

Equivalence Structures in Mathematical Models of Consciousness: A Preliminary Algebraic and Geometric Investigation

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Abstract

This exploratory paper proposes a mathematical programme for studying equivalence structures in models relevant to consciousness. It inherits four provisional explanatory orientations, Variation, Trajectory, Generativity, and Repertoire, and considers their cross-classification with ontological, relational, and manifestational registers. Equivalence is treated as relative to a specified domain, invariant, representational map, and modeling or experimental context. The intended investigation compares transformation-action equivalence, orbit and homotopy relations, equivalences of changing relational structures, and equivalences of admissible path families. It addresses preservation and information loss across algebraic, geometric, topological, categorical, and observational translations, emphasizing that identical manifestations may conceal relational differences and that mathematical correspondence does not independently establish phenomenal identity. The current draft develops foundational criteria, the four orientation-specific equivalences, cross-orientation and cross-register comparisons, algebraic and geometric preservation conditions, protocol-indexed empirical discrimination, the limits of phenomenal inference, and a comparative synthesis of research priorities. The sixteen-section main text and Appendices A–F are drafted; independent specialist review remains planned. No empirical evidence concerning consciousness is claimed.

Keywords: consciousness; equivalence relations; invariance; mathematical representation; dynamical systems; algebraic topology; relational generativity; trajectories; bisimulation; category theory.

Research status, methodological notice, and attribution

Status. This is an independent preliminary theoretical manuscript. Sections 1–16 have been drafted; Appendices A–F have been drafted; a targeted internal proof and bibliographic audit has been recorded; independent specialist review remains necessary. The paper advances definitions and a research programme, not validated mathematical or empirical claims about consciousness.

Foundational scope. Variation, Trajectory, Generativity, and Repertoire are inherited as four provisional explanatory orientations from the preceding consciousness manuscript. Ontological, relational, and manifestational registers are modeling distinctions whose correspondence requires explicit assumptions. The twelve cells in their cross-classification are an indexing convention and do not specify natural kinds or a claim of novelty.

Ontological and temporal restraint. A formal model, mathematical invariant, epistemic observation, historical reconstruction, physical process, and phenomenal report are distinguished. Indexing, causal dependence, temporal parameterization, and modal admissibility are not conflated. No mathematical equivalence relation is assumed to entail phenomenal identity without a supplementary interpretation and supporting evidence.

Originality and literature. The inquiry employs established mathematics, including topology, dynamical systems, process equivalence, and category theory. The general question of theoretical commensurability in consciousness science is an active area of existing scholarship. The research contribution proposed here is a systematic conditional taxonomy and the conditional mathematical development of its preservation arguments and counterexamples.

Responsibility and AI assistance. The programme’s research direction belongs to the author. AI tools supported the drafting, source verification, and LaTeX preparation. The author retains responsibility for claims, proof checking, references and any publication decisions. This draft has not undergone peer review.

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1 Introduction

The mathematical description of a candidate consciousness-bearing system can change while specified properties of that system remain invariant. A continuous flow may be represented by sampled observations; a relational arrangement may be described by an abstract cell complex; a family of admissible processes may be replaced by an observed trace language; and two formal theories may employ different mathematical vocabularies to describe related phenomena. Whether such representations should be regarded as equivalent depends on the object of investigation, the transformations permitted between descriptions, and the properties selected for preservation. The present study develops a preliminary mathematical framework for making those commitments explicit. Its object is the structure of equivalence criteria applied to mathematical models used in consciousness research, with particular attention to what each criterion preserves and to distinctions concealed by otherwise successful translations.

A preceding exploratory study, *The Geometry, Topology, and Dynamics of Consciousness: Four Preliminary Formal Hypotheses*, distinguished four candidate explanatory orientations: Variation, Trajectory, Generativity, and Repertoire. Variation takes an organization of transformations or mechanisms as its candidate object; Trajectory concerns a selected path, orbit, or suitably defined relational history; Generativity concerns coupled changes in structures, constraints, distinctions, and transformation conditions; and Repertoire concerns a structured family of admissible histories. These orientations are neither stipulated to be mutually exclusive nor established as empirical accounts of conscious experience. The present paper adopts them as four families of objects upon which different mathematical equivalence

criteria can be imposed. It thereby shifts the central question from the choice of explanatory object to the preservation conditions under which descriptions of such an object may legitimately be compared.

The comparability of existing consciousness theories is an active methodological problem. Current programmes organize their explanations around different mechanisms and observational commitments [24]. Evers, Farisco, and Pennartz argue that assessments of theoretical commensurability benefit from explicit logical and empirical dimensions and from shared questions that expose meaningful disagreements [11]. This literature motivates a careful account of equivalence, although comparability, compatibility, and equivalence are distinct relations. Two proposals may be logically comparable while offering incompatible predictions; two models may agree on specified experimental outcomes while differing in their causal organizations; and a shared mathematical invariant may be insufficient to identify a physical substrate or a phenomenal property. Accordingly, this paper develops equivalence as a family of criterion-relative relations, rather than a single relation assumed to settle theoretical identity.

Mathematical precedents also provide reasons to distinguish the criteria. A homeomorphism identifies topological spaces while generally discarding their specified metrics; a diffeomorphism relates smooth manifolds while allowing different vector fields; a topological conjugacy of dynamical systems imposes stronger compatibility than a homeomorphism of carriers. In process theory, bisimulation compares transition structures in ways that trace-language equality may not capture [19, 20]. Category-theoretic equivalence can identify structural properties preserved by equivalences of categories without entailing literal identity of presentations [23]; theories of definitional and Morita equivalence make still other commitments concerning interpretability and the introduction of sorts [2]. These established constructions cannot simply be collected as synonyms for one universal equivalence. Each presupposes a domain of mathematical objects, a language of comparisons, and preservation conditions. Their philosophical interpretation requires further argument.

A second analytical dimension arises from three distinguishable registers of generative-relational inquiry, here called *ontological*, *relational*, and *manifestational*. In the present framework, the ontological register refers conditionally to an underlying system or realization assumed within a specified physical or metaphysical model. It is not a declaration that the ultimate ontology of experience has been recovered. The relational register concerns constituted arrangements, interactions, incidence, constraints, and historical organization as represented by a particular model. The manifestational register concerns selected observables, records, reports, or appearances under a declared interface. The register names indicate analytical roles, and no total order of reality, epistemic superiority, or temporal succession is built into their naming. A construction may contain maps between representations of these roles; such maps do not generally possess inverses and need not exhaust their source objects.

The proposed taxonomy crosses the four explanatory orientations with these three registers. For each orientation, the study will distinguish several candidate equivalences, potentially including equality or isomorphism of mechanisms, orbit equivalence, homotopy of selected paths, structure-preserving equivalence of indexed diagrams, and equality of admissible-history families. Each candidate must be defined within a domain where its comparison is meaningful. The resulting twelve *classification cells* are an organizational device rather than twelve independently discovered natural kinds. Each cell may contain several inequivalent criteria, and a criterion may naturally apply in more than one cell. A possible count of criteria therefore depends on the chosen mathematical vocabulary and granularity; the product of three registers and four orientations is not itself an estimate of computational complexity.

To express the intended indexing precisely, let $H = \{V, T, G, R\}$ denote the four orientations and let $L = \{O, L_r, M\}$ denote the ontological, relational, and manifestational registers. For an explicitly specified experimental or modeling context c , let $\mathcal{D}_c$ be a common domain of well-typed model descriptions. A selected criterion k supplies an invariant map into a comparison domain $\mathcal{I}_{h,\ell,k,c}$. The induced equivalence is defined in Equation 1.

$$A \equiv_{h,\ell,k}^c B \iff I_{h,\ell,k}^c(A) = I_{h,\ell,k}^c(B), \quad I_{h,\ell,k}^c : \mathcal{D}_c \longrightarrow \mathcal{I}_{h,\ell,k,c}. \quad (1)$$

When the codomain and equality relation in Equation 1 are well defined, the induced relation is reflexive, symmetric, and transitive on $\mathcal{D}_c$. The invariant map is a placeholder for an explicitly constructed

mathematical object; it has no independent physical or phenomenal interpretation. An invariant may take values in isomorphism classes, path languages, homology groups with selected comparison data, or response distributions, but the equality used in each case needs a definite specification. Comparisons between model descriptions of distinct types require an additional translation or a common target domain. In particular, the carrier of a dynamical law and a selected trajectory of that law cannot be compared by untyped identity.

Different criteria may be ordered by their distinguishing power. If I and J are two invariant maps on the same domain, equivalence according to I is stronger than equivalence according to J precisely when every I -equivalent pair is J -equivalent. This implication is expressed as kernel inclusion in Equation 2.

$$\equiv_I \subseteq \equiv_J \iff (I(A) = I(B) \implies J(A) = J(B)) \text{ for all } A, B \in \mathcal{D}_c. \quad (2)$$

Equation 2 defines an implication relation between fully specified criteria, not an intrinsic hierarchy of the four consciousness hypotheses. Two criteria may be incomparable when each distinguishes a pair that the other identifies. A translation may preserve one invariant while discarding another, creating an *equivalence remainder* that matters for a different research question. Approximate similarity also requires distinct treatment: imposing a positive threshold on a metric usually yields a symmetric and reflexive proximity relation that need not be transitive. Such proximity is useful for empirical work but should not silently inherit theorems requiring an equivalence relation.

