion. Equation 75 gives $(0, z)$ for arbitrary z by selecting $a = 1$ and $b = z$. An additional X -segment reaches any (u, z) . Choosing only X -segments realizes the stated common path in both descriptions. $\square$

Proposition 11.1 establishes a stronger distinction than observing different velocities along two curves: one and the same selected history is compatible with transformation systems having different globally attainable sets under the stated control assumptions. The result relies on allowing signed flow

segments and arbitrary concatenations. Restricting the controls to a forward cone, imposing resource budgets, or limiting the carrier can alter the reachable sets. In particular, a Lie bracket in the algebra of vector fields is not automatically an admissible physical mechanism in a one-sided system. Its interpretive relevance requires verifying the relevant controls, perturbations, and physical constraints.

Metric structure and observational dependence. A Riemannian metric g introduces lengths, angles, volume, and gradient structure on M . A distribution and its Lie brackets, by contrast, are defined from smooth fields without first selecting a metric. If $f : M \rightarrow N$ is a diffeomorphism, pushforward fields preserve Lie brackets. They preserve numerical lengths only when the target metric is correspondingly transported, or when f is an isometry for the metrics being compared. Thus a geometric statement about direction generation may survive a smooth change of representation while a claim about trajectory length need not. A dimensionality-reduction map that is not a diffeomorphism can discard directions entirely; projectability, as formalized by Proposition 5.1, becomes a substantive condition. For a consciousness-related use, the target property and its observational invariance must be selected before any geometric statistic is treated as a correlate.

11.2 Geometry and Topology of Paths and Flows

Local shape, global classes, and differential observables. A parameterized path $\gamma : I \rightarrow M$ can carry geometrically distinct information. Its speed and length refer to a metric and a parameterization class. Its endpoint-fixed homotopy class refers to the topology of M and chosen endpoint conditions. Integrals of differential forms provide further observables that are sensitive to how a path traverses the carrier. For a smooth one-form ω and a piecewise smooth path, the integral is defined in Equation 76.

$$\mathcal{I}_\omega(\gamma) = \int_\gamma \omega = \int_0^1 \omega_{\gamma(\lambda)}(\dot{\gamma}(\lambda)) d\lambda. \quad (76)$$

The integral in Equation 76 is invariant under orientation-preserving regular reparameterization, but its other invariance properties depend on ω . An exact form $\omega = dh$ gives $\mathcal{I}_\omega(\gamma) = h(\gamma(1)) - h(\gamma(0))$, so it records endpoint information. A closed, non-exact form can detect certain globally nontrivial cycles. These facts establish a mathematical spectrum between strictly local path characteristics, endpoint summaries, and topological period data [27, 19].

Proposition 11.2 (Closed-form periods of endpoint-fixed paths). *Let M be a smooth manifold and ω a closed smooth one-form. If two piecewise smooth paths $\gamma_0, \gamma_1 : I \rightarrow M$ are homotopic relative to their endpoints through a piecewise smooth homotopy, then $\mathcal{I}_\omega(\gamma_0) = \mathcal{I}_\omega(\gamma_1)$. For loops, the period depends only on their real homology class.*

Proof. Let $H : I \times I \rightarrow M$ denote the endpoint-fixed homotopy, with the parameter square oriented so that its boundary contains γ_1 and the reversal of γ_0 . The contributions of the fixed endpoint edges vanish. Stokes' theorem yields $\mathcal{I}_\omega(\gamma_1) - \mathcal{I}_\omega(\gamma_0) = \int_{I \times I} H^*(d\omega) = 0$. Integration of a closed form over a loop defines a homomorphism on $H_1(M; \mathbb{Z})$ with values in $\mathbb{R}$, which factors through real homology. $\square$

Proposition 11.2 supplies a path invariant and also delineates its incompleteness. Periods of closed one-forms do not generally distinguish loops whose homology classes agree but whose endpoint-fixed homotopy classes differ. They are insensitive to torsion classes after passage to real coefficients. An experiment that reconstructs only a small number of numerical periods may therefore identify certain path distinctions while leaving others unmeasured.

A circle and an observation-sensitive winding invariant. For $M = S^1 \subset \mathbb{R}^2$, the globally defined angular form $\eta = -y dx + x dy$ on the unit circle is closed. Consider based loops $\gamma_n(\lambda) = (\cos 2\pi n\lambda, \sin 2\pi n\lambda)$ for $n \in \mathbb{Z}$. Their periods, displayed in Equation 77, distinguish the winding classes.

$$\mathcal{I}_\eta(\gamma_n) = 2\pi n, \quad n \in \mathbb{Z}. \quad (77)$$

An endpoint-only observation sends every γ_n to the same based endpoint. A complete angle-sensitive record can preserve the winding number, provided continuity, sampling resolution, and reconstruction choices permit the class to be inferred. This gives a specific interpretation of a *trajectory-level* observable that cannot be reconstructed from endpoint equality alone. It also separates the geometric carrier from the evidential procedure used to infer the trajectory: an observed circular coordinate may be a useful neural-state representation without being a literal spatial circle in the brain. Cross-population studies of neural manifold topology motivate examining such representational relations, while retaining their observational limitations [41].

Directed paths and admissible deformation. Ordinary homotopy permits deformations through arbitrary continuous paths with specified endpoints. A physical or computational model may permit only paths tangent to a selected cone field, satisfying a differential inclusion, or following directed cellular transitions. A deformation through ordinary continuous paths may then leave the admissible repertoire. In directed topology, suitable directed homotopies and path categories retain chosen direction structures [18]. Consequently, two path descriptions can be equivalent as ordinary topological paths while inequivalent as constrained or directed histories. The distinction matters to each of the four hypotheses: the generator specifies allowed local change; the selected path has its own geometry; constraints constituting directionality contribute to generativity; and the admissible homotopies define a repertoire-sensitive topology. The presence of mathematical direction in a model is an explicitly imposed asymmetry and should not be conflated with a universal physical arrow of time.

11.3 CW-Complexes, Two-Complexes, and Relational Histories

Cells, attachment maps, and the unit of relational interpretation. Let K be a finite connected CW-complex of dimension at most two. Its cellular chain complex consists of free abelian groups generated by oriented cells, equipped with boundary maps $\partial_2 : C_2(K; \mathbb{Z}) \rightarrow C_1(K; \mathbb{Z})$ and $\partial_1 : C_1(K; \mathbb{Z}) \rightarrow C_0(K; \mathbb{Z})$. Cellular homology is computed from the kernels and images of these maps. The full homotopy type of a two-complex additionally depends on the attaching maps of its two-cells. In the relational interpretation adopted for illustration here, zero-cells designate events, one-cells designate couplings, and two-cells designate higher-order primitives whose precise semantics remain open. An enduring entity or subject may be represented through a structured subcomplex or a family of subcomplexes; the interpretation does not equate an isolated vertex with an agent.

Cellular incidence and attaching relations do not automatically encode directed causation or physically executable transitions. A topological edge can be traversed in either direction in an ordinary path space. If the coupling carries an orientation, a permitted traversal relation must be provided independently. The attachment of a two-cell identifies a loop as null-homotopic in the ordinary geometric realization; it does not in itself assert that a physical process traversing the loop is admissible or reversible. Treating faces as constitutive primitives therefore requires both an attachment specification and a domain-specific semantic interpretation. These qualifications are essential when the resulting complex is used to discuss relational history rather than merely topological shape.

Equal cellular chain complexes with different fundamental groups. A particularly informative example starts from the bouquet $A = S^1 \vee S^1$ with one vertex and two oriented one-cells a, b . Attach one two-cell either by the commutator word $aba^{-1}b^{-1}$ or by the constant map to the vertex. Denote the resulting two-complexes K_{com} and K_{triv} , respectively. The two constructions have the common cellular chain complex in Equation 78.

$$C_2 \cong \mathbb{Z} \xrightarrow{\partial_2=0} C_1 \cong \mathbb{Z}^2 \xrightarrow{\partial_1=0} C_0 \cong \mathbb{Z}. \quad (78)$$

The vanishing of ∂_2 for the commutator attachment follows because its abelianized winding counts around both one-cells are zero. The constant attachment also has zero cellular boundary. Thus both complexes have $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}^2$, and $H_2 \cong \mathbb{Z}$. Their fundamental groups nevertheless differ: K_{com}

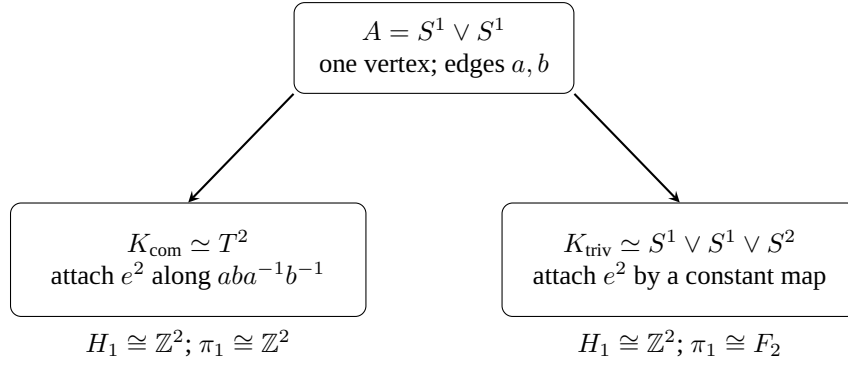


Figure 6: Different two-cell attaching maps yield the same cellular chain complex and homology groups but distinct fundamental groups. Arrows depict construction maps and do not designate physical temporal succession.

has the standard CW structure of the torus, with fundamental group $\mathbb{Z}^2$, while $K_{\text{triv}} \simeq S^1 \vee S^1 \vee S^2$ has free fundamental group F_2 [19]. Figure 6 records the constructions and the invariant levels involved.

Proposition 11.3 (Cellular homology does not determine the fundamental group of a two-complex). *There exist finite connected two-dimensional CW-complexes with isomorphic cellular chain complexes over $\mathbb{Z}$ in every degree and nonisomorphic fundamental groups.*

Proof. The complexes K_{com} and K_{triv} have identical cellular chain groups and boundary maps by Equation 78. Attaching a two-cell by the commutator imposes the relation $[a, b] = 1$ on the free group F_2 , giving $\pi_1(K_{\text{com}}) \cong \mathbb{Z}^2$. Attaching by a constant map imposes the trivial relation, leaving $\pi_1(K_{\text{triv}}) \cong F_2$. The latter group is nonabelian, so the groups are not isomorphic. $\square$

Proposition 11.3 establishes a concrete limit of homology-based reconstruction. If a proposed consciousness-relevant relational invariant depends on noncommuting path composition or other fundamental-group information, cellular homology alone may omit precisely the relevant distinction. Conversely, inferring any phenomenological role for fundamental groups requires further argument. The point is about the discriminating power of mathematical descriptions, not about which topological invariant consciousness depends upon.

Relational histories as typed cellular processes. A history recorded on a two-complex can be represented in several nonequivalent ways: as a continuous path in $|K|$, as a composable sequence of oriented edges, as a diagram of cellular subcomplexes with specified comparison maps, or as a labeled higher-dimensional configuration involving faces. The interpretation of a face as a primitive relation does not determine which of these records is adequate. A transition description may preserve event provenance and incidence information that a path in $|K|$ forgets. A homology class may forget both provenance and the order of traversals. These different losses should be made explicit in any empirical reconstruction from observations. A suitable relational model may require event labels, coupling types, face semantics, and directed admissibility in addition to the underlying CW structure.

11.4 Cellular Attachments, Homology, and Structural Change

The relative information carried by an attachment. Let $A = S^1 \vee S^1$ and let $K_w = A \cup_w e^2$ be obtained by attaching a two-cell along a word w in the generators a, b of $\pi_1(A)$. Denote the exponent sums of a, b in w by (m, n) . The long exact sequence of the pair (K_w, A) gives the attachment-sensitive segment displayed in Equation 79.

$$0 \longrightarrow H_2(K_w; \mathbb{Z}) \longrightarrow \mathbb{Z} \xrightarrow{\delta_w} \mathbb{Z}^2 \longrightarrow H_1(K_w; \mathbb{Z}) \longrightarrow 0, \quad \delta_w(1) = (m, n). \quad (79)$$

Table 12: Two cellular attachment orders with different intermediate homology. The branches are mathematical comparison paths; physical chronological interpretation requires an additional model.

Stage	Branch a^2 followed by b	Branch b followed by a^2
Initial complex	$A; H_1 \cong \mathbb{Z}^2$	$A; H_1 \cong \mathbb{Z}^2$
After first attachment	$H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$H_1 \cong \mathbb{Z}$
After second attachment	$H_1 \cong \mathbb{Z}/2\mathbb{Z}$	$H_1 \cong \mathbb{Z}/2\mathbb{Z}$

The homomorphism δ_w remembers only the abelianized winding vector. If $(m, n) \neq (0, 0)$ and $d = \gcd(|m|, |n|)$, its cokernel and kernel give Equation 80.

$$H_1(K_w; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}, \quad H_2(K_w; \mathbb{Z}) = 0. \quad (80)$$

For $(m, n) = (0, 0)$, the first and second homology groups instead are $\mathbb{Z}^2$ and $\mathbb{Z}$. Equations 79 and 80 show exactly which information about an attaching map enters ordinary integral homology at this dimension. They also explain why the two attachments in Proposition 11.3 cannot be distinguished by these groups: both abelianize to $(0, 0)$, even though their full attaching words have different algebraic effects.

Nonmonotone diagrams and path-dependent intermediate structure. Consider the two named attaching rules $w_a = a^2$ and $w_b = b$. From the same initial bouquet A , one description attaches w_a first and then w_b ; another attaches w_b first and then w_a . With standard cellular inclusion maps, the two branches have the same terminal complex up to cellular identification, but different intermediate first homology groups. The calculation is summarized in Table 12.

The terminal boundary matrix, in the bases supplied by the two named one-cells and two attached faces, is given in Equation 81.

$$\partial_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2. \quad (81)$$

The matrix in Equation 81 has trivial kernel and cokernel $\mathbb{Z}/2\mathbb{Z}$. The difference between the branches in Table 12 is therefore invisible to an inspection of terminal homology alone. It is visible in a diagram of intermediate complexes, together with their attachment identities and maps. Zigzag persistence provides a well-established method for studying homological information in diagrams that need not consist exclusively of forward inclusions [7]. Its use requires the actual diagram and the chosen field coefficients; integral torsion, such as the $\mathbb{Z}/2\mathbb{Z}$ term here, is not fully described by the usual barcode decomposition over a single field. A field with characteristic two and a field of other characteristic can give different homological signatures of this example.

The comparison illustrates an aspect of the Generativity Hypothesis that is distinct from both a final shape and a selected path within one shape. The represented attachment conditions determine how a repertoire of paths and distinctions is reorganized across the indexed descriptions. Nevertheless, a prescribed attachment sequence is still an external rule unless the model explains how that rule itself arises from represented mechanisms. In particular, labeling the branches as alternatives does not make them unrealized physical futures, and arranging them in a diagram does not establish a privileged causal ordering.

11.5 Topology and Geometry of Generative Repertoires

Boundary conditions change the topology of path families. Let M be a path-connected topological space, and equip $C(I, M)$ with the compact–open topology. For a selected basepoint $x_0 \in M$, define the based path family $P_{x_0}M$ by requiring only $\gamma(0) = x_0$. There is a direct contraction of the entire family to the constant path. The contraction is displayed in Equation 82.

$$H_s(\gamma)(\lambda) = \gamma((1-s)\lambda), \quad 0 \leq s, \lambda \leq 1. \quad (82)$$

For compact I , the map in Equation 82 is continuous in the compact–open topology, fixes the starting point, and takes each path to the constant path at x_0 when $s = 1$. Thus the unconstrained based path space is contractible. Requiring the endpoint to equal a specified x_1 changes the result: the contraction generally violates the endpoint condition. In the special case $x_1 = x_0$, the resulting based loop space $\Omega_{x_0} M$ has components indexed by $\pi_1(M, x_0)$. If $M = S^1$, there are infinitely many such components. Hence changing the admissibility predicate from a free endpoint to a fixed endpoint can reveal nontrivial topological organization without changing the carrier manifold.

Proposition 11.4 (Endpoint conditioning changes path-space topology). *For any pointed topological space (M, x_0) , the based path space $P_{x_0} M$, equipped with the compact–open topology, is contractible. The path components of its based loop subspace $\Omega_{x_0} M$ correspond to elements of $\pi_1(M, x_0)$; in particular, the loop space is disconnected when this group is nontrivial. Path-connectedness of the full carrier M is not required.*

Proof. Equation 82 contracts $P_{x_0} M$ by shortening each path toward its designated initial point; the contraction stays within the based path space even when M has other path components. Continuity in the compact–open topology follows from continuity of composition with the interval map $(u, t) \mapsto ut$, because $[0, 1]$ is locally compact Hausdorff. A continuous path in $\Omega_{x_0} M$ is an endpoint-fixed homotopy between based loops. Conversely, an endpoint-fixed homotopy gives a path in the compact–open loop space by the exponential law for the compact parameter square. Therefore the path components of $\Omega_{x_0} M$ are exactly $\pi_1(M, x_0)$. $\square$

Proposition 11.4 shows why topology of a repertoire is an attribute of the *specified family*, including boundary constraints, rather than solely of the carrier. The ordinary topology of M and the topology of an admissible subset of $C(I, M)$ answer different questions. A generator and initial boundary set define one family, while endpoint constraints, controls, and representation rules can define another. Repertoire statements must consequently report these choices explicitly.

A bounded-speed loop repertoire on the circle. Equip S^1 with its unit-radius arc-length metric. For a real bound $v \geq 0$, let $\Omega^v S^1$ consist of based absolutely continuous loops with a.e. metric speed at most v , parameterized on $I = [0, 1]$. Each loop has a unique continuous angular lift $\theta : I \rightarrow \mathbb{R}$ with $\theta(0) = 0$; its endpoint must equal $2\pi n$ for an integer n . Under the speed bound, the possible winding indices satisfy Equation 83.

$$|n| \leq \frac{v}{2\pi}, \quad n \in \mathbb{Z}, \quad \gamma(\lambda) = e^{i\theta(\lambda)}. \quad (83)$$

The bound in Equation 83 follows from $|\theta(1) - \theta(0)| \leq \int_0^1 |\dot{\theta}| d\lambda \leq v$. Conversely, each integer in the stated interval is realized by the constant-speed lift $\theta_n(\lambda) = 2\pi n\lambda$. The path-space classification is more informative than the endpoint set: every member ends at the same point, while different winding classes form disconnected components.

Proposition 11.5 (Components of a speed-constrained circle repertoire). *For $v \geq 0$, with the uniform topology on the based loop repertoire $\Omega^v S^1$, the path components are indexed by the integers n with $|n| \leq v/(2\pi)$. In particular, the component count is the integer in Equation 84.*

$$\#\pi_0(\Omega^v S^1) = 2 \left\lfloor \frac{v}{2\pi} \right\rfloor + 1. \quad (84)$$

Proof. The lifting property of the covering map $\mathbb{R} \rightarrow S^1$ makes winding number constant along continuous families of based loops, so distinct integers yield distinct components. Every admissible loop has a lift satisfying $\theta(0) = 0$, $\theta(1) = 2\pi n$, and the Lipschitz bound $|\theta(s) - \theta(t)| \leq v|s - t|$. For two lifts with the same n , their pointwise convex interpolation preserves all three conditions and depends continuously on the interpolation parameter in the uniform topology. Projecting the interpolated lifts to S^1 gives a path through admissible loops. Thus each allowed winding class is path-connected, and Equation 83 gives precisely the stated set and count. $\square$

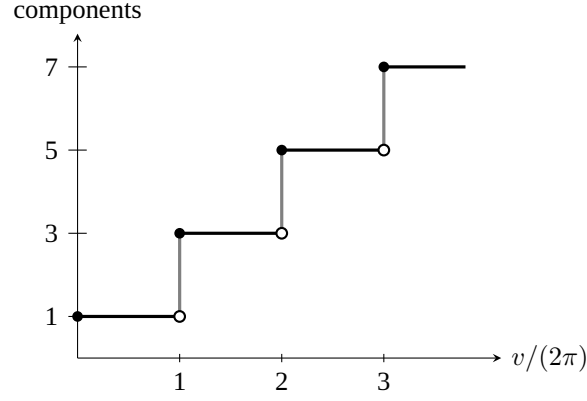


Figure 7: Path components of the bounded-speed based-loop repertoire on the unit circle, shown for $0 \leq v/(2\pi) < 4$. The step changes reflect admissibility thresholds; the carrier topology remains fixed.

Figure 7 illustrates the dependence established by Proposition 11.5. For $0 \leq v < 2\pi$ the repertoire has one component, for $2\pi \leq v < 4\pi$ it has three, and for $4\pi \leq v < 6\pi$ it has five. These are discontinuities in a *component-count statistic* as an admissibility bound varies. The underlying carrier circle is unchanged; the enlarged repertoire admits additional winding classes. The control budget v is a formal bound relative to the chosen interval and metric. Its variation neither establishes an ontological branching of physical futures nor implies a corresponding discontinuity in consciousness.

Relations between metric bounds and topological invariants. The preceding construction depends jointly on metric length, the compact parameter domain, and endpoint constraints. Replacing the unit circle by a circle of radius r changes the winding threshold from 2π to $2\pi r$ under an arc-length speed bound. Removing the speed bound permits every integer winding class. Allowing variable endpoints gives a different path space, which may be contractible as in Proposition 11.4. Passing to endpoint observations reduces all based loops to one endpoint value. Thus neither topology alone nor a scalar metric budget alone determines the structured repertoire. Comparisons across neural representations should state which metric, path regularity, sampling procedure, and admissibility conditions remain fixed. Unparameterized-path topologies and corresponding equivalence relations constitute a further mathematical issue beyond the present uniform-path example [10].

11.6 Variable Carrier Spaces and Representation Changes

Regular families and singular changes in fiber type. A smooth family of spaces is sometimes represented as the fibers of a map $p : E \rightarrow B$. If p defines a locally trivial fiber bundle, neighboring fibers are locally homeomorphic, and topology cannot change within a connected trivializing neighborhood. This is the limited topological constancy established in Proposition 9.2. More general maps can have singular fibers of different homotopy types. Consider the smooth total space and projection in Equation 85.

$$E = \{(x, y, s) \in \mathbb{R}^2 \times [0, \infty) : x^2 + y^2 = s\}, \quad p(x, y, s) = s. \quad (85)$$

The total space in Equation 85 is diffeomorphic to $\mathbb{R}^2$ through $(x, y) \mapsto (x, y, x^2 + y^2)$. Under this identification p is the radial-square function and has zero differential at the origin; hence it is not a submersion there. The regular-fiber description on $s > 0$ therefore cannot be extended as a smooth submersion across the singular fiber. For every $s > 0$, the fiber E_s is a circle; the fiber E_0 is a single point. Accordingly, $H_1(E_s; \mathbb{Z}) \cong \mathbb{Z}$ for $s > 0$ and $H_1(E_0; \mathbb{Z}) = 0$. A local trivialization near $s = 0$ is impossible because a circle is not homeomorphic to a point. The parameter s is a *level-set parameter* in the present construction. Assigning to it an interpretation as physical time, developmental stage, or historical continuation would require a separate dynamical law and measurement model.

For $s, t > 0$, radial scaling gives a natural homeomorphism $E_s \rightarrow E_t$ after the Euclidean embedding has been selected. Sending E_s to E_0 gives a continuous collapse map whose action on first homology is the zero map. The disappearance of the circular invariant describes a property of this explicitly selected comparison. It does not specify a physical mechanism that produces the collapse. In particular, a family of topological fibers, a system of comparison maps, and a generative law acting on or among the fibers are three separately defined mathematical objects.

Compatibility of cellular and smooth representations. Suppose a relational two-complex K_i is represented within a smooth or topological state space X_i by a continuous map $r_i : |K_i| \rightarrow X_i$. At another index j , let $r_j : |K_j| \rightarrow X_j$ be the corresponding representation. A proposed cellular comparison $q_{ij} : |K_i| \rightarrow |K_j|$ and a state-space comparison $h_{ij} : X_i \rightarrow X_j$ are strictly compatible when Equation 86 holds.

$$r_j \circ q_{ij} = h_{ij} \circ r_i. \quad (86)$$

Equation 86 gives a commutative square of typed maps. Without it, information carried by a cellular representation can be changed by the choice of comparison path. Even if it holds, there is no general inference that q_{ij} , h_{ij} , or either r is injective, a homotopy equivalence, or dynamically intertwining. Preservation of an admissible path family requires a further condition, stated in Equation 87.

$$(q_{ij})_*[\mathcal{R}_i] \subseteq \mathcal{R}_j, \quad (q_{ij})_*(\gamma) = q_{ij} \circ \gamma. \quad (87)$$

The inclusion in Equation 87 is an additional modeling condition on the repertoires, rather than a consequence of the underlying maps being continuous. If a mapping discards distinctions, its observed effects depend on which structural property is examined. Homology classes may merge under a quotient; admissible directed paths can become inadmissible under a map that reverses an orientation; and numerical metric statistics can change under a smooth representation that is not an isometry. A transformation equivalence, trajectory equivalence, topology-preserving comparison, and repertoire equivalence should therefore be checked separately, as the analysis in Section 5 already anticipates.

Measurement-relative geometric reconstruction. Experimental observations add another layer. Let $O_i : X_i \rightarrow Y_i$ be an observation or reconstruction map. The effective observation of a cellular history γ in $|K_i|$ is then $(O_i \circ r_i) \circ \gamma$, whenever these compositions are defined. The induced map may identify histories differing in homotopy class, in transverse perturbation response, in attachment provenance, or in their admissibility status. The equivalence class of models compatible with an observed record consequently depends on the observational interface. Geometric approaches to neural population dynamics and analyses of the topology of neural manifolds show that meaningful geometrical reconstruction can be pursued from neural measurements [40, 41]. A consciousness-specific inference additionally requires a defensible mapping from reconstructed structures to target phenomena, and an assessment of the competing explanations consistent with the same data.

Comparison of the four explanatory objects. Table 13 summarizes what the section's constructions establish and which distinctions a corresponding abstraction may lose. The table treats the hypotheses as analytical orientations and leaves their physical and phenomenal significance conditional.

The results above provide four particular comparisons. Proposition 11.1 shows that selected-path coincidence can accompany distinct generator organization and reachability. Proposition 11.2 exhibits a class of path observables whose topological content exceeds endpoint data while remaining coarser than full path structure. Proposition 11.3 shows that identical cellular homology can conceal different attaching relations and nonabelian path composition. Propositions 11.4 and 11.5 show that the topology of a repertoire depends on admissibility conditions and geometric budgets, even with a fixed carrier. These comparisons expose distinct mathematical questions rather than a mathematical ordering of the hypotheses by explanatory adequacy.

Table 13: Geometric and topological discriminants for the four preliminary consciousness hypotheses.

Orientation	Geometric or topological object	Distinction at risk under abstraction
Variation	Generated distribution, Lie brackets, perturbation directions, attainable transformations	A selected path may conceal transverse directions or permitted compositions
Trajectory	Parameterized path, differential-form periods, endpoint-fixed homotopy class	Endpoint and homology summaries may conceal trajectory or homotopy distinctions
Generativity	Attaching maps, constitutive constraints, diagrams of carriers and comparison morphisms	Terminal invariants may conceal attachment relations and intermediate constitution
Repertoire	Boundary-conditioned path family, component topology, metric or control bounds	Endpoint sets and coarse projections may identify distinct admissible path families

A physical investigation of consciousness would have to identify which relational carrier and observables are scientifically justified, what interventions make the relevant transformation or path-space properties estimable, how observations are mapped into the mathematical objects, and why a particular property should covary with or help explain a stated aspect of subjective experience. Under the present assumptions, the geometric and topological results support a differentiated formal research programme. They neither entail phenomenal sufficiency nor yield a preferred ontology of conscious subjects. Sections 12 and 13 further investigate algebraic preservation conditions and translation among representations, including cases in which apparently shared trajectories or homological invariants conceal incompatible generative structures.

12 Algebraic and Categorical Structures

Algebraic and categorical structures specify which aspects of a relational dynamics are retained when mechanisms, configurations, observational distinctions, and cellular carriers are compared. The present section examines the respective information carried by an action, its reachability relation, congruence quotient, indexed categorical description, and induced chain complex. The objective is to establish precise preservation and reconstruction criteria, together with counterexamples in which one representation agrees while another differs. The discussion proceeds from semigroup actions and observational congruences to categories of action models, coherent families of such models, and cellular maps. Its conclusions concern formally specified systems and do not by themselves identify the presence or character of consciousness.

The four hypotheses assign different candidate roles to these structures. Variation concerns the organization and composition of transformations; Trajectory concerns selected paths and factorizations; Generativity allows the models' carriers, mechanisms, admissibility rules, and distinguishing relations to enter the object of inquiry; and Repertoire concerns structured families of admissible paths or operations. Category-theoretic language is useful for separating the objects being compared from the conditions their comparisons preserve. Its standard meanings of category, functor, natural transformation, and equivalence follow established mathematical definitions [33]; the proposed application to consciousness is provisional. A comparison arrow does not acquire a physical temporal interpretation merely through its direction or compositional law.

12.1 Semigroup Actions and Transition Relations

Transformations and reachable configurations. Let $\alpha : M \times S \rightarrow S$ satisfy the left semigroup action law of Equation 19. When M lacks an identity, its unitization M^1 is used here, with the new identity acting as the identity endomap. The action specifies an endomap α_m for each mechanism and

records the way those endomaps compose. A coarser object retains only the existence of a mechanism taking one configuration to another. The reachability relation is defined in Equation 88.

$$R_\alpha = \bigcup_{m \in M^1} \text{graph}(\alpha_m) = \{(x, y) \in S \times S : \exists m \in M^1, y = \alpha(m, x)\}. \quad (88)$$

Equation 88 defines a reflexive and transitive relation: identity ensures reflexivity, and composition ensures transitivity. It may fail antisymmetry, so it is in general a preorder. Mutual reachability supplies one possible equivalence relation, which need not coincide with the congruence $\sim$ chosen in the metamodel. Reachability retains the existence of a composite but discards the mechanism realizing it, intermediate configurations, and alternative factorizations. Its use does not require a duration parameter, a global causal ordering, or a designated actual path.

A broader representation replaces maps by binary relations. In the category **Rel**, sets are objects, binary relations are morphisms, and composition existentially quantifies over intermediate states. Graphs of functions are special cases. One can replace the collection of mechanism-labeled graphs with their union, or treat a nondeterministic transition rule as a relation directly. This algebraic convenience carries no probability model and does not demonstrate that every related pair is a physically available intervention.

Proposition 12.1 (Common reachability with distinct collapse mechanisms). *Two monoid actions can have the same finite carrier and the same reachability relation while differing in whether their transformations collapse distinctions under equality congruence.*

Proof. Take $S = \{0, 1\}$. Let $M_G = \{\text{id}, \sigma\}$, where σ swaps the two states, and let $M_E = S^S$ be the full transformation monoid. Both monoids act by evaluation. Either action reaches either state from either starting state, hence $R_{\alpha_G} = R_{\alpha_E} = S \times S$. Every element of M_G is invertible, so its collapse relation under equality is empty. In M_E , the constant map c_0 with $c_0(0) = c_0(1) = 0$ has collapse set $\Delta_E(c_0) = \{(0, 1), (1, 0)\}$. $\square$

Proposition 12.1 establishes a concrete difference between transformation organization and the set of possible endpoints. Identical endpoint reachability does not determine whether noninjective transformations are admitted. In experimental applications, endpoint accessibility alone could therefore underdetermine a variation-based account; records of the applied mechanisms or interventions may be required. The finite example does not model phenomenal states.

Selected factorizations and transition categories. For a declared alphabet of generators $G \subseteq M$, a word $w = (g_1, \dots, g_k)$ and initial state x produce a discrete history with explicitly named intermediate configurations. Different words can represent the same composite $g_k \cdots g_1$ and the same endpoint, while differing in their intermediate states. The free category on a transition graph retains formal paths and their concatenations; an action-based realization sends each such path to a composite endomap or a reachable pair. This mapping can identify distinct histories. The extent of identification depends on the generator choice and the declared admissibility rules. Compositional placement in a word may be interpreted as procedural succession, but a relation between that order and physical time remains an additional modeling commitment.

12.2 Congruences, Quotients, and Distinguishability

Congruences induced by possible observations. An observation map $O : S \rightarrow Y$ determines $\ker O = \{(x, y) : O(x) = O(y)\}$. As Section 4.3 noted, this equivalence is not necessarily preserved by the transformations. For a monoid action, a finer relation can be defined by comparing the observations after every specified mechanism. Equation 89 gives this relation.

$$x \equiv_{O, M} y \iff O(\alpha(m, x)) = O(\alpha(m, y)) \quad \text{for each } m \in M. \quad (89)$$

The quantifier in Equation 89 ranges over the stipulated monoid, not over observations claimed already to have been performed. Its epistemic scope depends on whether the modeled mechanisms are available, testable, or merely counterfactual possibilities.

Proposition 12.2 (Largest action congruence within an observation kernel). *For a monoid action on S and a map $O : S \rightarrow Y$, the relation $\equiv_{O,M}$ of Equation 89 is the largest action congruence contained in $\ker O$, under inclusion of relations.*

Proof. The relation is an intersection of equivalence relations, hence an equivalence. The identity element ensures $\equiv_{O,M} \subseteq \ker O$. If $x \equiv_{O,M} y$ and $n \in M$, then for every $m \in M$ the action law yields $O(\alpha(m, \alpha(n, x))) = O(\alpha(mn, x)) = O(\alpha(mn, y)) = O(\alpha(m, \alpha(n, y)))$. Thus $\alpha(n, x) \equiv_{O,M} \alpha(n, y)$. Now let $E \subseteq \ker O$ be any action congruence. If xEy , action compatibility implies $\alpha(m, x)E\alpha(m, y)$ for each m , and $E \subseteq \ker O$ yields equality of their observations. Therefore $E \subseteq \equiv_{O,M}$. $\square$

Proposition 12.2 provides a canonical refinement once the action and observation map are specified. The refinement is the kernel of the jointly indexed observation signature in Equation 90.

$$\mathcal{O}_M : S \longrightarrow Y^M, \quad \mathcal{O}_M(x) = (O(\alpha(m, x)))_{m \in M}, \quad \ker \mathcal{O}_M = \equiv_{O,M}. \quad (90)$$

Equation 90 is a set-theoretic family of potential outputs. If an admitted family of transformations is enlarged while S and O remain fixed, more comparisons enter the signature, and its kernel can become finer. Finer discrimination in this formal sense remains relative to the admissible mechanism set and does not entail increased phenomenal discrimination.

A three-state observation example. Let $S = \{0, 1, 2\}$, with $O(0) = O(1) = A$ and $O(2) = B$. Define an idempotent transformation by $f(0) = 0$, $f(1) = 2$, and $f(2) = 2$, and take $M = \{\text{id}, f\}$. The observation kernel identifies 0 and 1, yet their transformed observations are A and B . Consequently the kernel is not a congruence, and the relation of Proposition 12.2 is equality. The same initial observation can thus be compatible with distinct outputs under a declared intervention; an autonomous action on the unrefined observation classes is not well-defined.

Quotients and commuting dynamics. For a genuine action congruence E , let $q_E : S \rightarrow S/E$ denote the quotient map and $\bar{\alpha}_m$ the induced transformation from Equation 21. Their exact compatibility is Equation 91.

$$q_E \circ \alpha_m = \bar{\alpha}_m \circ q_E \quad (m \in M). \quad (91)$$

Equation 91 retains the action on identified classes, while forgetting distinctions within classes. A second kind of information loss occurs if an induced quotient endomap identifies previously distinct quotient classes. The former loss is built into the representation map; the latter is a property of a specified mechanism acting in the quotient. Their respective inferential consequences differ, even if both are informally called “collapse.”

12.3 Categories of Represented Dynamical Systems

Objects, morphisms, and composition. A category of action descriptions establishes which translations qualify as structure-preserving. Define **ActCong** to have objects (S, M, α, E) , where M is a monoid acting unitaly on a set S (its identity acts as the identity map) and E is an action-compatible equivalence relation. This convention is stronger than the semigroup-composition axiom alone, and makes the one-object-category interpretation below well defined. A morphism $(h, \phi) : (S, M, \alpha, E) \rightarrow (T, N, \beta, F)$ consists of a function $h : S \rightarrow T$ and a unital monoid homomorphism $\phi : M \rightarrow N$ satisfying the two conditions in Equation 92.

$$h(\alpha(m, x)) = \beta(\phi(m), h(x)), \quad xEy \implies h(x)Fh(y). \quad (92)$$

The first condition in Equation 92 is action intertwining, and the second is forward preservation of congruence. These are strict, testable algebraic requirements on a proposed translation. Two composable morphisms (h, ϕ) and (k, ψ) have composite $(k \circ h, \psi \circ \phi)$. The intertwining equations and congruence implications are stable under composition, and identity pairs act as identities. Hence these objects and morphisms form a category. This construction specializes the cross-model equivariance relation of Section 9.2, with monoids used to make identities explicit.

The morphisms may change the carrier and the mechanism monoid; neither map is required to be injective or surjective. In particular, a surjective carrier map can identify source configurations, and a mechanism homomorphism can identify distinct source mechanisms. Reflection of congruence, invertibility, preservation of admissible paths, and compatibility with geometry are additional properties. A category with stronger morphism requirements could be chosen for specific explanatory questions, but such choices would change the meanings of invariance and equivalence.

A quotient functor. There is a quotient assignment $Q : \mathbf{ActCong} \rightarrow \mathbf{Set}$ which takes an action model to its set of congruence classes and a morphism to its induced function. Equation 93 specifies both assignments.

$$Q(S, M, \alpha, E) = S/E, \quad Q(h, \phi)([x]_E) = [h(x)]_F. \quad (93)$$

The second expression of Equation 93 is well-defined because morphisms preserve congruence. As an assignment into $\mathbf{Set}$, Q retains neither mechanism labels nor the induced quotient actions as structured objects, even though those actions informed its definition. An action-valued quotient construction could retain more information; the codomain of the functor is therefore consequential.

Proposition 12.3 (Quotient functor and nonreflection of isomorphism). *The assignment Q of Equation 93 is a functor. There exists a morphism whose image under Q is an isomorphism of sets, although the original morphism is not an isomorphism of action models.*

Proof. Preservation of identities is immediate. For composable morphisms (h, ϕ) and (k, ψ) , the induced composite sends $[x]$ to $[k(h(x))]$, as does $Q(k \circ h, \psi \circ \phi)$; thus Q preserves composition. For the second assertion take singleton mechanism monoids acting trivially on $S = \{0, 1\}$ and $T = \{*\}$, and give both carriers the indiscrete congruence. The unique function $h : S \rightarrow T$ and the unique monoid homomorphism form a morphism. Both quotients are singletons, so $Q(h, \phi)$ is bijective. The function h has no inverse, hence (h, ϕ) is not an isomorphism in $\mathbf{ActCong}$. $\square$ $\square$

Proposition 12.3 shows that Q fails to *reflect isomorphisms*. The source and target can become isomorphic after quotienting while retaining different configuration-level cardinalities. More generally, the existence of a well-behaved abstract representation is weaker than an equivalence between its richer sources. This gives a formal reason to avoid treating identical observable classes as proof of identical constitutive dynamics.

Monoid actions as one-object functors. For a monoid M , let BM denote its one-object category, with monoid elements as morphisms and multiplication as composition. An action α on S defines a functor $A : BM \rightarrow \mathbf{Set}$, assigning S to the unique object and α_m to each element m . A map intertwining two actions of the same monoid is precisely a natural transformation between their action functors. When the mechanism monoids differ, a homomorphism $\phi : M \rightarrow N$ determines a functor $B\phi : BM \rightarrow BN$. A comparison (h, ϕ) then gives a natural transformation $A \Rightarrow A' \circ B\phi$, where $A' : BN \rightarrow \mathbf{Set}$ is the target action functor.