A simple topological construction illustrates the need to keep these distinctions in view. Regard the two-torus as $\mathbb{T}^2 = S^1 \times S^1$, with loops parameterized by $s \in [0, 1]$. Let $p : \mathbb{T}^2 \rightarrow S^1$ project onto the first factor. The two based loops and their observed paths are given in Equation 3.

$$\begin{aligned} \gamma_0(s) &= (e^{2\pi i s}, 1), & \gamma_1(s) &= (e^{2\pi i s}, e^{2\pi i s}), \\ p \circ \gamma_0(s) &= e^{2\pi i s}, & p \circ \gamma_1(s) &= e^{2\pi i s}. \end{aligned} \quad (3)$$

The observed paths in Equation 3 agree at every value of the parameter. Their winding vectors in $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ are nevertheless $(1, 0)$ and $(1, 1)$, so the loops are not based homotopic within the specified carrier [14]. The construction therefore exhibits equivalence under the chosen manifestation map alongside a relational topological distinction. Its force is conditional: projection onto a first coordinate is a mathematical observation prescription, and the example itself contains no assertion about neural processes or phenomenal differences. Nor does the difference in torus homotopy classes suffice to establish ontological non-equivalence without a physical interpretation of the torus and its loops.

This example exhibits one direction of a wider problem. A measurement projection can merge distinctions present in its source representation. Conversely, observational protocols can also introduce boundaries, discretizations, or statistical summaries whose apparent invariants depend on the protocol. For example, equal distributions of individual measurements do not entail equality of joint trajectory laws, and a shared trace language need not preserve the branching possibilities of a transition system. These observations motivate explicit preservation theorems and counterexamples for each equivalence criterion rather than an inference from a visually similar trajectory or a numerically similar summary. Establishing what has been observed is a separate task from identifying the generative structure that produced the observation.

The four orientations imply different candidates for the preserved data. In the Variation orientation, conjugacy or equivariance of mechanisms may preserve the organization of permissible transformations, while weaker reachability equivalence retains only which states can be reached. In the Trajectory orientation, equality of parameterized paths, equality of unparameterized images, orbit equivalence, and homotopy preserve different information. In the Generativity orientation, the target of equivalence may be an indexed family of changing carriers, admissibility conditions, attaching maps, or a diagram of transformations, requiring a representation of how such structures are compared. In the Repertoire orientation, set equality of admissible histories, trace-language equality, path-space homeomorphism, and probabilistic equivalence are different candidate criteria. Whether an implication holds among these criteria depends on their formal definitions and on any imposed regularity or observation assumptions.

Historical and modal qualifications are especially important for the Generativity and Repertoire orientations. An index set describing a family of related structures is not automatically a physical timeline, and a map between members of the family is not necessarily a causal transition. Likewise, the presence of a path in an admissible repertoire does not imply that the path is waiting for realization or that its parameter represents a universal physical clock. The present paper treats causal partial orders, model indices, selected paths, laboratory sampling times, and possible histories as potentially related but mathematically distinct structures. Claims concerning historically constituted relational systems will specify how the relevant ordering or comparison is obtained rather than deriving it from ordinary tense.

Categorical language offers one route to organizing translations, provided its limitations remain visible. A functor may preserve compositional structure while identifying morphisms that were distinct in its source. An equivalence of categories concerns the relevant categorical organization; it may omit designated observables, metrics, or physical implementation constraints unless those are incorporated into the objects and morphisms. In contexts where model comparison matters, the choice of what the morphisms must preserve is consequently substantive. The same caution applies to differential-geometric and topological constructions: a diffeomorphism between carriers need not intertwine the vector fields defined on them, and isomorphic homology groups do not recover attaching words or fundamental groups in general. Thus preservation must be stated at the mathematical level at which it is claimed.

The manifestational register creates a further interpretive obligation. A relation between records, reports, or measured signals concerns those specified observations. Even an idealized exact match of complete response distributions cannot establish identity of subjective experience without an independently argued bridge between experience and the model. Philosophical discussions of subjective character have long identified limits in inferring first-person experience from third-person structural descriptions [21]. The present paper neither assumes that such limits are insurmountable nor infers phenomenal identity from observational equivalence. Its contribution is to identify precisely where an additional bridge assumption is required. Comparisons of models of other subjects must therefore remain conditional on the selected epistemic access and calibration procedures.

The formal results anticipated in the paper have a narrow but useful scope. First, they will establish implication relations among equivalence criteria under explicit preservation maps. Second, they will construct cases in which a claimed implication fails, including examples involving changes of geometric, topological, algebraic, or observational descriptions. Third, they will analyze how equivalence claims may change when moving between ontological, relational, and manifestational registers under specified model interfaces. Fourth, they will identify research designs under which different criteria can produce empirically distinguishable predictions. None of these tasks constitutes a demonstration that one of the four mathematical orientations is either necessary or sufficient for consciousness.

An investigation of equivalence in this setting also needs to address the explanatory and ethical use of classifications. Declaring two modeled subjects equivalent for an observational task may be convenient without licensing equivalence of their experiences, capacities, relations, interests, or claims to consideration. Conversely, differences detected at the level of a mathematical representation need not establish morally salient differences. The relevance of an equivalence remainder depends upon the question asked, the affected subjects, and the epistemic and practical consequences of disregarding a distinction. The present inquiry makes this dependence explicit rather than collapsing all forms of identity into a single relation.

The paper proceeds by establishing its scientific and mathematical background and clarifying the three registers of comparison. It then formalizes indexed equivalences and studies the four orientations separately, before examining cross-orientation translations, algebraic and categorical preservation conditions, geometric and topological invariants, relational constitution, observability, and statistical discriminability. Later sections address explanatory limitations and research implications, while technical appendices will collect definitions, proofs, counterexamples, and reproducible constructions. The investigation is exploratory: it seeks a disciplined vocabulary and a set of testable mathematical obligations for comparing consciousness models, while leaving ontological realization and phenomenal explanation open to further research.

2 Scientific and Mathematical Background

The purpose of this section is to establish the disciplinary and mathematical foundations needed to compare equivalence criteria in models proposed to be relevant to consciousness. The scientific motivation arises from the coexistence of theories with different explanatory targets, model variables, and experimental interfaces. The mathematical task is to distinguish sameness of theoretical content, sameness of transformation structure, sameness of selected or admissible histories, sameness of geometric or topological organization, and sameness of observations. Each notion has an established literature, yet its relevance depends on the objects and comparison maps being specified. We review these foundations, derive two elementary preservation results, and develop counterexamples that will serve as reference cases in later sections. The more systematic classification of model domains and the three analytical registers is reserved for Section 3.

2.1 Theory comparison and scientific commensurability

A comparison of scientific theories typically begins with their languages, models, intended applications, and evidential commitments. Two theories may employ different primitive vocabularies while specifying corresponding model structures. They may also share mathematical vocabulary yet differ in the observables that their terms denote, their causal commitments, or the procedures by which a claim is tested. Accordingly, *commensurability* concerns whether and how a meaningful comparison can be established; *equivalence* requires the additional specification of what a successful comparison preserves. A common explanatory question facilitates comparison without deciding the result.

The literature on the equivalence of formal theories offers several distinct criteria. Under conventional presentations of *definitional equivalence*, two suitably formulated logical theories admit translations by definitions that recover the relevant claims of each theory. *Morita equivalence* permits certain explicitly regulated introductions of new sorts, thereby providing a criterion that can relate theories whose primitive domains differ. *Categorical equivalence* instead compares categories of models with their selected structure-preserving maps. Barrett and Halvorson analyze the relationships among these notions and make clear that their logical requirements differ [2]. None of these criteria can be transferred to a neuroscientific model merely by observing that two descriptions are intertranslatable in ordinary language. The signature of the theory, the admissible morphisms, and the observational interpretation all affect the comparison.

The categorical case warrants particular care. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ that is full, faithful, and essentially surjective constitutes an equivalence of categories, up to the usual characterization. Fullness concerns surjectivity on each designated hom-set map, faithfulness its injectivity, and essential surjectivity the existence of an isomorphic representative in the image for each target object [23]. These are properties of the categories and their morphisms. If neural locations, parameter calibrations, implementation constraints, or particular observables are omitted from the category, equivalence of its objects does not restore those omitted data. In the present inquiry, an appropriately *enriched* category may incorporate specified observation and realization interfaces, but the resulting equivalence must then be established for that enriched category.

A useful methodological separation therefore concerns three questions. First, which models are admitted to the domain being compared? Second, which mathematical or empirical features are retained in the chosen description? Third, what additional interpretive claim links those features with a consciousness-related property? The first two questions can often be given mathematical answers. The third requires a physical or phenomenal bridge supported by further argument. The distinction parallels the caution in contemporary assessments of consciousness-theory commensurability, where common denominators are useful for comparison without guaranteeing integration or theoretical identity [11].

These precedents motivate the criterion-relative notation introduced in Section 1. An equivalence relation is meaningful only on a domain for which the stated comparison is defined. When one criterion distinguishes more pairs than another, its associated equivalence classes are finer; this need not determine which criterion is scientifically more informative. For example, a fine-grained representation of

microscopic degrees of freedom may be irrelevant to a narrowly specified experimental discrimination task, while a coarser representation may omit a variable needed for causal explanation. The adequacy of a criterion is consequently indexed to the research question and the stipulated interpretation of its domain.

2.2 Equivalence in dynamical and behavioral systems

Dynamical equivalence provides established examples of how comparison strength changes with the features retained. Consider discrete systems (X, F) and (Y, G) with maps $F : X \rightarrow X$ and $G : Y \rightarrow Y$. For topological dynamical systems, a *topological conjugacy* is a homeomorphism $h : X \rightarrow Y$ intertwining the dynamics. A *semiconjugacy* requires the same intertwining law but allows a continuous, frequently surjective, map h that need not have an inverse. Their common compatibility condition is given in Equation 4.

$$h \circ F = G \circ h. \quad (4)$$

The homeomorphism condition turns Equation 4 into a conjugacy; a noninjective h can identify different source states. The intertwining law preserves forward iteration under the chosen map but does not, by itself, preserve the ability to reconstruct a source state from its image. If h is a diffeomorphism between smooth carriers and also intertwines their smooth dynamics, the resulting smooth conjugacy preserves additional differentiable structure. The smooth structure of a carrier and the evolution specified upon that carrier are thus separate comparison targets.

One can observe this distinction on the circle. Let $R_\omega(z) = e^{i\omega}z$ on S^1 , let $G = R_{2\omega}$, and put $h(z) = z^2$. Equation 5 records an exact semiconjugacy.

$$h(R_\omega(z)) = R_{2\omega}(h(z)), \quad h(z) = h(-z). \quad (5)$$

The second equality in Equation 5 exhibits the noninjectivity of the covering map. Source points z and $-z$ are indistinguishable under this factor representation even though they are different points of the stipulated state space. Such information loss is a feature of the chosen model comparison, without itself demonstrating a loss of any experiential distinction.

For flows, further criteria concern time parameterization and geometric orbits. A transformation that maps complete oriented orbits to complete oriented orbits may allow orbit-dependent changes of parameter; a time-preserving conjugacy demands compatibility with the specified flow parameter. The distinction matters when a parameter is an experimental sampling coordinate, an intrinsic variable of a mathematical model, or a proposed physical time coordinate. Such roles require separate declarations. Likewise, equality of a selected orbit says little about the transverse vector field, local stability, available controls, or other reachable trajectories. A complete model of Variation can therefore carry distinctions discarded by a selected-trajectory representation.

The behavioral tradition in systems theory supplies a complementary viewpoint. Rather than selecting a particular internal state-space realization as the primary system object, one can define a system by its admissible trajectories of manifest variables, potentially subject to differential or difference constraints [22]. Different internal realizations may generate the same specified behavior, while differing in latent-state dimension or realization structure. This is useful for the Repertoire orientation: equality of a behavioral set is a legitimate exact equivalence criterion, conditional on the variable alphabet, admissibility conditions, interface, and temporal or other indexing domain. It is generally weaker than equivalence of particular generating laws.

A distinction arises even within behavioral comparison. The set of admissible paths, a probability measure on that set, and a selected path drawn from a measure are different data. Two systems may admit the same paths with different likelihoods; they may also induce the same probability law on observed paths through different hidden mechanisms. The relation among these representations will later require explicit observation maps and, where appropriate, specified sampling measures. At the present stage, no unique or privileged probabilistic measure on a physically proposed repertoire is presumed.

2.3 Trace and branching semantics

Process theory makes the difference between accessible histories and branching possibilities especially transparent. A labeled transition system consists of a state set S , an alphabet A , and labeled transitions $s \xrightarrow{a} s'$ with $a \in A$. A finite trace from a distinguished state is the sequence of labels encountered along one finite transition path. The *trace language* collects those sequences and forgets the internal branching structure of the paths that produce them. The linear-time–branching-time spectrum gives a systematic account of multiple criteria that preserve different amounts of behavioral structure [13, 19].

A strong bisimulation between labeled transition systems with a common alphabet is a relation R between their states such that, whenever $(s, t) \in R$, each transition from s can be matched by a transition carrying the same label from t to a related state, and conversely. The matching is local to each reachable pair; it retains a specified organization of alternatives unavailable from the trace language alone. We recall the standard preservation statement for finite traces because it establishes an implication needed in the later comparison of Variation and Repertoire.

Proposition 2.1 (Strong bisimulation preserves finite traces). *For two labeled transition systems over the same alphabet, suppose distinguished states s_0 and t_0 are related by a strong bisimulation. Then their sets of finite traces, including the empty trace, are equal.*

Proof. The empty trace belongs to both languages. Suppose a label sequence $a_1 \cdots a_n$ is realized by a transition path from s_0 . Starting with the related pair (s_0, t_0) , the bisimulation condition supplies a matching a_1 -transition from t_0 to a related target. Reapplying the condition successively yields a matching path labeled $a_1 \cdots a_n$. The reverse simulation clause supplies the opposite language inclusion. Thus the two finite-trace sets coincide. $\square$

The converse of Proposition 2.1 fails even for small finite systems. Consider two systems with labels a, b, c . In the first, a root p has two distinct a -transitions to p_b and p_c ; p_b has one b -transition to a terminal state, while p_c has one c -transition to a terminal state. In the second, a root q has one a -transition to q_{bc} , which has both a b -transition and a c -transition to terminal states. The two finite-trace languages are identical, as displayed in Equation 6.

$$\text{Tr}(p) = \text{Tr}(q) = \{\epsilon, a, ab, ac\}. \quad (6)$$

For a bisimulation relating p and q , each a -successor of p would have to be related to q_{bc} . The successor p_b cannot match the available c -transition of q_{bc} , and p_c cannot match its b -transition. Hence the roots are trace-equivalent but not strongly bisimilar. This example does not assert that branching equivalence is always more relevant than trace equivalence; it exhibits a distinction whose scientific importance depends on whether alternatives and their organization belong to the explanatory object.

Several further qualifications are necessary. Strong bisimulation depends on the transition labels and the granularity of the state description; hiding internal actions leads to different weak equivalences. Probability-bearing systems require relations that address transition probabilities or observable distributions. Infinite behavior may necessitate a separate criterion involving divergence, acceptance, fairness, or infinite traces. Moreover, a transition label is a formal symbol until it is interpreted as a physical or laboratory operation. In the consciousness setting, it would be unwarranted to read a label automatically as an event in lived experience, a neuronal mechanism, or an act of a constituted subject. The process-theoretic precedent is powerful precisely because it allows us to state the preservation criterion independently of such interpretations.

2.4 Geometric and topological criteria

Geometry and topology supply additional, partially incomparable criteria. For topological carriers, a homeomorphism preserves the full topological structure. A homotopy equivalence preserves a weaker homotopy-theoretic organization; the induced isomorphisms on homotopy and homology groups furnish

familiar invariants under their respective hypotheses [14]. Equality or isomorphism of selected homology groups is weaker still. For smooth manifolds, a diffeomorphism respects differentiable structure; a metric isometry additionally preserves the designated metric. These mathematical requirements compare the carrier itself. When the target is a dynamical process, additional compatibility with a vector field, transition rule, or path family is necessary.

The distinction between homology and the fundamental group is particularly useful for cellular representations. For a path-connected CW-complex K , the first integral homology is the abelianization of its fundamental group, as expressed in Equation 7.

$$H_1(K; \mathbb{Z}) \cong \pi_1(K) / [\pi_1(K), \pi_1(K)]. \quad (7)$$

Equation 7 explains one systematic loss of information: noncommuting loop relations can disappear on passing to first homology. For example, begin with a wedge of two circles and attach one two-cell along their commutator. The resulting torus has fundamental group $\mathbb{Z}^2$. Attach the two-cell instead by a constant map; the resulting space is homotopy equivalent to a wedge of two circles and a two-sphere and has free fundamental group F_2 . Both constructions have cellular boundary $\partial_2 = 0$ and the same integral homology groups in every degree: $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}^2$, and $H_2 \cong \mathbb{Z}$. Their fundamental groups differ, so their homotopy types differ. In a proposed relational two-complex, the attaching information may therefore carry distinctions invisible to cellular homology.

For the present research programme, the meaning of a relational two-complex must be made explicit. Its vertices may represent events, its edges couplings, and its faces chosen higher-order primitives. Their status as a mathematical encoding does not establish that the corresponding space is the physical spacetime, a neural state manifold, or the geometry of experience. Further, a homology class or a path component need not denote an observer, subject, or experiential episode. Those interpretations require constitutive and empirical arguments beyond the cellular computations.