This description separates changes to the action on a fixed mechanism category from translations that alter the mechanism category itself. It also clarifies why an ordinary natural transformation of fixed-index actions does not suffice when the transformation alphabets differ. In physical applications the mechanism category, its generators, and the legitimacy of their compositions require separate justification.

12.4 Functors, Natural Transformations, and Equivalences

Indexed generative descriptions. Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \rightarrow \mathbf{ActCong}$ a functor. Each $i \in \mathcal{I}$ names a represented action model; an arrow $u : i \rightarrow j$ specifies a coherent model comparison. The functorial laws are stated in Equation 94.

$$D(\text{id}_i) = \text{id}_{D(i)}, \quad D(v \circ u) = D(v) \circ D(u) \quad \text{for } u : i \rightarrow j, v : j \rightarrow k. \quad (94)$$

Equation 94 expresses compatibility of comparisons along different compositional routes. An index arrow may record a revision of theoretical resolution, a proposed intervention, a relation of structural dependence, or a model of process succession. Its physical meaning is not supplied by the axioms of a category. A diagram indexed by a poset does not automatically impose a globally valid chronology or make one object the actual present.

For two diagrams $D, E : \mathcal{I} \rightarrow \mathbf{ActCong}$, a natural transformation $\eta : D \Rightarrow E$ assigns a morphism $\eta_i : D(i) \rightarrow E(i)$ to each index. Coherence across comparison arrows is expressed by Equation 95.

$$E(u) \circ \eta_i = \eta_j \circ D(u) \quad \text{for each } u : i \rightarrow j. \quad (95)$$

The naturality condition of Equation 95 requires that translating a source model and then following a target comparison agree with following the source comparison and then translating. It is consequently stronger than a family of unrelated state maps. It does not guarantee that the translations are invertible, reflect distinctions, or describe the same physical substrate.

Proposition 12.4 (Naturality of quotient comparisons). *Let $D, E : \mathcal{I} \rightarrow \mathbf{ActCong}$ be functors and $\eta : D \Rightarrow E$ a natural transformation. Then $QD, QE : \mathcal{I} \rightarrow \mathbf{Set}$ are functors and the family $Q\eta_i$ is a natural transformation $Q\eta : QD \Rightarrow QE$. An isomorphism at each action-model component implies a natural isomorphism of quotient diagrams. The converse may fail.*

Proof. Composition of functors establishes the first assertion. Applying Q to Equation 95 and using functoriality gives $(QE)(u) \circ Q\eta_i = Q\eta_j \circ (QD)(u)$, proving naturality. Functors preserve isomorphisms, so componentwise action-model isomorphisms yield componentwise quotient isomorphisms and hence a natural isomorphism. For the converse, choose $\mathcal{I}$ to be the terminal one-object category and use the noninvertible morphism of Proposition 12.3; its quotient map is invertible. $\square$ $\square$

Proposition 12.4 makes the information-loss issue diagrammatic. Exact agreement of entire quotient diagrams may conceal carrier, mechanism, or congruence differences at each index. Even a natural isomorphism between diagrams of observed classes therefore need not establish an isomorphism of the associated generative descriptions. The categorical distinction becomes more consequential when observations are taken at different resolutions or across heterogeneous model families.

Levels of equivalence. Isomorphism of action models, natural isomorphism of fixed model diagrams, equivalence of categories, isomorphism of quotient diagrams, and equality of reachable endpoints express different claims. A categorical equivalence concerns a pair of functors between categories together with suitable natural isomorphisms; it does not automatically identify the physical interpretations of objects. A chosen functor may preserve compositional structure while forgetting distinctions central to one hypothesis. Table 14 summarizes the retained and potentially forgotten structures.

12.5 Cellular Maps, Chain Complexes, and Induced Morphisms

Cellular functoriality. Let K be a CW-complex with n -skeleton $K^{(n)}$. Its cellular chain groups are free abelian on the oriented cells in each dimension. The chain structure needed for the current comparison is given in Equation 96.

$$\cdots \longrightarrow C_2(K; \mathbb{Z}) \xrightarrow{\partial_2} C_1(K; \mathbb{Z}) \xrightarrow{\partial_1} C_0(K; \mathbb{Z}), \quad \partial_{n-1} \circ \partial_n = 0. \quad (96)$$

Table 14: Preservation and reconstruction boundaries for selected algebraic and categorical representations. The consciousness-related significance of these boundaries requires further domain assumptions.

Representation	Guaranteed information under its assumptions	Information not generally determined
Reachability relation	Existence of an admitted composite between states	Transformation labels, factorization history, collapse relations
Action congruence quotient	Composite action on equivalence classes	Differences within classes, source carrier cardinality
Action-model morphism	Intertwining and forward preservation of equivalence	Invertibility, distinction reflection, physical meaning
Natural transformation	Coherence across indexed comparisons	Componentwise bijectivity or phenomenal identity
Cellular chain map	Boundary compatibility and induced homology maps	Attaching words, fundamental groups, constrained repertoires

For CW-complexes with cellular maps, the assignment $K \mapsto C_*(K; \mathbb{Z})$ is functorial into the category of chain complexes of abelian groups [19]. A cellular map $f : K \rightarrow L$ induces homomorphisms $f_{\#,n} : C_n(K) \rightarrow C_n(L)$ satisfying Equation 97.

$$\partial_n^L \circ f_{\#,n} = f_{\#,n-1} \circ \partial_n^K, \quad H_n(f) : H_n(K; \mathbb{Z}) \longrightarrow H_n(L; \mathbb{Z}). \quad (97)$$

The chain-map identity in Equation 97 ensures that cellular cycles are sent to cycles and boundaries to boundaries, producing the induced homology homomorphism. A continuous map between CW-complexes may require cellular approximation before this construction is applied in the stated form. Neither chain compatibility nor homology functoriality, by itself, establishes a physical interpretation of a map between relational structures.

Proposition 12.5 (Nonfaithfulness of the cellular chain assignment). *There exist distinct cellular homeomorphisms of the same CW-complex that induce identical maps on its complete cellular chain complex.*

Proof. Let $K = S^1 = \mathbb{R}/\mathbb{Z}$ carry one zero-cell $[0]$ and one open one-cell. For $0 < \varepsilon < 1$, define $f_0([t]) = [t]$ and $f_\varepsilon([t]) = [t + \varepsilon \sin(2\pi t)/(2\pi)]$. The lifted map for f_ε has derivative $1 + \varepsilon \cos(2\pi t) > 0$ and satisfies $f_\varepsilon(t+1) = f_\varepsilon(t) + 1$. Therefore it defines an orientation-preserving circle homeomorphism fixing $[0]$. Both maps are cellular and induce the identity on the one zero-cell and the single oriented one-cell, so their cellular chain maps agree in each degree. Since the sinusoidal term is nonzero for some values of t , the continuous self-maps are distinct. $\square$

Proposition 12.5 shows that the cellular chain functor is not faithful on cellular maps: a full chain-map record can still fail to recover its generating topological map. Another loss of information was established in Proposition 11.3: two relational two-complexes have identical integral cellular chain complexes, yet their fundamental groups differ. The former result concerns different maps between the same carriers; the latter concerns distinct carriers with the same cellular chain data. These failures have different implications for reconstructing generative organization.

Attaching words and nonabelian constraints. For a connected two-complex formed by attaching two-cells to a wedge of circles, fundamental-group relations are specified by attaching words modulo normal closure. The cellular boundary ∂_2 retains their abelianized exponent sums, while the noncommutative arrangement of generators can affect the fundamental group. Thus an exact chain comparison may lose information about higher-order cellular attachment even when it preserves integral homology. If a relational model assigns interpretive roles to attaching maps, face incidence, or admissible coupling arrangements, those data should be retained independently of chain-level summaries. A cell complex is a mathematical representation of such roles; the construction alone does not identify neurons with vertices or faces with conscious events.

Preservation of constrained path families. A cellular map may also fail to preserve the dynamics admitted on the geometric realizations. Let $\mathcal{R}_K \subseteq C(I, |K|)$ and $\mathcal{R}_L \subseteq C(I, |L|)$ be model-relative admissible path families. A continuous map $f : |K| \rightarrow |L|$ is repertoire-preserving only when it satisfies Equation 98.

$$f_*\mathcal{R}_K \subseteq \mathcal{R}_L, \quad f_*(\gamma) = f \circ \gamma. \quad (98)$$

Equation 98 is separate from cellularity and chain-map compatibility. For example, take $K = L = S^1$ and f to be the identity cellular map. Assign the source and target the speed-limited based-loop repertoires of Proposition 11.5, with respective speed bounds $v_K > v_L > 0$. There are source loops that exceed the target bound, so the identity map fails the inclusion in Equation 98, even though it induces isomorphisms on cellular chains, fundamental groups, and homology. This illustrates why a morphism adequate to one hypothesis may be insufficient for another.

Formal comparison and explanatory interpretation. The algebraic constructions in this section help audit proposed explanatory invariants across the four hypotheses. A variation-based account requires explicit attention to actions and their generating mechanisms. A trajectory-based account needs the selected factorizations, paths, or homotopy data relevant to its explanandum. A generativity-based account may require changing action objects and comparisons of their constitutive relations. A repertoire-based account additionally needs declared admissibility conditions on whole families of histories. Exact preservation in one formal category supplies neither full reconstruction across these objects nor an empirical bridge to phenomenal experience.

Section 13 will examine translations among heterogeneous representations, including partial and approximate correspondences. The present section supplies strict mathematical benchmarks for that inquiry: action intertwining, quotient congruence, naturality, chain-map compatibility, and the separate preservation of admissible paths. Its counterexamples establish the need to specify which distinctions a proposed translation preserves, which it loses, and which remain unexamined by the available evidence.

13 Translation and Abstraction Among Generative Representations

A mathematical translation links descriptions constructed with different state carriers, transformation languages, observational codomains, or rules of admissibility. Its adequacy depends on the information that the receiving description is required to preserve. This section develops a comparative framework for translations among continuous dynamical systems, finite-dimensional approximations, labeled transition systems, and symbolic rewriting descriptions. It distinguishes exact semiconjugacy, simulation, bisimulation, sound overapproximation, and bounded approximation error. It then examines partial translations and the losses produced by many-to-one representations, before using generative grammar as a deliberately secondary example. The primary methodological purpose is to specify which claims about Variation, Trajectory, Generativity, or Repertoire are warranted by a particular representation bridge. Mathematical correspondence alone supplies no interpretation of subjective experience.

The word *translation* is used in a model-relative sense. A map may translate states, selected paths, admitted operations, observational predictions, or families of constitutive descriptions; these are different tasks. A correspondence can preserve a target observable while altering other mathematically meaningful distinctions. Accordingly, we specify the compared carriers, the chosen observations, and the structure-preservation conditions separately. The composition of translations may be ordered mathematically without introducing a privileged physical present or assuming a unique direction of historical becoming. The constructions build on the functorial and quotient-level distinctions of Section 12; their connection to consciousness remains a research hypothesis requiring physical and empirical mediation.

13.1 PDEs, Discrete Dynamics, and Rewriting Systems

Continuous dynamics and finite-dimensional reductions. A partial differential equation specifies admissible changes of a field under suitable regularity, domain, and boundary conditions. A discrete state-

space model instead specifies an update or transition on a possibly finite carrier. The relation between these descriptions may arise through an invariant restriction, projection, sampling, spatial discretization, time-stepping method, or an inferred effective model. The term *discretization* therefore names a family of procedures rather than a single representation-preserving map. Their analytical guarantees depend on the governing equation and on the approximation regime; for classical stability and convergence theory of linear finite-difference equations, see Lax and Richtmyer [26].

Consider the heat equation on the circle $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, with $\kappa > 0$, as an illustrative evolution equation on a fixed physical domain. A mathematical parameter $\tau \geq 0$ indexes its solution semigroup and receives a physical time interpretation only when the model is related to a specified physical process. The equation and its Fourier-mode solutions are given in Equation 99.

$$\partial_\tau u(\theta, \tau) = \kappa \partial_{\theta\theta} u(\theta, \tau), \quad u(\theta, \tau) = \sum_{k \in \mathbb{Z}} \hat{u}_k(0) e^{-\kappa k^2 \tau} e^{ik\theta}. \quad (99)$$

For initial data in $L^2(\mathbb{S}^1)$, Equation 99 is understood in the usual semigroup sense; additional regularity yields the corresponding classical interpretation. Let Φ_τ denote this evolution and let P_N retain the Fourier modes $|k| \leq N$. The finite-dimensional carrier $E_N = \text{im } P_N$ is invariant under the heat semigroup, with restricted evolution $\Phi_\tau^{(N)}$. Exact preservation is expressed by Equation 100.

$$P_N \circ \Phi_\tau = \Phi_\tau^{(N)} \circ P_N, \quad \Phi_\tau^{(N)} = \exp(\tau A_N), \quad A_N e^{ik\theta} = -\kappa k^2 e^{ik\theta}. \quad (100)$$

Here the same basis vectors are regarded as elements of L^2 or E_N as appropriate. Equation 100 is an exact semiconjugacy of the parameterized semigroup action. It is not generally an injective representation of the initial field: distinct high-frequency components can share the same projection. Thus exact preservation of projected dynamics coexists with loss of source distinctions.

Proposition 13.1 (Exact spectral translation and unresolved source modes). *For the heat semigroup on $L^2(\mathbb{S}^1)$, the Fourier projection P_N intertwines the full and restricted semigroups as in Equation 100. For every finite N , the projection is noninjective, and there exist distinct source solutions whose projected trajectories agree for every $\tau \geq 0$.*

Proof. Each Fourier basis vector $e^{ik\theta}$ is an eigenvector of the Laplacian with eigenvalue $-k^2$. Both $P_N \Phi_\tau u$ and $\Phi_\tau^{(N)} P_N u$ retain exactly the terms with $|k| \leq N$, with equal multipliers $e^{-\kappa k^2 \tau}$. This gives the intertwining identity. Select any initial field u and a nonzero Fourier mode $v = e^{i(N+1)\theta}$. Then $P_N u = P_N(u + v)$, while $u \neq u + v$. Intertwining shows that the projected trajectories coincide for all $\tau \geq 0$. $\square$

Proposition 13.1 separates exact projected prediction from recovery of the complete field. The finite-dimensional system is mathematically closed under this specific evolution, but its completeness is relative to the projected observables. If a candidate consciousness correlate depends on suppressed modes or their coupling to other dynamics, the example supplies no guarantee that the correlate survives the reduction. Conversely, any claim explicitly restricted to the retained Fourier coefficients can be assessed within the exact projected system.

Sampling and discretization error. A common spatial discretization of the same equation evaluates a field at J equally spaced grid points, with spacing $h = 2\pi/J$. For a sampled Fourier mode $e^{ik\theta}$, the standard centered second-difference operator has eigenvalue given by Equation 101.

$$\lambda_{k,h} = -\frac{4\kappa}{h^2} \sin^2\left(\frac{kh}{2}\right), \quad \lambda_k = -\kappa k^2, \quad \lambda_{k,h} = \lambda_k + O(h^2 k^4) \quad (h \rightarrow 0, k \text{ fixed}). \quad (101)$$

Equation 101 makes the representational difference visible even for one resolved nonconstant mode: the finite-difference and continuum decay rates are generally unequal at fixed positive h . Grid sampling also aliases modes whose indices differ by multiples of J . An explicit time integrator introduces another

approximation. For the N -mode generator A_N , forward Euler uses $F_\delta = I + \delta A_N$ in place of $\exp(\delta A_N)$; modal stability requires appropriate restrictions on δ relative to N and κ . Statements about convergence require a declared norm, refinement regime, initial regularity, and stability assumptions. This mathematical distinction should remain explicit when a neural field equation is translated into a computational simulation.

Rewriting as a distinct form of generation. A rewriting system $\mathcal{W} = (\mathcal{T}, \mathcal{R})$ has a carrier of terms, graphs, or cellular descriptions and a family of admitted rewrite rules. Its generating relation acts on representations, with each rewrite producing a locally specified successor or a family of possible successors. A sequence of rewrites retains derivational information that the endpoint alone may erase. In a graph or two-complex rewriting model, rules may additionally change the available cells and incidence data; the carrier of the induced geometric realization can therefore vary. The existence of such rewriting rules is a representational choice and does not assert that neuronal or experiential generation consists literally of symbolic productions.

One bridge from a rewriting model to a discrete dynamical description retains only an observable state $O(t)$ of each term $t \in \mathcal{T}$. When two distinct terms yield the same observation and admit successors with different observed behavior, the observable classes may fail to support an autonomous deterministic update. The observation-stabilized congruence of Proposition 12.2 offers one exact remedy when the specified rewrite mechanisms form an action. For nondeterministic rewriting, relational abstraction and simulation provide a more general setting.

13.2 Semiconjugacy, Simulation, and Bisimulation

Different degrees of dynamic correspondence. Let $F : X \rightarrow X$ and $G : Y \rightarrow Y$ be deterministic self-maps and let $h : X \rightarrow Y$ be a proposed state translation. An exact semiconjugacy is defined by the commuting relation in Equation 102.

$$h \circ F = G \circ h, \quad h(F^n x) = G^n(hx) \quad (n \in \mathbb{N}). \quad (102)$$

The second identity follows by induction from the first. The relation preserves the translated discrete orbit and its step indexing, subject to the modeled meaning of a step. If h is bijective and has a compatible inverse, the systems are conjugate. A many-to-one h permits exact reduced dynamics without state reconstruction; the projection in Proposition 13.1 is an example for parameterized semigroups. Semiconjugacy does not, by itself, preserve the description of every source mechanism if the full model has a family of transformations rather than a single F .

For nondeterministic systems, fix a common transition-label alphabet Λ , and write $x \xrightarrow{a}_X x'$ and $y \xrightarrow{a}_Y y'$ for $a \in \Lambda$. The labels describe declared operations or observations; any relation between their order and empirical time must be justified separately. A relation $B \subseteq X \times Y$ is a *forward simulation* when related source transitions have label-matched related target transitions, as specified in Equation 103.

$$(x, y) \in B, \quad x \xrightarrow{a}_X x' \implies \exists y' \in Y : y \xrightarrow{a}_Y y' \text{ and } (x', y') \in B. \quad (103)$$

A *strong bisimulation* also requires the converse matching condition for every target transition from a related pair. Standard process-equivalence terminology is used in its established mathematical sense [30]. Extra observation predicates, termination conditions, or continuous outputs can be included by augmenting the conditions on related states; none is automatic in Equation 103.

Proposition 13.2 (Simulation and finite trace preservation). *If B satisfies Equation 103 and $(x, y) \in B$, every finite label trace executable from x is executable from y . If B is a strong bisimulation, the sets of executable finite label traces from x and y coincide. Equality of these trace sets does not in general imply strong bisimilarity.*

Proof. For a finite source trace, apply Equation 103 successively to each transition, beginning with (x, y) . This constructs a target path with the same labels, proving inclusion. Under the reverse matching condition, the identical argument with source and target interchanged gives equality. For failure of the converse, consider the rooted systems in the paragraph below, which have equal trace sets but incompatible branching conditions. $\square$ $\square$

A branching counterexample. Consider System A with a root r and a unique a -transition $r \xrightarrow{a} p$; the state p has both $p \xrightarrow{b} \bullet$ and $p \xrightarrow{c} \bullet$, where the terminal symbols denote dead-end states. System B has a root s with two transitions $s \xrightarrow{a} q_b$ and $s \xrightarrow{a} q_c$; the former state has only a b -transition, and the latter only a c -transition. Both rooted systems permit exactly the finite label traces $\{\epsilon, a, ab, ac\}$. Their roots are not strongly bisimilar: matching $r \xrightarrow{a} p$ to either a -successor of s leaves one of the b or c transitions from p unmatched. Proposition 13.2 therefore distinguishes an observationally matching trace repertoire from the branching organization of choices within that repertoire.

A path-based explanatory candidate might depend on a selected trace and be insensitive to this example's branching distinction, once its explanandum is specified. A Variation or Repertoire candidate might depend on the choice structure or its intermediate availability. A Generativity candidate may require still richer information about how the admissible branches were constituted. The mathematical example supports this separation of questions; it does not rank the four hypotheses empirically.

Path-space correspondence and the role of choice. Strong bisimulation ensures matching of finite transition steps; a construction of matched infinite paths additionally invokes choices or compactness and branching assumptions appropriate to the path model. Moreover, label traces need not preserve durations, geometry, homotopy classes, or the topology of a continuous trajectory space. If the represented paths possess such structure, a bridge must carry explicit continuity, parameter, orientation, or homotopy conditions beyond Equation 103. The choice of a behavioral equivalence is thus part of the modeling assumptions, not an invariant fixed in advance by the term *translation*.

13.3 Structure-Preserving and Structure-Forgetting Maps

Exact maps and sound abstractions. A translation may preserve an operation exactly, preserve it up to an equivalence, or overapproximate its possible effects. To distinguish these cases, let $R_X \subseteq X \times X$ be a transition relation and $h : X \rightarrow Y$ a state abstraction. Write $\text{Post}_X(A)$ for the set of source successors of states in $A \subseteq X$. The direct-image and inverse-image operators defined by h are $\alpha_h(A) = h(A)$ and $\gamma_h(B) = h^{-1}(B)$. They form a Galois connection, characterized by Equation 104.

$$\alpha_h(A) \subseteq B \iff A \subseteq \gamma_h(B), \quad A \subseteq X, B \subseteq Y. \quad (104)$$

Equation 104 is a set-theoretic adjunction. Abstract interpretation more generally uses ordered abstract domains and sound approximations of concrete computations [13]. In the present elementary instance, the smallest relational transition rule on Y containing all images of source transitions has the successor operator in Equation 105.

$$\text{Post}^\sharp(B) = \alpha_h(\text{Post}_X(\gamma_h(B))), \quad \alpha_h(\text{Post}_X(A)) \subseteq \text{Post}^\sharp(\alpha_h(A)). \quad (105)$$

The inclusion of Equation 105 is a sound overapproximation of one-step reachable abstract states. Its right-hand side can include successors generated from a concrete state in the same abstract fiber as a reachable state, although that particular concrete state was never reached. This mechanism can introduce spurious abstract histories after successive steps.

Proposition 13.3 (Sound abstraction with a spurious abstract history). *There exist finite transition systems and state abstractions h for which Equation 105 is sound at every step, yet a two-step path in the abstract transition graph has no lift from a specified concrete initial state.*

Table 15: Translation conditions and their respective guarantees. A guarantee is relative to the mathematical objects and hypotheses listed in the same row.

Translation form	Retained mathematical information	Further information requiring separate conditions
Semiconjugacy	Translated deterministic orbits	Source-state injectivity, generator identities, physical time
Forward simulation	Inclusion of finite labeled trace sets	Converse matching, branching structure, geometric paths
Strong bisimulation	Matched labeled transition branching	Path-space topology, durations, homotopy, embodied semantics
Sound relational abstraction	Overapproximation of concrete successors	Liftability of abstract histories, precision of reachability
Cellular chain map	Boundary compatibility and induced homology	Attaching maps, homotopy type, admissible mechanisms
Approximate commuting map	Quantified finite-horizon error under assumptions	Exact structural equivalence, long-horizon fidelity

Proof. Let $X = \{p, q, r, s\}$, with exactly two nonidentity transitions $p \rightarrow q$ and $r \rightarrow s$. Let $Y = \{A, B, C\}$ and define $h(p) = A$, $h(q) = h(r) = B$, and $h(s) = C$. The induced abstract transitions are $A \rightarrow B$ and $B \rightarrow C$. Hence the abstract path $A \rightarrow B \rightarrow C$ exists. From the concrete initial state p , the only nontrivial successor is q , which has no successor. Therefore the abstract two-step path has no concrete lift beginning at p . The one-step inclusion remains sound by the construction in Equation 105. $\square$

Proposition 13.3 distinguishes guaranteed coverage of source behaviors from guaranteed realizability of each abstract behavior. An overapproximation can be useful for conservative exclusion or safety reasoning, but it does not warrant interpreting all abstract paths as actual or even jointly admissible source paths. In consciousness research, a coarse observable transition diagram may therefore represent a larger set of formal possibilities than the underlying model supports. Whether this loss matters depends on the hypothesis and on which observables have empirical support.

Structural invariants carried by a map. Each mathematical category supplies its own preservation test. A smooth map may intertwine vector fields, as in the projectability conditions of Section 5; a cellular map induces chain maps; a functor preserves identities and composition; a state translation satisfying Equation 102 preserves reduced orbit dynamics. None of these conditions ensures the others without additional structure. In particular, a continuous map that collapses a circle to a point sends all winding classes to one class, whereas a cellwise map with a chain-level correspondence need not preserve a prescribed family of control-admissible paths. Table 15 records the distinct obligations.

13.4 Partial, Approximate, and Non-Equivalent Correspondences

Restrictions of domain and admissible mechanisms. A translation can be exact only on a chosen invariant subspace, an experimentally accessible regime, or a subset of admissible transformations. Let $D \subseteq X$ be a declared translation domain. A partial semiconjugacy $h : D \rightarrow Y$ between F and G satisfies the same identity as Equation 102 for those $x \in D$ for which $F(x) \in D$. Forward invariance $F(D) \subseteq D$ permits arbitrary finite iteration; in its absence the comparison applies to path segments that remain in D . A translation between indexed constitutive models may likewise exist only for some objects and arrows of the index category. The absent domain is a scientific limitation, and an extension requires an additional construction rather than an assertion of generality.

For heterogeneous dynamical representations, a more realistic requirement may be a small residual rather than exact commutation. Suppose X and Y are metric spaces, $F : X \rightarrow X$, $G : Y \rightarrow Y$, and

$h : X \rightarrow Y$. The one-step defect is bounded by Equation 106, where G is L -Lipschitz on the relevant region.

$$d_Y(h(Fx), G(hx)) \leq \varepsilon, \quad d_Y(Gy, Gz) \leq Ld_Y(y, z). \quad (106)$$

The inequality supplies a local, norm-dependent translation guarantee. It does not alone control error over indefinitely many iterations. The finite-iteration result follows in Equation 107.

$$d_Y(h(F^n x), G^n(hx)) \leq \varepsilon \sum_{j=0}^{n-1} L^j = \begin{cases} \varepsilon \frac{L^n - 1}{L - 1}, & L \neq 1, \\ n\varepsilon, & L = 1. \end{cases} \quad (107)$$

Proposition 13.4 (Propagation of a one-step translation defect). *Assume Equation 106 holds along the F -orbit of x and that the L -Lipschitz bound applies to the target points compared at each iteration. Then Equation 107 holds for every integer $n \geq 1$ for which these assumptions are satisfied.*

Proof. Let $e_n = d_Y(h(F^n x), G^n(hx))$. By the triangle inequality, the one-step defect, and the Lipschitz bound, $e_{n+1} \leq \varepsilon + Le_n$, while $e_0 = 0$. Induction gives $e_n \leq \varepsilon \sum_{j=0}^{n-1} L^j$. Evaluation of the geometric sum proves Equation 107. $\square$ $\square$

The bound distinguishes contraction ($L < 1$), nonexpansion ($L = 1$), and potentially amplifying target dynamics ($L > 1$). These regimes describe the specified metric model. A small one-step discrepancy can be compatible with a large long-horizon discrepancy when perturbations amplify. In numerical PDE comparisons, consistency and stability hypotheses require separate examination, as illustrated by Equation 101; the classical linear theory [26] supplies one well-defined setting. Empirically estimated errors also depend on measurement uncertainty and system identification, which the present proposition does not include.

Non-equivalent mathematical objects. Sometimes the intended translation changes what counts as a state, event, or admissible history. A graph-rewriting process that creates a two-cell can be described by a changing CW-complex, whereas a fixed-dimensional neural state-space model can track only selected summaries of that event. A translation may preserve the resulting Betti numbers while losing the attaching word, or preserve a collection of observations while discarding the procedure generating them. Some comparisons may be meaningful only through a common third representation or through a span of partial maps rather than a total function. Such arrangements should identify the respective domains, equivalence relations, and nontranslated remainders.

13.5 Lost Distinctions, Semantic Adequacy, and Abstraction Costs

Information loss and reconstruction. For a map $h : A \rightarrow B$ between sets of mathematical descriptions, its fiber relation $\ker h$ identifies descriptions producing the same target representation. Relative to equality on A , the lost distinctions are the off-diagonal fiber pairs defined in Equation 108.

$$\mathcal{L}_h = \{(a, a') \in A \times A : a \neq a', h(a) = h(a')\}. \quad (108)$$

This loss is relative to the declared source carrier A . When A consists of states, paths, generators, or whole indexed models, the same expression measures different sorts of identification. A quotient can be fully adequate for an intended observational claim while losing distinctions important to another explanatory question. The loss relation therefore requires an explicitly named object type.

Proposition 13.5 (Monotonic loss under compositional abstraction). *For functions $h : A \rightarrow B$ and $k : B \rightarrow C$, the lost-distinction relations satisfy $\mathcal{L}_h \subseteq \mathcal{L}_{k \circ h}$. Moreover, for a function $v : A \rightarrow V$, there exists $\bar{v} : h(A) \rightarrow V$ with $v = \bar{v} \circ h$ if and only if v is constant on each fiber of h .*

Table 16: A structured audit of possible abstraction losses across the four hypothesis families. Each entry requires a model-relative adequacy test.

Hypothesis	Candidate information to preserve	Typical hidden distinctions
Variation	Admitted generators, composition, perturbation responses	Mechanism labels, transverse dynamics, noninjective transformations
Trajectory	Selected paths with relevant parameter and geometry	Intermediate states, winding classes, event ordering
Generativity	Constitutive data and maps among evolving descriptions	Changes of carrier, constraints, attaching maps, observational partitions
Repertoire	Whole admissible path families and their organization	Branching, forbidden paths, component topology, accessibility bounds

Proof. If $a \neq a'$ and $h(a) = h(a')$, applying k gives $k(h(a)) = k(h(a'))$, proving the inclusion. If $v = \bar{v} \circ h$, equal h -images have equal v -images. Conversely, if v is constant on each fiber, define $\bar{v}(h(a)) = v(a)$. Fiber constancy makes this definition independent of the chosen representative, and the factorization follows. $\square$ $\square$

Proposition 13.5 gives an exact test for whether a source property can be reconstructed from a translated representation. The use of $h(A)$, rather than all of B , matters: extension beyond the actual image may require separate assumptions. Composed translations cannot recover distinctions already identified by their earlier maps through deterministic postprocessing alone. However, different external measurements or interventions may supply new information not contained in the original map. This distinction between postprocessing and additional epistemic access is essential when comparing recorded data and modeled unobserved dynamics.

Semantic adequacy relative to a claim. A proposed translation is adequate only relative to a stated family of predicates, observables, or structural invariants. For example, retaining one-point observation sequences may suffice to compare marginal predictions yet be insufficient for joint correlations or intervention-conditioned repertoires. Preserving homology groups does not ensure preservation of fundamental groups, as Proposition 11.3 illustrates. Preserving finite traces does not ensure branching equivalence, as Proposition 13.2 illustrates. A semantic-adequacy statement should therefore provide the source property, target reconstruction rule, admissible interventions, and the regime over which the claim is evaluated.

A single scalar *abstraction cost* would require commensurable weights for losses that inhabit different mathematical and empirical domains. Without those weights, it is more informative to maintain a structured loss profile. Table 16 illustrates the minimum distinctions relevant to the four hypotheses. The table identifies possible losses rather than asserting that each must be phenomenally relevant.

Epistemic and phenomenal scope. A representation may be adequate for predicting a neurophysiological measurement yet inadequate for a theory of first-person experience. The formal loss profile neither determines which physiological variables carry phenomenal significance nor supplies a test for subjective report validity. These are matters for experimental design and theory comparison. A responsible explanatory programme should state which candidate property is operationalized, what apparatus observes it, and what rival structures could yield equivalent observations. Translation results then delimit what the chosen evidence can distinguish; they do not settle consciousness by mathematical convention.

13.6 Generative Grammar as a Secondary Illustrative Model

Generated expressions and derivational organization. Formal grammars provide an accessible instance in which a generated output language can agree across systems while the derivation structures

differ. Let a context-free grammar be $\mathcal{G} = (N, \Sigma, P, S)$, with nonterminals N , terminals Σ , productions P , and designated start symbol S . Its generated language records terminal strings derivable from S ; derivation trees record additional structural information. These objects have different equivalence criteria. The following pair is defined by Equation 109.

$$\begin{aligned} \mathcal{G}_1 : \quad & S \rightarrow AB \mid CD, \quad A \rightarrow a, B \rightarrow b, C \rightarrow a, D \rightarrow b, \\ \mathcal{G}_2 : \quad & S \rightarrow AB, \quad A \rightarrow a, B \rightarrow b. \end{aligned} \tag{109}$$

Both grammars generate the language $\{ab\}$. Grammar $\mathcal{G}_1$ has two derivation trees for its terminal string, distinguished by the first production, whereas $\mathcal{G}_2$ has one. Thus equality of terminal languages does not determine derivational multiplicity, and a translation that retains only generated strings loses a distinction in the production structure. If the productions are subsequently given different costs, causal roles, or constraints, that lost distinction may become relevant to a more demanding formal question.

Translation among grammatical, dynamical, and cellular descriptions. A grammar or rewriting system may be represented as a labeled transition system on derivation states; its transition relation then admits the simulation and bisimulation comparisons of Subsection 13.2. A transition model may also carry geometric information, such as embeddings of produced graphs or attachments of cells, if these are specified as additional data. Passing from grammatical derivations to terminal strings is analogous to forgetting some transition history, while passing from a graph rewrite to a homology summary is a different form of structural abstraction. The analogy demonstrates several available mathematical questions; it does not identify syntactic derivation, neural dynamics, and subjective experience.

Generative grammar remains secondary in the present programme for three reasons. First, the four hypotheses concern the mathematical organization of change and relational dynamics in much broader model classes, including continuous fields and evolving topological carriers. Second, a grammar normally presupposes chosen primitives and production rules, whereas the Generativity Hypothesis also asks how constitutive distinctions and constraints might vary. Third, the observed output of a grammar supplies no intrinsic interpretation as phenomenal experience. The usefulness of grammatical models lies in their precisely controlled examples of nonunique inference from outputs, trace-branching distinctions, and derivational information loss.

Integrating translation criteria with the four hypotheses. The present section establishes a graduated vocabulary for comparing generative representations. Exact spectral projection can preserve reduced dynamics while losing source modes; labeled-trace equivalence can coexist with distinct branching organizations; sound abstraction can introduce spurious abstract histories; and approximate intertwining gives a quantitative error bound only under stated metric and stability assumptions. These results show how equal outputs, matching paths, or structurally adequate simulations support different kinds of reconstruction. The four consciousness hypotheses may therefore require different translation guarantees even when they concern the same measured process.

The next section develops further illustrative constructions and counterexamples with explicitly specified dynamical and cellular carriers. The longer-term empirical question is which invariants and translation conditions remain informative when source descriptions are linked to measured neural processes, experimental perturbations, and carefully operationalized reports of experience. This question requires both sound mathematical comparison and independent physical evidence. The absence of a universal representation bridge is a limitation of the present preliminary investigation, and preserving that limitation is integral to its explanatory discipline.

14 Illustrative Constructions and Counterexamples

The mathematical comparison of Variation, Trajectory, Generativity, and Repertoire benefits from models whose objects, assumptions, and observation maps can be inspected directly. This section develops

six connected constructions: transverse dynamics near a common smooth trajectory; cellular attachments with computable changes in homology; constitutive comparisons among relational two-complexes; torus paths with observationally hidden winding; transformation monoids supporting identical unlabelled histories; and a quotient transition system whose admissible abstract histories exceed those inherited from its source. Each construction yields a precisely scoped formal result or counterexample. Five accompanying figures are produced by a reproducible Python programme; their plotted values derive from stipulated mathematical models and contain no human, neural, or phenomenal observations. The objective is to make the informational boundaries of the four hypotheses examinable while preparing the questions of reconstruction and model discrimination developed in Sections 13 and the subsequent investigation.

14.1 Smooth Flows on Manifolds

Common selected path and opposed local stability. Let $M = \mathbb{R}^2$ carry its standard smooth structure and Euclidean metric, with coordinates (x, y) . Fix a positive real constant λ and consider two complete smooth vector fields X_- and X_+ , their flows, and a shared integral curve. These objects are defined in Equation 110.

$$\begin{aligned} X_\sigma &= \partial_x + \sigma \lambda y \partial_y, & \Phi_\sigma^s(x, y) &= (x + s, e^{\sigma \lambda s} y), \\ \gamma(s) &= (s, 0), & \sigma &\in \{-1, +1\}, & s &\in [0, T]. \end{aligned} \quad (110)$$

The curve γ in Equation 110 is an integral curve for either choice of sign from the same starting state $(0, 0)$. The choice of s as a flow parameter supplies an ordered numerical coordinate within this model; it entails neither an independently given phenomenal present nor a general identification of mathematical flow with experienced succession. The different signs affect the dynamics transverse to γ , as specified by the variational equation in Equation 111.

$$\frac{d}{ds} \delta y = \sigma \lambda \delta y, \quad \delta y(s) = e^{\sigma \lambda s} \delta y(0), \quad D\Phi_\sigma^s = \begin{pmatrix} 1 & 0 \\ 0 & e^{\sigma \lambda s} \end{pmatrix}. \quad (111)$$

Proposition 14.1 (Selected-path equivalence with distinct transverse dynamics). *For the fields of Equation 110, the parameterized solution from $(0, 0)$ is identical on $[0, T]$ for both signs. For every $s > 0$, the derivatives of their flow maps at points on that path differ in the transverse direction. In particular, the magnitude of a nonzero sufficiently small transverse displacement decays under X_- and increases under X_+ for positive s .*

Proof. Substituting $y = 0$ in the explicit flows gives $\Phi_\sigma^s(0, 0) = (s, 0)$ for both signs. Differentiation of their second coordinates yields Equation 111. Since $\lambda s > 0$, the corresponding multipliers $e^{-\lambda s}$ and $e^{+\lambda s}$ are unequal and lie on opposite sides of one. $\square$

Proposition 14.1 formalizes an elementary underdetermination: a selected, perfectly recorded curve does not identify its local generator in directions away from that curve. Perturbation data carry information absent from the unperturbed history. In the present example the two dynamical systems have the same carrier, metric, selected path, and unrestricted smoothness class; only the transverse action differs. Their distinction therefore does not depend on complicated topological carriers. The observation is relevant to the Variation Hypothesis because different local transformation structures can agree on a selected history; it is relevant to the Trajectory Hypothesis because a trajectory-level description can remain correct within its stated object type while being insufficient to infer Variation.

Reproducible numerical approximation. Figure 8 compares exact solutions from the perturbed initial value $y(0) = 0.08$ with explicit midpoint integration, using $\lambda = 0.7$, $T = 3$, and step $h = 0.025$. For this linear example the midpoint update is stated in Equation 112.

$$y_{k+1} = \left(1 + \sigma \lambda h + \frac{\lambda^2 h^2}{2}\right) y_k, \quad y(kh) = y_0 e^{\sigma \lambda kh}. \quad (112)$$

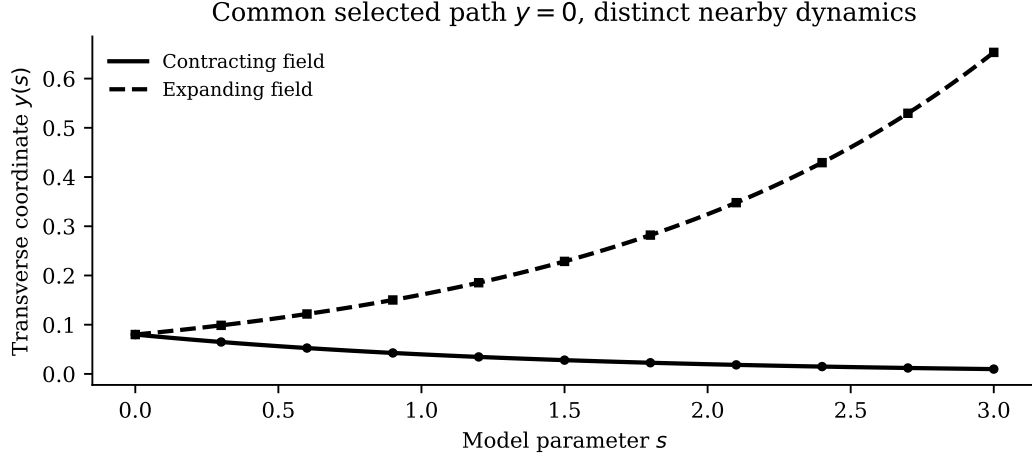


Figure 8: Exact transverse solutions (curves) and explicit-midpoint values (markers) near the shared integral curve $y = 0$. Contracting and expanding fields agree on the selected unperturbed trajectory; the experiment probes perturbation responses. The horizontal coordinate is a model flow parameter.

The exact solution in Equation 112 is available without simulation; the numerical experiment tests the plotting and approximation pipeline rather than discovering an unknown dynamical property. At the stated parameters the maximum absolute discrepancy over the sampled interval is below 3×10^{-4} for each sign, as checked in the bundled programme. Figure 8 therefore provides a visually inspectable perturbation distinction accompanying an analytic proof.

Dynamical and interpretive limits. Local Lyapunov behaviour, growth rates, and directional derivatives depend on the chosen model and metric conventions. For a neural system, a parameter fitted from recorded activity may describe an effective reconstructed state space rather than the microscopic physical state. The selection of perturbations and the reliability of measurement become part of the empirical interpretation. A difference in model stability does not, without a consciousness-specific bridge, establish a difference in experience. Conversely, identical unperturbed observation sequences give no grounds for asserting identity of the underlying transformation mechanisms.

14.2 Cellular Dynamics on Two-Complexes

A controlled attachment family. Let K_0 be a wedge $S_a^1 \vee S_b^1$, represented by one zero-cell v and two oriented one-cells a, b . Construct a family of connected CW-complexes of dimension at most two by attaching cells according to the words a^2 , b , and again a^2 , in the specified order. Write K_j for the complex after j attachments, so $K_0 \subset K_1 \subset K_2 \subset K_3$ is a filtration of mathematical descriptions. The repeated attaching word at the third step designates a distinct two-cell whose boundary has the same cellular degree as the first cell. The notation a^2 signifies twice winding around the loop a ; the attaching map is a topological specification and does not by itself encode an admissible physical traversal.

Every K_j has $C_0 \cong \mathbb{Z}$, $C_1 \cong \mathbb{Z}^2$, and $C_2 \cong \mathbb{Z}^j$, with $\partial_1 = 0$. The incidence degrees for the maximal complex are given in Equation 113, and the earlier boundary matrices consist of its first j columns.

$$D_3 = [\partial_2]_{K_3} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \quad D_0 : \mathbb{Z}^0 \longrightarrow \mathbb{Z}^2. \quad (113)$$

The matrix D_j consists of the first j columns of D_3 , for $j = 1, 2, 3$. The integer chain complex determines cellular homology through $H_1(K_j; \mathbb{Z}) = \text{coker } D_j$ and $H_2(K_j; \mathbb{Z}) = \ker D_j$.

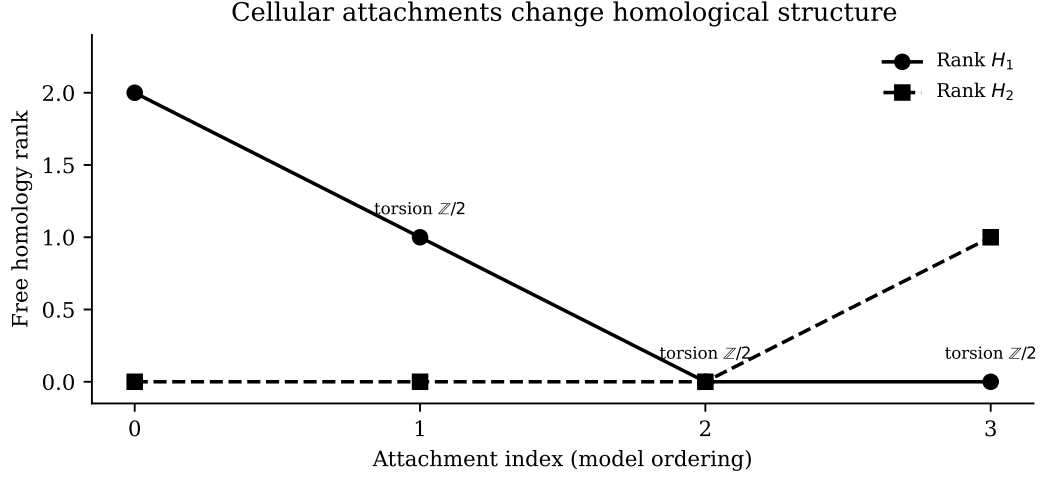


Figure 9: Free ranks of integral homology groups across an indexed sequence of cellular attachments. Smith normal form additionally detects $\mathbb{Z}/2$ torsion after the first attachment. The horizontal index orders formal model modifications and does not assert a privileged physical time coordinate.

Proposition 14.2 (Homology and torsion along an attachment family). *For the family K_0, K_1, K_2, K_3 described above, the first and second integral homology groups are the groups specified in Equation 114.*

	K_0	K_1	K_2	K_3
$H_1(-; \mathbb{Z})$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_2(-; \mathbb{Z})$	0	0	0	$\mathbb{Z}$

(114)

Proof. The first column $(2, 0)^T$ imposes the relation $2[a] = 0$ while leaving $[b]$ free. The second column $(0, 1)^T$ kills $[b]$. The third column duplicates the first boundary column, leaving the cokernel unchanged and generating a rank-one kernel represented by the formal difference between the first and third two-cells. Equivalently, integer Smith normal form has nonzero invariant factors (2) for D_1 and $(1, 2)$ for D_2, D_3 , with matrix ranks 1, 2, 2, respectively. Taking cokernels and kernels yields Equation 114. $\square$

The attachment family is presented in Figure 9, with exact homology verified independently by computer algebra. The figure reports free ranks of H_1 and H_2 ; the annotations retain the torsion summand that a plot of Betti numbers alone would conceal. The example therefore exhibits two distinct informational reductions. First, free homology ranks discard torsion, and second, a cellular boundary matrix can discard homotopy information about attaching words, as established earlier in Proposition 11.3. An investigator should specify whether a statistic captures full integral homology, rational homology, ordinary path classes, or actual permitted paths under additional constraints.

Cellular structure and admissible transformation. A cell attachment changes the carrier of possible paths and their homotopy classes. However, a dynamical description on K_j requires additional specifications, for example oriented transition rules on edges, control constraints, and permitted local operations around faces. Different directed systems can share one undirected two-complex. The Generativity Hypothesis may regard changes in incidence, attaching maps, and constraints as candidate constitutive changes; the Repertoire Hypothesis additionally asks which paths or directed histories those constraints permit. Thus a decrease in rank H_1 in Figure 9 does not independently imply a reduction in physically accessible histories or a change in neural function. The chain calculation concerns the chosen mathematical carrier, with empirical meaning contingent on interpretation and measurement.

14.3 Topology-Changing Relational Constructions

Attachment as an indexed constitutive operation. The filtration $K_0 \subset K_1 \subset K_2 \subset K_3$ also serves as an example of relational structural variation. Its vertices, edges, and faces designate events, couplings, and higher-dimensional primitives, respectively, under the stipulated relational reading. The repeated two-cell along a^2 illustrates why geometric appearance or first homology alone may fail to distinguish independently attached higher-order relations. In the chain model, the addition of the third face produces a new two-dimensional cycle even though the H_1 group remains unchanged. This is a concrete instance of generativity changing an available structural organization while leaving one selected invariant fixed.

The relative homology exact sequence supplies a more precise account of the first attachment. The connecting map for the pair (K_1, K_0) sends the generator of the relative two-cell to its attaching boundary, as specified in Equation 115.

$$\partial_{\text{rel}} : H_2(K_1, K_0; \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_1(K_0; \mathbb{Z}) \cong \mathbb{Z}^2, \quad 1 \longmapsto 2[a]. \quad (115)$$

The image of the map in Equation 115 records an exact constitutive relation introduced by the attaching cell. Its quotient contribution to $H_1(K_1)$ is $\mathbb{Z}/2$ torsion, while $[b]$ remains free. This connecting morphism contains information about the attachment that is more specific than an Euler characteristic or a single Betti number. In later versions of the research programme, chains of attachments can be enriched with incidence semantics, costs, constraints, and morphisms connecting different observed resolutions, but those additional structures must be supplied as model data.

Constitutive indexing and the status of order. An inclusion sequence provides a mathematically chosen partial ordering of model modifications. It does not establish that physical reality possesses a universal decomposition into earlier and later carriers. Alternative diagrams may encode incomparable modifications or multiple admissible continuations of a relational structure. In the present example, the effect of an attachment is interpreted through a specified morphism between complexes. The epistemic distinction is between represented structures and transformations among those structures, rather than between a fixed ontological past and a pre-existing future awaiting realization. This qualification matters when translating the construction into developmental, neural, social, or phenomenological settings.

A representational boundary. Cellular homology is invariant under suitable homotopy equivalences, while a relational theory might also require information about which event couplings were present, how attachments were enacted, and what transformations remained admissible. These latter properties generally depend on richer structured objects and their morphisms. The mathematical family K_j therefore illustrates a candidate representation of changing generative conditions. Its relation to an actual physical history remains an empirical and theoretical question; the terminology of emergence is warranted only when the generating description and its evidential support are explicit.

14.4 Non-Unique Trajectory Reconstruction

Observed circle paths and hidden torus winding. Let the complete modeled carrier be the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let the observation space be $S^1 = \mathbb{R}/\mathbb{Z}$. Define the observation map $h([x], [y]) = [x]$, where brackets denote equivalence modulo integers. The family of based torus loops and their common observation are specified in Equation 116.

$$\gamma_q(s) = ([s], [qs]), \quad h \circ \gamma_q(s) = [s], \quad [\gamma_q] = (1, q) \in \pi_1(T^2) \cong \mathbb{Z}^2, \quad q \in \mathbb{Z}, \quad s \in [0, 1]. \quad (116)$$

The loops in Equation 116 possess distinct endpoint-fixed homotopy classes for different q , while their complete continuous observed trajectories coincide exactly. Figure 10 displays lifts of three such loops to the universal cover $\mathbb{R}^2$. The lifted curves visualize the second-coordinate winding classes that the observation discards.

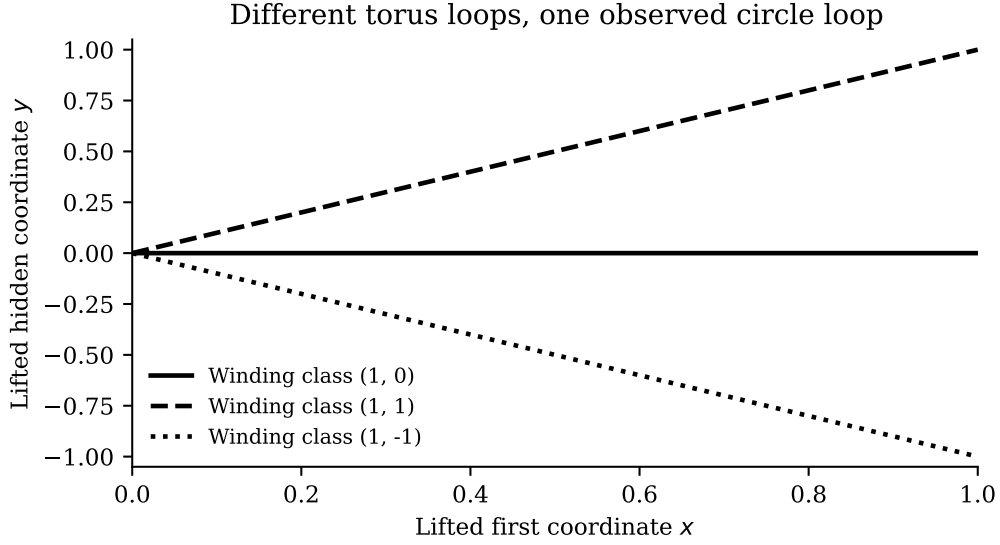


Figure 10: Lifted torus loops with winding classes $(1, 0)$, $(1, 1)$, and $(1, -1)$. Their first-coordinate projections coincide as continuous circle loops, although their full classes in $\pi_1(T^2)$ differ. The figure uses lifted coordinates for visualization.

Proposition 14.3 (Infinite path-class ambiguity under circle projection). *For the map $h : T^2 \rightarrow S^1$ specified above, a single observed based circle loop $s \mapsto [s]$ has preimages representing infinitely many distinct homotopy classes of based loops in T^2 . Consequently, even error-free continuous observation of that circle loop cannot determine its source winding class without additional information.*

Proof. Each γ_q in Equation 116 projects to the same observed loop. The fundamental group $\pi_1(T^2)$ is $\mathbb{Z}^2$, and the classes $(1, q)$ are pairwise distinct. Thus the observation fiber over the given path contains loops of infinitely many homotopy classes. $\square$

The ambiguity persists even with dense temporal sampling of the observed coordinate, since the observation map itself identifies the trajectories. Increasing sample frequency removes discretization uncertainty but cannot reconstruct information absent from the chosen sensor. A measurement of the second coordinate, a suitable coupling observable, or a justified source-dynamical restriction could reduce the compatible family; each such supplement changes the epistemic specification. The formal result concerns observability, and it offers no basis for attributing a particular winding class to subjective experience.

Representation and reconstruction criteria. The example distinguishes existence of a path in a source carrier, observation of its projection, and reconstruction of an equivalence class of source paths. In a neural manifold representation, the full state space may itself be estimated rather than independently known; inferential uncertainty can therefore arise both in the construction of T^2 and in the projection to recorded variables. An inverse problem should identify the set of compatible source histories, the imposed dynamical priors, and which path invariants remain identifiable. Proposition 14.3 demonstrates an exact topological nonidentifiability even before measurement noise is introduced.

14.5 Identical Repertoires with Distinct Generators

A pair of finite transformation monoids. Let $S = \{0, 1, 2\}$. Denote by $c_i : S \rightarrow S$ the constant map $c_i(x) = i$, and by id the identity. Consider the submonoid M_A generated by the identity and constants, and the full transformation monoid $M_B = S^S$. Their definitions, action-reachability relation,

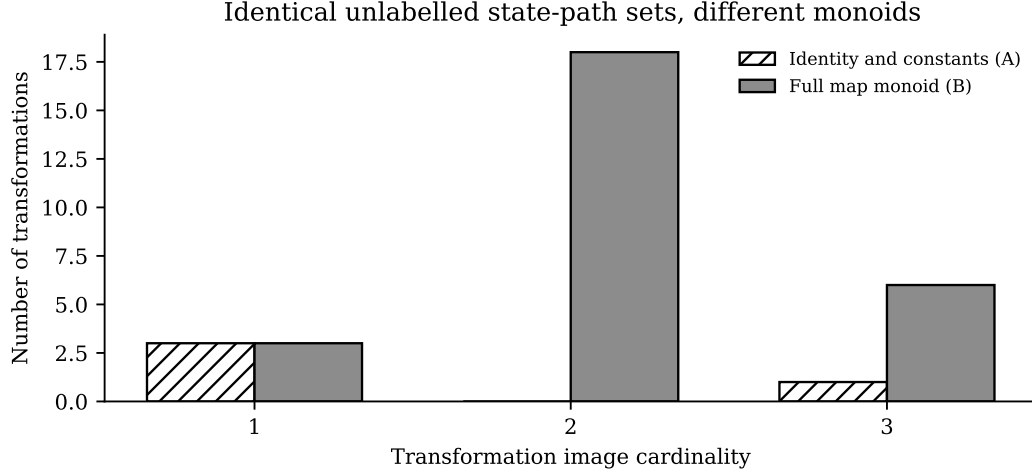


Figure 11: Exact rank distributions of two finite transformation monoids with the same complete one-step state relation and hence the same unlabelled finite state-path repertoires. The internal transformation inventories are different despite complete repertoire agreement at this observational level.

and unlabelled finite-state repertoires are given in Equation 117.

$$\begin{aligned}
 M_A &= \{\text{id}, c_0, c_1, c_2\}, & M_B &= S^S, \\
 E_M &= \{(x, m(x)) : x \in S, m \in M\}, & \mathcal{P}_M^{(n)} &= \{(x_0, \dots, x_n) : (x_{j-1}, x_j) \in E_M\}, \\
 E_{M_A} &= E_{M_B} = S \times S, & \mathcal{P}_{M_A}^{(n)} &= \mathcal{P}_{M_B}^{(n)} = S^{n+1} \quad (n \geq 0).
 \end{aligned} \tag{117}$$

In Equation 117, each formal step may apply an arbitrary monoid element, and histories retain only state sequences, omitting transformation labels. These modeling choices are decisive. Restricting moves to a smaller declared primitive generating set, preserving labels, recording operational costs, or imposing a composition budget would define a different repertoire. The present pair deliberately keeps the observation and step convention identical so that the remaining distinction concerns the internal transformation sets themselves.

Proposition 14.4 (Full path-repertoire equivalence with different transformation structure). *The monoids M_A and M_B in Equation 117 have the same unlabelled finite-state path repertoire at every length and from every starting state. They are not isomorphic monoids, and their action sets differ in the image ranks of their transformations.*

Proof. For any $x, y \in S$, the constant map $c_y \in M_A$ sends x to y . As $M_A \subset M_B$, both one-step reachability relations equal $S \times S$, which yields identical unrestricted state-word repertoires of every length. But $|M_A| = 4$, while $|M_B| = |S|^{|S|} = 27$, so they cannot be isomorphic. Moreover, a map of image cardinality two belongs to M_B whereas M_A contains only rank-one constants and the rank-three identity. $\square$

Figure 11 displays the exact enumeration of transformation ranks. In M_B , the three rank-one maps, eighteen rank-two maps, and six rank-three permutations exhaust the 27 transformations. The smaller monoid has three rank-one maps and one rank-three map. Even the entire family of unlabelled finite state sequences fails to distinguish these monoids under the declared step convention. This counterexample complements the torus example: one concerns the loss of topological information about paths, while the other concerns loss of action-level information despite full agreement of state-path repertoires.

Variation, Repertoire, and the granularity of admissibility. Proposition 14.4 does not state that the two systems have identical labelled trajectories or identical costs. The available action types and their

compositional structure are different, and one may intervene by choosing a particular map in M_B with rank two, an option unavailable in M_A . If admissibility refers to such intervention-labelled mechanisms, the repertoires would differ. Thus a Repertoire Hypothesis requires explicit specification of the elementary path object and its equivalence relation. Treating paths as unlabelled state words yields one equality; treating them as mechanism-labelled histories yields another, more discriminating question. The Variation Hypothesis gives reasons to retain generator information when the explanatory objective concerns transformation organization.

14.6 Quotient Dynamics and Information-Losing Abstractions

A deterministic source and a nondeterministic observed relation. Consider the source state set $S_c = \{a, b, c, d\}$ with the deterministic update $F(a) = c$, $F(c) = a$, $F(b) = d$, and $F(d) = b$. Let the observation map $h : S_c \rightarrow \{0, 1\}$ send a, b to 0 and c, d to 1. The direct-image observed transition relation is specified in Equation 118.

$$\begin{aligned} R_c &= \{(a, c), (c, a), (b, d), (d, b)\}, \\ h(a) &= h(b) = 0, \quad h(c) = h(d) = 1, \\ R_h &= \{(h(s), h(t)) : (s, t) \in R_c\} = \{(0, 1), (1, 0), (1, 1)\}. \end{aligned} \tag{118}$$

Every observed concrete one-step transition lies in R_h by construction. Nonetheless, successive abstract transitions can combine edges witnessed by different concrete states in the same observation fiber. For example, the abstract word $(0, 1, 1, 0)$ traverses edges of R_h , but it has no lift to a source trajectory, because the $1 \rightarrow 1$ step forces a source state d and the next observed $1 \rightarrow 0$ step would require c . This is a specific false-positive history introduced by forgetting which representative of a fiber supports a transition.

Finite-history enumeration. Write $\mathcal{O}_n$ for the distinct observation words of length n induced by the deterministic source trajectories from all possible source states, and $\mathcal{A}_n$ for the length- n paths of the abstract directed graph on $\{0, 1\}$. Their cardinalities admit a closed form. Taking the Fibonacci sequence to have $F_0 = 0$ and $F_1 = 1$, the counts are given in Equation 119.

$$\begin{aligned} |\mathcal{A}_n| &= F_{n+2} \quad (n \geq 1), \\ |\mathcal{O}_n| &= \begin{cases} 2, & n = 1, \\ 3, & n = 2, \\ 4, & n \geq 3, \end{cases} \quad |\mathcal{A}_n \setminus \mathcal{O}_n| = F_{n+2} - 4 \quad (n \geq 3). \end{aligned} \tag{119}$$

Proposition 14.5 (Growth of spurious abstract histories). *For the source update and observation of Equation 118, the path languages satisfy $\mathcal{O}_n \subseteq \mathcal{A}_n$ for every $n \geq 1$, and the counts of Equation 119 hold. Consequently, the fraction of abstract paths without concrete realizations tends to one as n grows.*

Proof. The inclusion follows because each observed source transition is in R_h . For the abstract relation, let $N_0(n)$ and $N_1(n)$ be the counts of length- n words ending in 0 and 1. The recurrence is $N_0(n+1) = N_1(n)$ and $N_1(n+1) = N_0(n) + N_1(n)$, with $N_0(1) = N_1(1) = 1$. Therefore $|\mathcal{A}_n| = F_{n+2}$. Each of the four source states follows one deterministic orbit. Their observed words become pairwise distinct by length three: they begin respectively with 010, 011, 101, and 111, and remain distinct under extension. Hence $|\mathcal{O}_n| = 4$ for $n \geq 3$. The fraction of nonrealizable words is then $1 - 4/F_{n+2}$, which tends to one as the Fibonacci denominator grows. $\square$

The exact counts in Equation 119 were independently checked by exhaustive enumeration through length eleven. Figure 12 shows the increasing gap between the observed concrete language and the unrestricted abstract path language. The logarithmic vertical axis makes the contrast visible across different word lengths. The mathematical implication is specific: one-step soundness of a direct-image transition abstraction does not guarantee completeness of its longer abstract histories. This example strengthens the two-step observation in Proposition 13.3 by providing an exact growth law.

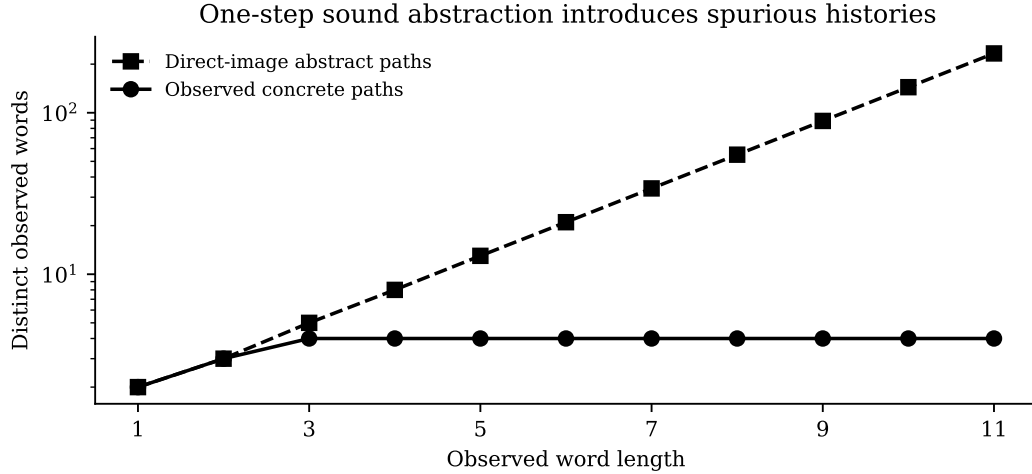


Figure 12: Numbers of concrete observable words and abstract graph words under a direct-image quotient. Exact exhaustive enumeration agrees with the closed-form counts in Equation 119; increasingly many abstract paths do not lift to the deterministic source model.

Table 17: Informational distinctions exhibited by the formal constructions in Section 14. Candidate discriminating evidence must be matched to the specified object and equivalence criterion.

Construction		Preserved under the stipulated observation	Information requiring stronger access
Transverse flows	smooth	Same parameterized selected solution	Nearby flow derivatives and perturbation response
Two-complex attachments	attach-	Some stable homology summaries across modifications	Attaching degrees, torsion, higher-dimensional cycles, path constraints
Torus projection		Same complete first-coordinate observed loop	Hidden winding and source homotopy class
Finite transformation monoids	transformation	Every unlabelled state word at every finite length	Transformation ranks, action labels, internal semigroup structure
Direct-image quotient		Source one-step observed transitions	Consistent lifting of long observed histories

Abstraction adequacy and physical interpretation. A quotient relation can be adequate for a one-step reachability question while unsuitable for longer-horizon trajectory reconstruction. A path-lifting condition, stronger simulation relation, history-sensitive state enrichment, or retained fiber information could address the discrepancy, depending on the intended claim. In a physical experiment, repeated sampled observations may supply a sequence of admissible recorded symbols while still leaving incompatible hidden source reconstructions. The computational example uses exact discrete updates and a declared observation partition; it contains no stochastic measurement model. The extrapolation from abstract-word proliferation to concrete empirical bias therefore requires additional assumptions about noise, interventions, system identification, and how model states relate to physiology.

Comparison of discriminating observations. Table 17 collects the explanatory object retained and the information hidden in each construction. The table gives a basis for designing a more explicit comparison of observation schemes, perturbations, and hypotheses in Sections 15 and 16. The examples show that equality at one level of representation can coexist with disagreement at another, while different experiments require different observational resources.

Reproducibility and epistemic standing. The editable source archive includes the Python file `section14_experiments.py` in the `experiments` directory, all five plotted figures in PDF and PNG form, five CSV data files, and an assertion report. The computational programme uses NumPy, Matplotlib, and SymPy. It calculates the Smith normal forms, enumerates finite transformations and paths, evaluates the explicit midpoint scheme, and checks the derived analytic values. All five figures illustrate stipulated mathematical objects. The numerical plots test the internal consistency of formal constructions; their role is clarification and falsification of implementation mistakes. They do not constitute physiological experiments or evidence in favor of any consciousness hypothesis.

The constructions jointly motivate a differentiated epistemic standard. Candidate consciousness correlates may depend on generator information, selected trajectories, constitutive transformations, or admissible-history families, and observational maps may preserve these at different levels of fidelity. An explanatory comparison must therefore declare the represented carrier, observational equivalence, formal invariants, perturbations or interventions, and the empirical criteria connecting modeled properties to consciousness-related observations. The next section examines reconstructibility and underdetermination from this starting point, retaining the distinction between model-relative mathematical implication and phenomenological explanation.

15 Epistemic Reconstruction and Model Underdetermination

The four hypotheses concern different mathematical objects, whereas the evidence available for their evaluation consists of records produced by specified measurement and intervention protocols. This section formulates the corresponding reconstruction problem. It first distinguishes exact identification of a model or property from identification only up to observational equivalence; it then examines dynamical observability, the reconstruction of trajectory topology, limits on identifying constitutive changes, and the inference of admissible-history repertoires from finite observations. Finally, it separates structural identifiability from stability and statistical discrimination under uncertainty. Throughout, a reconstruction claim is relative to a declared candidate class, observation protocol, and property of interest. The conditions supplied below are mathematical constraints on possible inference rather than empirical demonstrations of any hypothesis about conscious experience. Structural identifiability and observability have established mathematical traditions [4, 20, 34]; the present comparison uses these traditions as methodological resources without assuming that their customary state or parameter variables already characterize consciousness.

15.1 Experiments, Observational Equivalence, and Recoverable Properties

Objects and access conditions. Let Θ be a specified class of candidate descriptions. A member $\theta \in \Theta$ may denote a smooth vector field with constraints, a selected relational history, an indexed family of constitutive complexes, or a structured set of admissible paths. The interpretation of Θ must therefore be fixed before an identifiability claim is meaningful. Let $\mathcal{E}$ be a declared family of experiments, measurement arrangements, or observation contexts. For $e \in \mathcal{E}$, an idealized response map $R_e : \Theta \rightarrow \mathcal{Y}_e$ assigns the complete observational record predicted by a candidate under that arrangement. The joint response for a chosen design $D \subseteq \mathcal{E}$ is the map of Equation 120.

$$R_D : \Theta \longrightarrow \prod_{e \in D} \mathcal{Y}_e, \quad R_D(\theta) = (R_e(\theta))_{e \in D}, \quad \theta \equiv_D \theta' \iff R_D(\theta) = R_D(\theta'). \quad (120)$$

The product in Equation 120 is a record space, not a claim that the experiments can be performed simultaneously. Each e must declare which perturbations, preparations, readout channels, resolution limits, and comparisons are permitted. An idealized intervention is a protocol with a specified model-relative effect; the existence, ethics, and reproducibility of a corresponding physical intervention are separate empirical questions. The index D is a collection of accessible operations, not a distinguished temporal present or a universal sequence of epistemic progress.

A candidate description is *globally identifiable on D* when R_D is injective on Θ . In many applications, such injectivity is stronger than the intended explanatory question requires. Let $p : \Theta \rightarrow P$ describe the object whose recovery matters, for example a generator rank, an observed homotopy class, a homology invariant, or a repertoire boundary. We call p *identifiable on D* if it is constant on every fiber of R_D . For a specified equivalence relation $\approx$ on Θ representing accepted gauge or coordinate changes, identification *up to $\approx$* requires equality of observations to imply $\theta \approx \theta'$. An assertion of uniqueness must consequently specify whether it concerns representatives, equivalence classes, or selected invariants; coordinate redundancy alone does not establish that different mechanisms are observationally indistinguishable.

Proposition 15.1 (Recoverable properties and refinement of experimental access). *For a fixed design D , a map $p : \Theta \rightarrow P$ is identifiable on D if and only if there is a unique map $\bar{p} : R_D(\Theta) \rightarrow P$ such that $p = \bar{p} \circ R_D$. If $D \subseteq D'$, then $\equiv_{D'}$ is contained in $\equiv_D$. Consequently, every property identifiable on D is identifiable on D' , provided the same candidate class and exact response maps are retained.*

Proof. If p factors through R_D , equal response records have equal p -values. Conversely, fiber constancy makes $\bar{p}(R_D(\theta)) = p(\theta)$ well defined; uniqueness holds on the image. If $\theta \equiv_{D'} \theta'$, all component responses for the smaller index set D agree, giving $\theta \equiv_D \theta'$. Fiber constancy on the larger equivalence classes therefore implies fiber constancy on their subsets. $\square$

Proposition 15.1 makes the gain from additional observations precise without promising that any finite extension of D will achieve injectivity. It extends the fiber-factorization perspective of Proposition 13.5 by treating experiments as a variable family of epistemic operations rather than one fixed translation. If responses are probability laws instead of exact records, R_e should take values in a specified space of probability measures. Structural identification then means distinct candidates yield distinct distributions under the accessible designs. Actual samples create further estimation and discrimination problems addressed below.

Equivalence at different descriptive levels. Two physically distinct descriptions can have identical records in a chosen regime, while a stipulated equivalence of descriptions need not guarantee matching records when their observation interfaces differ. A physically meaningful identification problem must therefore state both a scientific account of relevant representation changes and an experimental account of what is measured. In the present taxonomy, observational equivalence may collapse generators, selected histories, changing attaching maps, or untested possible histories. The quotient $\Theta / \equiv_D$ is a description of the information supplied by D within Θ ; it has no automatic claim to describe an observer-independent set of physical or phenomenal kinds.

15.2 Dynamical Observability and Generator Reconstruction

Known dynamics and hidden initial states. Consider a finite-dimensional real linear system with a known matrix $A \in \mathbb{R}^{d \times d}$ and known output matrix $C \in \mathbb{R}^{m \times d}$. The evolution parameter $s \in [0, T]$ is a declared coordinate for solutions. The observed response and its finite-interval observability Gramian are specified in Equation 121.

$$x'(s) = Ax(s), \quad y(s) = Cx(s) = Ce^{As}x_0, \quad W_T = \int_0^T e^{A^\top s} C^\top C e^{As} ds. \quad (121)$$

A reconstruction of the initial state x_0 from the complete ideal record $y|_{[0, T]}$ is possible precisely when the map $x_0 \mapsto y(\cdot)$ is injective. The classical observability matrix is introduced in Equation 122.

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{d-1} \end{bmatrix}. \quad (122)$$

Proposition 15.2 (Finite-interval observability and a stability bound). *For every $T > 0$, the following statements are equivalent: (i) the output record in Equation 121 uniquely determines x_0 ; (ii) W_T is positive definite; and (iii) $\mathcal{O}(A, C)$ in Equation 122 has column rank d . If these conditions hold, two initial states x_0, x'_0 satisfy the bound in Equation 123.*

$$\|x_0 - x'_0\|_2 \leq \lambda_{\min}(W_T)^{-1/2} \|Ce^{A(\cdot)}x_0 - Ce^{A(\cdot)}x'_0\|_{L^2([0, T]; \mathbb{R}^m)}. \quad (123)$$

Proof. For $v \in \mathbb{R}^d$, $v^\top W_T v = \int_0^T \|Ce^{As}v\|_2^2 ds$. A zero integral means the continuous function $Ce^{As}v$ vanishes on $[0, T]$, and analyticity then gives $CA^k v = 0$ for every nonnegative integer k . By the Cayley–Hamilton theorem, these conditions are equivalent to $CA^k v = 0$ for $0 \leq k < d$, which is $\mathcal{O}(A, C)v = 0$. Thus the kernels of the response map, W_T , and $\mathcal{O}(A, C)$ agree. The three statements follow. For $v = x_0 - x'_0$, the integral identity and $v^\top W_T v \geq \lambda_{\min}(W_T)\|v\|_2^2$ yield Equation 123. $\square$

Proposition 15.2 is conditional on knowing A and C , exact model structure, and complete observation over a nondegenerate parameter interval. A formally invertible response may be practically unstable when $\lambda_{\min}(W_T)$ is small. Additional unmodeled noise, channel selection, sampling, and parameter uncertainty can further impair reconstruction. Neural state-space studies provide motivated contexts for such questions, yet interpreting an inferred latent coordinate as a conscious property requires separate experimental justification [40, 16].

Unknown dynamics and selected trajectories. The identification problem changes when the vector field is unknown. Suppose X is a smooth vector field on a smooth manifold M with local flow Φ_X^s . If ideal, parameterized, full-state flows $\Phi_X^s(x)$ are known for every x in an open region and for s near zero, then $X(x)$ is uniquely reconstructed by differentiation. This identity is stated in Equation 124.

$$X(x) = \left. \frac{d}{ds} \Phi_X^s(x) \right|_{s=0}. \quad (124)$$

Equation 124 presupposes a fixed parameter convention and access to an open family of initial states. A single observed integral curve only fixes a vector field along the curve and leaves its transverse behavior free. Section 14 gave two smooth flows with the same selected path and opposite local stability. An unknown observation map further weakens access: for $h(x, z) = x$ on $\mathbb{R}^2$, every vector field of the form $X_a = \partial_x + a(x, z)\partial_z$ has the same projected trajectories $h(\Phi_{X_a}^s(x, z)) = x + s$ wherever its local flow exists. The function a is unconstrained by these records. Thus observability of a state in a fixed model, reconstruction of an unknown generator, and identification of a class of mechanisms are separate problems.

Perturbations and causal access. A controlled system requires its admissible control alphabet, control-to-dynamics correspondence, and admissible preparation set. For an input u the response $R_{e,u}(\theta)$ is an additional component of the experiment family in Equation 120. Appropriately varied inputs can distinguish candidates that agree on unperturbed trajectories, as in the opposed transverse flows. This observation links Variation to interventional sensitivity and to nonlinear observability and controllability theory [20, 1]. A hypothetical instantaneous intervention on a latent neural coordinate has a different physical standing from a validated stimulation protocol; the formal possibility of applying u does not guarantee physical accessibility or ethical permissibility.

15.3 Topological Reconstruction of Trajectories

Observed paths and induced fundamental groups. For a based continuous map $h : (X, x_0) \rightarrow (Y, y_0)$ between path-connected spaces, every based source loop γ produces an observed loop $h \circ \gamma$. Up to based homotopy, this operation induces the homomorphism described in Equation 125.

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad h_*[\gamma] = [h \circ \gamma]. \quad (125)$$

The observation considered here is the *based homotopy class* of an observed loop, a deliberately coarse datum. It differs from a fully sampled parameterized path and from the output of an uncertain physical sensor.

Proposition 15.3 (Identifiability of source loop classes under observation). *Within the full set of based source loop classes, the observed class $[h \circ \gamma]$ uniquely determines $[\gamma]$ if and only if h_* in Equation 125 is injective. If $\ker h_*$ contains a nonidentity element, distinct source homotopy classes yield identical observed homotopy classes.*

Proof. If h_* is injective, equality of its output classes entails equality of the source classes. If h_* is not injective, choose $g \neq 1$ in its kernel. The source classes g and the identity have the same image under h_* but are different. $\square$

For the torus projection $h : S^1 \times S^1 \rightarrow S^1$ onto the first factor, the induced map under the standard identifications is $(p, q) \mapsto p$ on $\mathbb{Z}^2$. Its kernel consists of the hidden second-factor winding classes. The explicitly calculated torus examples in Sections 11 and 14 realize this loss with entire observed loop traces, a stronger failure than equality merely after homotopy reduction. By contrast, combining the two coordinate readouts gives an injective map to $S^1 \times S^1$ and distinguishes both winding numbers in the stipulated noiseless model. This difference is an exact statement about the selected observation interface, not about conscious access to topological invariants.

Cellular summaries and missing attaching information. For a CW-complex K , a cellular chain complex retains cells, boundary homomorphisms, and their induced homology. Its image may omit attaching-word information and the fundamental group. Proposition 11.3 constructs two complexes with identical cellular chain complexes but distinct fundamental groups. Thus an inference pipeline that records only cellular chain data or Betti-number summaries cannot distinguish their attaching structures, even if all those summaries are known exactly. More detailed measures can still be insufficient when they are insensitive to the specific invariant of interest. A topology-based investigation needs an explicit reconstruction target: cell counts, incidence, full attaching maps, homology with torsion, fundamental groups, directed admissible paths, or changing carrier diagrams.

Geometric embedding and sampling. A manifold reconstructed from finite coordinates depends on embedding assumptions and the quality of the measurement map. Even an exact topological embedding of a source space into a measurement space does not automatically identify the physical generators, metric scales, admissible control bounds, or phenomenological meaning assigned to source coordinates. Conversely, an observation map can induce injective maps on a selected invariant while remaining non-injective on individual states. These possibilities imply that the epistemic adequacy of a geometric reconstruction should be expressed relative to a stated category of objects and mappings. The mathematical distinctions documented in Section 13 constrain attempts to infer one kind of organization from another.

15.4 Constitutive Changes and Historically Indexed Evidence

Indexed descriptions and incomplete access. The Generativity Hypothesis permits constitutive data to vary across an indexed family of models. Let I be a small category describing a chosen relational or compositional ordering and $\mathcal{K} : I \rightarrow \mathbf{CW}$ a diagram of cellular spaces with declared cellular maps. This use of an index category does not posit universal chronological succession. Let $J \hookrightarrow I$ be a subcategory whose objects and arrows are accessible under a stipulated epistemic protocol. Observations restricted to J can identify the restricted diagram $\mathcal{K}|_J$ under sufficiently rich measurement, while extensions of that diagram to I may remain nonunique. A claim about the construction of future admissible objects demands assumptions on the extension class or measurements of the relevant constitutive transformations.