The topology of a carrier and the topology of a repertoire defined over it also differ. A fixed circle may admit based loops of multiple winding classes, while a speed or energy bound can exclude some classes and alter the components of the admissible path family. The carrier remains topologically unchanged during such a change of mathematical constraint. Conversely, a family of cellular attachments may change the carrier's homotopy type, yet a chosen observation may retain only a constant or topologically trivial projection. A topological equivalence claim must therefore identify *which space* is being compared: the physical or epistemic carrier, the realization of a relational complex, the space of selected paths, or the collection of admissible paths.

Topological information is also sensitive to the choice of comparison map. A continuous projection need not induce an injective map on fundamental groups, and a quotient can collapse homotopy classes. A chain-level translation may preserve homology while omitting attaching relations. A smooth-to-discrete discretization can approximately preserve solutions over a finite horizon while failing to preserve the underlying topology or exact semigroup structure. Later sections will distinguish the preservation obligations appropriate to these transformations rather than treating smooth, cellular, and observational spaces as interchangeable geometric descriptions.

2.5 Statistical equivalence and experimental information

Scientific comparisons typically encounter records or measurements rather than complete mathematical state descriptions. An observation protocol may be deterministic, stochastic, subject to noise, or dependent on an intervention. In a finite statistical experiment, let Θ denote the model or condition parameter and let $K(y \mid \theta)$ be a Markov kernel describing the measurement distribution over outcomes $y \in Y$. An experiment with outputs in Z is obtained by *garbling* the first experiment if a parameter-independent channel $T(z \mid y)$ satisfies Equation 8.

$$L(z \mid \theta) = \sum_{y \in Y} K(y \mid \theta) T(z \mid y). \quad (8)$$

This factorization is one standard basis for comparing informativeness of statistical experiments, following Blackwell’s decision-theoretic approach [4]. Under Equation 8, decisions that can be implemented from Z can also be simulated from Y by first applying T . Thus the K -experiment is at least as informative for decision problems allowing such randomized procedures, with the same prior and loss specification. The converse characterization of experiment comparison requires the relevant decision-theoretic framework; it is not inferred merely from one error statistic.

A standard data-processing consequence makes this information ordering quantitative. For distributions P and Q on the output space Y , write PT and QT for the distributions after the common channel T .

Proposition 2.2 (Contraction of total variation under a measurement channel). *For probability distributions P, Q on a finite set Y and a Markov kernel T from Y to a finite set Z , total-variation distance obeys Equation 9.*

$$\text{TV}(PT, QT) \leq \text{TV}(P, Q). \quad (9)$$

Proof. Write $d(y) = P(y) - Q(y)$. The triangle inequality and row-stochasticity of T give $\sum_z |\sum_y d(y)T(z | y)| \leq \sum_y |d(y)| \sum_z T(z | y) = \sum_y |d(y)|$. Dividing by two yields Equation 9. $\square$

Proposition 2.2 concerns the ideal distributions available under the stipulated experiment. Its interpretation differs from the statistical power of a finite experiment: positive total-variation distance does not guarantee reliable model discrimination with a small sample. Moreover, an observation interface may alter the experimental conditions rather than merely postprocess a fixed output distribution. In that case, treating the new experiment as a channel T applied to the old one is an additional assumption and requires justification.

Even equality of one-point observation laws need not imply equality of history laws. Consider two binary processes observed at two indexed positions. Under model P , the output is (Z, Z) for a fair binary variable Z . Under model Q , the output is (Z_1, Z_2) for two independent fair binary variables. Both models assign a Bernoulli($1/2$) distribution at each single position, but their joint distributions differ. Equation 10 records a distinguishing event.

$$\Pr_P(Y_1 = Y_2) = 1, \quad \Pr_Q(Y_1 = Y_2) = \frac{1}{2}. \quad (10)$$

Thus equality of marginal manifestation criteria may coexist with inequivalence of observed-trajectory measures. The reverse inference is also conditional: equality of complete observed path laws concerns the selected observation interface and does not identify hidden generative mechanisms or unobserved physical structure. The indices in this example label observation positions; the interpretation of their ordering as physical time is an additional modeling specification.

The distinction among identifiability, equivalence, and finite-sample testing will be important later. A parameter or structural property is identifiable relative to a class of models when equality of the ideal observation laws implies equality of that property. A finite experiment can fail to discriminate theoretically distinguishable distributions because of sample size, noise, confounding, or model misspecification. Causal comparisons further require specifying interventions, their targets, and the conditions that license causal identification. These distinctions prevent a criterion based on measured resemblance from being promoted silently into an equivalence claim about physical constitution or subjective experience.

2.6 Consciousness theories and explanatory interfaces

Theories of consciousness differ in the phenomena they target, the mechanisms they propose, and the measurements they regard as discriminating. The review by Seth and Bayne distinguishes higher-order, global-workspace, re-entry and predictive-processing, and integrated-information approaches, describing differences in theoretical and neurobiological commitments [24]. Evers, Farisco, and Pennartz emphasize the need for carefully selected dimensions of logical and empirical comparability rather than

presuming one common denominator across theories [11]. The present paper inherits this methodological problem but does not recast those existing scientific theories into four purportedly equivalent versions of one mechanism.

Variation, Trajectory, Generativity, and Repertoire describe candidate *formal explanatory orientations*. They need not compete as exclusive kinds of physical system, nor do they coincide respectively with any four named neuroscientific theories. An integrated-information calculation, an effective-connectivity graph, a neural state trajectory, a perturbational-response family, and an experimentally accessible report distribution may arise within one research programme, but they inhabit different mathematical domains. It is therefore possible to compare two models in their trajectory representations while retaining an unresolved difference in their causal or relational representations. Such a comparison is scientifically useful when its scope is maintained.

The explanatory question becomes especially delicate at the boundary between measured data and experience. A physical account may specify a neural state space, a transition mechanism, and observation channels to behavioral or physiological records. The assertion that a selected mathematical invariant is constitutive of phenomenal structure is a further claim, potentially involving psychophysical principles, calibration, and empirical predictions. The subjectivity problem discussed by Nagel remains relevant to the epistemic access available for comparison [21]. The mathematical equivalence of recorded outputs alone does not supply the missing phenomenal interpretation, while a difference among formal invariants alone does not establish that two subjects have different experiences.

Three analytical registers, introduced in the Introduction and developed formally in the following section, keep these levels distinct. The *ontological register* considers conditional physical realization; the *relational register* considers constituted couplings, incidence and transformation organization; the *manifestational register* considers observable records and reports. These are methodological registers, neither an exhaustive ontology nor a hierarchy of epistemic worth. A model may carry information spanning several registers, and the maps connecting them may be many-to-one, partial, probabilistic, or unavailable. The twelve-cell cross-classification formed by the three registers and four orientations will therefore function as an index of possible questions, not as a claim that all twelve cells possess independently measurable phenomena or unique equivalence criteria.

2.7 Comparative criteria and research obligations

The preceding traditions support a comparative discipline based on declared mathematical domains and explicit preservation obligations. Table 1 summarizes representative criteria, the information they retain, and an example of information they may omit. Its rows are illustrative rather than exhaustive: exact properties of a criterion depend on the definitions and assumptions used in a particular model class.

Table 1 also exposes an important asymmetry between mathematics and interpretation. An exact theorem may establish that a translation preserves a particular algebraic or observational structure. A scientific claim about consciousness additionally asks whether that structure corresponds to a physically realized mechanism or a phenomenal feature and what observations could bear upon the proposal. This requires an explicit bridge from the formal domain to a well-specified research setting. Conversely, phenomenological or experimental considerations may motivate distinctions absent from an initially chosen mathematical representation. The classification can accordingly be revised when further evidence or conceptual analysis shows that its objects or maps are inadequate.

Several research obligations follow. Each proposed equivalence should specify the typed domain of comparison, the invariant or relation used, and any maps mediating models expressed in different languages. A statement about cross-orientation preservation should identify which information can be recovered after translation and furnish a proof or counterexample. An empirical comparison should specify available observables, interventions where relevant, and the distinction between exact identification and finite-sample discrimination. Finally, any proposed inference to phenomenal equivalence requires a separately defended interpretation of experience. These obligations supply the motivation for Section 3, where model domains and register interfaces are defined more systematically.

Table 1: Representative mathematical equivalence criteria and their preservation scope.