A common structural prefix and divergent attachments. Take the initial complex $K_0 = S^1 \vee S^1$ with oriented generators a, b , and consider two extensions formed by attaching one two-cell: in one extension the attaching word is the commutator $aba^{-1}b^{-1}$; in the other the attaching loop is null-homotopic. Both share the exact initial carrier K_0 . Their cellular boundary maps in degree two vanish, and their cellular homology groups coincide, while their fundamental groups are, respectively, $\mathbb{Z}^2$ and the free group F_2 by the example established in Proposition 11.3. Observation restricted to the initial complex, even with full initial-cell information, cannot determine which attachment was made in the extension. Observation of the resulting cellular chain complexes still cannot distinguish the two extensions. Data about the attaching map, or an appropriate invariant sensitive to it, would be required.

This construction separates three epistemic tasks. The first is reconstructing structures in an actually studied subdiagram. The second is inferring a constitutive transformation connecting studied structures. The third is constraining admissible extensions that have not been tested or realized in the stated model. The third task concerns model-relative modality, not a collection of already existing future events. Assumptions about extension laws, locality, conservation, compositional constraints, and accessible interventions may narrow the possibilities; those assumptions become part of the explanatory burden of the Generativity Hypothesis.

Historical selection and changing observability. Epistemic contexts may themselves change: the available vocabulary, measuring instrument, or equivalence relation at index i can differ from that at index j . Let $h_i : K_i \rightarrow Y_i$ and $h_j : K_j \rightarrow Y_j$ denote corresponding observation maps. Comparisons between records require an explicit translation between their target spaces, and preferably a commuting comparison with the constitutive map when one exists. The naturality analysis of Section 12 provides one representation of this requirement. A later description can distinguish patterns hidden by an earlier one without implying that those patterns were absent from the studied process. Conversely, changing the observational classification can create apparent topological or dynamical discontinuities whose physical status requires independent validation. Historical indices should specify what is being ordered: records, experiments, model revisions, or physical events.

15.5 Repertoire Identification and Untested Admissibility

Possibility is an additional claim. A selected path provides evidence that a particular modeled behavior occurred under an applicable representation and protocol. It does not enumerate every behavior the model admits. Even a collection of observed histories can leave the full Repertoire Hypothesis underdetermined unless the candidate class and completeness of testing are constrained. Let Σ be an alphabet of elementary transitions and $L \subseteq \Sigma^*$ a prefix-closed language of finite admissible words. Word length here measures combinatorial composition; it should not be read as an externally supplied physical time coordinate.

For a positive integer N , define two languages as in Equation 126.

$$L_{\text{all}} = \{0, 1\}^*, \quad L_N = \{w \in \{0, 1\}^* : w \text{ has no substring } 1^{N+1}\}. \quad (126)$$

Both are prefix-closed and include the empty word. Both can be represented by finite-state transition systems with designated initial states and an admissibility restriction, yet they define different sets of possible histories.

Proposition 15.4 (Underdetermination of a repertoire from bounded-length access). *For every finite $N \geq 1$, the two repertoires in Equation 126 have identical restrictions to all words of length at most N , but differ at length $N + 1$. Therefore even exhaustive observation of admissible words through a fixed finite length need not identify the complete prefix-closed regular language.*

Proof. A word with length at most N cannot contain a run of $N + 1$ consecutive symbols 1, so every such binary word belongs to both languages. The word 1^{N+1} belongs to L_{all} and is excluded from L_N . The condition of avoiding a fixed finite forbidden word is recognized by a finite automaton that records the trailing run length of 1 symbols; thus both languages lie in the stated regular-language class. $\square$

Proposition 15.4 concerns exhaustive access to all admissible finite paths below a bound, which is substantially stronger than observing a small empirical sample. Its result persists under this strong assumption because untested continuation behavior remains unconstrained. Stronger reconstruction statements require restricted candidate classes, known finite-state dimension, generating rules, exhaustive admissibility queries, or experimentally warranted closure principles. Interventional repertoires likewise depend on the set of permissible inputs: a control can enlarge the family of paths only in the model-relative sense specified by the control system. The different generator monoids of Proposition 14.4 also show that recovering every unlabelled path need not recover the transformation system generating those paths.

Lower and upper epistemic envelopes. Suppose a model class $\Theta_D \subseteq \Theta$ remains compatible with the records from design D , and each candidate θ specifies a repertoire $\mathcal{R}(\theta)$. The common and possible repertoire envelopes relative to this candidate class are defined in Equation 127.

$$\mathcal{R}_D^\cap = \bigcap_{\theta \in \Theta_D} \mathcal{R}(\theta), \quad \mathcal{R}_D^\cup = \bigcup_{\theta \in \Theta_D} \mathcal{R}(\theta), \quad \mathcal{R}_D^\cap \subseteq \mathcal{R}(\theta) \subseteq \mathcal{R}_D^\cup \quad (\theta \in \Theta_D). \quad (127)$$

Equation 127 is set-theoretic and presupposes that the path objects across candidates have been placed in a common comparison space. The intersection contains histories admitted by every currently compatible candidate, while the union contains histories admitted by at least one. Neither bound should be interpreted as a frequency distribution or realized history. New records may shrink Θ_D and consequently expand the intersection or contract the union; these changes reflect refinement of epistemic compatibility rather than an automatic change in the physical repertoire. With candidate-dependent carriers or history types, intersections and unions require a justified translation into a shared space, or a diagrammatic replacement preserving heterogeneous path types.

15.6 Statistical Discrimination and Robustness of Reconstruction

Law-level versus sample-level identification. The preceding results assume ideal response functions or complete idealized paths. A stochastic protocol assigns a probability law P_θ^D on a measurable record space, incorporating a declared noise and sampling model. Exact statistical identification of θ demands that distinct target-relevant candidates induce distinguishable laws; it does not ensure that a feasible sample distinguishes them reliably. A structural equivalence class can be small while the corresponding distributions are too close for robust inference at an available sample size. Conversely, two candidates can be statistically distinguishable under an intervention even when their unperturbed observational laws coincide. Here, as in the deterministic case, the experimental design is part of the reconstruction problem.

For two specified candidates with joint record laws P and Q , define their total variation distance and the testing error quantity in Equation 128.

$$\text{TV}(P, Q) = \sup_B |P(B) - Q(B)|, \quad \inf_\varphi \frac{P(\varphi = 1) + Q(\varphi = 0)}{2} = \frac{1 - \text{TV}(P, Q)}{2}. \quad (128)$$

In Equation 128, the infimum is over measurable binary tests φ , with label 0 assigned to P and 1 to Q , and the two candidates have equal prior weights. The total variation is taken over the complete joint data produced by the declared design; the formula does not assume independent samples.

Proposition 15.5 (Binary discrimination bound). *For any two probability laws P, Q on a common measurable record space, the optimal equal-prior average error among measurable binary tests is the right-hand side of Equation 128.*

Proof. For a test with decision region $B = \{\varphi = 1\}$, its average error is $\frac{1}{2}[P(B) + Q(B^c)] = \frac{1}{2}[1 - (Q(B) - P(B))]$. The supremum of $Q(B) - P(B)$ over measurable B equals $\text{TV}(P, Q)$ because P and Q are probability measures and taking complements exchanges the sign of their difference. Taking the infimum over tests gives Equation 128. $\square$

Table 18: Reconstruction obligations for four candidate explanatory objects. Identifiability is relative to a candidate class, observation equivalence, experimental access, and declared physical interpretation.

Candidate object	Access that could discriminate models	Ambiguity under weaker access
Variation	Perturbed responses, generator composition, full-state local flows, accessible control regimes	Different transverse dynamics, generator inventories, or latent directions with common measured paths
Trajectory	Appropriately resolved paths, reparameterization protocol, homotopy-sensitive sensors	Hidden winding classes or distinct source paths with equal projected records
Generativity	Attaching information, changing constraints, comparable indexed observation schemes	Distinct constitutive extensions sharing structural prefixes or homology summaries
Repertoire	Admissibility queries, intervention-conditioned histories, justified closure or complexity bounds	Untested continuations and distinct generator systems sharing an unlabelled path language

The exact expression in Proposition 15.5 implies error $1/2$ for identical laws and smaller achievable error when the laws are distinguishable. It makes no claim about which hypothesis is true and no assumption that conscious reports directly measure subjective experience. Robustness additionally depends on misspecification, nonstationarity of instruments or substrates, selection bias, intervention side effects, and the meaning of the recorded labels. Empirical work on neural correlates supplies a distinct level of evidence and warrants separate scrutiny of report, task performance, and background conditions [23, 36]. The mathematical discrimination of candidate models cannot independently adjudicate whether a recovered property constitutes or causes phenomenal experience.

15.7 A Reconstruction Protocol Across the Four Hypotheses

Objects and adequacy conditions. Table 18 summarizes the reconstruction targets that emerge from the formal results. Each row specifies a kind of evidence that may improve identification, together with an ambiguity that survives weaker access. The table does not rank the hypotheses by empirical plausibility; the quality and relevance of actual measurements remain an open scientific question.

A workable reconstruction argument should identify the modeled object, state the permitted equivalence relation, enumerate the actual observation and intervention interfaces, and give a factorization, observability, lifting, or discrimination condition appropriate to that object. It should then assess whether inference is stable to finite data, errors, and alternative model structures. These obligations are cumulative but not interchangeable. For example, a full-rank observability matrix identifies the latent initial state of a specified linear dynamical model while leaving open the identification of unknown neuronal couplings, the uniqueness of a relational two-complex, and the existence of a particular phenomenal interpretation.

The remaining empirical interface. For consciousness research, a reconstructed formal quantity must ultimately be connected to an independently specified target of explanation: reportable content, perceptual discrimination, arousal, attentional control, a physiological correlate, or a philosophical characterization of phenomenal experience. These targets have different inferential statuses. A putative causal relation requires intervention and control for confounding under the relevant design; a constitutive or identity relation additionally requires arguments not supplied by observability alone. The present results therefore support a disciplined order of inquiry: articulate the candidate mathematical distinction; determine how an observation scheme could separate it from alternatives; assess its empirical connection to a consciousness-related target; and separately evaluate its explanatory interpretation. Section 15 establishes the formal reconstruction obligations needed for the subsequent examination of physical interpretation and empirical discrimination.

16 Physical Interpretation and Empirical Discrimination

The preceding formal investigation has distinguished transformation mechanisms, selected relational histories, evolving constitutive organizations, and admissible-history repertoires. Empirical evaluation requires additional commitments: a physical system, a specified measurement interface, an experimentally accessible contrast, and an account of how the relevant measurements bear on experience. This section establishes those interfaces while preserving the explanatory independence of the four mathematical hypotheses. It considers neural measurements, perturbational and observational paradigms, the physical interpretation of geometric and topological constructions, and a set of contrastive study designs. Four conditional propositions and a synthetic reconstruction example clarify what interventional observations can and cannot determine. The section closes with criteria for comparative testing and the scope of phenomenal interpretation. Consciousness theories differ in their explanatory targets and empirical commitments [36, 23]; the four orientations investigated here are exploratory mathematical research programmes, not replacements for existing theories or empirically established causal mechanisms.

16.1 Physical Systems, Measurement Interfaces, and Phenomenal Targets

Physical process and observable record. For a preparation and context c , let Ω_c be a measurable space of complete physical-process descriptions, with no presupposition that the elements of Ω_c form trajectories on a smooth manifold. An experimentally specified operation a belongs to a set $\mathcal{A}_c$ of permitted operations. The operation may be a stimulus, a perturbation, a change in task conditions, or a clinically authorized intervention; its physical meaning and selectivity require independent justification. A model assigns a probability law μ_c^a on Ω_c under that operation, while a measurable recording-and-assessment map $\Pi_c : \Omega_c \rightarrow \mathcal{Y}_c \times \mathcal{R}_c$ produces a physiological record Y and an auxiliary assessment R . The resulting observable law is defined in Equation 129.

$$P_c^a = (\Pi_c)_\# \mu_c^a, \quad P_c^a(B) = \mu_c^a(\Pi_c^{-1}(B)), \quad B \in \mathcal{B}(\mathcal{Y}_c \times \mathcal{R}_c). \quad (129)$$

The pushforward in Equation 129 separates the hypothesized physical process from the law of observable records. For neural studies, Y may comprise EEG, MEG, fMRI, intracranial recordings, or stimulation responses, with different spatial and temporal resolutions. The supplementary variable R may be a verbal report, task performance, eye movement, a clinical score, or a separately inferred target. Each choice supplies a different observational interface. A report-based condition offers phenomenological access under a defensible report protocol, but also engages the mechanisms of perception, memory, decision, and motor reporting. No-report approaches seek to reduce some of these contributions, while retaining inferential assumptions about what their measurements indicate [39].

The candidate phenomenon must be specified at a comparable level of precision. Studies of conscious *level*, the presence or absence of a particular *content*, subjective temporal organization, and access to information address distinct targets. A clinical measure distinguishing wakefulness from anaesthesia can be informative for one target while supplying little discrimination among hypotheses of content-specific phenomenal organization. Similarly, a behavioral failure to report may reflect limitations of access or motor execution. In physically inaccessible cases, the inference from indirect indicators to phenomenology inherits additional uncertainty.

Layers of interpretation. At least four assertions require separate justification. The first states that a fitted geometric or dynamical model adequately summarizes the measured signal. The second interprets the fitted representation as carrying information about a physical mechanism. The third links a property of that mechanism causally or counterfactually to an operational consciousness-related outcome. The fourth proposes an explanatory relation to subjective experience itself. Validation at an earlier layer may contribute to later analysis, yet the latter layers introduce new assumptions. In particular, the presence of a mathematically rich state-space manifold or complex path repertoire in a recording does not itself fix the phenomenal interpretation of the subject or condition.

The context label c encompasses electrode or imaging configuration, sampling and preprocessing, participant characteristics, task, and a declared physical or physiological regime. Equality of two reconstructed geometries across contexts is consequently meaningful only relative to specified comparison maps. An arbitrary embedding learned separately in two experimental conditions may change distances, dimension estimates, and apparent topological signatures without an underlying alteration of neural organization. Conversely, stable observations may conceal changes orthogonal to the measured interface, as established mathematically in Sections 11–15.

16.2 Neural Dynamics and Geometric-Topological Observables

Population-state reconstructions. Neural state-space methods represent simultaneous activity from multiple neurons or recording channels by a coordinate vector $y(s) \in \mathbb{R}^m$, sometimes followed by dimension reduction or an inferred latent state $z(s) \in M_{\text{rec}}$. Such techniques have proved useful for studying neural population dynamics, including motor computation and control [40, 16]. Their use in consciousness research is an empirical proposal requiring consciousness-specific comparisons. Smoothness, local dimension, tangent structure, and recovered flow fields belong first to M_{rec} , which is an epistemic model conditioned on sampling and fitting assumptions. The reconstructed manifold is not thereby identified with neural tissue, physical spacetime, or phenomenal space.

For a calibrated reconstruction $\hat{z}_c$, a proposed Variation observable may be a family of local response maps, estimated Jacobians, or an experimentally constrained distribution of responses to input perturbations. A Trajectory observable may instead be a selected path $\hat{z}_c(s)$, together with specified geometric or topological attributes. A Generativity observable must address the organization that produces or modifies these descriptions: evidence could include changes in fitted constraints, coupling architecture, admissible cell attachments, or representation correspondences across controlled contexts. A Repertoire observable concerns a population of possible recorded paths generated under a declared family of inputs and initial conditions, with its geometry or topology conditioned on those operations. These four proposals refer to different mathematical estimands even when built from the same electrophysiological observations.

Higher-order interactions and cellular constructions. Graph, simplicial, and CW-complex representations can organize high-dimensional neural interactions. Clique complexes constructed from correlation networks, for example, have been used to reveal geometric organization in neural data [17]. Higher-order network models provide further structural alternatives to pairwise graph descriptions [3]. A cellular realization with vertices interpreted as events, edges as couplings, and faces as unspecified higher-order primitives requires a separate observation-to-incidence rule. The meaning of a face must be fixed by the physical or inferential construction; attaching a two-cell to a measured interaction network is an operation on a mathematical representation, not direct observation of an event in physical spacetime.

Homology groups, persistence diagrams, and estimates of recurrent geometric structure can be examined for reproducibility across resampling, thresholds, electrode selections, and independent measurements. Their numerical stability and sensitivity to preprocessing are part of the test. Preservation of a homology summary under a comparison does not identify the full attaching maps or the fundamental group, as demonstrated in Section 11. In particular, a change in recorded simplicial or cellular topology may arise from a change in measurement density, graph-construction threshold, signal-to-noise ratio, or modeled physical coupling. A claim of constitutive physical change therefore requires alternative explanations to be tested using additional modalities, controlled perturbations, or explicit cross-context correspondences.

Protocol-indexed candidate estimands. To compare physical evidence without pretending that the hypotheses already have unique numerical biomarkers, let $\vartheta_k(\theta; c, a)$ denote a declared candidate feature derived from a description θ , context c , and operation a , for $k \in \{V, T, G, R\}$ corresponding to Variation, Trajectory, Generativity, and Repertoire. A comparison of candidate models θ and θ' is possible only after selecting observables and protocols under which their consequences differ. The practical interface

Table 19: Candidate physical and observational interfaces for the four preliminary hypotheses. Each proxy requires an explicit reconstruction method and independent phenomenological validation.

Orientation	Candidate estimand	Distinguishing access and principal ambiguity
Variation	Perturbation-response operator, local generator, or constrained directional response	Controlled input changes; off-path directions and stimulation selectivity can remain hidden
Trajectory	Calibrated selected path, geometric invariants, or loop classes	Repeatable time-resolved records; latent coordinate and projection ambiguity
Generativity	Indexed coupling and constraint descriptions; transformations of constitutive representations	Cross-context or adaptation comparisons; state changes and preprocessing can mimic constitutional change
Repertoire	Reachable-path family or experimentally sampled support under a declared intervention set	Systematic variation of inputs and initial states; an unvisited path is not thereby inadmissible

of each hypothesis is summarized in Table 19. Entries designate proposed estimands and discriminating procedures, rather than confirmed biomarkers of consciousness.

16.3 Perturbational Complexity, Spontaneous Activity, and Consciousness-Related Conditions

Perturbational responses. Transcranial magnetic stimulation combined with EEG has been used to study the extent and diversity of cortical responses under different physiological conditions. Massimini and colleagues reported altered cortical effective connectivity during non-rapid-eye-movement sleep relative to wakefulness [29]. Casali and colleagues subsequently introduced a perturbational complexity index (PCI) intended to capture properties of the cortical response while reducing dependence on sensory processing and behavioral report [8]. Casarotto and colleagues investigated the use of an independently validated index for stratification of unresponsive patients [9]. These contributions establish important empirical programmes for evaluating responsiveness and differentiation across conditions. They do not demonstrate that PCI measures one of the four proposed mathematical objects, that one numerical threshold characterizes every kind of subjective experience, or that a complexity score is equivalent to a complete mechanism of consciousness.

The distinction between responsiveness and the response repertoire is important. A single stimulation site and intensity samples a restricted portion of the physical input space. Variation concerns the action or local structure associated with transformations; Repertoire concerns a specified family of admissible histories across interventions and preparations. A scalar complexity statistic of one evoked response may summarize aspects of the resulting activity without reconstructing either object. The statistical and physical assumptions needed to generalize to a larger input family belong to the experimental model, rather than to the name of the index.

Spontaneous activity. Schartner and colleagues compared multiple complexity statistics of spontaneous multidimensional EEG during propofol anaesthesia, reporting decreases in several such statistics under their studied conditions [35]. The distinction from TMS-evoked complexity matters: spontaneous activity, stimulation-evoked activity, compression-based sequence statistics, and entropy of channel coalitions are different measurement constructions. Their empirical relationships may constrain model building, but one scalar summary may be compatible with many physically and mathematically distinct generators. Complexity alone can also vary with sampling rate, power spectrum, channel coverage, artefact removal, and selected observation windows. These dependencies should be exposed in analysis and sensitivity reports.

Adversarial predictions and theoretical diversity. A preregistered adversarial collaboration directly compared selected predictions of integrated information theory (IIT) and global neuronal workspace theory (GNWT) using fMRI, MEG, and intracranial EEG in 256 participants [12]. Its results were compatible with some predictions of both frameworks while challenging specified commitments of each. Importantly, this study primarily tested divergent predictions of proposed biological implementations; a failure of a tested implementation does not by itself logically refute every formulation of a theory's mathematical or computational core. Such work demonstrates the importance of defining divergent predictions, analysis plans, and interpretation criteria prior to observing the relevant data. The four hypotheses of the present manuscript are classified by formal explanatory object rather than by the anatomical localization or explanatory axioms of IIT or GNWT. Their comparison with either established theory therefore requires a separate, explicit mapping. A result favoring a prediction of an existing theory cannot automatically be relabeled as evidence for one of the four orientations.

16.4 Perturbation, Causal Access, and Hidden Generators

Identifiable causal contrasts. A perturbational experiment can distinguish more mechanisms than a passive observation protocol when it physically changes the conditions controlling their observable responses. It does not follow that every intervention accesses the distinctions of interest. Let $A \in \{0, 1\}$ be a randomized experimental assignment and let $Y(0), Y(1)$ be integrable potential outcomes of an explicitly specified physiological readout. Under consistency, random assignment independent of the potential outcomes, and absence of interference between experimental units, the observable average contrast is identified by Equation 130.

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}[Y \mid A = 1] - \mathbb{E}[Y \mid A = 0]. \quad (130)$$

Proposition 16.1 (Randomized identification of a recorded response contrast). *Suppose the two potential outcomes of an integrable readout Y are well defined, the observed readout satisfies $Y = Y(A)$, assignment probabilities are positive, and A is independent of $(Y(0), Y(1))$. Then Equation 130 holds. Its identified quantity is the average causal effect of assignment on the specified readout, relative to the experiment and population.*

Proof. Consistency implies $\mathbb{E}[Y \mid A = a] = \mathbb{E}[Y(a) \mid A = a]$ for either assigned value. Randomization identifies each conditional expectation with $\mathbb{E}[Y(a)]$. Taking the difference yields Equation 130. $\square$

Proposition 16.1 does not identify the effect of a neural mechanism on phenomenal consciousness. The measured effect may be mediated through sensory stimulation, arousal, motor preparation, attention, or reporting channels. The mechanism of delivery, sham or control procedures, blinding where practicable, physiological safety, and measurement of collateral effects are therefore parts of the scientific definition of the intervention. Random assignment is also insufficient if treatment versions vary in uncontrolled ways or the relevant no-interference and consistency assumptions fail.

A controlled coupling example. The effect of experimental access can be shown without inventing neural data. Consider a two-dimensional family of linear systems with a hidden coordinate v and one recorded coordinate u . Their model-parameter evolution and observation law are stated in Equation 131.

$$\frac{du}{ds} = -u + av, \quad \frac{dv}{ds} = -\lambda v, \quad u(0) = 0, \quad v(0) = 1, \quad y(s) = u(s), \quad (131)$$

Here $a \geq 0$ is a controllable coupling parameter and $\lambda > 0$ is an unknown decay rate. The coordinate s is the declared evolution parameter for this illustrative system. A zero coupling hides the second coordinate from the readout even when λ is changed. If $a > 0$, the readout is given by the exact expression in Equation 132.

$$y_{a,\lambda}(s) = \begin{cases} a \frac{e^{-\lambda s} - e^{-s}}{1 - \lambda}, & \lambda \neq 1, \\ ase^{-s}, & \lambda = 1. \end{cases} \quad (132)$$

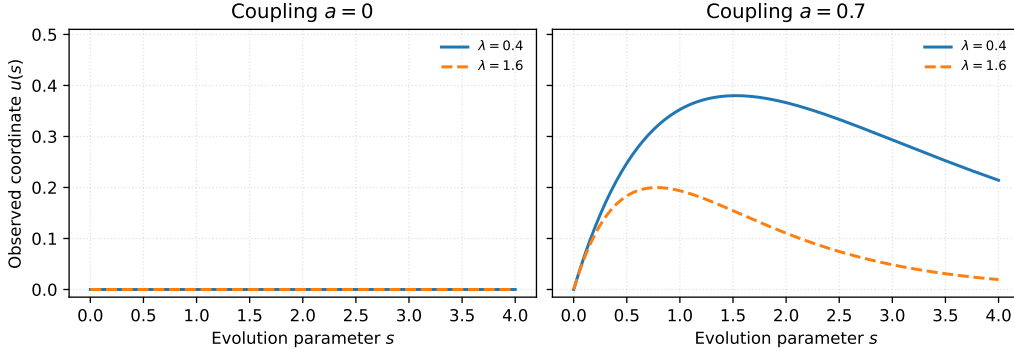


Figure 13: Synthetic controlled-coupling example. For $a = 0$ (left), different hidden decay rates produce the same observable zero path. For $a = 0.7$ (right), their exact observable responses separate. The figure is generated analytically by the included Python script and contains no human or neural recordings.

Proposition 16.2 (Hidden decay rate exposed by controlled coupling). *In the exact model of Equation 131, the full recorded history is identically zero for every $\lambda > 0$ when $a = 0$. If a known coupling $a > 0$ is introduced while the model and its initial conditions remain fixed, an exact continuous record in a neighborhood of $s = 0$ uniquely determines λ from the second derivative of the readout.*

Proof. The solution of the second equation is $v(s) = e^{-\lambda s}$; variation of constants gives Equation 132. For $a = 0$ and $u(0) = 0$, the observable u vanishes for every λ . When $a > 0$, differentiation of the first equation at the initial coordinate gives $y'(0) = a$ and $y''(0) = -a(1 + \lambda)$. Thus $\lambda = -y''(0)/a - 1$ is uniquely determined under the stated assumptions. $\square$

Figure 13 plots the exact curves for $\lambda = 0.4$ and $\lambda = 1.6$ with a common initial state. The two systems are observationally indistinguishable under $a = 0$, while their responses separate for the nonzero coupling. Proposition 16.2 clarifies why an experimentally varied input can reveal generator-level information concealed by a selected unperturbed trajectory. It also exposes the assumptions on which the conclusion depends: control of a , known initial states, a correct two-dimensional model, and access to sufficiently precise derivatives. Recovering a second derivative from sampled noisy data is often unstable; the displayed identifiability argument therefore does not guarantee practical parameter estimation. The coupling is an abstract control and does not purport to model a selective TMS, pharmacological, or clinical intervention.

Interventions constrained by an observation quotient. Perturbation does not overcome underdetermination when each admissible operation respects precisely the same observational equivalence classes. To express this, let X be a set of source states, $h : X \rightarrow Y$ a recording map, $\{T_a : X \rightarrow X\}_{a \in \mathcal{A}}$ a collection of source transformations, and $\{S_a : Y \rightarrow Y\}_{a \in \mathcal{A}}$ recorded-state transformations. The compatibility requirement is stated in Equation 133.

$$h \circ T_a = S_a \circ h \quad \text{for each } a \in \mathcal{A}. \quad (133)$$

Proposition 16.3 (Interventional indistinguishability under an equivariant recording map). *Under Equation 133, source states $x, x' \in X$ satisfying $h(x) = h(x')$ produce identical recorded states after every finite sequence of admissible transformations, provided both are subjected to the same sequence and the same deterministic source model.*

Proof. For a sequence of length zero, equality holds by hypothesis. Assume it holds after a finite sequence ending in source states z, z' . Applying one further operation a gives $h(T_a z) = S_a(h(z)) = S_a(h(z')) = h(T_a z')$. Induction proves the result for every finite admissible sequence. $\square$

Proposition 16.3 identifies a structural obstacle, rather than an absolute epistemic prohibition. Expanding the recording channels, changing the admissible intervention family, or testing the validity of the equivariance assumption may expose previously collapsed distinctions. If the actual physical system includes stochastic transitions, a probabilistic kernel-level analogue is needed, and the deterministic result should not be applied without modification. Distinguishing the effects of controlled operations from the mere number of stimulation conditions is important for economical experimental design.

16.5 Measurement Coarsening and Cross-Protocol Comparability

Loss of statistical discrimination. A measurement pipeline often filters a signal, compresses a recording, estimates latent coordinates, thresholds a correlation, or converts a complex response to a scalar. Such operations may preserve a useful observable while identifying outcomes relevant to another hypothesis. Let P_0, P_1 be the distributions of a full recorded variable under two physically specified conditions. For a measurable deterministic reduction g let $g_{\#}P_i$ be the corresponding reduced distributions. The discrimination bound is given in Equation 134.

$$\text{TV}(g_{\#}P_0, g_{\#}P_1) \leq \text{TV}(P_0, P_1). \quad (134)$$

Proposition 16.4 (Statistical distinction cannot increase under deterministic coarsening). *For probability measures P_0, P_1 on a measurable space and a measurable map g , Equation 134 holds. Equality holds whenever g is a measurable bijection between the relevant measurable spaces with measurable inverse.*

Proof. By the definition of total variation, the left-hand side is the supremum over measurable target sets B of $|P_0(g^{-1}(B)) - P_1(g^{-1}(B))|$. The collection of preimages $g^{-1}(B)$ is a subset of the measurable source sets, and the inequality follows. If g and its inverse are measurable bijections, every measurable source set is the preimage of a measurable target set, giving equality. $\square$

In combination with the equal-prior testing result of Proposition 15.5, Proposition 16.4 shows why a lossy summary cannot improve the ideal discrimination available from its parent recording when the candidate distributions are specified and the full statistical decision problem is available. An actual analysis of finite noisy samples may benefit from variance reduction through compression; this practical benefit does not contradict the mathematical inequality. Equally, a change of neural coordinates that is invertible at the record level retains ideal statistical discriminability, but may alter metric-dependent numerical descriptors. Preservation of distributional discrimination is therefore distinct from preservation of geometry, topology, or a physically meaningful mechanism.

Sensitivity to the analysis pipeline. A cross-condition comparison should state the raw measurement space, coordinate transforms, filtering and dimensionality reduction, topological construction rule, and statistical analysis. The stability of a result can be evaluated under prespecified plausible choices of preprocessing, measurement density, and model complexity. Disagreement among modalities is not necessarily a failure; each modality may observe distinct projections of the underlying activity. A multimodal study can become more informative when its model explicitly identifies the common physical variables and the separate channels. Without a cross-modal observation model, concatenating recordings does not automatically produce a more faithful latent spacetime or a complete history.

16.6 Comparative Experimental Designs for the Four Hypotheses

A common comparative programme. A scientifically discriminating study requires candidate versions of the four hypotheses with predictions expressed over a common outcome domain. Their broad definitions overlap: a generator can determine histories and repertoires under fixed conditions, and constitutive change can alter each of those objects. A comparison that finds predictive signal in all four representations is therefore compatible with overlap and cannot by itself select a unique ontological explanation. Meaningful discrimination requires pairs of models, interventions, or controlled contexts for which specified features diverge while alternative explanatory variables are sufficiently constrained.

Table 20: Illustrative discriminating contrasts. These are proposed study designs, not conducted experiments or unique predictions derived from the four broad hypotheses.

Orientation	Controlled contrast	Potentially informative result	Principal competing account
Variation	Same observed baseline; calibrated perturbations in distinct input directions	Response operators differ although baseline records coincide	Stimulation strength, hidden preparation changes, or unmodeled input paths
Trajectory	Matched coarse summaries; compare individual path geometry or repeated content-related evolution	Selected path feature generalizes to independent reports or calibrated content estimates	Reporting, attention, time alignment, or arbitrary latent embedding
Generativity	Controlled learning/adaptation or regime change with repeated recording and model fitting	Evidence for changed constraint or coupling structure beyond state shifts	Measurement drift, nuisance variables, or nonidentifiable models
Repertoire	Multiple preparations and systematically varied admissible inputs	Difference in accessible path classes despite matched selected paths	Limited sampling, finite recording resolution, or unequal control access

A feasible programme could employ two kinds of datasets within ethically permitted conditions. The first would collect high-resolution time-resolved neural measurements under repeatable conditions and calibrated reports, including no-report variants when their interpretation is justified. The second would collect controlled perturbation responses or naturally available task/input variation, with independent validation data and registered analysis plans. Clinical states, sleep stages, and anaesthesia can offer informative contrasts, but their pharmacological, physiological, and clinical differences cannot be assumed to manipulate consciousness alone. Research in patients with disorders of consciousness requires especially cautious interpretation of absent reports and dedicated clinical safeguards [9].

Table 20 specifies example contrasts and possible outcomes for each orientation. A predicted failure mode is included because a high-performing classifier is insufficient as an explanation if its features also track simpler confounds such as stimulus energy or channel quality.

Variation. A concrete variant of the Variation Hypothesis might predict that a perturbation-response operator, estimated over sufficiently diverse input directions, distinguishes an operationally specified set of conscious and nonconscious conditions after matching ordinary baseline recordings. The key comparison is between a fitted transformation law and a restricted observation of its selected outputs. Experimental designs should characterize pulse amplitudes, spatial spread, stimulus artefacts, and the reproducibility of inferred responses across sessions. A positive result would support a relationship between a transformation-level feature and the stated outcome. A mechanistic claim requires further intervention specificity; a phenomenal claim remains a distinct hypothesis.

Trajectory. A concrete Trajectory model might assign an operational outcome distribution to calibrated properties of selected neural paths: residence in regions of a reconstructed state space, recurrence structure, or a representation-stable geometric statistic. Its predictions should be tested on unseen data while controlling for attention, reporting, and input properties. Homotopy classifications require stable carrier or cross-condition maps; in moving and learned latent spaces, even equality of path endpoints can lack a coordinate-independent meaning. The proposal becomes informative when it predicts an aspect of measured experience that matched static or generator-only summary models fail to capture under specified conditions.

Generativity. A concrete Generativity model could posit a measurable relationship between a consciousness-related outcome and changes in physical or effective generative organization across conditions. One possible operationalization would compare estimated coupling constraints or higher-order interaction patterns before and after a controlled adaptation protocol. An indexed relational two-complex

may represent inferred events and couplings, with faces added according to a preregistered higher-order rule. The study must distinguish changes in inferred cell incidence from changes in the physical organization responsible for the recordings. Model selection, missing units, altered sensor sensitivity, and changes in task strategy can mimic generative reorganization. Evidence for an endogenous law of constitutive change would require stronger experiments than a sequence of fitted network snapshots.

Repertoire. A concrete Repertoire model might associate an outcome with the geometry, effective volume, or topological organization of paths reachable under a physically defined collection of allowed perturbations and preparations. Estimation requires repeated trials and control of sampling bias, including uncertainty for rarely observed histories. If two conditions share a highly reproducible selected trajectory yet differ under systematic perturbational exploration, the result may favor a repertoire-level explanatory feature relative to that matched-trajectory model. It would remain compatible with a broader Variation model whose generators determine the same repertoire. Claims about impossible histories require declared dynamical assumptions; failure to record a path during finite sampling supplies a weaker observation.

Contrastive preregistration. A comparison should register the target phenomenon, model classes, operations, observational resolution, candidate predictors, control variables, exclusion rules, statistical estimands, decision thresholds, and the interpretation of positive and negative results. At least one contrast should be constructed in which the competing operational hypotheses yield meaningfully separated predictive distributions. A finding that discriminates four statistical proxies does not automatically discriminate four philosophical views. The adversarial design of the Cogitate collaboration provides a useful methodological precedent for specifying divergent predictions and independent tests, while its particular theory commitments should be evaluated on their own terms [12].

16.7 Evidence Grading, Reproducibility, and Phenomenal Interpretation

Levels of warranted conclusion. A graded conclusion can distinguish (i) reproducibility of a mathematical or signal-derived feature, (ii) association with a clearly measured consciousness-related condition or content, (iii) successful out-of-sample discrimination under specified confound controls, (iv) experimentally supported causal influence on a physiological or report-based target, and (v) a theory-supported explanatory relation to phenomenal experience. Each level adds inferential requirements. Even when an intervention changes both a candidate geometric property and a report, the evidence may leave their causal ordering, mediation, and physical constitution underdetermined. A separate interpretive argument is required to infer that the intervention alters experience through the candidate property.

Neural dynamics and topological analysis should also retain open interfaces to evidence that exceeds their mathematical description. First-person reports, structured phenomenological comparisons, psychophysical performance, physiological monitoring, and clinical assessment are not interchangeable measurement channels. Methodological pluralism requires explicit reasons for using each channel and transparency about its blind spots. In populations unable to provide reliable reports, uncertainty should be communicated rather than converted into a categorical attribution or denial of experience.

Practical audit requirements. A reproducible test would preserve raw-data provenance, the measurement and intervention protocols, preregistered outcomes, analysis code, model versions, exact preprocessing transformations, uncertainty estimates, and admissible alternative explanations. For mathematical representations, it should also record the construction of the carrier, the semantics assigned to vertices, edges, and faces, the definition of a path and its parameter, and which invariants survive the chosen transformations. Where numerical reconstructions of complex dynamical structure are unstable, uncertainty must be propagated to the corresponding comparative claims. Synthetic controls and null simulations can test a pipeline's behavior, but cannot substitute for validation on physical recordings.

The present section has offered empirically interpretable candidate objects, conditional identification results, and designs for discriminating operational versions of the four hypotheses. It has not established that one orientation is necessary or sufficient for subjective experience, nor that the broad fourfold taxonomy is exhaustive or mutually exclusive. In the next section, explanatory limitations can be investigated more directly: the inferential distance between observed neural organization, formal equivalence, physical constitution, and a proposed account of phenomenality remains a substantive part of the research programme.

17 Explanatory Limits of Formal Models of Consciousness

The preceding sections developed four candidate mathematical objects for consciousness research and specified conditions under which they can be compared, reconstructed, and tested. This section examines the additional explanatory commitments required to connect those objects to subjective experience. Its objective is to delimit what follows from mathematical description, physical interpretation, observation, and intervention, and to identify the premises that would be needed for stronger phenomenal conclusions. The analysis combines conceptual distinctions, elementary impossibility results relative to specified information, and countermodels of causal and measurement inference. The results concern the adequacy of stated representations; they do not establish an impossibility theorem for a future science of consciousness.

17.1 Explanatory Targets and Relations

An explicit explanandum. A proposed explanation acquires its scope from the phenomenon it addresses. In consciousness research, an operational contrast between wakefulness and general anaesthesia, the recoverability of a visual content from recordings, the availability of information for report and deliberation, and the subjective character of an experience are related but distinct targets. Block's distinction between phenomenal consciousness and access consciousness is one influential articulation of their potential divergence [5]. Nagel's account of subjective character, Levine's explanatory-gap discussion, and Chalmers's formulation of the hard problem clarify why understanding information processing or physical organization may leave a further question about experience [31, 28, 11]. These are substantive philosophical arguments; their conclusions and proposed solutions remain contested. The present paper adopts the distinction among explanatory questions without presupposing a particular metaphysical answer.

Four modes of explanatory success are especially relevant. A *representational account* supplies mathematical objects with an interpretation of their components. A *predictive account* associates those objects with independently assessed outcomes under stated protocols. A *causal account* specifies interventions and an identified dependence among physical or measured variables. A *phenomenal account* proposes and justifies a connection to the occurrence, qualitative organization, or unity of experience. Such modes can interact, and a single model may contribute to several; they impose different standards of support. A strong numerical predictor can therefore be scientifically useful while leaving the proposed phenomenal interpretation unresolved.

The four candidate objects. Within the present taxonomy, Variation concerns mechanisms and admissible transformations; Trajectory concerns selected processes and relationally ordered histories; Generativity concerns transformations of structures, constraints, and the conditions of further generation; and Repertoire concerns organized families of admissible histories. The fourfold organization does not itself stipulate which aspects of experience, if any, correspond to these mathematical distinctions. Moreover, a chosen representation may encode the very same observed phenomena through several mathematical descriptions. Consequently, a phenomenal explanation must say which features of a proposed geometry, topology, or dynamics matter, what would count as their preservation, and why those features bear on the target phenomenal distinction.