Criterion	Preserved data under stated conditions	Potentially omitted data
Definitional/Morita equivalence	Logical content under specified translations and permitted definitional constructions	Implementation, designated measurements, or experiential interpretation
Categorical equivalence	Object–morphism organization of the chosen categories	Extra structure excluded from objects and morphisms
Dynamical conjugacy	Carrier topology and iterated dynamics through a homeomorphic intertwiner	Physical interpretation of states; additional unmodeled observables
Semiconjugacy	Projected deterministic dynamics under an intertwining map	Source distinctions merged by a noninjective map
Strong bisimulation	Corresponding labeled branching possibilities	Features absent from the transition labels or state semantics
Finite-trace equivalence	Realizable finite label sequences	Branching organization and potentially infinite-behavior distinctions
Homology-group iso-morphism	Specified homology groups, for chosen coefficients	Fundamental-group relations and attaching information
Statistical garbling comparison	Simulability of measurement-based decisions by a channel	Source information lost in downstream randomization

3 Model Domains and the Three Registers

This section establishes the mathematical domains within which equivalence claims can be interpreted across three analytical registers: ontological, relational, and manifestational. The objective is to specify what occupies each register, how representations and measurements may connect them, and which assertions about preservation follow from those connections. The four orientations introduced in Section 1 furnish different choices of explanatory object within each register; they do not independently prescribe a map between registers. We first construct a typed, context-dependent interface, then establish factorization and comparison results, extend the construction to probabilistic observations, and examine a cellular counterexample. The resulting apparatus supplies a disciplined vocabulary for later equivalence criteria without presuming an ontology of subjective experience.

3.1 Registers as roles in a contextual model

The term *ontological register* denotes the domain of realizations or implementing processes *postulated by a particular model*. In an applied neural model, its variables may describe physical degrees of freedom, interaction mechanisms, or experimentally controlled conditions. Their inclusion in the ontological register records the model’s intended target and explanatory commitment; it does not certify either that the model is physically correct or that its states determine phenomenal states. Accordingly, two ontological models may disagree about primitive variables while remaining subject to comparison through a declared interface. In a purely abstract construction, the ontological domain can remain an uninterpreted candidate-realization space.

The *relational register* describes constituted organization selected from or attributed to a candidate realization: incidence structures, couplings, transformation actions, constraints, and more complex families of relational configurations. A smooth neural-state manifold and the geometric realization of a cellular two-complex may both serve relational modeling purposes, but they are different kinds of space and require different morphisms. In the two-complex convention adopted from the preceding manuscript, vertices represent events, edges represent couplings, and faces represent locally stipulated higher-order primitives. A putative entity or subject is a candidate historically or relationally constituted suborganization, rather than a primitive identification of a vertex with an agent.

The *manifestational register* collects the outcomes available under a selected interface, including

instrument records, observable responses, reports, and reconstructed data products. A report belongs to an experimental and interpretive procedure, and its semantic assignment can depend on reporting conventions, calibration, and the subject's available actions. The existence of an observable distinction therefore need not imply a distinct physical mechanism, just as equality of recorded outcomes need not imply relational or physical identity. Philosophical work on scientific representation emphasizes the role of interpretation and modeling practice in relating mathematical descriptions to target systems [12]; discussions of commensurability in consciousness research likewise motivate explicit comparison interfaces [11].

These three registers are *analytical roles*. A measured electrical waveform is physically realized while also serving as an observation; a graph may be a reconstructed observable or a theoretically posited mechanism, depending on its role in the comparison. The roles are consequently assigned relative to a modeling context, not to unchanging classes of worldly things. Nor do the registers represent ordered epochs, a universal temporal direction, or a hierarchy in which one methodological tradition is automatically prior to another.

3.2 Typed domains and conditional representation interfaces

Fix a context c that specifies a model class, declared physical interpretation, relevant scale, admissible transformations, and observation protocol. Let X_c be a set of candidate configurations or realizations, R_c a set of typed relational descriptions, and Y_c a set of manifest records. A simple deterministic interface is a pair of functions $\rho_c : X_c \rightarrow R_c$ and $\omega_c : R_c \rightarrow Y_c$. Its composite $m_c = \omega_c \circ \rho_c$ is displayed in Equation 11.

$$X_c \xrightarrow{\rho_c} R_c \xrightarrow{\omega_c} Y_c, \quad m_c = \omega_c \circ \rho_c : X_c \longrightarrow Y_c. \quad (11)$$

Equation 11 is a *conditional representational chain*, not an assertion that every physical measurement is first generated by one comprehensive relational encoding. Its maps must be supplied rather than inferred from the register names. For example, ρ_c might assign a cell complex to a system's reconstructed interaction history and ω_c might retain a vector of graph statistics. In this example, R_c is a class of cellular descriptions and Y_c is a record space; neither space is identified with physical spacetime or with phenomenal character. Partial maps are sometimes more realistic: a relational reconstruction may be undefined outside the measurements' identifiable domain. In that case the present constructions apply on an explicitly restricted common domain on which all required maps are defined.

Even when the individual maps of Equation 11 are given, their structural meaning depends on the types involved. For a dynamical application, compatibility with transformations requires an additional commuting relation. For a cellular application, a purported translation may need to preserve attaching maps rather than merely cell counts. For a probabilistic observation, a function into an outcome set may be replaced by a Markov kernel. Thus, the elementary chain serves as a common point of comparison without exhausting the types of interfaces needed later in the paper.

For fixed c , each deterministic interface induces equivalence relations on X_c by equality of its images. The three relations displayed in Equation 12 illustrate the systematic loss of distinctions in a compositional encoding.

$$\Delta_{X_c} \subseteq \ker(\rho_c) \subseteq \ker(m_c), \quad \ker(f) := \{(x, x') : f(x) = f(x')\}. \quad (12)$$

Here Δ_{X_c} is the equality relation, and the inclusions in Equation 12 express relative *distinguishability* rather than a metaphysical claim that manifestations are less real. If ρ_c or ω_c is injective on the relevant image, the corresponding inclusion may be equality. A description with fewer distinctions may nevertheless be better suited to a specified experimental question. These kernels are equivalence relations because they are inverse images of equality on their target sets. A relation based instead on a nonzero metric tolerance need not be transitive, as emphasized in Section 1.

3.3 Recoverability of a property across registers

A central task is to decide whether a model property $q : X_c \rightarrow Q$ can be reconstructed from manifest records. For instance, q may extract a topological invariant of a relational representation, a dynamical parameter, a class of available transformations, or a stipulated physical feature. The following factorization result makes the recoverability condition exact. It concerns recovery by a deterministic function of the stipulated record, without assuming that noisy finite data allow reliable estimation.

Proposition 3.1 (Record-relative factorization). *Let X, Y, Q be sets, $m : X \rightarrow Y$ a specified record map, and $q : X \rightarrow Q$ a property. There exists a unique function $\bar{q} : m(X) \rightarrow Q$ with $q = \bar{q} \circ m$ when m is regarded as a map to its image if and only if $\ker(m) \subseteq \ker(q)$.*

Proof. If $q = \bar{q} \circ m$, then equality of two m -images implies equality of their q -values, yielding the asserted kernel inclusion. Conversely, assume the inclusion and define $\bar{q}(m(x)) := q(x)$. If $m(x) = m(x')$, the assumption gives $q(x) = q(x')$, so this definition is independent of the representative. Every point of $m(X)$ has a preimage, making $\bar{q}$ uniquely determined on $m(X)$. $\square$

The criterion of Proposition 3.1, applied to the composite m_c of Equation 11, is stated in Equation 13.

$$q \text{ is exactly recoverable from } m_c(X_c) \iff \ker(m_c) \subseteq \ker(q). \quad (13)$$

Equation 13 distinguishes three issues often conflated in debates about theoretical equivalence. A property might fail to descend through an observation interface; it might descend but be statistically difficult to estimate; or it might descend as a mathematical feature without possessing the physical or phenomenal interpretation provisionally assigned to it. The proposition resolves only the first issue. Conversely, when the interface omits differences irrelevant to q , it can permit exact recovery of q even while identifying many other source configurations. The relevant criterion is consequently property-specific, rather than a demand to recover every microscopic detail.

Recoverability may also be needed at the intermediate relational register. For a property $u : R_c \rightarrow U$, exact reconstruction from a manifest record $y = \omega_c(r)$ requires $\ker(\omega_c) \subseteq \ker(u)$ on the selected relational domain. Reconstruction of a property defined directly on X_c requires the composite criterion in Equation 13. These two claims differ because an ontological property need not factor through the relational description in the first place. We will use this distinction when comparing criteria in different orientation–register cells.

3.4 Direct measurements and commuting comparisons

Some observations of a candidate physical system are collected through a channel not originally specified by a relational reconstruction. Let $e_c : X_c \rightarrow Y_c$ be this direct measurement map, while ρ_c and ω_c retain their meanings from Equation 11. The statement that e_c is entirely mediated by the chosen relational description is a testable or stipulated factorization claim, displayed in Equation 14.

$$e_c = \omega_c \circ \rho_c. \quad (14)$$

When Equation 14 holds, a change in realization that leaves ρ_c fixed also leaves e_c fixed. This is a precise sufficiency relation for the given measurement, not a claim that all relevant measurements can be reconstructed from the relational representation. A necessary and sufficient condition for *some* factorization of e_c through ρ_c is immediate from Proposition 3.1; it does not require the proposed ω_c to be the correct one.

Proposition 3.2 (Mediation by a selected relational description). *Let $\rho : X \rightarrow R$ and $e : X \rightarrow Y$ be functions. There exists a unique map $\tilde{e} : \rho(X) \rightarrow Y$ satisfying $e = \tilde{e} \circ \rho$ if and only if $\ker(\rho) \subseteq \ker(e)$. An independently proposed map $\omega : R \rightarrow Y$ satisfies $e = \omega \circ \rho$ exactly when its restriction to $\rho(X)$ equals $\tilde{e}$.*

Proof. The first equivalence is Proposition 3.1 with $m = \rho$ and $q = e$. For the second assertion, both maps agree on $\rho(X)$ if the factorization holds; conversely, agreement there implies $e(x) = \omega(\rho(x))$ for every $x \in X$. $\square$

Proposition 3.2 highlights a difficulty in interpreting relational descriptions as explanatory intermediaries. The mere existence of a factorization is a mathematical statement about which distinctions the chosen variables retain; it does not establish a causal mediation pathway. A causal proposal additionally requires intervention semantics, appropriate controls or structural equations, and a physical interpretation of the intermediate variables. The distinction is relevant when a neural network statistic predicts a report but has not been shown to mediate the physical process generating that report.

A different comparison concerns two modeling contexts. Suppose c and d have respective domains (X_c, R_c, Y_c) and (X_d, R_d, Y_d) , with specified translation maps h_X, h_R, h_Y between corresponding domains. Commutativity of the two register interfaces is given in Equation 15.

$$\rho_d \circ h_X = h_R \circ \rho_c, \quad \omega_d \circ h_R = h_Y \circ \omega_c, \quad m_d \circ h_X = h_Y \circ m_c. \quad (15)$$

The final identity in Equation 15 follows from the first two. To retain the chosen observational equivalence in *both directions*, a translation must also preserve the relevant distinctions in its manifestational interface.

Proposition 3.3 (Transport of manifestation equivalence). *Assume the commutative conditions in Equation 15. If h_Y is injective on $m_c(X_c)$, then for all $x, x' \in X_c$,*

$$m_c(x) = m_c(x') \iff m_d(h_X(x)) = m_d(h_X(x')). \quad (16)$$

If h_Y is not injective on that image, the implication from left to right still holds, while the converse may fail.

Proof. Commutativity makes the two translated record values equal to $h_Y(m_c(x))$ and $h_Y(m_c(x'))$. Equality of the source records implies equality after applying h_Y . Injectivity of the restriction gives the converse. If the restriction is noninjective, two distinct source record values may map to one target record value. $\square$

Equation 16 expresses precisely the two-directional record criterion of Proposition 3.3. The proposition concerns equivalence of *records* under two linked interfaces. In particular, it makes no assertion that the translations h_X or h_R are bijective, dynamically equivariant, or topology-preserving. Stronger claims about equivalence of mechanisms, selected histories, constitutive structures, or repertoires require independently specified preservation requirements. A diagram may commute at the level of measurements while failing to preserve the processes those measurements summarize.

3.5 Probabilistic manifestation and observational experiments

Actual observation protocols frequently produce probability distributions rather than deterministic records. For finite sets R_c and Y_c , a Markov kernel $K_c(y \mid r)$ assigns a probability distribution of measurement outcomes to each relational description $r \in R_c$. Given a distribution μ on R_c , the resulting manifest distribution is defined in Equation 17.

$$(\mu K_c)(y) = \sum_{r \in R_c} \mu(r) K_c(y \mid r), \quad \sum_{y \in Y_c} K_c(y \mid r) = 1. \quad (17)$$

Equality of the distributions in Equation 17 is an exact *law-level* manifestational equivalence under the declared experimental interface. It differs from equality of individual sampled records and from the finite-sample decision problem. Following the statistical comparison of experiments developed by Blackwell, an additional observation channel can be considered an information-degrading randomization of the original experiment under specified conditions [4]. Section 2 established contraction of total-variation distance under such a common kernel. Here the question is which complete source distributions can be recovered.

Proposition 3.4 (Injectivity of a finite measurement channel). *For finite sets R, Y , let K be a row-stochastic $|R| \times |Y|$ matrix. The affine map from probability distributions μ on R to their observed distributions μK is injective if and only if K has full row rank $|R|$ over $\mathbb{R}$.*

Proof. Full row rank makes multiplication by K injective on all real row vectors, hence on the probability simplex. Conversely, suppose K lacks full row rank. There is a nonzero real row vector v with $vK = 0$. Since K is row-stochastic, $K\mathbf{1} = \mathbf{1}$; therefore $v\mathbf{1} = vK\mathbf{1} = 0$. Choose a probability vector μ_0 whose coordinates are all strictly positive. For sufficiently small $\varepsilon > 0$, the distinct vectors $\mu_{\pm} = \mu_0 \pm \varepsilon v$ remain probability vectors. Their observed distributions agree because $vK = 0$. Thus the map is not injective. $\square$

Proposition 3.4 gives a complete criterion for recovering an *arbitrary probability distribution* on the given finite source set from its exact measured law. It does not imply that a rank-deficient channel fails to identify every source property: some functions of μ may still be recoverable. Nor does full row rank ensure good statistical conditioning or reasonable sample complexity. Two nearby observed distributions can be hard to discriminate with feasible observations, and a slowly varying physical process need not supply independent samples. These matters require a statistical model for the full joint record law, as developed later in Section 13.

For deterministic relational-to-manifestational maps, the corresponding stochastic matrix has one unit entry in each row. Such a matrix has full row rank exactly when the deterministic map is injective. The probabilistic proposition therefore extends the deterministic distinction-preservation problem while introducing new concerns about measurement noise and the conditioning of inverse inference. A generative theory of consciousness would require still more: a defensible link between source distributions, physical mechanisms, relational constitution, and the interpretation of any phenomenal report.

3.6 Relational topology beyond a manifest statistic

A finite construction illustrates why the register distinction matters even without a dynamical law. Define a relational graph as a one-dimensional CW-complex whose vertices are modeled events and whose edges are modeled couplings. Consider two graphs, C_6 , a six-cycle, and $C_3 \sqcup C_3$, a disjoint union of two triangles. Each graph has six vertices, six edges, and degree multiset $(2, 2, 2, 2, 2, 2)$. Define a manifestational interface ω that records only that multiset. It identifies the graphs, although their connectedness and their first homology differ, as displayed in Equation 18.

$$\begin{aligned} \omega(C_6) &= \omega(C_3 \sqcup C_3) = (2, 2, 2, 2, 2, 2), \\ H_0(C_6; \mathbb{Z}) &\cong \mathbb{Z}, & H_1(C_6; \mathbb{Z}) &\cong \mathbb{Z}, \\ H_0(C_3 \sqcup C_3; \mathbb{Z}) &\cong \mathbb{Z}^2, & H_1(C_3 \sqcup C_3; \mathbb{Z}) &\cong \mathbb{Z}^2. \end{aligned} \tag{18}$$

The graph pair is depicted in Figure 1. The homology calculation in Equation 18 follows from the cell counts and incidence of each connected component: a connected cycle has one degree-one homology generator, while disjoint union gives direct sums. The observation map has two graph representatives in one equivalence class, but a property selecting the number of connected components is not constant on that class. Proposition 3.1 therefore rules out exact reconstruction of that property from the degree multiset alone.

Several limitations are important. The graphs have not been asserted to represent two actually realized nervous systems. No mapping from physical systems to these graph descriptions has been defined, and their different homology groups do not imply different experiences. The example establishes a precise statement about a proposed *relational* invariant and one chosen *manifestational* interface. Other observations, such as direct tests of connectivity, could distinguish the graphs. Thus, manifestational equivalence is relative to the measurement and comparison context, rather than a universal inability to observe their difference.

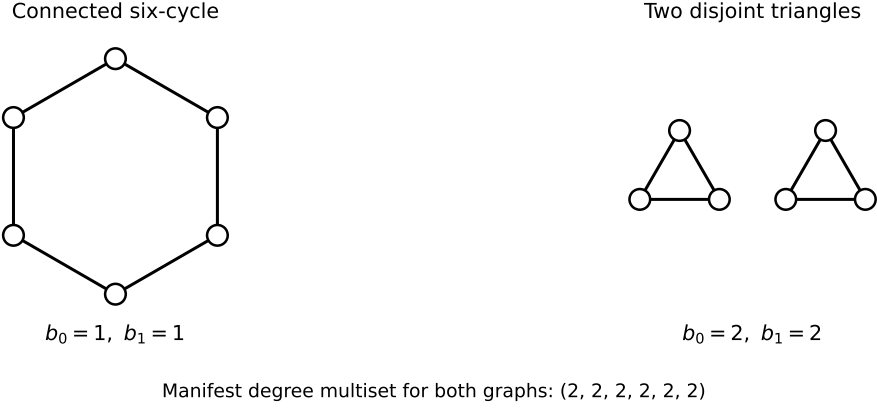


Figure 1: Two event-coupling graphs with identical degree multisets and distinct cellular topology. The left graph is a connected six-cycle; the right graph is the disjoint union of two triangles. Both are abstract relational descriptions, and the shared recorded statistic is only the degree multiset.