17.2 Structural Invariance and Phenomenal Bridges

Representation equivalence. Let $\mathcal{P}$ be a declared class of physically interpreted systems or system-conditions. Let $R : \mathcal{P} \rightarrow \mathcal{Z}$ extract the formal representation used in a proposed consciousness account, and let E_R denote equality of those representations. Depending on the intended theory, R may retain a smooth flow up to smooth conjugacy, a cellular complex up to a specified equivalence, a path family up to reparameterization, a quotient of transition systems, or only selected invariants. Such choices are substantive. A diffeomorphism can preserve smooth structure while altering a metric unless the metric is transported, and a chain-level equivalence may conceal homotopy information that a richer cellular model preserves.

A phenomenal-bridge proposal introduces a candidate target space $\mathcal{Q}$ and a relationship between physically interpreted systems and target properties. When a total, single-valued assignment $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ is assumed for the purposes of analysis, the factorization requirement in Equation 135 expresses sufficiency of the declared representation for this target.

$$\Phi = b \circ R, \quad b : \text{im}(R) \longrightarrow \mathcal{Q}. \quad (135)$$

This equation is a *proposed condition*, not an empirical finding. $\mathcal{Q}$ need not carry a numerical scale or identify experiences across subjects by an assumed universal coordinate. If one declines to posit a total phenomenal assignment, the same issue can be examined through partial assignments, relations, or comparative judgments; additional domain assumptions are then necessary.

Proposition 17.1 (Representation-relative obstruction to a phenomenal bridge). *Let $R : \mathcal{P} \rightarrow \mathcal{Z}$ and $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ be maps of sets. A map b satisfying Equation 135 exists if and only if Φ is constant on every fiber of R .*

Proof. If $\Phi = b \circ R$ and $R(p) = R(p')$, then $\Phi(p) = b(R(p)) = b(R(p')) = \Phi(p')$. Conversely, assume constancy on fibers. For $z \in \text{im}(R)$, choose any p with $R(p) = z$ and define $b(z) = \Phi(p)$. Constancy makes this independent of the choice, and composition gives $\Phi = b \circ R$. $\square$

Proposition 17.1 specifies a testable *form* of the desired adequacy claim only when a target assignment and the relevant physical contrasts can be independently constrained. Its mathematical proof makes no assertion that phenomenal assignments exist as total functions, or that a particular R violates the constancy condition. In particular, exhibiting physically different systems with identical homology is insufficient to demonstrate that they differ phenomenally. The proposition instead makes visible the additional premise required when homology, an attractor classification, a path invariant, or a numerical complexity measure is advanced as a sufficient phenomenal descriptor.

Invariance is relative to chosen structure. Suppose a theory defines a real-valued geometric quantity using path length on a Riemannian manifold (M, g) . Replacing g by $c^2 g$ for $c > 0$ retains the topology and smooth atlas while multiplying every path length by c . Whether this change is a coordinate reexpression, a physically meaningful change of metric, or an inadmissible transformation must be stated by the physical interpretation. Similarly, the first homology of a relational two-complex can remain unchanged when its fundamental group differs, as shown in Section 11; a proposed experience-related invariant of one structure need not be an invariant of the other. Mathematical equivalence is therefore always an equivalence *with respect to specified retained structure*.

Sufficiency, necessity, and explanatory relevance. A successful factorization would show, within the assumed domain and target assignment, that R is sufficient to determine Φ . Necessity has a different meaning: the absence of an R -property must preclude a specified phenomenal property, under appropriate comparison rules. Either claim may demand exclusion of coincidental correlations, control of physical realization, and an account of changes in observation protocols. Even a stable bridge can be explanatory in different senses: as a lawlike dependence, a reduction, a constitutive account, or a causal

account. The present four hypotheses leave these options open. A correlation between R and a report is not, by definition, a validated phenomenal bridge.

17.3 Physical Realization, Intervention, and Causal Mediation

Distinct experimental claims. A geometrical representation estimated from neural recordings may track a behavioral outcome under several conditions, respond to an intervention, or enter a mechanistic explanation of a recorded effect. The first establishes an association under a model. A well-designed intervention may identify how changing a physical input affects an outcome. Mediation requires an additional claim: the outcome changes *through* the nominated geometric or dynamical feature. The meaning of intervening on that feature must also be established if the feature is a statistic computed from many physical states. Section 16 supplies identification results for randomized assignment and measured response contrasts; the present question concerns mediation through an explanatory candidate.

Proposition 17.2 (Distinct causal mediations with identical assigned-treatment data). *There exist deterministic structural causal models with the same joint distribution of randomized assignment Z , candidate feature F , and recorded response Y , and the same response to every intervention on Z , but different responses to an intervention on F that holds Z fixed.*

Proof. Let Z be Bernoulli with probability $1/2$ and let $F = Z$ in both models. In Model A set $Y = F$; in Model B set $Y = Z$. Under ordinary sampling and under either intervention $\text{do}(Z = z)$, both models yield $(Z, F, Y) = (z, z, z)$. Under the hypothetical intervention $\text{do}(F = f)$ with $Z = z$ fixed, Model A yields $Y = f$, whereas Model B yields $Y = z$. Thus the available treatment-response data agree but the mediation mechanisms differ. $\square$

The two models in Proposition 17.2 are deliberately elementary. They establish a logical distinction between a randomized effect and identification of a mediating pathway. A practical experiment may gain identifying information through independent variation of F , supplementary measurements, or explicit and credible causal restrictions. It remains essential to ask whether a geometric summary such as a reconstructed homology class is independently manipulable, or whether a feasible experiment only intervenes on neural substrate components of which that summary is an effect. Treating a software-calculated statistic as a freely manipulable physical variable would conceal an important part of the proposed mechanism.

Organization and physical implementation. Formal simulations may reproduce selected trajectories, input–output relations, or finite repertoires of another implementation. Such preservation is informative for the corresponding functional account. Physical identity, however, requires a separately declared criterion of realization, and phenomenal identity requires still further commitments. Arguments such as the “unfolding” critique dispute whether input–output equivalence can adjudicate consciousness theories that distinguish internal causal organizations [14]. The critique raises a real methodological problem of discriminatory evidence, while competing theories can question which evidence and which notion of realization should be privileged. Our abstraction results in Section 13 clarify one mathematical part of the issue: a translation can preserve outward traces while losing distinctions among mechanisms or counterfactual transitions. They do not settle the accompanying metaphysical disagreement.

17.4 Latent Phenomenology and the Calibration of Reports

Reporting as a measurement process. Let X denote a modeled physical condition, Q a provisional latent phenomenal variable, and Y an observable report or behavioral measure. An observation model can be described using a conditional kernel K for Q given X and a reporting kernel L for Y given (X, Q) . The resulting report law is given in Equation 136.

$$P(Y \in B \mid X = x) = \int_{\mathcal{Q}} L(B \mid x, q) K(\mathrm{d}q \mid x), \quad (136)$$

for measurable report sets B . This is a provisional statistical representation of inferential interfaces. It is not a claim that experiences are ordinarily directly observed random variables with a known objective scale. The kernel L may vary across language, communicative practices, context, capacities for report, and the measurement instrument. Inference about Q from Y therefore depends on calibration assumptions as well as on the observed distribution.

Proposition 17.3 (Latent-description symmetry of report laws). *Let $\sigma : Q \rightarrow Q'$ be a measurable bijection with measurable inverse. Define K' as the pushforward of K by σ and define $L'(B | x, q') = L(B | x, \sigma^{-1}(q'))$. Then the two latent descriptions yield the same observable report law for every x and every measurable report set B .*

Proof. Integration with respect to a pushforward kernel is integration of the pulled-back integrand with respect to the original kernel. Substituting the definitions into Equation 136 gives $\int_{Q'} L'(B | x, q') K'(dq' | x) = \int_Q L(B | x, q) K(dq | x)$. $\square$

Proposition 17.3 is a familiar kind of latent-label symmetry. By itself it establishes a redundancy of *unanchored descriptions*; it does not establish the metaphysical possibility of inverted experiences, nor that any two actual experiences are phenomenally identical. Semantic constraints, repeatable first-person comparisons, task-specific psychophysics, intersubjective calibration, and physical mechanisms may restrict the admissible σ . The result makes an epistemic obligation precise: a putative experience descriptor must be anchored by more than an arbitrary latent label if its qualitative interpretation is intended to be stable.

Phenomenological access and its limits. Reports can convey genuine evidence about experience and can be distorted by memory, attention, language, expectation, or inability to respond. A no-report protocol changes the measurement dependencies, while introducing its own assumptions about the relation of indirect measures to perceptual experience [39]. Structured first-person methods, behavioral comparisons, and physiology provide partially distinct modes of access. Triangulation should investigate concordance and disagreement, including when a study's operational consciousness labels inherit judgments from an earlier assessment. An inability to express an experience does not itself establish the absence of experience. Conversely, agreement of linguistic outputs does not alone establish identity of their subjective sources.

Other systems and interpretive symmetry. The difficulty becomes more conspicuous when the system under investigation uses a different sensory apparatus, reporting practice, organization, or historically generated distinction system. Nagel's discussion exemplifies the distance between a third-person account of a sensory capacity and an understanding of its subjective character [31]. A generative representation inferred from another system may recover reliable regularities, transformations, and constraints without recovering the way the system differentiates or expresses its own experience. The obstacle includes experimental limitations and interpretive differences, alongside the philosophical question of whether and how the proposed physical description constitutes phenomenality. It therefore motivates graded claims of epistemic access rather than an all-or-nothing attribution.

17.5 Relational Constitution of Subjects and Experiential Unity

Observers as candidate historical organizations. In the present two-complex vocabulary, vertices are events and edges are couplings; faces encode higher-order primitives whose meaning must be declared for each model. A subject is consequently not identified with a primitive vertex. One possible research construction represents a candidate observer by a historically organized family of subcomplexes or other relational substructures, with specified persistence and coupling conditions. Such a mathematical specification can aid the study of observer formation and sensorimotor integration. Its success in tracking stable behavior does not independently establish the onset, boundaries, or unity of a phenomenal subject.

To make the preservation requirement concrete, consider an indexed diagram $\{K_i, f_{ij}\}$ of relational complexes and a selected family of candidate observer subcomplexes $S_i \subseteq K_i$. A compatibility condition for a specified set of comparison morphisms is given in Equation 137.

$$f_{ij}(S_i) \subseteq S_j \quad \text{for each declared comparison } f_{ij} : K_i \rightarrow K_j. \quad (137)$$

Equation 137 is a necessary bookkeeping rule for this particular persistence convention, not a general criterion of personal or phenomenal identity. Equality, inclusion, partial matching, splitting, and merging of observer candidates demand distinct interpretive choices. Moreover, the index i may encode observational contexts, histories, or formal comparison stages; it does not supply a universal clock or a privileged present. The appropriate meaning of an ordering must be specified by the generating relations.

Unity and multiplicity. Connectedness of a complex, vanishing first homology, a strongly connected causal graph, and ergodicity of a flow are different mathematical properties. None has been independently established here as necessary or sufficient for the unity of experience. Connected neural tissue can include strongly differentiated processes; disconnected components of a reconstructed graph can arise from missing sensors or thresholding. If experiential unity is an intended target, the manuscript must specify what unity means phenomenologically, which physical conditions are compared, and why the selected invariant is expected to track it. Accounts proposing one experiencer, several experiencers, or a changing boundary of subject constitution remain distinct until evidence and bridging assumptions discriminate them.

Observer participation and epistemic dependence. A measuring observer and an observed system can participate in coupled physical and interpretive relations. Experimental intervention, reporting, and data annotation may modify the interactions through which a scientific record is produced. This observation does not entail that scientific results are purely subjective or that stable comparisons are unavailable. It demands an explicit account of the observation interface, its historical construction, and its effects when those effects influence the target phenomenon. Within a relational programme, the constitution of the investigator's categories is also an epistemic issue, rather than an automatic feature of the modeled system.

17.6 Disagreement among Explanatory Strategies

Different uses of formal structure. One family of proposals investigates functional availability, global access, or information broadcasting. Another relates experience to intrinsic causal organization; other accounts privilege recurrent sensory processing, higher-order representation, or predictive organization. These programmes may agree on measured neural signatures while disagreeing about necessity, sufficiency, physical substrate, or the role of phenomenology [36, 23, 38]. A dynamical or topological reformulation does not automatically reconcile their commitments. It can, however, make their preserved objects and competing predictions easier to state.

The distinction between phenomenal experience and access functions is particularly important here. An empirical success in predicting access may be relevant to an account of phenomenality, but the inferential step depends on a further theoretical argument [5]. Similarly, a model of intrinsic causal structure may seek a more direct bridge to experience, yet must justify its choices of causal grain, physical units, counterfactual repertoire, and phenomenological interpretation. Recent adversarial empirical comparison illustrates the value of stating divergent predictions while also exposing the interpretive work needed after mixed results [12]. The formal fourfold taxonomy should accommodate these disagreements as substantive research questions.

Table 21 collects the extra burdens associated with claims of phenomenal relevance. These burdens arise even when the corresponding mathematical model is internally rigorous and empirically predictive. Such a diagnosis should guide model development, rather than be treated as an argument for abandoning mathematical work.

Table 21: Explanatory obligations and appropriate limits of inference for the four formal orientations. The entries concern candidate research programmes, not validated mappings to phenomenal experience.

Orientation	Additional explanatory obligation	Representative inferential limitation
Variation	Justify which transformation operators and counterfactual interventions are physically constitutive or phenomenally relevant	Input–output fit can leave internal transformation organization unresolved
Trajectory	Fix the physically relevant path representation and its invariances, and relate selected histories to experience	Similar paths can arise from different laws or from observation projections
Generativity	Specify subject formation and the physical laws or mechanisms of changing structures and constraints	Structural revisions in fitted models need not be endogenous physical generation
Repertoire	Define admissibility, the scope of counterfactual comparison, and a principled connection between path families and experience	Finite exploration cannot generally establish the complete admissible family

17.7 A Graded Agenda for Phenomenal Explanation

A sequence of evidential commitments. A disciplined research programme can separate the following achievements: (i) internally valid mathematical characterization; (ii) a physically specified realization and calibrated recording interface; (iii) representation-stable empirical association; (iv) discriminating, adequately controlled interventions; (v) justified claims about the identity, unity, or qualitative structure of experience. The sequence is conceptual and methodological rather than a universal chronological order. A research programme may develop several levels together or discover new forms of evidence that reorganize them. Later claims should nevertheless identify the additional assumptions on which they depend.

A proposed adequacy checklist. For any strong consciousness-related claim, the intended explanandum should be named, the physical reference class made explicit, and the chosen notion of structural equivalence defended. The investigator should disclose observation and reporting assumptions, identify possible confounds, provide meaningful countermodels, distinguish intervention on a physical cause from intervention on a descriptive statistic, and report which phenomenal premises remain untested. For relational complexes, the semantics of events, couplings, faces, and candidate subject boundaries require special care. For geometries of path spaces, the definitions of admissibility, parameter, and invariant must accompany the empirical interpretation. The exercise of expressing these commitments is itself productive: it renders a formal model open to criticism and systematic comparison.

The scope of the present results. Proposition 17.1 proves a conditional factorization criterion; Proposition 17.2 supplies a causal countermodel; Proposition 17.3 establishes observational invariance under a change of latent description. Their mathematical status is distinct from philosophical arguments concerning the explanatory gap and from empirical evidence concerning neural correlates of consciousness. The propositions do not demonstrate that an adequately supported bridge from physical organization to experience is impossible. They clarify which information and justificatory premises are absent when such a bridge is asserted directly from a model’s geometry, topology, dynamics, or predictive accuracy.

The four hypotheses thus remain open research orientations with differing mathematical strengths and explanatory responsibilities. Formal constructions may illuminate what mechanisms can generate, what particular processes are selected, how relational constitutions are transformed, and which histories are admissible under specified conditions. The relation between these discoveries and the existence or character of subjective experience requires further conceptual analysis and empirical investigation. Section 17 supplies the critical boundary for the discussion of the programme’s contributions and limitations in the following section.

18 Comparative Discussion of the Four Hypotheses

The mathematical development of the four hypotheses has produced a comparative account of transformation prescriptions, selected histories, constitutive generative organization, and admissible-history families. The role of the present section is to assess what has been established, identify the dependencies among those objects, and formulate research decisions that follow from their differences. The discussion proceeds from the information retained by alternative representations to the empirical conditions under which those differences could matter, and then to the additional explanatory obligations involved in subjective experience. It combines deductions from the preceding formal constructions with explicitly provisional methodological proposals. Its conclusions concern the scope and use of the taxonomy rather than the truth of any hypothesis about consciousness.

18.1 Explanatory Objects and Conditional Dependencies

A taxonomy of explanatory emphasis. Variation, Trajectory, Generativity, and Repertoire single out distinguishable objects within represented dynamical systems. A Variation model concerns the organization of admissible changes, such as the action of a semigroup, a family of vector fields, or an intervention-dependent transition relation. A Trajectory model concerns a selected path or history, with its declared ordering, parameterization, and possible geometric or topological invariants. A Generativity model concerns the constitution and transformation of change mechanisms, structures, constraints, and distinctions, potentially across heterogeneous carriers. A Repertoire model concerns a family of possible histories relative to stated admissibility conditions, including any topology, metric, accessibility relation, or measure imposed on that family. These definitions identify the kinds of information a theory proposes to use. They leave the proposed association with conscious experience to an independently motivated explanatory bridge.

The distinction is sharper than a change in vocabulary. The transverse-stability constructions in Sections 11 and 14 show that a selected trajectory can coincide in two models whose nearby transformation behavior differs. The finite transformation-monoid construction in Section 14 shows that the same unlabelled path repertoire can coexist with different generating monoids. Conversely, admissible paths on a fixed carrier can change when a permitted-control bound changes, even though the carrier's topology remains fixed. These cases establish formal separations among declared data structures. They supply neither a ranking of scientific usefulness nor evidence that conscious experience differs between the mathematical systems.

Context-dependent determination. Under sufficiently strong modeling assumptions, some objects can be derived from others. A well-posed deterministic evolution law, an initial state, and a specified input schedule determine a selected solution on its interval of existence. A fully specified transition mechanism, a set of initial conditions, an admissible-input policy, and a time horizon determine a corresponding history repertoire. Consequently, under a declared reference class, the mathematical data of Variation can contribute to the construction of Trajectory and Repertoire objects. The reverse recovery of a unique generating law generally fails with restricted observations, as the counterexamples in Sections 13 and 15 demonstrate. The relation is also sensitive to what the word *Variation* includes: a field without initial data or control constraints does not itself select a unique realized history or complete repertoire.

Generativity presents a distinct issue. An indexed collection of models records constitutive variation, but an index alone supplies neither a physical history of revisions nor a law governing the changes of its own models. Where a constitutive evolution law, correspondences between carriers, and local dynamical mechanisms are specified, the model may determine selected histories and admissible continuations relative to those assumptions. Where only snapshots of fitted networks are available, there may be considerable uncertainty about whether an apparent constitutive change reflects a physical reorganization, an observation change, or the investigator's representational convention. This distinction is particularly important for consciousness research using learned latent geometries and dynamic neural connectivity.

Relative informational sufficiency. A useful local language for these dependencies is a recoverability preorder on descriptions of a fixed reference class of physical or formal systems. Let $\mathcal{P}_c$ be a stated reference class under context c , and let A and B be two representations defined on that class. We say that B is sufficient to recover A on $\mathcal{P}_c$ when the factorization in Equation 138 exists.

$$A \preceq_{\mathcal{P}_c} B \iff \exists f : \text{im}(B|_{\mathcal{P}_c}) \longrightarrow \text{im}(A|_{\mathcal{P}_c}) \text{ such that } A|_{\mathcal{P}_c} = f \circ B|_{\mathcal{P}_c}. \quad (138)$$

For set-valued descriptions, Equation 138 is interpreted only after the relevant values have been typed as elements of explicitly defined spaces. The relation is reflexive and transitive, and distinct representations can be mutually recoverable without being syntactically identical. Crucially, the relation depends on $\mathcal{P}_c$; changing the admissible models, observation precision, intervention policies, or horizon can destroy a previously valid factorization. A richer description in this sense need not be easier to estimate, physically more fundamental, or phenomenally more informative. The same formal relation underlies the representation-relative criterion in Proposition 17.1, but that proposition would require an independently justified phenomenal target before yielding a claim about experience.

A methodological consequence follows. Comparisons among the four hypotheses should state the physical reference class and the operations available to the investigator before claiming that one object subsumes another. Theoretical subsumption under complete mathematical knowledge and empirical superiority under finite measurements answer different questions.

18.2 Geometric and Topological Information Across Representations

Representation-dependent invariants. Differential geometry, algebraic topology, and dynamical systems provide several informative but partially independent descriptions of relational organization. A vector field on a smooth manifold requires a differentiable carrier; its flow additionally depends on domain and existence conditions. Homology of a CW-complex retains abelian information about cycles and boundaries, while the fundamental group can distinguish nonabelian loop organization invisible to cellular homology. A path-space topology is further dependent on the chosen class of paths and the topology used to compare them. These mathematical choices become consequential whenever a theory proposes that a particular geometric or topological invariant is relevant to consciousness.

The constructions in Sections 11 and 14 provide diagnostic counterexamples to unrestricted transfer of interpretations. Two two-complexes can have identical cellular homology and different fundamental groups. Different winding histories on a torus can have the same projection onto an observed circle. A quotient transition graph can contain histories for which no concrete lift exists. These examples indicate specific information that is lost, retained, or added by the corresponding representation map. None identifies a consciousness-bearing invariant. Their value lies in specifying the resolution and equivalence relation at which a future empirical or explanatory claim would have to operate.

For a relational two-complex, the distinction between modeled events, couplings, and higher-order faces should also be maintained. Vertices can represent events, edges couplings, and faces stipulated relational primitives; the semantics of a face require local definition. A candidate subject would be represented, if at all, through an organization or historically constituted subcomplex together with rules of identification across changes. Treating a vertex as an already individuated observer would insert the sought constitution into the primitive model. Similarly, the physical spacetime in which a neural process occurs, the state manifold used to model neural activity, and a geometric realization of a cellular complex require separate maps before their topological properties can be compared.

Translation with declared losses. The abstraction analyses in Sections 12 and 13 show that a structure-preserving map is always structure-preserving with respect to specified data. A semiconjugacy may preserve evolution of observed states while identifying distinct source states; a simulation can preserve eligible traces while failing to reflect branching; a cellular chain map can forget homotopy information. Numerical approximations can add an error budget whose long-horizon effect depends on stability and

the length of the represented process. Consequently, agreement among representations at one interface does not automatically establish agreement at other interfaces.

A productive response is to treat each translation as a declared research object. Its audit should specify the source and target carrier types, dynamics, observations, allowable paths, constitutive indices, and intended invariants; it should also state what information is projected away and whether a reverse reconstruction is known. The audit permits limited, verifiable interoperability without assuming a canonical mathematical language for every candidate consciousness theory. This remains compatible with using complementary mathematical descriptions in the same investigation when their respective scopes are clear.

Structural change and temporal interpretation. A changing network statistic, an altered estimated manifold, and a genuine change of a modeled cellular carrier are different events in a formal analysis. For example, a sequence of complexes formed through attaching maps can change cellular homology; a sequence of observations taken through different sensors can also lead an estimator to return different complexes. Interpretation requires a physically and methodologically warranted correspondence between successive representations. The index of a family need not be physical time, and an order of model transformations need not automatically define causal order. The paper’s avoidance of a privileged past–present–future partition is consistent with studying historical constitution through explicit ordering and comparison maps. It also obliges the investigator to distinguish an epistemic revision of a representation from a proposed physical evolution of the represented system.

18.3 Empirical Discrimination and the Design of Comparative Models

Discriminating specified versions. The broad fourfold taxonomy generates questions and candidate explanatory locations. It does not generate unique empirical distributions by itself. A meaningful test therefore compares concrete models in which the state variables, intervention classes, measurement maps, outcome definitions, and relevant parameter ranges have been specified. For example, a matched-baseline experiment can ask whether independent perturbations reveal transformation-level differences; a matched-trajectory experiment can ask whether systematic interventions distinguish repertoires; a matched-static-structure experiment can ask whether a proposed constitutive-change law improves prediction of a measured outcome under adaptation. Each contrast tests particular operational commitments and is compatible with more than one broad orientation.

To state one level of the problem precisely, suppose two operational models H_i and H_j supply probability laws for a prespecified recorded outcome Y under a feasible intervention u and physical context c . Their ideal statistical separation is described in Equation 139.

$$d_{ij}(u, c) = \text{TV}(P_{H_i}(Y \in \cdot \mid \text{do}(U = u), c), P_{H_j}(Y \in \cdot \mid \text{do}(U = u), c)). \quad (139)$$

The intervention notation in Equation 139 presupposes that U denotes a meaningful, feasible physical intervention and that each model has a well-defined interventional probability law. Where $d_{ij}(u, c) = 0$ across the feasible protocols, observations drawn solely from those outcome laws cannot distinguish the two operational models within the stated comparison. Where separation is positive, finite sample size, nuisance variation, and uncertainty in fitted models determine practical discriminability. The total-variation framework in Sections 15 and 16 also implies that a deterministic coarsening of the common recorded outcome cannot improve this ideal pairwise separation. This observation concerns statistical tests of operational models, not direct verification of subjective experience.

Matched contrasts and their limits. For Variation, the main technical challenge is identifying a sufficiently rich intervention-response operator without confounding changes in stimulation efficacy, preparation, or recording quality. For Trajectory, a reconstructed path may change under arbitrary choices of embedding or time alignment; stable predictions should therefore be assessed under a declared family of permissible coordinate and parameter transformations. For Generativity, changes of estimated coupling,

Table 22: Comparative research commitments for operational versions of the four orientations. Each proposal requires a separately specified outcome and model class; failures constrain the specified instantiation.

Orientation	Contrast requiring additional data	Evidence favoring the specified model	Potential reason to revise the model
Variation	Matched selected histories with distinct controlled perturbation directions	Response-level feature predicts held-out outcomes after matching baseline statistics	Estimated generator is unstable or fails under input changes
Trajectory	Matched coarse generators or summaries with distinguishable ordered paths	Path invariant generalizes under predeclared allowable reparameterizations	Association disappears under valid coordinate changes or report controls
Generativity	Matched states with independently verified alteration of constitutive constraints	Explicit constitutive-transition model predicts subsequent outcomes better than a fixed-law alternative	Fitted structural changes track sensor drift or flexible model selection
Repertoire	Matched selected paths with systematically different input-accessible histories	Admissible-family statistic predicts across controlled input and preparation variations	Repertoire estimate is driven by missing trials or changing admissibility rules

incidence, or constraints should be distinguished from state-dependent transitions within a fixed model and from altered measurement operators. For Repertoire, a family of sampled histories is only a partial empirical image of the admissible set; inference about its unobserved boundary requires modeling assumptions and uncertainty intervals.

Table 22 summarizes possible decisive contrasts for concrete instantiations of the four orientations. The failure conditions in this table concern specific measured proposals, rather than refutation of the abstract taxonomy.

Comparative rather than isolated validation. Model discrimination benefits from the joint reporting of predictive performance, calibration, effect sizes, uncertainty, and the structure of failures. A geometrical biomarker that predicts an operationally defined clinical state can be useful even if it does not distinguish the four mechanisms or establish phenomenal sufficiency. Conversely, an experiment designed to separate mechanisms may have modest predictive performance for a clinical target because the experiment addresses a different estimand. The TMS–EEG perturbational literature and subsequent consciousness-related complexity indices provide examples of informative measurements whose interpretation depends on protocol, population, and explanatory purpose [29, 8, 9]. Similarly, comparative tests of IIT and GNWT illustrate the importance of specified, divergent predictions without making either of those theories identical to an orientation in the present taxonomy [12]. These empirical literatures motivate disciplined comparison; they do not validate the proposed fourfold classification.

18.4 Generative Organization, Historical Constitution, and Observer Boundaries

Constitutive and effective dynamics. The Generativity Hypothesis brings an additional question into focus: a physical process may change the conditions under which subsequent processes are generated. A formally adequate model must distinguish an ordinary evolution of states under fixed rules, a change in parameter values inside an otherwise fixed model class, and a revision of the constitutive objects or permissible relations themselves. The distinction is relative to modeling scale. A microscopic model with fixed governing laws could represent what appears, at a coarser scale, as the emergence of a new effective constraint. That possibility makes an effective constitutive-change model useful without implying fundamental creation of physical laws.

Historically indexed relational constructions are especially promising as a means of recording these changes, provided that source events, couplings, higher-order primitives, and comparison maps are defined. The mathematical role of a two-complex is to preserve incidence and attachment information that

a simple unordered list of states may omit. A sequence of cellular transformations can then express formation or removal of paths and constraints in a reproducible model. Its scientific interpretation depends on whether the cells correspond to defensible physical measurements and whether the inferred changes persist under alternative sampling and reconstruction conventions. The author's metamodel provides a minimal vocabulary of change, structure, constraint, and difference [22]; an endogenous law of constitutive evolution remains a substantive extension of that metamodel.

Candidate observers and persistence. A theory of consciousness also needs to specify the system to which an experience is attributed. Boundaries chosen by an investigator, by a measurement apparatus, by an anatomical convention, or by a dynamical decomposition may differ. Treating subjects as historically constituted relational systems creates the opportunity to examine persistence across changes of their constituent events and couplings; it also requires explicit criteria for continuity and identity. A proposed family of observer subcomplexes might carry embeddings, overlaps, or correspondence maps, but these maps alone would establish neither personal identity nor unity of experience. Such claims require additional physical constraints and, for phenomenal conclusions, an explanatory account of the experiences attributed to the candidate observer.

The question of boundaries also affects the other orientations. A transformation operator is an operator on a *chosen* domain. A trajectory is a path in a *chosen* carrier. A repertoire is a family defined by a *chosen* admissibility relation. When the candidate observer's boundary changes, the meaning of those three objects may change as well. Consequently, observer constitution is a cross-cutting modeling problem, although the Generativity Hypothesis makes that dependence particularly visible. This is one reason a rigorous research programme should study observer identification together with, rather than only after, the proposed consciousness-related dynamical feature.

18.5 Phenomenal Explanation and Epistemic Boundaries

Formal success and the target of consciousness. The preceding sections establish rigorous facts about mathematical representations under explicit assumptions. They also describe empirical programmes for investigating neural measurements and operationally defined conscious conditions. The inferential step from either accomplishment to subjective experience requires further premises. The representation-relative factorization condition of Proposition 17.1 makes one such premise explicit: a proposed phenomenal assignment would have to be constant on the fibers of a supposedly sufficient representation. This formal statement identifies the location of an explanatory commitment without supplying a validated assignment of experience. An account limited to discrimination between recorded states need not make such a sufficiency claim, and it should state its narrower explanandum accordingly.

The philosophical literature distinguishes subjective character, phenomenal versus access consciousness, and the status of the explanatory gap [31, 28, 5, 11]. These debates bear directly on whether and how physical organization can explain experience. The present analysis remains compatible with multiple positions on those debates because its mathematical separations do not settle metaphysical necessity, reduction, or constitution. In particular, the ability to devise distinct physical or mathematical descriptions with identical observations does not imply that they have distinct experiences. Likewise, finding that a high-dimensional neural model improves a clinical classifier does not independently establish that its dimension, topology, or path entropy is a constitutive phenomenal property.

Measurement and other-system inference. Reports, behavioral responses, physiological recordings, and experimental interventions are different routes to evidence. None should silently be treated as an unmediated reading of another subject's experience. Section 17 exhibits latent-report models whose observational predictions remain invariant under a relabeling of unanchored latent descriptions. The mathematical symmetry exposes an identification problem; it does not entail that subjective experiences can be freely permuted in reality. Structured phenomenological methods, cross-context calibration, and independent evidence may constrain some ambiguities, but the scope of their inference must be stated.

This concern is especially important in noncommunicative patients and populations for which reporting capacities differ from those of the calibration group.

The causal question introduces another layer. Experimentally manipulating a neural input and observing a change in a report can identify an effect on that report under the design's assumptions. Attributing the effect to an intervening geometric property requires identification of the mediation pathway and a physical interpretation of the property. Proposition 17.2 provides an elementary countermodel to an unrestricted inference from randomized assignment to causal mediation. A comprehensive theory would need to relate the physical manipulation, modeled dynamical transformation, measurement interface, and proposed phenomenal consequence in one adequately constrained account.

18.6 Integration, Limitations, and Research Priorities

Comparative achievement of the programme. The paper has developed a common mathematical vocabulary for four complementary orientations without deriving their equivalence. Its constructive results identify cases in which selected histories, generators, topology, branching, and admissible repertoires retain different information. Its reconstruction results specify how that information depends on observations, interventions, and representation maps. Its explanatory analysis separates these achievements from physical realization and phenomenal sufficiency. The principal result at the present stage is therefore a *disciplined comparison framework*: researchers can state what kind of organization their account concerns, what their observables can recover, what transformations preserve the proposed quantity, and which additional assumptions are needed to interpret the quantity as consciousness-related.

The framework has significant limitations. The source metamodel is an abstract description, and its semigroup structure imposes no uniquely neural realization. The cellular and differential constructions are selective mathematical examples rather than a complete physical theory. They do not provide a canonical map between observed electrophysiology and higher-order relational primitives. The formalism offers illustrative model families but does not yet yield an experimentally validated endogenous law of changing constitutive organization. Several observability and path-repertoire results rely on ideal measurements, known model classes, or controlled inputs that may be unavailable in living systems. Existing empirical findings cited here were produced for different questions and do not directly validate the proposed fourfold taxonomy. No original human or animal data, first-person measurements, or consciousness-related causal interventions have been collected for this paper.

Research priorities with explicit endpoints. A first priority is to develop small, physically specified models in which all four orientations can be constructed from the same data-generating process. The output should include generators, selected paths, admissible histories, and any changing constraints, with separate invariance and uncertainty audits. Such models would expose which comparisons rely on mathematical assumptions and which rely on measurements. A second priority is to identify pairs of operational models whose predictions diverge under feasible interventions while ordinary recordings agree. Success would consist of a predeclared separation in measured distributions and confirmation on independent data; failure would refine the model class rather than refute an entire philosophical orientation.

A third priority concerns changing representations and subject boundaries. Longitudinal experiments and simulations could compare fixed-carrier dynamics with explicit constitutive revisions, separating physical reorganization from estimation drift by using stable reference measurements and justified correspondence maps. A fourth priority is to connect the proposed organizational quantities to well-defined targets, such as independently calibrated reports, clinical responsiveness, or perceptual content, while maintaining distinct evidence standards for phenomenality. Finally, the formal and computational material should be audited by specialists in differential geometry, topology, control, mathematical neuroscience, philosophy of mind, and causal inference. Appendices A–F are intended to consolidate assumptions, computations, proofs, representation case studies, and reproducibility details; they remain to be completed.

Interpretive outlook. The four hypotheses can guide a staged programme of mathematical construction, observational recovery, physical testing, and explanatory interpretation. Their usefulness depends on the degree to which specific versions yield stable and informative distinctions under real measurement conditions. The programme remains open to the possibility that a successful consciousness account will combine several orientations, add explanatory objects absent from the present taxonomy, or identify a different mathematical structure altogether. Preserving that openness is consistent with the formal analysis: the established results concern relations among defined models, while the constitution and character of subjective experience remain research questions requiring their own warrants.

19 Conclusion

The present investigation has developed four preliminary mathematical orientations for the study of consciousness and examined the conditions under which their explanatory objects can be distinguished. Variation concerns the organization of transformations and their constraints; Trajectory concerns a selected path or relationally ordered history; Generativity concerns the coupled constitution and evolution of change mechanisms, structures, constraints, and distinctions; and Repertoire concerns a structured family of admissible histories. Their separation is a choice of explanatory emphasis within a declared model class, with scope for overlap and integration. The result of the investigation is a set of conditional mathematical comparisons and a more explicit account of their evidential obligations. The relation of these objects to subjective experience remains a substantive scientific and philosophical question.

Mathematical distinctions established within the model classes. The constructions in Sections 11–14 show why several apparently similar descriptions retain different information. A selected integral curve may be shared by vector fields whose transverse behavior differs. Finite transformation systems may generate the same unlabelled repertoire while possessing different internal action structures. Projected histories can conceal winding classes, and cellular complexes with identical homology can carry different fundamental groups. Changing admissibility bounds can modify the component structure of a path repertoire even when its underlying carrier is fixed. The quotient constructions further show how an abstract transition relation can admit histories that have no concrete realization from the stipulated initial conditions. Each result has a precise domain of validity; their combined significance is that the mathematical object used to characterize a process affects which aspects of its organization remain available for explanation.

The algebraic and categorical comparisons in Sections 12 and 13 make the same point in terms of preservation. A morphism can preserve a selected evolution, language, chain complex, or observable distribution while losing distinctions recorded at another level. Conversely, a representation may be adequate for a particular prediction even when it cannot reconstruct the source model. Equivalence is therefore meaningful relative to the structure and property under examination. This observation provides a methodological requirement for the four hypotheses: an alleged invariant of consciousness-related organization should be accompanied by a specification of its carrier, admissible transformations, observational interface, and preservation class. Without such a specification, comparisons between smooth flows, cellular dynamics, transition systems, and empirical reconstructions remain interpretively ambiguous.

From formal comparison to empirical inquiry. The distinction among explanatory objects creates opportunities for experimental comparison, provided that particular physical models and measurements are specified. Section 15 identifies the information that an observation protocol can recover and exhibits cases in which ideal reconstruction fails. Section 16 separates observational discrimination, intervention effects, and the identification of proposed causal mediators. These analyses indicate that the relevant empirical unit is an operational model equipped with its physical scope, predictions, observation map, and uncertainty characterization. Broad labels such as Variation or Repertoire acquire empirical content when they constrain that unit in ways that yield divergent, feasible predictions. Even an experimentally

supported prediction would initially establish a relation to the measured target and the tested population; further argument would be needed to assign phenomenal significance to the inferred quantity.

An informative next investigation would instantiate the four orientations within one physically constrained generative model. Its transformation laws, selected histories, admissible-path families, and changing constitutive data could then be compared through the same observation and intervention protocols. Explicit countermodels would reveal whether apparent differences are identifiable, and calibrated uncertainty would indicate whether those differences remain meaningful at the available measurement resolution. Such a design would permit a localized result: a specified model variant may gain or lose support relative to an alternative under stipulated conditions. It would leave room for revisions to the selected carrier, observer boundary, and representation map, and for experimental results favoring a combination of orientations or an explanatory object outside the present taxonomy.

Historical constitution and the status of the observer. For the Generativity Hypothesis, a further challenge is the representation of structural change as an endogenous physical process. Indexed families of manifolds, evolving constraints, and cellular attachment maps describe alternatives and correspondences; a dynamical account must also explain what produces the changes, which structures persist through them, and what observations can distinguish physical reorganization from changes in reconstruction. This obligation is especially important when the candidate subject of experience is treated as a historically constituted relational organization. Vertices, couplings, and higher-dimensional primitives in a two-complex require an explicit interpretation before they can be related to neural or other physical systems. A candidate observer would need principled criteria of boundary, persistence, and unity, with the evidence for those criteria evaluated separately from claims concerning experience.

The treatment of temporality remains conditional throughout this programme. Path parameters, causal orders, laboratory time, and reconstructed historical sequences play different roles. The set of admissible histories is defined by the chosen physical or formal constraints; its members need not be interpreted as events awaiting realization from a privileged present. Likewise, algebraic noninvertibility, loss of a quotient distinction, and irreversible physical processes have different definitions. Keeping these distinctions explicit permits geometric and topological methods to contribute to consciousness research without importing an unargued ontology of time or reality.

Phenomenal explanation and the limits of inference. Section 17 demonstrates that the sufficiency of a mathematical representation for an assigned phenomenal property depends on an additional bridge condition. A successful neural classification, recovery theorem, or perturbational prediction may be important scientific evidence while leaving open the question of why a given physical organization should be associated with the character of experience. The present formal results do not settle that question. Inferences about another system's experience additionally depend on the calibration of reports, physiological measurements, model assumptions, and the suitability of the tested population. These dependencies matter in particular when direct reports are limited or unavailable. The paper consequently treats mathematical adequacy, empirical reliability, causal identification, and phenomenal explanation as distinct achievements requiring distinct warrants.

The theoretical position adopted here is therefore intentionally provisional. The four hypotheses are neither collectively exhaustive nor mutually exclusive; they are research orientations with separable formal commitments. The examples use idealized dynamical systems, cellular models, and reproducible synthetic computations, and the paper offers no original measurement of consciousness or experimentally supported psychophysical law. A completed programme would require explicit physical instantiations, robust observation and intervention designs, analysis across relevant populations and conditions, and independent examination of any proposed bridge to experience. Appendices A–F document the assumptions, cellular and geometric constructions, supplementary proofs, representation case studies, and computational reproducibility procedures. These materials improve the traceability of the mathematical claims while leaving empirical and phenomenal questions open to further investigation.

The present contribution is thus a disciplined basis for asking which aspect of a system's organization a consciousness theory seeks to explain, how that aspect is represented, and what evidence could distinguish it from related alternatives. By making these commitments explicit, the four orientations can support cumulative comparisons across geometry, topology, dynamics, and consciousness science without presupposing the outcome of those comparisons.

A Formal Definitions and Assumptions

This appendix records the mathematical typing, regularity conditions, and interpretation boundaries used in the comparative investigation. Its purpose is to make the principal constructions reproducible at the level of definitions, to identify assumptions that vary between examples, and to distinguish properties established by a formal result from properties requiring a separate physical or phenomenological interpretation. The appendix first specifies fixed-carrier actions and their quotients, then smooth and cellular carriers, histories and heterogeneous model families, representation morphisms, and observation-relative reconstruction. The final subsection states a compact set of assumptions to be checked when the four hypotheses are applied to a common physical candidate. The definitions are conditional conventions; they do not assert that one common mathematical carrier is fundamental to consciousness.

A.1 Domains, Types, and the Scope of Mathematical Objects

A *model description* in this paper is a specified collection of mathematical carriers, operations, admissibility conditions, and interpretation maps. Its equality and equivalence relations are defined within that collection. A physical instantiation is an additional assignment connecting the description to a candidate process, measurement practice, and explanatory target. The fixed-carrier description of Section 4, the smooth representations of Section 5, and the indexed constructions of Section 9 provide alternative instances of this general policy. A smooth state space can encode dimension-reduced neural measurements without being a physical spacetime; the realization of a relational cell complex can encode incidence and attachment information without identifying its vertices with independently constituted subjects.

The logical status of a map depends on its domain and codomain. A function $r : Z \rightarrow X$ can represent a deterministic assignment of descriptions, whereas a relation $R \subseteq Z \times X$ allows multiple descriptions for one source configuration. A Markov kernel $K : Z \rightsquigarrow Y$ assigns a probability measure on Y to each source element, after measurable structures and the required kernel measurability have been specified. None of these mathematical signatures entails that the source object is accessible to an observer. In particular, the composite representation chain in Equation 10 is an optional model construction, with the context index c identifying the conventions of observation.

For later reference, we keep the four explanatory roles typed as in Equation 140. These are selected objects of investigation rather than claims that their collections constitute canonical functors on one universal category.

$$\begin{aligned} V &: \text{transformation descriptions,} \\ T &: \text{selected paths and relational histories,} \\ G &: \text{indexed constitutive models and comparisons,} \\ R &: \text{structured admissible-history families.} \end{aligned} \tag{140}$$

Equation 140 restates the typing discipline underlying Equation 41. A choice of topology on R , for example, is additional to the specification of a path family; a choice of comparison maps for G is additional to the specification of a collection of models. A candidate physical system can admit several useful and partially incompatible representations, with the relevant relation of preservation determined separately for each pair.

4718 A.2 Transformation Actions, Constraints, and Quotients

4719 **Definition A.1** (Fixed-carrier action description). A fixed-carrier action description is a tuple $\mathfrak{D} =$
 4720 $(S, M, \alpha, \sim)$ in which S is a nonempty set, M is a semigroup, $\alpha : M \times S \rightarrow S$ satisfies the left-
 4721 action composition law, and $\sim$ is an equivalence relation on S . The relation $\sim$ is an *action congruence*
 4722 when it is invariant under each $m \in M$. The pair of requirements is displayed in Equation 141.

$$4723 \quad \begin{aligned} \alpha(mn, x) &= \alpha(m, \alpha(n, x)), \\ x \sim y &\implies \alpha(m, x) \sim \alpha(m, y) \quad (m, n \in M; x, y \in S). \end{aligned} \quad (141)$$

4724 The first line of Equation 141 defines a left action; the second gives the condition needed for an
 4725 induced action on $S/\sim$. An identity element is convenient but not required: an arbitrary semigroup
 4726 can be extended by a formal identity acting as the identity transformation on S . When the original
 4727 semigroup already has an algebraic identity e , the composition axiom need not imply $\alpha(e, x) = x$.
 4728 Thus a *unital monoid action*, and in particular the reversible group action used in Proposition 4.4, must
 4729 explicitly satisfy $\alpha(e, x) = x$. When the existing action is nonunital, a formal external identity may be
 4730 adjoined without identifying it with e . Later observation-signature arguments quantify over the chosen
 4731 unital extension to include the initial observation. A time-dependent differential equation or stochastic
 4732 transition relation is not automatically a semigroup action; an action arises under additional existence,
 4733 uniqueness, composition, and domain assumptions appropriate to the chosen solution concept.

4734 The class $[x]$ denotes the equivalence class of x and $q : S \rightarrow S/\sim$ the quotient map. An action
 4735 congruence makes the prescription in Equation 142 independent of representative.

$$4736 \quad \bar{\alpha}(m, [x]) = [\alpha(m, x)], \quad q \circ \alpha_m = \bar{\alpha}_m \circ q. \quad (142)$$

4737 Equation 142 has a specific information-preservation content: the action commutes with the chosen quo-
 4738 tient. It permits identification of distinct source configurations and does not in general recover the source
 4739 action from the quotient. For a topological or smooth action description, continuity or differentiability
 4740 must be supplied separately; the set-theoretic result does not automatically supply a smooth quotient
 4741 manifold.

4742 An observation $O : S \rightarrow Y$ produces the kernel relation $x \equiv_O y$ when $O(x) = O(y)$. This relation
 4743 can fail to be an action congruence. The stabilized observational relation used in Section 12 is given in
 4744 Equation 143.

$$4745 \quad x \approx_{O, M} y \iff O(\alpha(m, x)) = O(\alpha(m, y)) \text{ for every } m \in M^1. \quad (143)$$

4746 **Proposition A.2** (Maximal invariant equivalence compatible with an observation). *For a fixed-carrier*
 4747 *semigroup action and deterministic observation O , the relation in Equation 143 is the largest action*
 4748 *congruence contained in $\equiv_O$, ordered by inclusion of relations.*

4749 *Proof.* For each $m \in M^1$, equality of $O \circ \alpha_m$ defines an equivalence relation. Their intersection is
 4750 therefore an equivalence relation. Because $1 \in M^1$, it is contained in the kernel of O . If $x \approx_{O, M} y$
 4751 and $n \in M$, then for every $m \in M^1$ the action law gives $O(\alpha(m, \alpha(n, x))) = O(\alpha(mn, x)) =$
 4752 $O(\alpha(mn, y)) = O(\alpha(m, \alpha(n, y)))$. Thus the relation is an action congruence. Conversely, if $R \subseteq \equiv_O$
 4753 is an action congruence, then xRy implies $\alpha(m, x)R\alpha(m, y)$ for each $m \in M^1$, and hence equality of
 4754 their observations. Consequently $R \subseteq \approx_{O, M}$. $\square$

4755 Proposition A.2 is an exact set-theoretic statement. It does not assert that the experiments needed to
 4756 realize every mechanism in M^1 are available, safe, or physically implementable. The relation also de-
 4757 pends on the declared scope of S and the actions included in M . Restricting the admissible interventions
 4758 can enlarge the observational equivalence classes.

4759 The collapse relation $\Delta(m)$ of Section 4 is defined only after an action congruence has been selected.
 4760 It records pairs distinguishable in $S/\sim$ that become identified after the specified action. Its monotonic
 4761 behavior under composition concerns this quotient-level loss of distinction. Physical irreversibility, a
 4762 thermodynamic arrow, and actual change in a conscious subject require independently stated concepts
 4763 and evidence.

4764 A.3 Smooth Dynamics, Paths, and Parameterization

4765 A smooth example begins with a finite-dimensional Hausdorff, second-countable smooth manifold X ,
 4766 optionally with boundary if the corresponding tangency and boundary conditions are specified. A vector
 4767 field $f \in \Gamma(TX)$ assigns tangent vectors to configurations and defines an ordinary differential equation
 4768 for a chosen parameter λ . Smoothness implies local existence and uniqueness of maximal integral curves.
 4769 A global flow defined for every real parameter requires completeness; without that assumption one works
 4770 with local flows and their domains of definition. A time-dependent field is naturally treated by a two-
 4771 parameter evolution family or an augmented autonomous system, with the distinction declared explicitly.

4772 The local smooth evolution condition is recorded in Equation 144.

$$4773 \quad \frac{d}{d\lambda} \gamma_x(\lambda) = f(\gamma_x(\lambda)), \quad \gamma_x(0) = x, \quad \lambda \in I_x. \quad (144)$$

4774 The interval I_x in Equation 144 is the solution domain for the chosen initial configuration. It is a math-
 4775 ematical parameter interval; its identification with clock time, physical duration, or a directed history
 4776 is a separate modeling choice. Vector-field trajectories, unparameterized curves, endpoint-constrained
 4777 homotopy classes, and measured sequences are distinct kinds of path data. Where orientation is relevant,
 4778 equivalence by reparameterization should specify whether the allowed reparameterizations preserve ori-
 4779 entation and regularity.

4780 Controlled dynamics may be described through a family of vector fields f_u indexed by admissible
 4781 controls $u \in U$. Their *reachable set* depends on the permitted control sequences, state constraints, and
 4782 domain of existence. A selected solution of one control schedule can coincide across two control systems
 4783 even when their reachable sets differ. For differential inclusions, a chosen class of absolutely continuous
 4784 paths and a measurable or otherwise justified solution concept replaces the uniqueness assumptions of
 4785 ordinary smooth flows. For stochastic systems, the object to be compared may be a probability law on
 4786 path space, and equality of sample paths is replaced by a declared criterion on laws.

4787 **Definition A.3** (Admissible and selected paths). For a carrier X and a parameter domain I with a speci-
 4788 fied topology or order, let $\mathcal{P}(I, X)$ be a chosen space of maps $I \rightarrow X$. An admissibility predicate $A_{D,c}$
 4789 on that space determines the model- and context-relative repertoire $\mathcal{H}_{D,c} = \{\gamma \in \mathcal{P}(I, X) : A_{D,c}(\gamma)\}$.
 4790 A selected path is a member $\gamma \in \mathcal{H}_{D,c}$ accompanied by an explicitly stated selection or evidential con-
 4791 dition.

4792 Definition A.3 allows geometric paths, solution histories, observation histories, and constrained tra-
 4793 jectories to be discussed with their domains visible. Its predicate makes no assertion that each admis-
 4794 sible element will be physically realized. In particular, the word *selected* can mean mathematically
 4795 distinguished, sampled, reconstructed, or physically evidenced; the intended sense is specified by the
 4796 associated protocol rather than by the path symbol alone.

4797 For a compact interval I and metric space (X, d_X) , continuous paths can carry the uniform topology
 4798 with metric defined in Equation 145.

$$4799 \quad d_\infty(\gamma, \eta) = \sup_{\lambda \in I} d_X(\gamma(\lambda), \eta(\lambda)), \quad \gamma, \eta \in C(I, X). \quad (145)$$

4800 Equation 145 is one admissible topology on a repertoire; a C^1 , Sobolev, Skorokhod, or compact-open
 4801 topology may be more appropriate to another model. Statements about compactness, connected compo-
 4802 nents, convergence, or robustness depend on that choice. For example, compactness of an equicontinuous
 4803 constrained family requires the relevant completeness, compact-range, and closure hypotheses; it does
 4804 not follow from the presence of a speed bound on an arbitrary carrier.

4805 A continuous map $h : X \rightarrow Y$ induces composition on continuous path spaces, but a restriction
 4806 $h_* : \mathcal{H}_X \rightarrow \mathcal{H}_Y$ is well defined only if the target admissibility conditions accept every projected path.
 4807 A partial observation may be defined only on a subset of histories, and a recording process with mem-
 4808 ory may act on whole histories rather than pointwise on X . These distinctions underlie the observed-
 4809 repertoire constructions in Section 10.

A.4 Cell Complexes, Homology, and Relational Interpretation

A CW-complex is a topological space equipped with cells attached inductively by specified characteristic and attaching maps, endowed with the weak topology and closure-finiteness conditions. A two-dimensional CW-complex K has cells of dimensions zero, one, and two, together with the complete attaching information. The shorthand *two-complex* in the manuscript denotes this cellular object unless a more restrictive combinatorial complex is declared. A finite two-complex admits a cellular chain complex with free abelian groups on oriented cells and boundary homomorphisms determined by attaching degrees. Equation 146 fixes the corresponding notation.

$$C_2(K; \mathbb{Z}) \xrightarrow{\partial_2} C_1(K; \mathbb{Z}) \xrightarrow{\partial_1} C_0(K; \mathbb{Z}), \quad H_n(K; \mathbb{Z}) = \ker \partial_n / \text{im } \partial_{n+1}. \quad (146)$$

The quotient in Equation 146 is well defined because $\partial_n \partial_{n+1} = 0$, with the usual zero groups at dimensions outside the complex. Incidence matrices and their Smith normal forms provide calculations of integral cellular homology for finite complexes. Cellular homology is an invariant of the realization up to appropriate equivalence, yet chain matrices alone can omit information carried by attaching homotopy classes and the fundamental group. A cellular description that stores only boundary ranks is therefore a strictly reduced record compared with the full CW attaching data.

The relational interpretation adopted for illustrative GR models assigns *events* to vertices, *couplings* to edges, and context-dependent higher-order primitives to faces. These assignments are semantic annotations on an already specified mathematical object. A vertex does not, merely by being a zero-cell, constitute an agent, subject, or instant of physical time. An entity may be represented through a structured subcomplex and rules of constitution, persistence, and interpretation, but such rules require explicit definitions. An attaching map is a topological construction and does not independently supply a physical generator, causal law, or psychophysical bridge.

When a family of complexes changes across an index category, the corresponding homology groups can be compared only after the connecting maps have been supplied. A pair of groups with the same abstract isomorphism type does not identify the historical or causal relations between the represented complexes. Relative homology applies to a specified pair (K, L) with $L \subseteq K$ (or, more generally, a suitable subspace) and describes that pair; it does not by itself assert an actual temporal transition from L to K . The concrete attachment calculations and their limits are developed in Sections 11 and 14. The cellular-construction calculations and exact boundary-matrix audit are documented in Appendix B.

A.5 Indexed Constitutive Models and the Meaning of Structural Change

The Generativity Hypothesis permits the carrier, transformation law, constraints, admissible distinctions, or representational apparatus to vary within a family. A bare indexed collection $\{\mathcal{D}_j\}_{j \in J}$ stores descriptions but does not specify their relations. To encode compatible comparisons, one can choose a small category $\mathcal{J}$ of indices and a functor into a declared category **Mod** of model descriptions and structure-preserving morphisms. This convention is summarized in Equation 147.

$$\mathbf{D} : \mathcal{J} \longrightarrow \mathbf{Mod}, \quad j \longmapsto \mathcal{D}_j, \quad (a : j \rightarrow k) \longmapsto \mathbf{D}(a). \quad (147)$$

Functoriality in Equation 147 requires identity arrows to map to identity morphisms and composable arrows to map to the corresponding compositions. The model category must be specified before one can assert that an arrow preserves a carrier topology, semigroup action, congruence, cellular attaching map, observation operator, or path repertoire. If only correspondences rather than functions are available, the comparison structure must instead be defined in an appropriate category or bicategorical setting; strict functoriality is not silently assumed.

An index arrow can represent a comparison between models, an admissible structural operation, an empirical calibration, or an analytically stipulated relation. Its existence does not by itself assign it the meaning of physical succession, causation, or generative production. To claim endogenous structural

evolution, a theory must additionally specify a physically interpretable rule selecting or generating admissible changes of constitutive descriptions, criteria for persistence of relevant objects, and evidence that the rule applies in the target system. This requirement is stronger than the existence of the family in Equation 147.

For an explicit elementary example, let $\mathbf{X} : \mathcal{J} \rightarrow \mathbf{Top}$ be a diagram of topological carriers. Its disjoint union of fibers is described in Equation 148.

$$E_0 = \coprod_{j \in \text{Ob } \mathcal{J}} \{j\} \times X_j, \quad X_j = \mathbf{X}(j). \quad (148)$$

Equation 148 supplies labeled states belonging to distinct fibers. The disjoint-union topology by itself provides no continuous path between different connected fibers. One may build a mapping telescope, a homotopy colimit, a stratified total space, or a combinatorial category of transported states, but each construction entails further choices and can preserve different data. This is the specific mathematical reason that a variable-carrier history cannot be inferred from an unstructured inventory of topological spaces.

The same restraint governs observation across changing representations. An instrument or epistemic operator may be indexed along with the carrier. A diagram of observation maps is comparable through commuting squares only after source and target comparison maps have been supplied; failed commutation can represent a change in measurement meaning as well as a difference in the modeled physical process. Empirical claims of structural transformation therefore require procedures for separating changes in the target from changes in its observation and reconstruction.

A.6 Representation Morphisms and Recoverable Information

The word *translation* in the main text denotes a specified correspondence between mathematical descriptions with an identified preservation target. For two actions of the same semigroup M on X and Y , an intertwining map satisfies Equation 149.

$$h(\alpha_X(m, x)) = \alpha_Y(m, h(x)), \quad m \in M, x \in X. \quad (149)$$

Equation 149 retains the indexed action relation, but injectivity, surjectivity, continuity, smoothness, preservation of probability laws, and observational adequacy are distinct assumptions. When the acting semigroups differ, a morphism requires an additional homomorphism of the mechanism descriptions and the corresponding typed equivariance equation. For discrete transition systems, a forward simulation and a bisimulation preserve different classes of behavior; for differential models, approximate semiconjugacy needs an explicit error metric and domain. These cases are analyzed in Sections 12 and 13.

For a function $h : X \rightarrow Y$, define its identification relation $\ker h$ by equality of image. In a composable chain $X \xrightarrow{h} Y \xrightarrow{k} Z$, any pair already identified by h is also identified by $k \circ h$. A property $P : X \rightarrow W$ can be reconstructed as a function of $h(x)$ only if it is constant on those identified pairs. The proposition below combines the two elementary conditions in a form used repeatedly in the paper.

Proposition A.4 (Factorization of recoverable properties through a representation). *Let $h : X \rightarrow Y$ and $P : X \rightarrow W$ be functions between sets. There exists a function $\hat{P} : h(X) \rightarrow W$ with $P = \hat{P} \circ h$ if and only if $P(x) = P(x')$ whenever $h(x) = h(x')$. If $k : Y \rightarrow Z$ is any function, then $\ker h \subseteq \ker(k \circ h)$.*

Proof. If $P = \hat{P} \circ h$, equal h -images have equal P -values. Conversely, define $\hat{P}(h(x)) = P(x)$ on $h(X)$; constancy of P on fibers of h makes this definition independent of representative. For the final assertion, $h(x) = h(x')$ implies $k(h(x)) = k(h(x'))$ by functionality of k . $\square$

Proposition A.4 concerns algebraic recoverability on the image $h(X)$. Extension of $\hat{P}$ to every element of Y , continuity of the factor, or stability under noise requires further hypotheses. Consequently, a claim that one representation recovers a consciousness-related invariant must identify the property, the representation class, and the topology or statistical procedure under which reconstruction is proposed.

The analogous proposition for a stipulated phenomenal assignment in Section 17 remains conditional on such an assignment; factorization alone establishes no relation between physical structure and experience.

For continuous maps between topological spaces, preservation of an induced homology class and preservation of a fundamental-group element are distinct claims. For a finite-dimensional discretization, reproduction of selected trajectories at a specified numerical tolerance does not alone establish equality of continuous-time generators. A comparison that preserves observable outcomes may fail to preserve branching counterfactuals or the full set of admissible histories. These distinctions explain the restricted use of terms such as *equivalent*, *faithful*, and *adequate* throughout the manuscript.

A.7 Observation Protocols, Identifiability, and Physical Claims

A deterministic state observation $O_c : X_c \rightarrow Y_c$ is distinguished from an experiment indexed by a condition or intervention u . For a statistical model class Θ , a protocol family $\mathcal{U}$, and a law $\mathcal{L}(\theta, u)$ on a specified measurable data space, experimental equivalence is equality of all predicted laws under the available protocols. Equation 150 records this construction.

$$\theta \equiv_{\mathcal{U}} \theta' \iff \mathcal{L}(\theta, u) = \mathcal{L}(\theta', u) \text{ for every } u \in \mathcal{U}. \quad (150)$$

The equivalence in Equation 150 is relative to the data type, calibration, observation window, admissible interventions, and uncertainty model. It permits exact mathematical statements about identifiability, while finite samples pose an additional statistical discrimination problem. Increasing the protocol class can refine the equivalence classes, provided the additional protocols are well defined and their laws are specified consistently. A claim of causal identification needs intervention semantics and assumptions about assignment, interference, and measurement. In particular, a randomized contrast may identify an effect on recorded responses while leaving a proposed mediator or the occurrence of phenomenal experience underdetermined.

Definition A.5 (Property identifiability within an experimental model class). A property $P : \Theta \rightarrow W$ is identifiable under $\mathcal{U}$ if it is constant on every equivalence class of Equation 150. It is *stably recoverable* only after a chosen metric or statistical loss, a reconstruction map or estimator, and suitable continuity or risk bounds have additionally been specified.

Definition A.5 separates exact identifiability from estimation accuracy. The first is a logical relation between models and their ideal data laws. The second concerns sensitivity to finite measurements, noise, protocol variation, and the geometry of the model family. A parameter can be identifiable in the ideal sense while estimation becomes ill-conditioned over a particular parameter range. Conversely, two distinct complete-state models can be adequate for the same measured target when the distinctions between them do not affect that target. These cases are consistent with the reconstruction analysis of Section 15.

The hypothetical phenomenal relation $\mathcal{B}_c \subseteq X_c \times \text{Phen}_c$ and any deterministic phenomenal assignment $\phi_c : X_c \rightarrow \text{Phen}_c$ are additional interpretive commitments. A report distribution, a behavioral classification, a perturbational index, and a stipulated phenomenal property occupy distinct evidential roles. Their association can be investigated; the mathematics of identifiability does not supply an empirical bridge between them. Likewise, a candidate observer represented by a historically constituted relational subcomplex requires explicit criteria of constitution, boundary, and persistence before the model can make a well-typed prediction about the same subject across conditions.

A.8 Assumption Register for Comparative Applications

The paper's mathematical claims are local to their declared domains. Table 23 collects the requirements most likely to be suppressed when moving between the four explanatory orientations. Its entries distinguish assumptions needed to form a mathematical expression from additional hypotheses needed for physical or phenomenal interpretation. They are not presented as conditions satisfied by every theory or every neural system.

Table 23: Assumption register for mathematical constructions and empirical applications of the four orientations. Each entry describes a requirement to check, rather than a universal feature of consciousness.

Construction	Mathematical requirements	Additional interpretive or evidential requirements
Semigroup action and quotient	Associative mechanism composition; action law; invariant equivalence relation	Physical meaning and admissibility of mechanisms; meaning of collapsed distinctions
Smooth generator and flow	Smooth carrier and field; local solutions; completeness for global flows	Parameter-to-time interpretation; physical state and intervention correspondence
Selected path and homotopy	Specified domain, continuity and equivalence class; endpoints where fixed	Evidence of selection or realization; significance of topology for the target
CW two-complex and homology	Cells, attaching maps, orientations and coefficient ring	Semantics of events, couplings and faces; target of relational constitution
Indexed model family	Index category, object models and typed comparison arrows	Origin of structural change, persistence criteria and causal interpretation
Admissible repertoire	Path space, admissibility predicate and selected topology or measure	Physical justification of the admissible set; observational coverage
Representation translation	Source/target types, map and explicit preservation property	Measurement comparability and reason for the selected invariants
Experimental reconstruction	Model class, observation laws, interventions and target property	Calibration, sampling, confounding controls and phenomenal bridge

Table 23 also provides a practical check for a shared physical instantiation of Variation, Trajectory, Generativity, and Repertoire. A study should specify its physical domain and representation maps; define the transformation prescriptions, history selection and admissibility rules; state whether the carriers and constraints are fixed or indexed; identify the invariants expected to survive translation and measurement; and specify the measurable and phenomenal targets separately. If these conditions cannot be met for a particular model, the resulting limitation should be recorded as part of the research result rather than suppressed by changing the meaning of a symbol.

Temporality, history, and modal language. Across the definitions, the index j , path parameter λ , composition order in M , causal relation, and recorded clock time are distinct structures. A category of descriptions may have arrows without a physical arrow of time; an admissible path family may contain mathematical alternatives without encoding unrealized future events; a semigroup may lose quotient distinctions without modeling thermodynamic irreversibility. A proposed continuation operator or endogenous update rule must therefore state its own domain, generative mechanism, admissibility conditions, and physical interpretation. This discipline is important for consciousness investigations in which observer constitution and the means of observing a process may themselves be modeled as historically dependent.

Remaining formal obligations. The propositions collected in this appendix identify valid quotient constructions and recoverability criteria; they do not establish that the four hypotheses exhaust relevant explanatory objects. The following appendices develop explicit cellular constructions, detailed geometric computations, supplementary proofs, representation case studies, and computational reproducibility. A final scientific account would additionally require physically grounded model selection, experimental discrimination against serious alternatives, and a justified relation between measured activity and subjective experience. The resulting research programme is conditional, revisable, and open to mathematical structures beyond the four preliminary orientations.

B Two-Complex and CW-Complex Constructions

This appendix gives explicit constructions behind the cellular comparisons used in Sections 5, 9, 11, and 14. Its first objective is to retain the attaching information that is often suppressed by drawings or by passage to a cellular boundary matrix. Its second is to compute the topological consequences of selected attachments, coefficient systems, subcomplex inclusions, and quotient operations. The calculations are elementary and are presented with a level of detail that permits independent verification. A CW structure is a chosen mathematical representation; neither its cell dimensions nor its attachment order supplies an ontology of consciousness, a physical temporal order, or an intrinsic account of generative transformation.

B.1 Attaching Data and Cellular Presentations

Let A be a connected graph with one distinguished vertex v and oriented loop edges a and b . It is homeomorphic to $S^1 \vee S^1$, and its based fundamental group is the free group $F(a, b)$. A based edge word w determines a continuous attaching map $\varphi_w : S^1 \rightarrow A$ after choosing an edgewise parameterization. The resulting two-dimensional CW-complex is specified by the pushout in Equation 151.

$$K_w = A \cup_{\varphi_w} D^2 = (A \amalg D^2) / \{z \sim \varphi_w(z) : z \in \partial D^2\}. \quad (151)$$

Equation 151 requires the attaching map as part of the construction. A two-cell can traverse a one-cell repeatedly and with opposite orientations; a general CW attachment therefore need not be a simplicial triangle or the boundary of a regular polygon embedded in the resulting space. Subdivision may improve visualization, but it must be accompanied by the corresponding comparison of spaces and attaching data. The construction is standard in the topology of CW-complexes [19].

Choose an orientation of each cell. In degree one the cellular boundary ∂_1 vanishes, because both endpoints of each loop are v . Write $m_a(w)$ and $m_b(w)$ for the signed exponent sums of a and b in w . The relative and cellular descriptions are connected by Equation 152.

$$\begin{aligned} C_2(K_w; \mathbb{Z}) &= \mathbb{Z}\langle f_w \rangle, & C_1(K_w; \mathbb{Z}) &= \mathbb{Z}\langle a, b \rangle, \\ \partial_2(f_w) &= m_a(w)a + m_b(w)b, & \partial_1 &= 0. \end{aligned} \quad (152)$$

Equation 152 captures abelianized winding information about the attaching loop. It does not remember the order in which the loop traverses a and b . In particular, the commutator $aba^{-1}b^{-1}$ and the constant attaching map each have exponent vector $(0, 0)$ even though their topological effects differ at the level of nonabelian fundamental groups. The corresponding counterexample in Proposition 11.3 is therefore a special case of the general distinction developed below.

Proposition B.1 (One-relator two-complex invariants). *For K_w as in Equation 151, let $g = \gcd(|m_a(w)|, |m_b(w)|)$ when the exponent vector is nonzero. Then $\pi_1(K_w, v)$ is the group with presentation $\langle a, b \mid w \rangle$. If the exponent vector is nonzero, $H_1(K_w; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/g\mathbb{Z}$ and $H_2(K_w; \mathbb{Z}) = 0$. If it is zero, $H_1(K_w; \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(K_w; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. The van Kampen theorem applied to the two-cell attachment adds the normal relation represented by w to $F(a, b)$, yielding the stated presentation of π_1 . For integral cellular homology, Equation 152 is multiplication by the column $(m_a(w), m_b(w))^T$. A nonzero integer column has trivial kernel, and an invertible integral change of basis takes its image to the subgroup generated by $(g, 0)^T$. The cokernel is $\mathbb{Z}/g\mathbb{Z} \oplus \mathbb{Z}$. For a zero column, the second cellular differential vanishes, giving the remaining groups. $\square$

Proposition B.1 distinguishes the fundamental-group presentation from its abelian approximation. Two words with the same exponent vector have the same cellular differential and integral cellular homology, while their fundamental groups can differ. With $w = [a, b]$, one obtains the familiar torus presentation; with the constant word one obtains $A \vee S^2$. The equality of their cellular chain data does not entail a cellular homeomorphism or a homotopy equivalence. In a relational interpretation, the chosen attaching relation encodes a potentially important aspect of how higher-order primitives couple to one-dimensional relational paths.

B.2 Relative Attachments and Obstructions to Representation Maps

The pair (K_w, A) contains one relative two-cell. Excision or the relative cellular chain complex gives $H_2(K_w, A; \mathbb{Z}) \cong \mathbb{Z}$ and vanishing relative homology in other degrees. Its long exact sequence contains the segment in Equation 153.

$$0 \longrightarrow H_2(K_w; \mathbb{Z}) \longrightarrow \mathbb{Z} \xrightarrow{\delta_w} \mathbb{Z}^2 \longrightarrow H_1(K_w; \mathbb{Z}) \longrightarrow 0, \quad \delta_w(1) = \begin{pmatrix} m_a(w) \\ m_b(w) \end{pmatrix}. \quad (153)$$

Equation 153 makes the attachment a relative construction: the connecting homomorphism records the abelianized boundary of the new cell. The same sequence can be applied to successive inclusions of subcomplexes, and the inclusion-induced maps record more information than a list of homology groups at isolated stages.

A second question concerns extension of a map of the one-skeleton to the newly attached two-cell. This is a different preservation requirement from commutation of cellular boundary matrices.

Proposition B.2 (Attaching-loop extension criterion). *Let Y be a topological space and let $f : A \rightarrow Y$ be continuous. A continuous extension $\bar{f} : K_w \rightarrow Y$ satisfying $\bar{f}|_A = f$ exists if and only if the loop $f \circ \varphi_w : S^1 \rightarrow Y$ is null-homotopic.*

Proof. An extension restricts on the characteristic disk D^2 to a map whose boundary is $f \circ \varphi_w$; that boundary loop is therefore null-homotopic. Conversely, a null-homotopy of $f \circ \varphi_w$ determines a continuous filling of its boundary by a disk. Together with f on A , the filling descends through the pushout quotient in Equation 151 to a continuous map on K_w . $\square$

Path-connectedness of Y is unnecessary in Proposition B.2: the image of the connected attaching circle lies in one path component, and a disk filling is entirely determined by its null-homotopy there. The criterion has an immediate consequence for the translation of generative representations. Take $K_{[a,b]}$ as source and $K_1 = A \vee S^2$ as target, with the identity map on their shared one-skeleton A . Their cellular chain complexes admit the identity chain map, since both have $\partial_2 = 0$. Yet the identity on A has no continuous extension from $K_{[a,b]}$ to K_1 : the commutator $[a, b]$ remains nontrivial in $\pi_1(K_1) \cong F(a, b)$. Thus the chain map is not realized by a continuous map that fixes the specified one-skeleton. This example makes precise a limit of functorial reduction discussed in Section 12. It does not preclude continuous maps with different restrictions to the one-skeleton.

A correspondence among relational cellular models should consequently state whether it preserves cell counts, incidence groups, attaching maps up to homotopy, fundamental-group information, or only an induced chain map. These requirements define different categories of admissible representation morphisms. A physical interpretation additionally needs to specify which relational correspondences the chosen maps represent.

B.3 Multiple Attachments and Integral Torsion

For r oriented two-cells attached to A , let $w_1, \dots, w_r$ be their attaching words. The cellular differential is the integer matrix whose j th column contains the two exponent sums of w_j , as specified in Equation 154.

$$\partial_2 : \mathbb{Z}^r \longrightarrow \mathbb{Z}^2, \quad D(w_1, \dots, w_r) = \begin{pmatrix} m_a(w_1) & \cdots & m_a(w_r) \\ m_b(w_1) & \cdots & m_b(w_r) \end{pmatrix}. \quad (154)$$

The integral homology in this one-vertex, two-loop setting follows from the kernel and cokernel of the matrix in Equation 154. Smith normal form detects the invariant factors of the cokernel. Computing only the rank over $\mathbb{Q}$ or the absolute determinant of a square matrix may discard different kinds of torsion information.

Consider the two attachment collections $\mathcal{W}_0 = (a^2, b^2)$ and $\mathcal{W}_1 = (a^2, ab^2)$. They each use one vertex, two one-cells, and two two-cells. Their boundary matrices, respectively, appear in Equation 155.

$$D_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad |\det D_0| = |\det D_1| = 4. \quad (155)$$

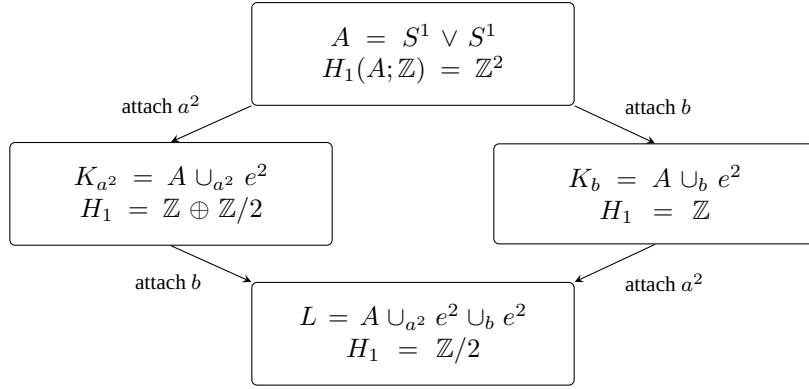


Figure 14: A diamond of CW attachments with the same common terminal complex and distinct intermediate homology. The arrows are subcomplex inclusions in a chosen diagram, not a claim about physical temporal succession. All homology groups displayed have integer coefficients.

Proposition B.3 (Identical cell counts and torsion order with different invariant factors). *The two CW-complexes specified in Equation 155 have trivial integral H_2 and finite first homology of order four, but their first homology groups are, respectively, $(\mathbb{Z}/2\mathbb{Z})^2$ and $\mathbb{Z}/4\mathbb{Z}$.*

Proof. Both determinants are nonzero, so each second cellular differential is injective over $\mathbb{Z}$ and hence $H_2 = 0$. The diagonal matrix D_0 directly gives the cokernel $(\mathbb{Z}/2\mathbb{Z})^2$. For D_1 , the abelian generators a, b obey the relations $2a = 0$ and $a + 2b = 0$. Eliminating a gives $4b = 0$, with b generating the quotient. Equivalently, the Smith invariant factors of D_1 are 1 and 4, whereas those of D_0 are 2 and 2. $\square$

Proposition B.3 exhibits a distinction that neither rational Betti numbers nor the order of the torsion subgroup reveals. In both constructions, $H_1(-; \mathbb{Q}) = 0$ and $H_2(-; \mathbb{Q}) = 0$. Integral homology, by contrast, differentiates the organization of torsion. Even this refined invariant does not reconstruct the full attaching words or all homotopy information. The matrix calculations are included in the reproducible audit script `experiments/appendixB_cellular_audit.py`.

B.4 Attachment Diagrams, Intermediate Spaces, and Coefficient Dependence

Let L be the complex obtained by adjoining disks along both a^2 and b . There are two inclusion diagrams from the common graph A to L : one through K_{a^2} and another through K_b . Their intermediate homology groups and their common terminal group are shown in Figure 14. The diagrams specify inclusion maps among separately defined CW-complexes. Their arrow directions do not alone impose a physical temporal interpretation.

The cellular boundary of the common terminal complex is the diagonal matrix in Equation 156.

$$D_L = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_1(L; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_2(L; \mathbb{Z}) = 0. \quad (156)$$

In Figure 14, the left intermediate group retains a free generator represented by b together with a two-torsion generator represented by a . Its map into $H_1(L; \mathbb{Z})$ kills the free b -summand and preserves the two-torsion class. The right intermediate group is generated by a , and its map to $H_1(L; \mathbb{Z})$ reduces this generator modulo two. Thus equal terminal complexes can be reached through diagrams having different intermediate groups and different induced maps. If the arrows are assigned a historical interpretation in an applied theory, that assignment adds information beyond the unordered terminal cell complex.

There is a separate coefficient effect. Over a field $\mathbb{F}_p$, multiplication by two is invertible when p is odd and is zero when $p = 2$. Consequently, the same integral boundary matrix can have different ranks after change of coefficients. Table 24 makes this dependence explicit; all entries use the displayed CW structures, with one vertex and two one-cells throughout.

Table 24: Coefficient-dependent homology in the attachment diamond. The $\mathbb{F}_2$ columns are dimensions of vector spaces, whereas the integral column records the abelian group, including torsion.

Complex	$H_1(-; \mathbb{Z})$	$\dim_{\mathbb{F}_2} H_1$	$\dim_{\mathbb{F}_2} H_2$
A	$\mathbb{Z}^2$	2	0
K_{a^2}	$\mathbb{Z} \oplus \mathbb{Z}/2$	2	1
K_b	$\mathbb{Z}$	1	0
L	$\mathbb{Z}/2$	1	1

Proposition B.4 (Mod-two two-cycles without an integral second-homology generator). *The complexes K_{a^2} and L have $H_2(-; \mathbb{Z}) = 0$, but each has one-dimensional $H_2(-; \mathbb{F}_2)$. For either complex, $H_2(-; \mathbb{Q}) = 0$.*

Proof. The integral second cellular differential of K_{a^2} has the single nonzero column $(2, 0)^T$ and is injective. The differential of L is D_L from Equation 156 and is also injective. Modulo two, each differential has a one-dimensional kernel: the sole two-cell column of the first complex vanishes, and D_L reduces to a rank-one matrix. Over $\mathbb{Q}$ both original integral matrices have full column rank, so their second homology vanishes. $\square$

The second-homology effect of Proposition B.4 is compatible with the universal coefficient theorem: a two-torsion term in integral H_1 can contribute to mod-two H_2 . It is a mathematical coefficient effect and should not be interpreted as the creation of an additional physical two-dimensional object. More generally, standard persistence barcodes over a single field retain information selected by that field; they do not by themselves encode integral torsion or the attaching words. When a filtration or a zigzag diagram is used to model relational history, its maps and its coefficient system are part of the formal hypothesis rather than neutral bookkeeping choices.