Table 2: Illustrative model objects for the four explanatory orientations across three analytical registers. Each entry requires a declared context, carrier, and interpretation.

Orientation	Ontological: proposed realization	Relational: selected organization	Manifestational: available record
Variation	Physical intervention/transition mechanisms under a stipulated implementation model	Transformation semigroup, vector fields, or coupling-dependent operators	Perturbation-response signatures or recorded transition labels
Trajectory	A stipulated realized process or episode with declared parameter/ordering	Selected orbit, path in a state manifold, or cellular-history representation	Recorded signal path, report sequence, or trajectory estimate
Generativity	Candidate physical processes that modify conditions of subsequent permitted transformations	Indexed or endogenously generated structures, constraints, attaching data, and their morphisms	Time-indexed reconstructed networks, change statistics, or reports of structural transitions
Repertoire	Physically admissible processes under stipulated conditions and available controls	Path space, labeled trace language, or family of admissible relational histories	Accessible record language, perturbation outcomes, or observed path law

3.7 The four orientations within the three registers

The four-orientation taxonomy can now be placed in the typed register framework. Table 2 provides one representative object in each of the twelve classification cells. Each row is conditional on a research program selecting the relevant meanings, and the entries need not be independent. For example, an experimentally reconstructed neural trajectory can serve as a manifestational record and, with further assumptions, as an estimate of a relational-state trajectory. The classification records which role an object plays in a *given comparison*; it does not dictate a unique philosophical identity for the object.

The types in Table 2 also suggest distinct *equivalence obligations*. Variation might demand an intertwiner between actions rather than agreement on one measured response. Trajectory may require equality of parameterized paths, orbit equivalence, or homotopy relative to an explicitly named carrier. Generativity calls for a comparison of structures together with the constitutive maps or transformation rules that relate them; isomorphic terminal complexes need not reproduce the same admissibility-changing process. Repertoire equivalence requires a choice among equality of admissible paths, equality of their observed images, and equality of probability measures over them. Every notion must specify its own domains and comparison maps before it can be inferred from a cross-register observation.

In particular, a relational description can participate in more than one orientation. An action on a CW-

complex, a selected path through its configurations, a rule modifying its attaching maps, and a space of permitted paths are mathematically distinct objects. Whether a particular experiment can reconstruct any of them depends on the selected interface and property. Even within one orientation–register cell, multiple criteria can be incomparable: homology-group equivalence and equality of geometric path length, for example, answer different questions and neither generally determines the other. Section 4 will formalize this plurality through typed invariant maps and relations among their kernels.

3.8 Comparison scope, missing interfaces, and phenomenal interpretation

The preceding constructions support several methodological commitments. First, a comparison requires a *common domain*, or a specified translation into a common comparison domain. Equality of graph-degree multisets is meaningful when a degree statistic has been defined for both graphs; equality between a physical system and a homology group is not a well-typed equivalence criterion. Second, the chosen interface must be acknowledged as part of the criterion. If two contexts use different sensors, calibrations, sampling windows, or intervention alphabets, an apparent difference between records need not reflect a difference in the proposed underlying mechanism. Conversely, agreeing records under a coarse protocol may conceal a distinction that a refined protocol could reveal.

Third, missing maps and noninjective maps pose different epistemic difficulties. A known noninjective interface supplies an explicit family of source distinctions that the representation identifies. An unavailable or uncertain interface precludes using a kernel-inclusion theorem to draw a conclusion about the physical target. A hypothesized physical-to-relational map might be multiply realizable, depend on model selection, or change under new evidence. Ontological comparison accordingly remains conditional on the declared physical interpretation and cannot be secured by naming the register alone.

Fourth, the manifestational register includes possible first-person reports but is not identical to *phenomenality*. Reports are produced through concrete physical, communicative, and interpretive conditions. Treating a report as evidence about experience may be scientifically appropriate under a justified protocol, but equality of reports does not provide a mathematical criterion for equality of subjective experience. Nagel’s discussion of the subjective character of experience helps locate the philosophical issue without determining any formal result here [21]. A proposed psychophysical correspondence would need to be separately stipulated, justified, and examined under empirical and epistemic constraints.

The section’s formal conclusion is therefore bounded. For deterministic or probabilistic interfaces, one can define exact equivalences relative to records, specify when properties factor through those records, and determine conditions for transporting such relations between modeling contexts. These results clarify what can be claimed when the mathematical objects occupy ontological, relational, or manifestational roles. They do not establish that a selected register contains all information relevant to consciousness or that there exists one privileged mathematical representation of another subject. The next section develops equivalence criteria and their implications within these typed domains, preserving the distinction between mathematically established relations and their physical or phenomenal interpretation.

4 Indexed Equivalence Relations and Comparison Criteria

The classification introduced in Sections 1 and 3 requires a mathematical account of relations among equivalence criteria themselves. The objective of the present section is to make the strength, combination, transfer, and contextual dependence of such criteria explicit. Our starting point is a typed common domain of model descriptions equipped with invariant maps; we then develop the order structure of their kernels, examine translations and changes of comparison domain, and construct cases of incomparability. Finally, we separate exact equivalence from metric proximity and from finite-sample evidential agreement. The results use elementary set theory and familiar principles of invariance; their proposed application to consciousness remains a methodological proposal, without an assumed phenomenal identity relation.

4.1 Typed comparison classes and invariant assignments

Fix a modeling context c , an explanatory orientation $h \in H = \{V, T, G, R\}$, and a register $\ell \in L = \{O, L_r, M\}$. The context includes the admissible descriptions, a level of granularity, and any physical or observational interpretation. Let $\mathcal{D}_{h,\ell}^c$ be a set of *well-typed* descriptions appropriate to these specifications. The context index may be omitted only when all comparisons take place on one declared domain. For a criterion label k , an invariant assignment $I_{h,\ell,k}^c : \mathcal{D}_{h,\ell}^c \rightarrow \mathcal{I}_{h,\ell,k,c}$ selects the data to be retained. The target is a set, possibly a set of isomorphism classes or a set of other mathematical invariants. If the intended target consists of objects defined only up to isomorphism, the target must be quotiented by that isomorphism or supplied with a suitable set of canonical representatives before literal equality is applied. The criterion-indexed equivalence relation is defined in Equation 19; it develops the initial notation of Equation 1.

$$A \equiv_{h,\ell,k}^c B \iff I_{h,\ell,k}^c(A) = I_{h,\ell,k}^c(B), \quad E_{h,\ell,k}^c := \ker I_{h,\ell,k}^c. \quad (19)$$

The equality in Equation 19 is meaningful only on the stipulated type and comparison domain. For example, a transformation monoid, a chosen trajectory, an indexed family of cell complexes, and a set of admissible trajectories belong to potentially different types. Calling two such objects *equivalent* without a common translation or designated shared representation would leave the comparison undefined. Likewise, two CW-complexes with isomorphic homology groups may be considered equivalent by the isomorphism class of their integral homology, yet the homology groups' equality as individually presented sets is generally an inappropriate test. This is one reason to record a criterion's codomain and the equality or isomorphism convention used there [14].

A criterion also has a *scope*. A finite observation window, an intervention class, a subset of initial conditions, and an admissible collection of attaching maps can each delimit the domain or the available comparisons. When the scope changes, the criterion itself may change. This dependence is explicit in the index c ; it is not a claim that the objects of a physical theory cease to exist outside a particular observer's measurements. The relations defined by Equation 19 satisfy reflexivity, symmetry, and transitivity because equality in the target does. For descriptions related only by a partially defined invariant, an equivalence arises on the domain where that invariant is defined; extension beyond that domain requires an additional convention.

4.2 Criterion refinement and the lattice of indistinguishability

For the rest of this subsection let D be a fixed set of descriptions, and let $I : D \rightarrow U$ and $J : D \rightarrow V$ be two invariant maps. We call I *at least as discriminating as* J when $\ker I \subseteq \ker J$. This order refers to equality classes on D . It neither ranks explanatory quality nor claims that one criterion supplies a more adequate account of consciousness. For example, an invariant that remembers all coordinates is maximally discriminating on a finite encoding, while an experimental invariant that forgets irrelevant nuisance variables may support a more interpretable question. The implication direction is shown in Equation 20.

$$I \succeq_D J \iff \ker I \subseteq \ker J \iff [I(A) = I(B) \Rightarrow J(A) = J(B)]_{A,B \in D}. \quad (20)$$

Equation 20 defines a preorder on criteria. Two different codomain descriptions can induce the same kernel, and hence are equally discriminating on D . For instance, assigning the symbols 0, 1 to two partition classes or renaming them a, b changes an invariant's presentation while retaining its equivalence classes. The preorder becomes a partial order after identifying invariants that induce the same kernel. On this quotient, we may work directly with the lattice $\text{Eq}(D)$ of equivalence relations ordered by inclusion. Its least element is the diagonal Δ_D , and its greatest element is $D \times D$. The latter identifies every description in D and should not be confused with a statement of phenomenal or physical identity.