B.5 Contraction of Subcomplexes and Information-Losing Quotients

The quotient operation that sends the one-skeleton A to a point provides an exact example of representation-induced loss. Suppose $K = A \cup_{\varphi_1} e_1^2 \cup \cdots \cup_{\varphi_r} e_r^2$ is formed by attaching r two-cells to A , without adding additional cells. All boundary points of each attaching disk map into A . The quotient therefore has the form in Equation 157.

$$K/A \cong \bigvee_{j=1}^r S^2, \quad (157)$$

where the wedge consists of r spheres with a shared basepoint. The homeomorphism in Equation 157 is a statement about the quotient CW structure. The full attaching words may be arbitrary, and the quotient records only the number of attached two-cells in this construction.

Proposition B.5 (One-skeleton contraction conceals attaching relations). *For two complexes built by attaching the same number r of two-cells to a connected common one-skeleton A , their quotients by A are homeomorphic to the same wedge of r two-spheres, independently of their attaching maps. In particular, $K_{[a,b]}/A \cong K_1/A \cong S^2$ although $\pi_1(K_{[a,b]}) \cong \mathbb{Z}^2$ and $\pi_1(K_1) \cong F(a, b)$.*

Proof. In the quotient of the pushout construction, every point of A is identified with one basepoint. Thus the boundary S^1 of each characteristic disk is identified to this basepoint, giving $D^2/S^1 \cong S^2$. The interiors of different characteristic disks remain disjoint, and their quotient spheres meet only at the basepoint. This yields the wedge homeomorphism. The final comparison follows from the respective fundamental-group presentations of Proposition B.1. $\square$

Proposition B.5 makes the information loss of a quotient geometrically explicit. Even where a quotient is an adequate representation of a selected relative invariant, it need not preserve loops in the one-skeleton, nonabelian attachment relations, or a repertoire of paths constrained to traverse selected edges. Nor does a quotient operation specify a dynamical mechanism that physically erases the contracted relations. The relevant preservation target must therefore be named before a quotient is used to compare candidate generative structures or experiences.

B.6 Relational Semantics, Observer Subcomplexes, and Comparison Diagrams

The constructions above can support several different semantics. In the relational interpretation motivating this paper, zero-cells denote events, one-cells denote couplings, and two-cells denote higher-order primitives whose exact meaning depends on the selected theory. A zero-cell need not denote an autonomous subject, and a two-cell need not denote a psychological experience. The attaching map specifies how a chosen higher-dimensional primitive meets the lower skeleton. It is a mathematical incidence and homotopy datum until a physically or epistemically meaningful interpretation has been added.

An entity-like or observer-like structure may be proposed as a subcomplex $O \subseteq K$ satisfying additional relational and dynamical conditions. Topological inclusion alone supplies a subspace and inclusion-induced maps; it does not establish which histories belong to the same observer or whether the proposed subcomplex is persistent under structural transformations. If the constitutive rules change, the representation may use a diagram $K_i \xrightarrow{f_{ij}} K_j$, a span of correspondences, or a diagram of partial maps. The maps must specify their domain, the attaching information they preserve, and the status of proposed observer correspondences. In this setting, the attaching-loop extension criterion in Proposition B.2 becomes a concrete obstruction to certain proposed translations of relational organization.

A coarse representation of the model may retain the cellular chain complex; a finer representation may additionally retain attaching words up to appropriate homotopy, a cellular inclusion diagram, the fundamental-group action, or an admissible directed path system. Each enhancement retains a different class of distinctions. For example, the complexes of Proposition B.3 are distinguished by integral homology, while the complexes of Proposition B.2 require information beyond an identity chain map. Under a dynamics-sensitive interpretation, even a complete homotopy type may leave transformation rates, admissible operations, and selected trajectories unspecified.

B.7 Construction Records and Reproducibility Boundaries

The finite examples in this appendix can be recorded as a tuple $\mathcal{C} = (V, E, F, s, t, \Phi, \omega, \kappa)$: finite sets of oriented cells, source and target maps for one-cells, an attaching-word assignment Φ for each two-cell, orientation conventions ω , and a specified coefficient system κ . In a one-vertex presentation such as A , the source and target maps are constant. To compute integral cellular homology, one abelianizes the attaching words into the matrix of Equation 154 and uses exact integer arithmetic. To compute the fundamental group, one retains the full words, including their order and inverses. To compare historically indexed structures, one must additionally store the diagram of maps and any proposed selection or update rule.

The accompanying script `experiments/appendixB_cellular_audit.py` constructs the matrices for the paired-torsion example and the attachment diamond; calculates Smith normal forms; and checks ranks over $\mathbb{Q}$ and $\mathbb{F}_2$. Its generated CSV file records the results without replacing the proofs. The script does not compute nonabelian group isomorphism, determine an ontology of cells, or infer phenomenal experience. The exactness of the small cellular calculations makes them useful tests of a mathematical representation layer, while the physical significance of those calculations remains an empirical and theoretical question.

The constructions clarify a general methodological point for the four hypotheses. Variation requires transformations and their composition; Trajectory requires a specified class or selection of paths; Generativity may include changes of the carrier and of the attaching or comparison rules; and Repertoire

requires the admissible family of such paths or relational processes. Ordinary homology, relative homology, fundamental groups, attaching maps, and inclusion diagrams contribute different information to those representations. Their use in consciousness research should therefore be guided by the information required for a stated explanatory or experimental claim.

C Geometric and Topological Computations

This appendix develops explicit calculations that connect differential geometry, the topology of selected paths, the geometry of admissible families, and cellular approximations. Its role is computational and diagnostic: to establish what several commonly used invariants actually measure, to identify the assumptions under which a passage between representations preserves an invariant, and to provide reproducible checks of selected formulae. The first two constructions concern flow composition and constrained loops; the subsequent calculations address differential-form periods, curvature, discretization, and a changing family of carrier subsets. All parameters introduced here index prescribed mathematical operations or descriptions. Their use alone supplies neither physical succession nor an endogenous mechanism of structural evolution.

C.1 Flow Commutators and Generated Displacements

The distribution generated instantaneously by a set of smooth vector fields can be smaller than the set of directions accessed through admissible compositions of their flows. Consider the globally defined fields $X = \partial_x$ and $Y = x\partial_y$ on $\mathbb{R}^2$, also used in Section 11. Their flows are explicit, as recorded in Equation 158.

$$\Phi_X^h(x, y) = (x + h, y), \quad \Phi_Y^h(x, y) = (x, y + hx). \quad (158)$$

Both flow maps in Equation 158 exist for every real h . The availability of negative control segments is an additional admissibility assumption, rather than a consequence of a selected physical experiment. Define a four-segment commutator by Equation 159.

$$\mathcal{C}_h = \Phi_Y^{-h} \circ \Phi_X^{-h} \circ \Phi_Y^h \circ \Phi_X^h. \quad (159)$$

The rightmost map in Equation 159 acts first. A direct calculation yields an exact bracket displacement without taking a small-step limit.

Proposition C.1 (Exact displacement of a four-segment flow commutator). *For the complete vector fields X and Y above, $[X, Y] = \partial_y$ and $\mathcal{C}_h(x, y) = (x, y + h^2)$ for each $h \in \mathbb{R}$. Consequently, the transverse displacement divided by h^2 is exactly one when $h \neq 0$.*

Proof. Starting at (x, y) , the successive images under Φ_X^h , Φ_Y^h , Φ_X^{-h} , and Φ_Y^{-h} are $(x + h, y)$, $(x + h, y + h(x + h))$, $(x, y + h(x + h))$, and $(x, y + h^2)$. The Lie bracket follows by differentiating the coefficient x of Y in the X direction. The displacement assertion follows from the composition. $\square$

Proposition C.1 is a finite-control computation with stronger content than observing the horizontal path generated by X . The latter is shared with descriptions that offer no vertical control. The former depends on the admissibility of four particular transformations and reveals a structural feature of Variation and of the resulting Repertoire. In a system with only forward controls, this four-segment construction may be excluded; its applicability must then be re-examined using that system's admissible concatenations. Figure 15 includes the exact displacement against h^2 .

C.2 Energy Geometry of Based-Loop Repertoires

A family of paths can change when its metric or resource constraint changes, even when the topology of its carrier remains fixed. Endow S^1 with the Riemannian metric $g_a = a^2 d\theta^2$, where $a > 0$ is constant.

Fix a parameter interval $[0, T]$ with $T > 0$, and consider based Sobolev loops $\gamma \in H^1([0, T], S^1)$ satisfying $\gamma(0) = \gamma(T) = 1$. Every such loop has a continuous lift $\theta : [0, T] \rightarrow \mathbb{R}$, chosen with $\theta(0) = 0$, and an integer winding n characterized by $\theta(T) = 2\pi n$. Define the Dirichlet energy and its sublevel repertoire in Equation 160.

$$E_a(\gamma) = \frac{a^2}{2} \int_0^T |\dot{\theta}(s)|^2 ds, \quad \mathcal{L}_a(E_0) = \{\gamma : E_a(\gamma) \leq E_0\}, \quad E_0 \geq 0. \quad (160)$$

The energy in Equation 160 is independent of the lift once its initial value has been fixed, and it is a model-imposed path functional. It need not coincide with metabolic, thermodynamic, or neuronal energy.

Proposition C.2 (Energy threshold and connected components of a loop repertoire). *Equip $\mathcal{L}_a(E_0)$ with the topology inherited from $H^1([0, T], S^1)$. Its nonempty path components are indexed by those $n \in \mathbb{Z}$ satisfying $2\pi^2 a^2 n^2 / T \leq E_0$. For each such n , the minimum energy is $2\pi^2 a^2 n^2 / T$; in particular, the component count is given in Equation 161.*

$$\#\pi_0(\mathcal{L}_a(E_0)) = 2 \left\lfloor \sqrt{\frac{E_0 T}{2\pi^2 a^2}} \right\rfloor + 1. \quad (161)$$

Proof. The Cauchy–Schwarz inequality gives $\int_0^T |\dot{\theta}|^2 ds \geq T^{-1} |\theta(T) - \theta(0)|^2 = 4\pi^2 n^2 / T$. Equality is attained by $\theta_n(s) = 2\pi n s / T$, proving the stated minimum and the admissibility threshold. For a fixed winding integer, lifts with the same endpoints form an affine subset of $H^1([0, T], \mathbb{R})$. Every circle-valued H^1 loop in one dimension has an absolutely continuous real lift of this kind; choosing its initial value fixes the lift uniquely. This follows locally from the covering map and the absolute continuity of the embedded circle-valued representative. Linear interpolation between two lifts preserves their endpoints, and convexity of the squared L^2 norm of the derivative keeps the interpolated energy at most the larger endpoint energy. Thus each nonempty winding sector of the sublevel set is path connected. A continuous path of H^1 loops varies continuously in the uniform topology by the one-dimensional Sobolev embedding; winding is constant along such a path. Distinct winding sectors are therefore different components, and counting the permitted integers proves Equation 161. $\square$

Proposition C.2 refines the bounded-speed example in Section 11 through an integral rather than a pointwise constraint. It exhibits a simple mechanism by which the topology of an admissible-history family changes at discrete thresholds while the underlying circle remains homeomorphic to itself. For fixed E_0 and T , a change of metric scale a can alter the permitted winding sectors. Conversely, a change of E_0 can alter the same repertoire even with fixed metric. These are different parameter manipulations and need not have the same physical interpretation. Figure 16 plots the exact component counts.

C.3 Differential-Form Periods and Stability of Winding Reconstruction

A differential form can measure a path property that survives continuous deformation under stated conditions. Write $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ and let $\eta = -y dx + x dy$ be its smooth angular one-form. It is the globally defined restriction of a one-form on $\mathbb{R}^2$, although the angular coordinate θ is not globally single valued. For a piecewise C^1 based loop γ , its integral period is given in Equation 162.

$$\int_{\gamma} \eta = 2\pi \text{ wind}(\gamma), \quad \text{wind}(\gamma) \in \mathbb{Z}. \quad (162)$$

Equation 162 follows by pulling η back through a lift θ and integrating $d\theta$. It is also a cohomological pairing between a degree-one class and a degree-one cycle. In this one-dimensional example, the period detects the complete homotopy class of a based loop. In higher-dimensional or nonabelian settings, periods of closed one-forms generally detect only abelianized information, as the attachment calculations of Appendix B emphasize.

Proposition C.3 (Uniform robustness of the winding class). *Let $\gamma_0, \gamma_1 : [0, T] \rightarrow S^1 \subset \mathbb{R}^2$ be continuous based loops. If $\sup_{s \in [0, T]} \|\gamma_0(s) - \gamma_1(s)\| < 2$, then γ_0 and γ_1 are based-homotopic and have identical winding number. When both are piecewise C^1 , their periods in Equation 162 agree.*

Proof. Under the stated bound, corresponding image points are never antipodal. For $u \in [0, 1]$, the vector $(1 - u)\gamma_0(s) + u\gamma_1(s)$ therefore never vanishes. Dividing this vector by its Euclidean norm defines a continuous homotopy through circle-valued loops. Both loops have the same designated basepoint, which the normalized interpolation fixes. Winding is invariant under based homotopy, and the period identity then gives the final assertion. $\square$

The strict inequality in Proposition C.3 is a sharp universal threshold: the constant based loop $\gamma_0(s) = 1$ and the once-winding loop $\gamma_1(s) = \exp(2\pi i s/T)$ have different winding numbers but satisfy $\sup_s \|\gamma_0(s) - \gamma_1(s)\| = 2$. Thus replacing the strict bound by ≤ 2 would make the proposition false. The strict bound is a sufficient observation-space condition, measured in the known embedding of S^1 into $\mathbb{R}^2$. It requires simultaneous alignment of the loop parameters, a common basepoint, and observations covering the entire circle-valued path. An observed projection that erases a circle coordinate can conceal winding regardless of measurement precision in the retained coordinates, as shown by the torus example in Sections 11 and 15. The proposition therefore concerns stable reconstruction *within a specified representation*, not access to an unobserved phenomenal trajectory.

C.4 Variable Metrics, Curvature, and Stable Carrier Topology

Intrinsic geometry may vary while ordinary topological invariants remain fixed. Consider the cylinder $M = [-\pi, \pi] \times S^1$ with coordinates (r, θ) and the metric family specified in Equation 163.

$$g_\varepsilon = dr^2 + a_\varepsilon(r)^2 d\theta^2, \quad a_\varepsilon(r) = 1 + \varepsilon \cos r, \quad |\varepsilon| < 1. \quad (163)$$

The bound in Equation 163 guarantees positivity of the metric coefficient. The space M and its homotopy type remain unchanged across the family. In contrast, the circumference of a constant- r circle is $2\pi a_\varepsilon(r)$, and the associated curvature varies with both r and ε .

Proposition C.4 (Gaussian curvature of the warped cylinder). *For the metric in Equation 163, the Gaussian curvature and its total curvature integral are given in Equation 164.*

$$K_\varepsilon(r) = -\frac{a_\varepsilon''(r)}{a_\varepsilon(r)} = \frac{\varepsilon \cos r}{1 + \varepsilon \cos r}, \quad \int_M K_\varepsilon dA_\varepsilon = 0. \quad (164)$$

Proof. For $g = dr^2 + a(r)^2 d\theta^2$, the nonzero Christoffel symbols in these coordinates are $\Gamma_{\theta\theta}^r = -aa'$ and $\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = a'/a$. The corresponding Gaussian curvature is $-a''/a$. Substitution of a_ε proves the pointwise formula. Since $dA_\varepsilon = a_\varepsilon(r) dr d\theta$, the total integral equals $-2\pi[a'_\varepsilon(\pi) - a'_\varepsilon(-\pi)] = 0$. $\square$

Proposition C.4 illustrates the limits of using topological classification to reconstruct a metric description. The pointwise curvature may be positive at one axial position and negative at another, with zero total curvature and unchanged first Betti number. The boundary circles also have $a'_\varepsilon(\pm\pi) = 0$ and are geodesics in this metric, consistently with the boundary version of Gauss–Bonnet. The curvature calculations provide a possible analytic model of geometry-changing descriptions; they do not identify any measured neural manifold with this particular warped cylinder. The analytic curve in Figure 15 is checked independently by numerical differentiation in the included audit script.

C.5 Cellular Cochains and a Discrete-to-Continuous Period Bridge

A concrete bridge between a smooth path invariant and a cellular representation can be constructed by integrating one-forms along oriented edges. Partition the circle into $N \geq 4$ edges, with vertices at $\theta_j = 2\pi j/N$ and cyclic indexing. If f is a real zero-cochain, its cellular coboundary is $(d_0 f)_j = f_{j+1} - f_j$.

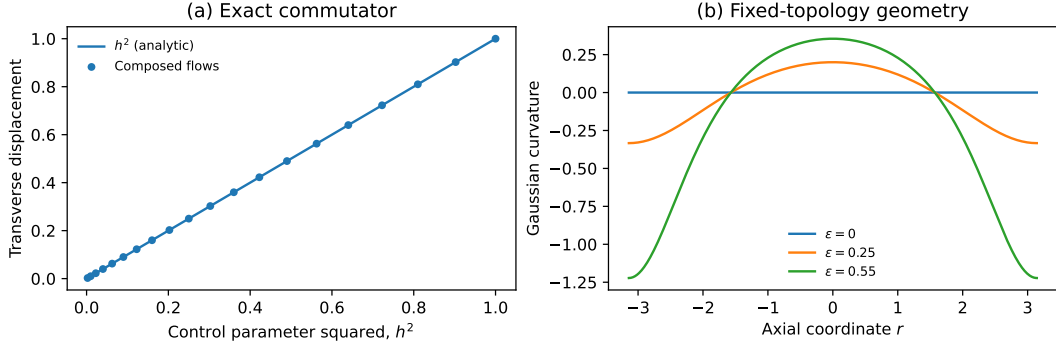


Figure 15: Exact flow commutator displacement and warped-cylinder curvature. Left: the transverse displacement of the four-segment flow composition is h^2 for the indicated control lengths. Right: analytic Gaussian curvature for three values of the metric parameter ε , demonstrating geometrical variation on a fixed topological cylinder. Both panels represent synthetic mathematical constructions.

The circular sum of these differences vanishes. For the smooth one-form $\omega_\alpha = (1 + \alpha \cos \theta) d\theta$, where $\alpha \in \mathbb{R}$, edge integration defines the one-cochain in Equation 165.

$$(I_N \omega_\alpha)_j = \int_{\theta_j}^{\theta_{j+1}} (1 + \alpha \cos \theta) d\theta = \frac{2\pi}{N} + \alpha(\sin \theta_{j+1} - \sin \theta_j), \quad \sum_{j=0}^{N-1} (I_N \omega_\alpha)_j = 2\pi. \quad (165)$$

The equality of the total periods in Equation 165 is independent of the mesh size N . The pointwise distribution of the cochain values retains additional local information, which the global period alone forgets. Unlike a simple vertex-sampling procedure, edge integration here preserves the smooth period exactly.

For even N , subdivide a circle with $N/2$ coarse edges into N fine edges. Let R restrict fine zero-cochains to the even-indexed vertices, and let A aggregate consecutive fine one-cochain values into a coarse edge. These are linear maps between specified cochain groups.

Proposition C.5 (Exact compatibility of subdivision aggregation with the coboundary). *For the cyclic cochain complexes just defined, $A d_0^{\text{fine}} = d_0^{\text{coarse}} R$. Aggregation also preserves the sum of all edge values. Consequently, under the natural identifications $H^1(S^1; \mathbb{R}) \cong \mathbb{R}$ by total period, the induced map of first cohomology is an isomorphism.*

Proof. On coarse edge k , aggregation of a fine coboundary equals $(f_{2k+1} - f_{2k}) + (f_{2k+2} - f_{2k+1}) = f_{2k+2} - f_{2k}$, which is the coarse coboundary of Rf . Aggregating consecutive edge values preserves their total circular sum by reindexing. Each first cohomology group is the quotient of the full one-cochain group by the image of its zero-cochain coboundary; its total-period map is an isomorphism with $\mathbb{R}$, since a cochain with zero total can be expressed as successive differences around the cycle. Preservation of the total period therefore implies the asserted induced isomorphism. $\square$

Proposition C.5 identifies an exact bridge within the chosen circular cellulations. It preserves a cohomological period and the commutation of selected cochain maps. It generally loses the fine-scale distribution among edges and does not supply a translation of full smooth dynamics, neural mechanisms, or higher-order attaching maps. The small computational audit records the maximum floating-point residual in the commuting-square identity and compares the period before and after aggregation. The example complements the integral cellular homology and attachment-word constructions in Appendix B.

C.6 Changing Constraint Subsets and a Topological Threshold

A changing collection of admissible configurations can alter its topological type without a change in the ambient manifold. Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and define a family of compact subsets using

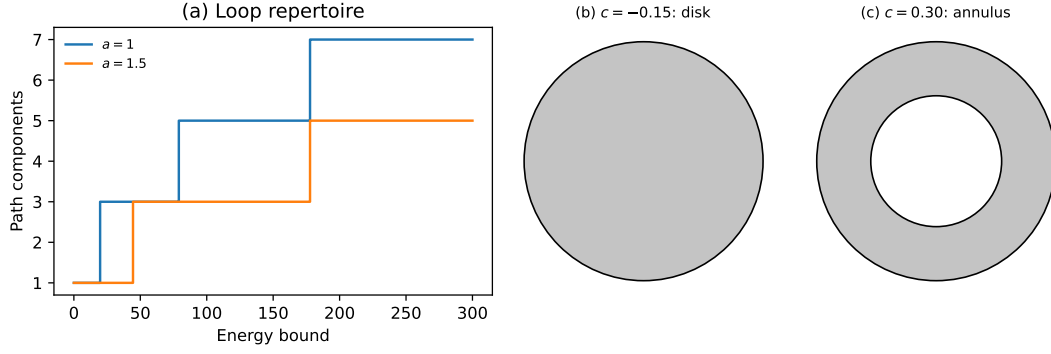


Figure 16: Geometry-dependent loop repertoires and constraint-induced changes of carrier topology. Left: exact component counts from Equation 161 for two circle-metric scales at fixed $T = 1$; the steps are analytic thresholds, not numerical discontinuities. Middle and right: disk and annulus admissibility subsets on the same ambient disk, corresponding to $c = -0.15$ and $c = 0.30$ in Equation 166. Shading represents mathematical membership only.

the constraint threshold $c < 1$. The sets appear in Equation 166.

$$F_c = \{(x, y) \in D^2 : x^2 + y^2 \geq c\}. \quad (166)$$

For $c \leq 0$ the inequality in Equation 166 is vacuous, so the admissible subset is the whole disk. For $0 < c < 1$ it excludes an open disk about the origin. The constraint therefore distinguishes a contractible carrier subset from one with a one-dimensional hole.

Proposition C.6 (Homology jump of a radial constraint family). *With integral coefficients, $H_1(F_c; \mathbb{Z}) = 0$ when $c \leq 0$ and $H_1(F_c; \mathbb{Z}) \cong \mathbb{Z}$ when $0 < c < 1$; the reduced homology groups in all other degrees vanish throughout these ranges. The inclusions $F_{c_2} \hookrightarrow F_{c_1}$ for $c_1 < c_2$ need not be homotopy equivalences when the two parameters straddle zero.*

Proof. For $c \leq 0$, $F_c = D^2$ is contractible. For $0 < c < 1$, polar coordinates identify F_c with the closed annulus $[\sqrt{c}, 1] \times S^1$, which deformation-retracts onto a circle. The corresponding homology groups follow. When $c_1 \leq 0 < c_2$, the induced map on H_1 sends the nonzero group $\mathbb{Z}$ into the zero group, precluding a homotopy equivalence. $\square$

The parameter c in Proposition C.6 indexes a nested family of constraint sets, ordered by inclusion in the reverse direction of threshold magnitude. It is not a time variable and its threshold crossing need not represent a physically realized event. A constitutive interpretation would require a model of how the admissibility constraint itself is generated or changed. An epistemic interpretation might instead regard c as a measurement or model-selection threshold applied to an unchanged process. These interpretations can produce the same formal family while making different physical claims. The set examples in Figure 16 display both sides of the topological threshold.

C.7 Invariants, Computation Records, and Interpretation Boundaries

The constructions above distinguish multiple noninterchangeable uses of geometric and topological information. The analytic invariants and their principal omissions are summarized in Table 25. A comparison of consciousness hypotheses should declare the model class, the admissibility rule, and the observation or representation map before selecting one of these computations.

The accompanying file `experiments/appendixC_geometric_audit.py` generates Figures 15 and 16 together with machine-readable outputs. Its tests compare commutator displacements with their analytic formula, curvature with centered finite differences, the exact loop-sector count with direct integer enumeration, and cochain periods and commuting squares with numerical tolerances. The tests

Table 25: Geometric and topological computations, preserved information, and limitations.

Construction	Information established	Information not supplied
Flow commutator	Exact displacement under specified reversible control compositions	Availability of these operations in a physical system
Loop energy sectors	Winding classes admitted by a metric and energy bound	Physical meaning of the metric or imposed energy
Closed-form period	Winding invariant and a sufficient robustness margin	Hidden coordinates lost by observation
Warped cylinder	Pointwise intrinsic curvature with fixed carrier topology	Intrinsic history of geometric constitution
Cochain aggregation	Exact period and coboundary compatibility	Local detail and nonlinear transformation laws
Radial constraint family	Homology change at a threshold	Endogenous dynamics of constraint selection

validate specific calculations in this appendix; they cannot substitute for proof of the general propositions or for the empirical evaluation of consciousness hypotheses. The qualitative boundaries described in Table 25 therefore remain part of the mathematical result’s appropriate interpretation.

D Supplementary Propositions and Proofs

This appendix supplies proof-level criteria connecting the action, trajectory, generativity, and repertoire descriptions developed in Sections 12–15. Its purpose is to identify exactly which properties survive specified observation and translation operations, and to quantify selected reconstruction limitations. The arguments range over finite transition systems, continuous dynamical systems, cellular complexes, and abstract representation maps. Their assumptions are stated separately because one kind of equivalence cannot generally substitute for another. No argument presupposes that mathematical equivalence entails sameness of physical realization or conscious experience.

D.1 Finite Observation Depth and Stabilized Action Congruences

Consider a finite nonempty set S , a finite collection G of total transformations of S , and the transformation monoid $M = \langle G \rangle$ with identity e . Write $\alpha(w, x)$ for the action of a finite word w in the generators, using the convention that word multiplication agrees with the action composition in Equation 19. For an observation $O : S \rightarrow Y$, define a nested sequence of finite-depth observational equivalences. Equation 167 includes the empty word at depth zero.

$$x \equiv_n y \iff O(\alpha(w, x)) = O(\alpha(w, y)) \quad \text{for all words } w \in G^{\leq n}. \quad (167)$$

Here word length is a compositional depth of the chosen generating alphabet. It carries no intrinsic temporal duration. The intersection of the relations $\equiv_n$ is the stabilized action congruence $\equiv_{O,M}$ of Equation 89.

Proposition D.1 (Finite-depth stabilization of distinguishability). *Let b_0 be the number of equivalence classes of $\equiv_0$ on the finite set S . Then the sequence in Equation 167 is decreasing and becomes constant at some depth $N \leq |S| - b_0$. At that depth, $\equiv_N = \equiv_{O,M}$, the largest action congruence contained in $\ker O$.*

Proof. Every test admitted at depth n is also admitted at depth $n + 1$, so $\equiv_{n+1} \subseteq \equiv_n$. A strict inclusion of equivalence relations on a finite set increases the number of their equivalence classes by at least one. At most $|S| - b_0$ strict refinements can therefore occur. It remains to exclude a refinement after an intervening plateau. Suppose $\equiv_{n+1} = \equiv_n$. If $x \equiv_n y$ and $g \in G$, then every word v of length at most n satisfies

5376 $O(\alpha(v, \alpha(g, x))) = O(\alpha(vg, x)) = O(\alpha(vg, y)) = O(\alpha(v, \alpha(g, y)))$. Thus $\alpha(g, x) \equiv_n \alpha(g, y)$. The
5377 relation $\equiv_n$ is preserved by each generator and hence by all of M . The empty-word test ensures $\equiv_n \subseteq$
5378 $\ker O$. Consequently $\equiv_n$ is an action congruence, and its maximality follows from Proposition 12.2
5379 together with the inclusion $\equiv_{O, M} \subseteq \equiv_n$. Hence $\equiv_n = \equiv_{O, M}$ and every subsequent relation is equal to
5380 it. As any plateau is terminal, the total number of proper refinement steps bounds the first stabilization
5381 depth. $\square$

5382 Proposition D.1 establishes a finite upper bound for distinguishing configuration pairs when the com-
5383 plete finite transition table and the observation map are known. It does not guarantee that experimental
5384 access to all words of the required length is practically available. The bound also depends on retaining
5385 the chosen set of states and transformations throughout the comparison; an indexed constitutive change
5386 need not satisfy those fixed-carrier assumptions.

5387 A change of observational resolution affects the stabilized relation in a controlled direction. Suppose
5388 $O_1 : S \rightarrow Y_1$ and $O_2 : S \rightarrow Y_2$ satisfy $O_2 = k \circ O_1$ for a map $k : Y_1 \rightarrow Y_2$. For every depth n , equality
5389 of the O_1 outputs of all tested words implies equality of their O_2 outputs. The resulting inclusion is
5390 recorded in Equation 168.

$$5391 \quad \equiv_{O_1, M} \subseteq \equiv_{O_2, M}, \quad \equiv_n^{(1)} \subseteq \equiv_n^{(2)}. \quad (168)$$

5392 Equation 168 identifies a direction of information loss under output coarsening. The reverse implication
5393 can hold in special cases, but needs its own argument. Even a complete collection of modeled actions
5394 may fail to distinguish states whose entire output signatures coincide.

5395 D.2 Path Lifting and the Fidelity of Abstract Transition Systems

5396 A transition relation describes individual admissible steps, whereas the associated repertoire also depends
5397 on their compatibility under concatenation. Let S and Q be sets of concrete and abstract states, respec-
5398 tively, and let $h : S \rightarrow Q$ be a surjection. Fix an alphabet A of step labels. Write $x \xrightarrow{a}_S x'$ and $q \xrightarrow{a}_Q q'$
5399 for the two labeled transition relations. A *sound* map h preserves concrete steps: $x \xrightarrow{a}_S x'$ entails
5400 $h(x) \xrightarrow{a}_Q h(x')$. To ensure that an abstract path can be implemented from each declared concrete repre-
5401 sentative, an additional condition is needed. We formulate the back-lifting condition in Equation 169.

$$5402 \quad \begin{array}{c} h(x) = q, \\ q \xrightarrow{a}_Q q' \end{array} \implies \exists x' \in S : x \xrightarrow{a}_S x', h(x') = q'. \quad (169)$$

5403 **Proposition D.2** (Finite-path lifting from every representative). *Assume that $h : S \rightarrow Q$ is sound. Every*
5404 *finite abstract labeled path beginning at $h(x_0)$ admits a concrete lift beginning at each prescribed repre-*
5405 *sentative x_0 if and only if Equation 169 holds for every abstract step and every concrete representative of*
5406 *its source. Under these conditions the finite labeled trace repertoires from x_0 and from $h(x_0)$ coincide.*

5407 *Proof.* Assume Equation 169. An abstract path of length zero lifts to its chosen x_0 . If a prefix of length
5408 j lifts to $x_0, \dots, x_j$, apply Equation 169 to x_j and the $(j + 1)$ st abstract step. This gives x_{j+1} with
5409 the required abstract image and label. Induction proves path lifting for every finite length. Conversely,
5410 apply the asserted lifting property to an arbitrary abstract path of length one and to any x in the fiber of
5411 its source; the resulting concrete edge supplies Equation 169. Soundness maps each concrete path to an
5412 abstract path, while the lifting property realizes each abstract path from the prescribed x_0 . The two finite
5413 labeled trace sets therefore agree. $\square$

5414 The back-lifting condition is stronger than the inclusion of concrete transitions in the abstract relation.
5415 A quotient built by existentially combining transitions from different representatives may allow an ab-
5416 stract history whose successive edges rely on incompatible underlying representatives. Proposition 13.3
5417 supplies an explicit such example. Proposition D.2 identifies a sufficient and necessary remedy when
5418 every representative must be a valid starting point. It does not imply preservation of the mechanism
5419 monoid or of the original geometry: distinct concrete paths can have the same abstract labels and state
5420 images.

D.3 Cellular Chain Equivalence and Topological Reconstruction

The cellular comparison in Appendix B exhibits information lost by passing from attaching maps to cellular boundary matrices. A complementary positive result specifies what a stronger chain-level equivalence preserves. Let (C_*, ∂_C) and (D_*, ∂_D) be chain complexes of abelian groups, and let $f : C_* \rightarrow D_*$ and $g : D_* \rightarrow C_*$ be chain maps. A chain homotopy H between gf and id_C is a family $H_k : C_k \rightarrow C_{k+1}$ satisfying Equation 170 in each degree, with a corresponding equation for a homotopy from fg to id_D .

$$(gf)_k - \text{id}_{C_k} = \partial_{C,k+1} H_k + H_{k-1} \partial_{C,k}. \quad (170)$$

Proposition D.3 (Homological sufficiency of a chain-homotopy equivalence). *If f and g are chain maps and both gf and fg are chain-homotopic to their respective identity maps, then f induces an isomorphism $H_k(C_*) \cong H_k(D_*)$ in every degree, with inverse induced by g .*

Proof. For a cycle $z \in C_k$ with $\partial_C z = 0$, Equation 170 gives $(gf)_k z - z = \partial_{C,k+1}(H_k z)$, a boundary. Thus the induced homology map of gf is the identity. The analogous homotopy on D_* shows that the induced map of fg is also the identity. Functoriality of homology then gives $g_* f_* = \text{id}_{H_*(C)}$ and $f_* g_* = \text{id}_{H_*(D)}$, proving the claim. $\square$

The conclusion of Proposition D.3 is explicitly homological. It requires neither homeomorphism of the original cellular realizations nor equivalence of their fundamental groups. The complexes in Proposition 11.3 show that even equality of the entire cellular chain complexes can coexist with different nonabelian fundamental groups. Accordingly, a chain-based recognition architecture should declare the level of structure it seeks to preserve before using homological equality as an equivalence test.

D.4 Observed Loop Classes and Dynamical Repertoire Stability

The loop reconstruction criterion of Proposition 15.3 becomes computationally explicit for homomorphisms between tori. Write $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$. An integer matrix $A \in \mathbb{Z}^{n \times m}$ defines a map $h_A : \mathbb{T}^m \rightarrow \mathbb{T}^n$ by $h_A([\theta]) = [A\theta]$. The induced fundamental-group map is identified with the integer homomorphism in Equation 171.

$$(h_A)_* : \mathbb{Z}^m \longrightarrow \mathbb{Z}^n, \quad v \longmapsto Av. \quad (171)$$

Proposition D.4 (Loop-class observability for integer torus maps). *The based homotopy class of every loop in $\mathbb{T}^m$ is determined by the based homotopy class of its image under h_A precisely when $\text{rank}_{\mathbb{Q}} A = m$. If the rank is smaller, distinct based loop classes have the same image class. Moreover, if two observed based loops in $\mathbb{T}^n$ are represented as n circle-valued coordinate paths whose corresponding chordal distances are uniformly less than 2, their observed homotopy classes agree; under the full-column-rank condition, so do their source classes.*

Proof. The standard universal covering $\mathbb{R}^m \rightarrow \mathbb{T}^m$ identifies each based loop class with its integer endpoint displacement. Applying h_A multiplies this displacement by A , giving Equation 171. An integer matrix has trivial kernel on $\mathbb{Z}^m$ exactly when its columns are linearly independent over $\mathbb{Q}$: a nonzero rational kernel vector can be rescaled to a nonzero integer one. If A has a nonzero integer kernel vector, a loop realizing that vector and the constant loop have identical image classes. For the stability assertion, the chordal-distance hypothesis excludes antipodal pairs in each observed circle coordinate. Coordinatewise normalized linear interpolation supplies a based homotopy between the observed loops, as in Proposition C.3. Injectivity of $(h_A)_*$ then gives equality of source classes. $\square$

Proposition D.4 combines an exact algebraic obstruction with an observation-dependent robustness condition. The observation map is fixed and known, the loops are aligned and based, and the chordal condition applies to the entire observed paths. General measurements with unknown distortion or partial sampling demand different stability assumptions.

A separate stability question concerns dynamical repertoires. Let $K \subseteq \mathbb{R}^d$ be compact, let $B \subseteq K$ be nonempty and compact, and let two autonomous vector fields f, g be jointly L -Lipschitz on K with

5466 $\sup_{x \in K} \|f(x) - g(x)\| \leq \varepsilon$. Assume their unique solutions from each point of B exist and remain in K
 5467 for the prescribed finite interval $[0, T]$. Write $\mathcal{R}_f(B, T)$ and $\mathcal{R}_g(B, T)$ for the families of full solution
 5468 paths, equipped with the uniform path metric.

5469 **Proposition D.5** (Finite-horizon stability of deterministic path repertoires). *Under the preceding assump-*
 5470 *tions, the Hausdorff distance between the two path repertoires satisfies Equation 172, with the stated*
 5471 *continuous extension at $L = 0$.*

$$5472 \quad d_H^\infty(\mathcal{R}_f(B, T), \mathcal{R}_g(B, T)) \leq \begin{cases} \varepsilon \frac{e^{LT} - 1}{L}, & L > 0, \\ \varepsilon T, & L = 0. \end{cases} \quad (172)$$

5473 *Proof.* Fix $x_0 \in B$ and let $x_f(t)$ and $x_g(t)$ be solutions from this common initial state. Their integral
 5474 equations and the Lipschitz assumption yield $\|x_f(t) - x_g(t)\| \leq \int_0^t [L\|x_f(s) - x_g(s)\| + \varepsilon] ds$. The
 5475 integral form of Gronwall's inequality bounds the difference by $\varepsilon(e^{Lt} - 1)/L$ when $L > 0$, and by εt
 5476 when $L = 0$. Taking the supremum over $t \in [0, T]$ gives the right-hand side of Equation 172. Every f -
 5477 trajectory from B has the corresponding g -trajectory from the same initial point within this distance, and
 5478 conversely. Both directed Hausdorff distances obey the same bound. Compactness of B and continuous
 5479 dependence of solutions on initial data give compactness of both repertoire images in the uniform path
 5480 space, so the Hausdorff distance is well defined. $\square$

5481 Uniform proximity in Proposition D.5 concerns metric closeness of path collections, while topol-
 5482 ogy is a separate property. Arbitrarily close compact subsets can have different numbers of connected
 5483 components. Consequently the bound does not guarantee stability of the topology of the repertoires; a
 5484 topological stability claim would require further regularity, isolation, or equivalence conditions. The re-
 5485 sult also does not identify the admissible repertoires of systems with different control rules or changing
 5486 carrier types.

5487 D.5 Statistical Coarsening and Quantitative Reconstruction Limits

5488 A measurement may provide an exact output distribution while concealing distinctions needed for model
 5489 identification. Let X and Y be finite sets, let P, Q be probability laws on X , and let $g : X \rightarrow Y$ be deter-
 5490 ministic. Define the signed mass difference $d(x) = P(x) - Q(x)$. Total variation of the uncompressed
 5491 distributions and of their pushforwards is computed by Equation 173.

$$5492 \quad \text{TV}(P, Q) = \frac{1}{2} \sum_{x \in X} |d(x)|, \quad \text{TV}(g_\# P, g_\# Q) = \frac{1}{2} \sum_{y \in Y} \left| \sum_{x \in g^{-1}(y)} d(x) \right|. \quad (173)$$

5493 **Proposition D.6** (Exact equality condition for deterministic statistical coarsening). *For the finite distri-*
 5494 *butions in Equation 173, equality $\text{TV}(g_\# P, g_\# Q) = \text{TV}(P, Q)$ holds if and only if, on each nonempty*
 5495 *fiber of g , the nonzero values of $d(x)$ have a common sign. Otherwise the total-variation discrimination*
 5496 *strictly decreases.*

5497 *Proof.* The triangle inequality applied separately to each fiber yields the usual data-processing inequality.
 5498 Equality holds exactly when the triangle inequality is an equality on every fiber. For a finite collection
 5499 of real numbers, the absolute value of their sum equals the sum of their absolute values if and only if all
 5500 its nonzero elements have the same sign. If any fiber contains both a positive and a negative value, the
 5501 inequality on that fiber is strict, and summation establishes a strict total-variation decrease. $\square$

5502 Proposition D.6 refines the contraction principle in Section 16. Losslessness depends on the *pair*
 5503 *of probability laws being discriminated*, as well as on the compression map. A measurement may pre-
 5504 serve discrimination between one pair of candidate models and erase it between another pair. The result
 5505 concerns their distributions under a stipulated protocol, rather than the correctness of a phenomenal in-
 5506 terpretation.

Finally, exact representation-relative sufficiency can be strengthened to a quantitative obstruction. Let X be a nonempty set, $h : X \rightarrow Y$ a representation map, and $p : X \rightarrow \mathbb{R}$ a bounded scalar target property. The target is stipulated for mathematical analysis; the existence of a calibrated phenomenal scalar p is an additional empirical and conceptual question. For each $y \in h(X)$, define the oscillation of p over the fiber $h^{-1}(y)$. The minimax reconstruction problem is stated in Equation 174.

$$\inf_{r: h(X) \rightarrow \mathbb{R}} \sup_{x \in X} |p(x) - r(h(x))| = \frac{1}{2} \sup_{y \in h(X)} \left[\sup_{h(x)=y} p(x) - \inf_{h(x)=y} p(x) \right]. \quad (174)$$

Proposition D.7 (Optimal uniform error of scalar reconstruction). *For every nonempty X , representation $h : X \rightarrow Y$, and bounded real-valued $p : X \rightarrow \mathbb{R}$, Equation 174 holds. In particular, exact recovery of p through h is possible if and only if every fiber of h has zero p -oscillation.*

Proof. Fix a nonempty fiber F_y . For any scalar estimate c , the supremum of $|p(x) - c|$ on F_y is at least half the difference between the supremum and infimum of p on that fiber. This follows by considering values arbitrarily close to the two extrema and applying the triangle inequality. Hence the left-hand side of Equation 174 is bounded below by its right-hand side. For the reverse inequality, choose $r(y)$ to be the midpoint of the supremum and infimum of p on F_y . Boundedness of p ensures these endpoints are finite, and every value on F_y lies within half its oscillation of the midpoint. Taking the supremum over all fibers yields the matching upper bound. Vanishing of the optimal error is therefore equivalent to p being constant on every fiber, consistently with Proposition 17.1. $\square$

Equation 174 quantifies a deterministic representation loss relative to a specified scalar property. It does not estimate such a property from real consciousness data, and it does not imply that subjective experience must admit a scalar representation. Extensions to vector-valued, metric-valued, or relational targets require different center-radius constructions, which cannot be replaced uncritically by the real-valued midpoint argument.

D.6 Indexed Observation Compatibility and the Proof Audit

A historically indexed family of model representations may compare distinct carriers and distinct observation spaces. Consider a constitutive map $f : S_i \rightarrow S_j$ between two modeled carriers, with observations $h_i : S_i \rightarrow Y_i$ and $h_j : S_j \rightarrow Y_j$. A comparison of their observed outputs can be defined on the image $h_i(S_i)$ only when the ambiguity of h_i is respected by the composite $h_j \circ f$. Equation 175 expresses the desired commuting relation.

$$k : h_i(S_i) \longrightarrow Y_j, \quad k \circ h_i = h_j \circ f. \quad (175)$$

Proposition D.8 (Compatibility criterion for changing observational interfaces). *There exists a unique map k satisfying Equation 175 if and only if $\ker h_i \subseteq \ker(h_j \circ f)$. The statement is set-theoretic; preservation of topology, measure, differentiability, or physical interpretation requires additional hypotheses.*

Proof. If k exists and $h_i(x) = h_i(x')$, then $h_j(f(x)) = k(h_i(x)) = k(h_i(x')) = h_j(f(x'))$, giving the inclusion of kernels. Conversely, assume the inclusion. Define $k(h_i(x)) = h_j(f(x))$. Any two representatives of the same h_i -value have the same $h_j \circ f$ -value by assumption, so the definition is unambiguous. It satisfies Equation 175 and is unique because every element in its domain has the form $h_i(x)$. $\square$

The compatibility criterion in Proposition D.8 makes the additional structure required for an indexed comparison explicit. It does not say that the map f is a physically realized history, that its index specifies a privileged present, or that compatible observational descriptions preserve phenomenological identity. A purported historical translation can fail because an earlier observational classification identifies states that the later classification distinguishes after the stipulated constitutive map.

Table 26: Assumptions and information targets in the supplementary proofs. Each row refers to a mathematical comparison, not a criterion of consciousness.

Mathematical setting	Preserved or recoverable property	Scope-limiting assumptions
Finite monoid action	Stabilized observational classes	Complete fixed finite state table and generators
Labeled transition abstraction	Every finite abstract path lifts	Back-lifting from each concrete representative
Chain complexes	Homology groups in every degree	Chain maps and mutual chain homotopies
Torus observation	Based loop class	Known integer map of full column rank
Smooth deterministic flows	Finite-horizon repertoire proximity	Common initial set, invariant compact region, Lipschitz regularity
Finite statistical coarsening	Total-variation discrimination	Fiberwise sign compatibility of candidate law differences
Scalar reconstruction	Optimal uniform property error	A specified bounded real-valued target
Indexed observation maps	Well-defined output comparison	Kernel inclusion for the constitutive map

Table 26 collects the specific preservation or reconstruction target for each supplementary result. It also identifies the additional assumptions that are needed before the result can contribute to an empirical or phenomenal explanation.

The proofs in Appendix D concern the level of structure indicated in Table 26. An observable equivalence can be mathematically exact while a generator, an attaching map, or a phenomenal description remains underdetermined. Conversely, additional measurements, interventions, and physical bridge assumptions may narrow the admissible interpretations. A complete scientific account would need to justify those additions for the systems under study, rather than infer them solely from the algebraic, geometric, or statistical constructions.

E Abstraction and Representation Case Studies

This appendix develops concrete translation problems in which two descriptions can agree on selected observations yet disagree on an explanatory object relevant to the four hypotheses. Its role is to supplement the abstract preservation criteria of Section 13 and Appendix D with worked cases that identify the source carrier, observational interface, receiving description, preserved information, and lost distinctions. The cases concern sampled flows, nonlinear spectral projection, finite-state observational memory, cellular attaching words, stochastic summaries of deterministic histories, and indexed constitutive choices. Each construction is synthetic. The order of the cases organizes mathematical comparisons and carries no claim about a physically privileged temporal succession.

E.1 Sampling Equivalence of Continuous Circle Flows

Consider the circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, equipped with the smooth vector field $X_\omega = \omega \partial_\theta$ for $\omega \in \mathbb{R}$. A fixed sampling interval $\Delta > 0$ defines an observable discrete transformation $F_{\omega,\Delta} : S^1 \rightarrow S^1$. Its dependence on the continuous generator is given by Equation 176.

$$\Phi_\omega^t([\theta]) = [\theta + \omega t], \quad F_{\omega,\Delta}([\theta]) = [\theta + \omega \Delta]. \quad (176)$$

Proposition E.1 (Sampling aliases of rotational generators). *For fixed $\Delta > 0$, two rotational generators X_ω and $X_{\omega'}$ induce the same sampled transformation $F_{\omega,\Delta} = F_{\omega',\Delta}$ if and only if $\omega' - \omega = 2\pi k/\Delta$ for some $k \in \mathbb{Z}$. When $k \neq 0$, the continuous flows disagree at some intermediate parameter values, although every integer-multiple sample agrees for every initial point.*

Proof. Equality of the two rotations is equivalent to $[(\omega' - \omega)\Delta] = [0]$ in S^1 , which holds precisely under the stated integer condition. Composition gives $F_{\omega,\Delta}^j = F_{\omega',\Delta}^j$ for each nonnegative integer j . When $k \neq 0$, the relative angle at $t = \Delta/(2|k|)$ equals $\pm\pi$, so the continuous flows differ at that parameter value. $\square$

The equivalence class of admissible continuous generators under a given sampling interface is therefore the coset displayed in Equation 177.

$$[\omega]_{\Delta} = \left\{ \omega + \frac{2\pi k}{\Delta} : k \in \mathbb{Z} \right\}. \quad (177)$$

A prior restriction $\omega \in (-\pi/\Delta, \pi/\Delta)$ makes the sampled rotation injective on that restricted parameter set. Such a bound is additional model information. At the sampled level, selected states and unlabelled sampled repertoires coincide; the continuous Variation generator, the parameterized intersample Trajectory, and the number of complete windings over a sampled interval can differ. This example concerns a stipulated sampling parameter and does not endow that parameter with an ontologically primitive arrow of time.

E.2 Spectral Projection and the Failure of Autonomous Closure

An exact projection identity for the linear heat equation appears in Section 13. The complementary nonlinear construction in this subsection isolates the limits of that identity. On the periodic circle, consider the locally well-posed semilinear evolution $\partial_t u = \kappa \partial_{xx} u + \beta u^2$, where $\kappa > 0$, $\beta \in \mathbb{R}$ and the initial data are smooth. Let $P_1 u$ denote the $\cos x$ Fourier coefficient, normalized by $\pi^{-1} \int_0^{2\pi} u(x) \cos x \, dx$. We compare the initial states $u_{A,B}(x) = A \cos x + B \cos 2x$ for fixed A and varying B . The derivative of the retained coefficient at the initial parameter value is given by Equation 178.

$$\left. \frac{d}{dt} P_1 u(t) \right|_{u(0)=u_{A,B}} = -\kappa A + \beta AB. \quad (178)$$

Proposition E.2 (Failure of exact one-mode Markov closure). *If $A\beta \neq 0$ and two initial states have the same nonzero A but distinct second-mode coefficients $B_1 \neq B_2$, there is no single autonomous scalar vector field G whose solution $a(t)$ satisfies $a(t) = P_1 u(t)$ for every smooth solution of the stated PDE with initial data in the family $u_{A,B}$.*

Proof. The two projected initial values both equal A . By expanding $u_{A,B}^2$ and using $2 \cos x \cos 2x = \cos 3x + \cos x$, the first-mode component of the nonlinear term is βAB , proving Equation 178. If a common autonomous scalar law $\dot{a} = G(a)$ reproduced both projected solutions, each initial derivative would equal $G(A)$. The computed derivatives differ by $\beta A(B_1 - B_2) \neq 0$, giving a contradiction. $\square$

The full smooth function u and its local flow form the source representation, while $P_1 u \in \mathbb{R}$ is the compressed representation. The retained mode is a well-defined observation of every solution, yet in the nonlinear system its derivative depends on a discarded mode. An exact receiving description can retain additional coefficients, store an appropriate memory state, or define a nonlocal functional of the observed history; the one-dimensional autonomous closure specifically fails under the proposition's assumptions. This distinguishes exact path projection from exact translation of the generator and illustrates why an empirically convenient state variable can be insufficient for a proposed Variation or Generativity mechanism. It is an analytic PDE case, without any inference about neural equations.

E.3 Observational Memory and Refinement of a Finite-State Quotient

Let $S = \{s_0, s_1, s_2, s_3\}$ carry the cyclic bijection $F(s_j) = s_{j+1 \pmod{4}}$. The observation alphabet is $Y = \{0, 1, 2\}$, with $h(s_0) = h(s_2) = 0$, $h(s_1) = 1$, and $h(s_3) = 2$. The coarse transition relation induced by h contains the arrows $0 \rightarrow 1$, $1 \rightarrow 0$, $0 \rightarrow 2$, and $2 \rightarrow 0$. This relational overapproximation admits

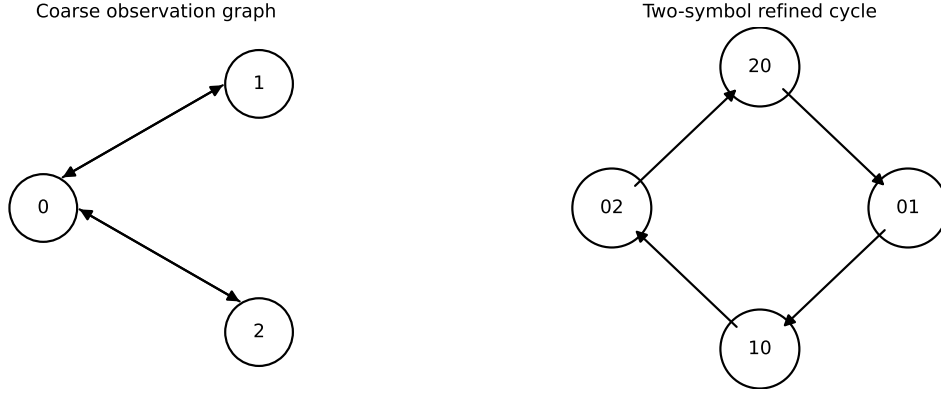


Figure 17: Coarse observational transitions and a two-symbol state refinement. The coarse graph admits source-infeasible concatenations of observed edges; the refined four-cycle is conjugate to the stipulated source dynamics. Arrowheads indicate the composition order of the discrete model, not an ontological time direction.

the word $(0, 1, 0, 1)$, whereas the source cycle, starting at its unique 0-state followed by 1, necessarily produces $(0, 1, 0, 2)$. Thus the coarse graph admits a spurious trajectory segment.

Because F is invertible, an observation window can be defined without imposing a preferred physical interpretation of the ordering convention. Equation 179 introduces a two-symbol map recording the previous and current outputs of the stipulated discrete dynamics.

$$H_2(s) = (h(F^{-1}s), h(s)), \quad H_2(S) = \{(2, 0), (0, 1), (1, 0), (0, 2)\}. \quad (179)$$

Proposition E.3 (Exact reconstruction by a two-symbol observational window). *For the above four-state cycle, H_2 is a bijection onto its image. Its induced transformation $\tilde{F} = H_2 \circ F \circ H_2^{-1}$ is a deterministic four-cycle. Consequently its full finite path repertoire is in bijective correspondence with that of (S, F) .*

Proof. The four images listed in Equation 179 are distinct. Hence H_2 has a two-sided inverse on $H_2(S)$, and $\tilde{F}$ is well defined. Substituting the four states yields $(2, 0) \rightarrow (0, 1) \rightarrow (1, 0) \rightarrow (0, 2) \rightarrow (2, 0)$. The identity $H_2 \circ F = \tilde{F} \circ H_2$ extends to each iterate by induction and yields a bijection of source and refined paths. $\square$

Figure 17 contrasts the coarse relation and the refined graph. The added observation window restores the lost distinction between the two source states labeled 0 for this finite model. Its success depends on the invertibility, labels, and finite observability of the example; a finite observation history does not in general reconstruct an arbitrary hidden system, and the refined pairs should not be equated automatically with conscious memory or experience.

E.4 Cellular Chains and the Obstruction from Attaching Words

Two-dimensional relational models can carry attaching information invisible to their cellular chain complexes. Consider the wedge $W = S_a^1 \vee S_b^1$ with fundamental group $F(a, b)$, and define $K_1 = W \cup_{[a,b]} e^2$ and $K_2 = W \cup_{[a,b]^2} e^2$, where $[a, b] = aba^{-1}b^{-1}$. Both attaching words have zero exponent sum in a and b . Accordingly the cellular boundary $\partial_2 : C_2 \cong \mathbb{Z} \rightarrow C_1 \cong \mathbb{Z}^2$ is zero for both, and the two cellular chain complexes are identical in the chosen bases. The resulting comparison is recorded in Equation 180.

$$H_1(K_1; \mathbb{Z}) \cong H_1(K_2; \mathbb{Z}) \cong \mathbb{Z}^2, \quad H_2(K_1; \mathbb{Z}) \cong H_2(K_2; \mathbb{Z}) \cong \mathbb{Z}. \quad (180)$$

Proposition E.4 (Directional obstruction despite matching cellular homology). *The identity map on W extends to a continuous map $K_2 \rightarrow K_1$, but it admits no continuous extension $K_1 \rightarrow K_2$. In particular, identity chain maps between the identical cellular chain complexes fail to certify realization of both directions by cellular maps agreeing with the specified identity on the one-skeleton.*

Proof. In $\pi_1(K_1) \cong F(a, b)/\langle\langle [a, b] \rangle\rangle \cong \mathbb{Z}^2$, the word $[a, b]^2$ is trivial. Thus the identity on W sends the attaching loop of K_2 to a null-homotopic loop in K_1 and extends over the two-cell. In contrast, $[a, b]$ is nontrivial in $\pi_1(K_2) = F(a, b)/\langle\langle [a, b]^2 \rangle\rangle$. To see this, map a and b to the quaternion units i, j in Q_8 . Their commutator equals -1 , which is nonidentity and has order two. The relation $[a, b]^2 = 1$ is therefore satisfied, so the map factors through $\pi_1(K_2)$ while detecting the nontrivial commutator. The attaching loop of K_1 is consequently non-null-homotopic in K_2 , obstructing extension of the identity on W . $\square$

The cellular-chain translation preserves the homology groups in Equation 180, but forgets the distinct relations imposed in the fundamental groups and the directional obstruction. This is a more restrictive question than abstractly comparing whether either pair of spaces admits *some* continuous map: the restriction to the identity on W is part of the representation problem. The geometric realizations of two-complexes serve as mathematical carriers; identifying their cells with physical events, couplings, or higher-order primitives requires an independent semantics. The result amplifies Appendix B by making the asymmetry of extension explicit.

E.5 Equal Marginal Statistics and Unequal Trajectory Languages

Observation can also erase order-sensitive distinctions without coarsening the instantaneous state alphabet. Let both source models be deterministic four-state cycles with the uniform invariant probability distribution on their states. In model A , the observation labels around the cycle are 0, 0, 1, 1; in model B , they are 0, 1, 0, 1. At a uniformly sampled state in either model the one-point distribution satisfies $P(Y = 0) = P(Y = 1) = 1/2$. However, the probabilities of consecutive equality, and the admitted two-symbol words, differ as in Equation 181.

$$\Pr_A(Y_{j+1} = Y_j) = \frac{1}{2}, \quad \Pr_B(Y_{j+1} = Y_j) = 0, \quad \mathcal{W}_A^{(2)} = \{00, 01, 11, 10\}, \quad \mathcal{W}_B^{(2)} = \{01, 10\}. \quad (181)$$

The probabilities in Equation 181 are exact counts over the four cyclic starting states; they use the designated invariant law and stipulated discrete ordering. The common one-point distribution is sufficient for some marginal prediction tasks. It is insufficient for identifying the observed two-step trajectory law or its word repertoire. Longer windows refine the evidential interface even when the instantaneous observable and its probability law are unchanged. This case is distinct from the spurious-history phenomenon of Subsection E.3: here each model supplies genuine but different observed trajectories. In a scientific setting, the selection and justification of a stationary law would itself require empirical scrutiny.

E.6 Indexed Constitutive Choices and Noncommuting Observations

The final case considers comparison across an indexed family of distinct cellular constructions, without assuming that its indices form a physically realized universal chronology. Start with a common circular one-skeleton $W = S^1$. The construction datum $d \in \{1, 2\}$ specifies attachment of a two-cell by a map of degree d , producing $K_d = W \cup_d e^2$. The initial observation q records only the common one-skeleton type. A second observation r records the first integral homology group of the completed complex. Equation 182 states the resulting information mismatch.

$$q(1) = q(2) = [S^1], \quad r(1) = H_1(K_1; \mathbb{Z}) = 0, \quad r(2) = H_1(K_2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}. \quad (182)$$

There is no single map k defined on the one-point image of q for which $r = k \circ q$, because its two values would be required to agree. Retaining the attachment degree as part of the earlier observational datum restores this elementary factorization. The example specializes the indexed observation-compatibility criterion in Proposition D.8. It demonstrates an epistemic limit of representing generative conditions solely through a common structural carrier: the data specifying the available constitutive operation may be relevant to the structure obtained. Calling the attachment a *generative process* adds an interpretation; the family of complexes alone specifies mathematical alternatives without establishing the physical process selecting or realizing one.

Case and receiving representation	Verified preservation	Lost or inaccessible structure
Sampled rotation $F_{\omega, \Delta}$	Entire sampled transformation and its iterates	Continuous rotational generator and intersample winding
Nonlinear one-mode projection $P_1 u$	Current first Fourier coefficient	Autonomous projected generator; dependence on discarded modes
Finite-state quotient h and refinement H_2	Coarse observed edges; full cycle after two-symbol refinement	Source feasibility of coarse concatenated paths; restored in this example
Cellular chain construction	Cell counts, boundaries, and integral homology	Attaching relators, fundamental-group information, one-skeleton extension
Stationary one-point law	Marginal label frequencies	Ordered word language and short-lag correlations
Common one-skeleton observation	Initial carrier type of both attachments	Attachment degree and resulting first homology

Table 27: Representation preservation and information loss in six synthetic translation case studies. The target of each translation determines the meaning of fidelity.

E.7 Preservation Ledger and Comparative Scope

The six case studies permit a target-relative audit of translation fidelity. Table 27 identifies the receiving representation, a preserved property, and a distinction erased or left inaccessible by that representation. The classification is relative to the cases' declared domains and does not rank the four hypotheses universally.

Across the cases, recoverability requires a specified target property, admissible source class, and observational or representational map. Distinct mechanisms may be observationally equivalent at one resolution and discriminable at another. A trajectory family may be preserved even when its generator is not recoverable; conversely a compressed transition relation may invent paths that lack a source realization. Topological invariants, temporal correlations, and constitutive attachment data occupy different mathematical levels. These results provide disciplined test problems for the exploratory Variation, Trajectory, Generativity, and Repertoire hypotheses. They do not establish a theory of consciousness from mathematical equivalences, and they leave open which invariants could be physically measured, causally relevant, or phenomenally informative.

F Numerical Experiments and Reproducibility

This appendix supplies an executable research record for the synthetic constructions used throughout the manuscript. Its objectives are to connect each numerical or finite computation to its mathematical claim, distinguish exact checks from finite-precision approximations, examine numerical resolution, and specify sufficient instructions to reproduce the supplied illustrations. The appendices on cellular complexes, geometry, supplementary proofs, and representation case studies respectively provide the underlying analytic results (Appendices B–E). The computations here are controlled tests of those constructions. They contain neither recordings of nervous systems nor behavioral reports nor measures of subjective experience. Their parameter coordinates and enumeration order are conventions of the chosen models; they carry no assumption of an ontologically privileged present or universal temporal direction.

F.1 Computational Provenance and Test Organization

Every executable experiment is retained under `experiments/`, with generated numerical tables under `experiments/results/` and figures under `figures/`. The root script `experiments/run_all_audits.py` executes the seven experiment scripts and stores the runtime version record, selected file digests, and their status reports. Table 28 maps the scripts to their explanatory objects. The table distinguishes the mathematical statement being probed from the particular finite calculation. Computationally finding

Table 28: Executable audit inventory and the mathematical information subjected to verification. Each file resides in `experiments/`.

Script	Principal construction	Validation type
<code>section14_experiments.py</code>	Flows, cellular homology, hidden winding, monoid comparison, quotient paths	Synthetic numerical and finite-state checks
<code>section16_coupling.py</code>	Perturbational revelation of a hidden linear mode	Closed-form response evaluation
<code>appendixB_cellular_audit.py</code>	Cellular boundaries and torsion	Exact integer algebra
<code>appendixC_geometric_audit.py</code>	Flow commutators, curvature, loops, cochain aggregation	Analytic identities and finite-resolution checks
<code>appendixD_proof_audit.py</code>	Observation stabilization, lifting, statistical and fiber criteria	Exhaustive finite cases
<code>appendixE_case_studies.py</code>	Sampling aliases, Fourier closure, observational memory, attaching relations	Finite and numerical case checks
<code>appendixF_numerical_audit.py</code>	Convergence, quadrature resolution, spurious paths, Smith forms and information loss	Supplementary accuracy and exact-arithmetic checks

agreement for selected parameters supports an implementation check; it does not furnish an inductive proof of an unrestricted theorem.

The archive also includes `appendixF_reproduction_manifest.json`, recording the Python and package versions used for the reference execution, SHA-256 digests of the audit scripts and CSV outputs, and individual subprocess status logs. The manifest is a local provenance record, while the source archive’s `SHA256SUMS.txt` provides an additional integrity inventory. A byte-identical PDF or PNG is not required for numerical reproducibility because fonts, rendering backends, and document metadata can affect these outputs. In contrast, their underlying formulae, stored tabular values, stated tolerances, and finite algebraic invariants provide the primary validation targets.

F.2 Execution Procedure, Determinism, and Dependencies

The reference runs used CPython 3.13.5, NumPy 2.3.5, Matplotlib 3.10.8, and SymPy 1.14.0. These values document a tested computational environment; compatibility with other versions requires an independent check. Matplotlib uses a noninteractive rendering backend. All computations in the scripts are generated from explicitly declared mathematical input values; no network retrieval, stochastic initialization, trained model, or external dataset is required. A local implementation can reproduce the results by extracting the editable source archive and executing the following commands from its root directory.

```
python experiments/run_all_audits.py
make pdf
```

The first command reruns all seven scripts and replaces their generated tables, status reports, and figures with newly computed versions. It then writes `appendixF_all_audits.txt` and a JSON manifest. The second command invokes the provided `Makefile` and requires a functioning XeLaTeX, BibTeX, and `latexmk` toolchain. For a more restrictive reproduction attempt, an investigator can create a clean Python environment with the four package versions listed above, retain the platform and interpreter details, and compare the resulting CSV values by exact equality where integral or rational operations are used and within the declared numerical tolerances elsewhere. If a comparison fails, the corresponding script and its input parameter block determine the local diagnostic target. Script exit status and numerical assertions are independent of any LaTeX compilation result.

An important dependency distinction concerns computer algebra. Smith normal form and rational probability calculations are performed with exact integer or rational arithmetic. Their reported invariants can be compared exactly. Float-based flow integration, trigonometric sampling, Fourier quadrature, and

curvature approximation instead require error controls. Figure generation is ancillary to either sort of check.

F.3 Convergence of a Synthetic Dynamical Integrator

A first numerical resolution test uses the scalar initial-value problem $\dot{y} = \lambda y$, $y(0) = y_0$, on a prescribed parameter interval of length T . It isolates discretization error without confounding model mismatch, uncertain initial data, or observational noise. The exact value at the endpoint and the explicit midpoint update are given in Equation 183.

$$y(T) = y_0 e^{\lambda T}, \quad y_{j+1} = y_j \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} \right), \quad h = \frac{T}{N}. \quad (183)$$

Proposition F.1 (Second-order terminal convergence of the midpoint test). *For fixed real λ , $T > 0$, and y_0 , the discrete solution in Equation 183 has terminal error $|y_N - y(T)| = O(N^{-2})$ as $N \rightarrow \infty$. When $y_0 \lambda \neq 0$, its leading signed relative error is given by Equation 184.*

$$\frac{y_N - y(T)}{y(T)} = -\frac{\lambda^3 T^3}{6N^2} + O(N^{-3}). \quad (184)$$

Proof. With $z = \lambda T/N$, the amplification factor of the update is $R(z) = 1 + z + z^2/2$. For sufficiently large N , $R(z) > 0$ and its logarithm has the expansion $\log R(z) = z - z^3/6 + O(z^4)$. Thus $N \log R(\lambda T/N) = \lambda T - \lambda^3 T^3/(6N^2) + O(N^{-3})$. Exponentiation and the fixed value of $y(T)$ give Equation 184 and the stated order. The zero cases follow directly from the update. $\square$

The executable sweep sets $T = 3$, $y_0 = 0.08$, and $\lambda \in \{-1, 1\}$, with $N \in \{12, 24, 48, 96, 192, 384\}$. The observed rate between adjacent refinements is measured using Equation 185.

$$p_N = \log_2 \left(\frac{|y_N - y(T)|}{|y_{2N} - y(T)|} \right). \quad (185)$$

The two finest-grid rates are approximately 2.009 for $\lambda = -1$ and 1.991 for $\lambda = 1$. These values agree with the second-order prediction of Proposition F.1 over the tested resolutions. The errors are calculated against the analytic solution, so this check addresses integration accuracy in a known model. It does not test identification of the physical generator from data or convergence of an arbitrary neural dynamical reconstruction. The full sweep and exact parameter values are in `appendixF_rk2_convergence.csv`.

F.4 Spectral Resolution and Hidden-Mode Dependence

The nonlinear Fourier example of Appendix E shows that a projected state variable can fail to satisfy an autonomous evolution law. Here the concern is more specific: numerical quadrature should accurately reproduce the analytic coefficient before the observed dependence on discarded modes is interpreted. For $u(x) = A \cos x + B \cos 2x$, the coefficient of $\cos x$ in the right-hand side of $\partial_t u = \kappa \partial_{xx} u + \beta u^2$ is $-\kappa A + \beta AB$, as in Equation 178. The grid-based estimator and its analytic comparison are defined in Equation 186.

$$\hat{a}'_N = \frac{2}{N} \sum_{j=0}^{N-1} (\kappa u_{xx}(x_j) + \beta u(x_j)^2) \cos x_j, \quad x_j = \frac{2\pi j}{N}. \quad (186)$$

The integrand of Equation 186 is a trigonometric polynomial with maximum harmonic order five. Uniform periodic quadrature sums every nonconstant Fourier mode to zero when its harmonic index is not a multiple of N . Consequently, for this exactly specified input family and exact arithmetic, $\hat{a}'_N = -\kappa A + \beta AB$ whenever $N \geq 6$. The statement applies to the instantaneous projected derivative of this polynomial input; it is not a convergence theorem for the nonlinear PDE over an arbitrary integration interval.

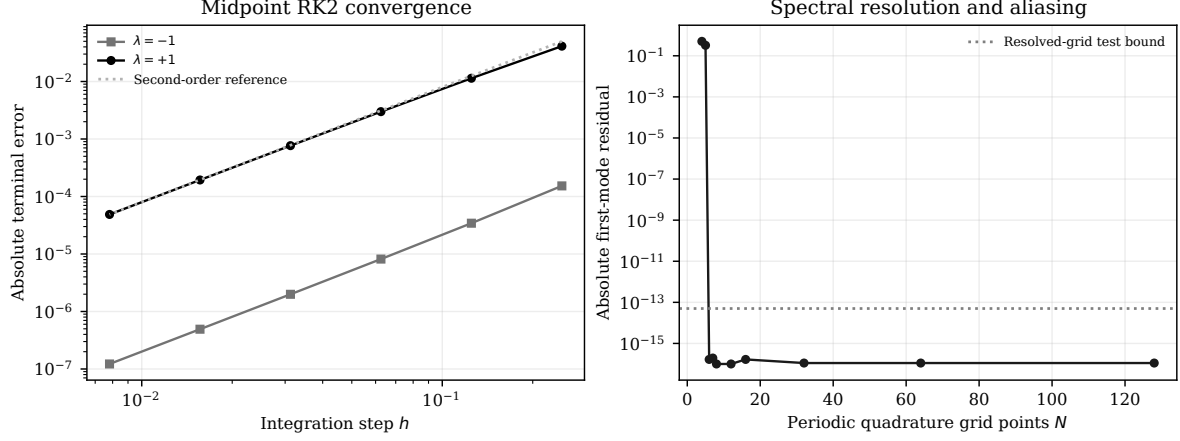


Figure 18: Supplementary numerical validation on synthetic mathematical systems. Left: midpoint-RK2 terminal error for contracting and expanding scalar dynamics, with a second-order reference. Right: residual of periodic quadrature for a nonlinear Fourier-mode derivative; deliberately coarse grids alias modes while sufficiently resolved grids approach floating-point roundoff.

The supplementary sweep sets $A = 0.7$, $B = -0.9$, $\kappa = 0.4$, and $\beta = -0.8$. The analytic target is 0.224. Deliberately under-resolved grids with $N = 4$ and $N = 5$ return 0.728 and -0.100 , respectively, due to harmonic aliasing. For the tested $N \geq 6$, the absolute residual is below 5×10^{-14} . Figure 18 reports this sweep alongside the midpoint integrator convergence. It visually separates a resolution-dependent quadrature failure from the approximately second-order dynamical discretization error.

F.5 Finite-History Enumeration and Representation Fidelity

The four-state cycle and three-symbol observation of Appendix E.3 provide a finite benchmark in which every source-generated word can be enumerated exactly. Its coarse observation graph alternates the symbol 0 with one of $\{1, 2\}$. Let L_n^{src} and L_n^{coarse} denote the sets of distinct source and coarse-graph words containing n symbols, respectively. The following result quantifies the abundance of abstract words lacking source realizations.

Proposition F.2 (Exact growth of the finite-path overapproximation). *For the specified four-cycle and its coarse three-symbol graph, the numbers of length- n words satisfy Equation 187, where m is a non-negative integer. For $n \geq 2$, the number of spurious coarse words is $|L_n^{\text{coarse}}| - 4$.*

$$|L_{2m+1}^{\text{coarse}}| = 3 \cdot 2^m, \quad |L_{2m}^{\text{coarse}}| = 2^{m+1} \ (m \geq 1), \quad |L_n^{\text{src}}| = 4 \ (n \geq 2). \quad (187)$$

Proof. A coarse word alternates between 0 and a nonzero symbol. For $n = 2m + 1$, words beginning with 0 have m independently chosen nonzero positions, while words beginning with a nonzero symbol have $m + 1$ choices of a nonzero position. This gives $2^m + 2^{m+1} = 3 \cdot 2^m$. For $n = 2m$, each of the two starting patterns has m positions with two independent choices, giving 2^{m+1} . The source system is a deterministic four-cycle. Its four possible initial states produce distinct observed words whenever $n \geq 2$, while every such word follows the allowed coarse arrows. Subtracting the four realizable words gives the spurious count. $\square$

The exact enumerator checks every word of lengths 1 through 16. At length 16, L_{16}^{coarse} has 512 words and L_{16}^{src} has four, leaving 508 spurious words, or 99.21875% of the coarse language at that length. The example isolates the distinction between preserving individual admissible edges and preserving concatenation of complete realizable histories. Enumerating a finite deterministic cycle is computationally tractable; extending the method to unrestricted dynamical systems, unbounded languages, or data-inferred transition graphs introduces substantial questions of computational complexity and epistemic reliability.

Table 29: Validation levels, completed checks, and their interpretive boundaries.

Validation level	Representative completed check	Remaining question
Exact finite algebra	Smith invariants, transition-word counts, rational total variation	Adequacy of the physical/cellular model
Analytic reference	Scalar-flow solution, projected Fourier derivative	Model misspecification and applicability
Resolution study	Midpoint convergence, trigonometric quadrature aliasing	Stability on other equations, geometries and parameter domains
Executable provenance	Seven scripts, generated CSV files, environment snapshot and SHA-256 record	Cross-platform portability and independent code review
Empirical discrimination	No empirical experiment is reported	Measurement, intervention, confounding, and competing predictions
Phenomenal explanation	No phenomenal adequacy test is reported	Justified bridge between physical model and experience

F.6 Exact Algebraic Checks and Statistical Coarsening

Finite-precision numerics and exact algebra answer different verification questions. The integer boundary matrices from Appendix B are revisited using Smith normal form. The diagonal presentation $\text{diag}(2, 2)$ and the coupled presentation with columns $(2, 0)^T$ and $(1, 2)^T$ both have determinant four, but their nonunit Smith invariants are $(2, 2)$ and (4) , respectively. Thus exact integer homology distinguishes $(\mathbb{Z}/2)^2$ from $\mathbb{Z}/4$. The audit computes the Smith forms over $\mathbb{Z}$ and checks the corresponding invariant factors without floating-point rank thresholds. It corroborates the finite presentations; the full topological interpretation additionally depends on the specified attaching maps and their cellular chain models.

A finite statistical compression supplies a complementary test. Take two stipulated probability vectors on four states, $P = (0.5, 0.1, 0.3, 0.1)$ and $Q = (0.1, 0.4, 0.1, 0.4)$, and identify states 1, 2 in one observation class and states 3, 4 in another. The exact total-variation values are given in Equation 188.

$$\text{TV}(P, Q) = \frac{3}{5}, \quad \text{TV}(g_{\#}P, g_{\#}Q) = \frac{1}{10}. \quad (188)$$

The rational-arithmetic check verifies Equation 188 without statistical estimation error. The strict contraction agrees with the equality criterion proved in Proposition D.6: positive and negative signed differences coexist within the same observation fibers. The example measures the loss of distinguishability between two specified distributions. It does not determine whether the compressed categories have any phenomenal content or whether a physical recording protocol realizes the stipulated observation map.

F.7 Validity Boundaries and Extensions of the Reproducibility Protocol

Table 29 summarizes which types of checks have been undertaken and what remains beyond their scope. A reproduction report should preserve the particular numerical tolerances and input families, state any changes of library versions, record failures as well as passes, and separate computational stability from inferential claims. The synthetic tests are deterministic within the implementations used, but machine-dependent rounding can affect the last digits of floating-point results. Model discrimination in empirical neuroscience would additionally require measurement calibration, sampling-design documentation, possible confound controls, identifiable intervention targets, held-out data, and uncertainty estimates tailored to the physical system.

Three methodological limits are especially important. First, a successful finite audit can expose an implementation error or supply a counterexample, while general validity depends on the analytic proofs and assumptions given elsewhere in the manuscript. Second, observable equivalence is indexed by the

recording and translation interface; exact reproduction of an output can coexist with divergent underlying mechanisms, topological types, or repertoires. Third, mathematical and computational representations here are intentionally heterogeneous. A single numerical error metric is therefore insufficient to compare the relevance of all four hypotheses. Future physically grounded work should specify a separate reconstruction target, experimentally accessible invariant, expected confounds, and falsification protocol for each candidate explanatory object. Under those requirements, the present archive offers reproducible formal examples and a transparent starting point for further investigation, rather than a purported computational demonstration of consciousness.

Source and research provenance

The author’s manuscript *A Metamodel for Ontologically Heterogeneous Social Dynamics, Part I: The Highest Abstract Layer* is an author-provided primary source. External references were checked against publisher, scholarly, or author records before insertion; the source archive contains a bibliographic verification ledger indicating the scope of the checks. The formal constructions use declared mathematical assumptions and representations without inferring a unique physical ontology or phenomenal sufficiency.

Sections 1–10 develop the epistemic foundations, common metamodel, mathematical representations, and four preliminary hypotheses. Sections 11–14 provide geometric, algebraic, categorical, abstraction-based, and reproducible computational comparisons. Section 15 examines ideal reconstruction, dynamical observability, loop-class identifiability, constitutive underdetermination, bounded-prefix repertoire ambiguity, and statistical discrimination. Section 16 develops physical and empirical interfaces, formal limits of intervention and coarsened measurement, and a synthetic perturbation example. Section 17 examines phenomenal bridge assumptions, limits of causal mediation claims, latent-report calibration, and observer constitution. Section 18 integrates representational comparisons, empirical discrimination strategies, historical constitution, phenomenal limitations, and research priorities. Section 19 states the conditional outcomes of the formal inquiry and the additional requirements of an empirical and phenomenal account. Appendix A consolidates model types, regularity and quotient assumptions, admissibility, representational recoverability, and empirical interpretation boundaries. Appendix B provides explicit attaching-map, relative homology, extension-obstruction, torsion, coefficient-dependence, and quotient constructions, together with an exact-arithmetic audit. Appendix C provides flow-commutator, constrained-loop energy, winding-period, warped-curvature, cochain-aggregation, and constraint-threshold computations with a reproducible numerical audit. Appendix D establishes finite-depth distinguishability, lifting of abstract histories, chain-homotopy preservation, torus loop observability, deterministic path-repertoire stability, and precise reconstruction and information-loss criteria. Appendix E develops six case studies of sampled-flow aliasing, nonlinear spectral closure, finite-observation memory, cellular attaching relations, marginal statistics, and indexed construction choices. Appendix F consolidates the executable experimental inventory, environment provenance, midpoint-flow convergence study, spectral-resolution audit, finite-language overapproximation counts, exact algebraic checks, and reproducibility limitations. All numerical examples are synthetic; the present draft reports no neural, behavioral, or phenomenological experiments.

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