Two invariant assignments can be combined by recording both results. Conversely, two equivalence relations can be jointly relaxed to the smallest equivalence containing each. Both operations are stated

765 in Equation 21.

$$\begin{aligned}
766 \quad E_I \wedge E_J &= E_I \cap E_J = \ker(I, J), & E_I &:= \ker I, & E_J &:= \ker J, \\
E_I \vee E_J &= \text{EqCl}(E_I \cup E_J), & & & &
\end{aligned} \tag{21}$$

767 where EqCl denotes the least equivalence relation containing the indicated binary relation. The meet
768 in Equation 21 represents the criterion that regards two descriptions as equivalent only when both indi-
769 vidual criteria do. The join relates descriptions connected by a finite chain of steps in either of the two
770 source equivalences. Crucially, the union of two equivalence relations can fail transitivity and therefore
771 need not itself be an equivalence relation.

772 **Proposition 4.1** (Joint criteria and equivalence closure). *For any set D and invariant maps $I : D \rightarrow U$,
773 $J : D \rightarrow V$, the kernel of the product invariant $(I, J) : D \rightarrow U \times V$ is the intersection of their kernels.
774 The equivalence closure of $\ker I \cup \ker J$ is their least common upper bound in $\text{Eq}(D)$. A union of two
775 equivalence relations is itself an equivalence relation precisely when it is transitive.*

776 *Proof.* Two pairs of invariant values are equal exactly when their first components agree and their second
777 components agree; this proves the intersection identity. The reflexive, symmetric, transitive closure of
778 $E_I \cup E_J$ is an equivalence relation containing both. Any equivalence relation containing each must
779 contain every finite chain of their steps, so it contains that closure. Since the union already contains the
780 diagonal and is symmetric, it is an equivalence relation exactly when it is transitive. $\square$

781 The elementary lattice structure matters because terms such as *agreement on either criterion*, *agree-*
782 *ment on both criteria*, and *agreement under a specified family of tests* encode different logical operations.
783 A conjunction of observational tests corresponds to the meet of their kernels. A disjunction of pairwise
784 observed matches may not be stable under composition of comparisons. Equivalence classes that arise
785 from such an unstable disjunction can be artifacts of the comparison procedure rather than coherent
786 classes of mathematical models.

787 4.3 Factorization, data retention, and observational refinement

788 Proposition 3.1 established a general factorization condition for recovering a property from observations.
789 We use that principle here to distinguish an ordering by discriminating power from a choice of representa-
790 tional encoding. A criterion I that is at least as discriminating as J supports a unique recovery map from
791 the *image* of I to the image of J , even if the codomain of I contains unused values. The factorization is
792 stated in Equation 22.

$$793 \quad \ker I \subseteq \ker J \iff \exists! g : I(D) \rightarrow J(D) \text{ such that } J = g \circ I. \tag{22}$$

794 This result follows from Proposition 3.1 by taking the record map to be I and the recovered property
795 to be J . The map g in Equation 22 need not admit an inverse: preserving a coarser criterion is compatible
796 with erasing distinctions detected by the finer one. If both kernel inclusions hold, g is a bijection of
797 images, although the original codomains need not be isomorphic and the map need not preserve any
798 geometric or dynamical structure omitted from the criteria. Thus an equivalence of partitions is weaker
799 than isomorphism of structured codomains.

800 For a concrete example, let $D = \{0, 1\}^2$, $I(u, v) = (u, v)$, $J_1(u, v) = u$, and $J_2(u, v) = v$. Each
801 coordinate is a quotient of the complete encoding; neither coordinate determines the other. Neverthe-
802 less the combined invariant (J_1, J_2) exactly recovers I . This shows that two individually incomplete
803 observational channels can jointly retain a property invisible from either channel alone. In applications,
804 dependence between channels and constraints on their joint accessibility matter: a product invariant pre-
805 supposes that both results can be associated with the same description under a compatible protocol. Data
806 fusion across incompatible measurement contexts requires additional alignment assumptions.

For a family of contexts or protocols indexed by a set C_0 , suppose invariant maps $I_c : D \rightarrow U_c$ are all defined on one common domain D . The joint invariant I_{C_0} takes values in the Cartesian product of the target sets, and its kernel is given in Equation 23.

$$I_{C_0}(A) := (I_c(A))_{c \in C_0}, \quad \ker I_{C_0} = \bigcap_{c \in C_0} \ker I_c. \quad (23)$$

Equation 23 establishes a monotonicity result: adding compatible tests can refine, or leave unchanged, observational equivalence classes. For a nonempty family it does not make those classes *true ontological kinds*. An experiment can only distinguish features encoded in its comparison map, and measurements may introduce noise, bias, or reporting limitations. When protocols alter the physical process, alignment on a single D may fail; the correct objects then include interventions or contexts, and their comparability must be established instead of silently assumed. This distinction is especially relevant to neural perturbation and report paradigms [24, 11].

4.4 Transfer of equivalence between representation domains

A criterion can be transported along a map from one description domain to another, provided its source and target types are clear. Let $F : D \rightarrow D'$ represent a change of description, and let $I : D \rightarrow U$, $I' : D' \rightarrow U'$ be invariants whose values are related by a specified map $a : U \rightarrow U'$. A preservation square is the equation $I' \circ F = a \circ I$. This commuting requirement may describe data processing, a projection of a dynamical representation, or a comparison of cellular invariants under a chosen mapping. It does not follow simply from naming F a translation.

Proposition 4.2 (Preservation and reflection by a comparison square). *Let D, D', U, U' be sets and $F : D \rightarrow D'$, $I : D \rightarrow U$, $I' : D' \rightarrow U'$, and $a : U \rightarrow U'$ be functions satisfying $I' \circ F = a \circ I$. Then I -equivalent descriptions have I' -equivalent images under F . If the restriction of a to $I(D)$ is injective, the converse also holds for pairs in D .*

Proof. If $I(A) = I(B)$, application of a gives $a(I(A)) = a(I(B))$, hence $I'(F(A)) = I'(F(B))$. Conversely, if those two I' -values agree, commutativity gives $a(I(A)) = a(I(B))$. Injectivity on the actually attained values $I(D)$ implies $I(A) = I(B)$. $\square$

The two implications in Proposition 4.2 separate *preservation* from *reflection*. Preservation says that equivalence already established at the source remains true under the translated criterion. Reflection says that apparent equivalence after translation implies equivalence at the source. An information-losing a can preserve without reflecting equivalence. An injective coding of invariant values reflects it on the relevant image even if F is not injective on all information present in the original descriptions. If the comparison is meant to preserve a richer dynamical or categorical structure, commutativity at the level of selected invariant values is only one of the required obligations [23].

A closely related issue arises when criteria are compared after restricting attention to a subset of models or pulling back descriptions through a parametrization $f : E \rightarrow D$. A spurious implication may appear if some distinguishing examples lie outside the image of f . The exact condition is as follows.

Proposition 4.3 (Criterion refinement under domain pullback). *Let $I : D \rightarrow U$, $J : D \rightarrow V$, and $f : E \rightarrow D$. If $\ker I \subseteq \ker J$, then $\ker(I \circ f) \subseteq \ker(J \circ f)$. Conversely, if f is surjective, inclusion of the pulled-back kernels implies $\ker I \subseteq \ker J$.*

Proof. If $I(f(e)) = I(f(e'))$, the source kernel inclusion yields $J(f(e)) = J(f(e'))$. For the converse, take any $A, B \in D$ satisfying $I(A) = I(B)$. Surjectivity supplies $e, e' \in E$ with $f(e) = A$ and $f(e') = B$; pulled-back inclusion then gives $J(A) = J(B)$. $\square$

Surjectivity in Proposition 4.3 is sufficient, and it is necessary for the unconditional converse across arbitrary invariant pairs. To see the danger of a smaller sample, take $D = \{a, b, c\}$ with $I(a) = I(b) \neq$

Table 3: Incomparable graph invariants on a fixed four-vertex comparison class. Graph A is a path on four vertices, B is a triangle with an isolated vertex, and C is a four-cycle.

Graph	Combinatorial description	$e(G)$	$b_0(G)$	$b_1(G)$
A	Path with four vertices	3	1	0
B	Triangle and isolated vertex	3	2	1
C	Four-cycle	4	1	1

$I(c)$ and $J(a) \neq J(b) = J(c)$. The restriction to $\{b, c\}$ makes I injective and therefore its kernel the diagonal, which is included in every equivalence relation. On the full D , however, $I(a) = I(b)$ while $J(a) \neq J(b)$. Thus refinement measured only on a selected model subset need not extend to an unrestricted model class. This issue is conceptual as well as statistical: a criterion can appear strictly informative merely because the sample omits the descriptions on which it fails.

When D and D' have unrelated types, there may be no function F that carries every object of one domain into the other while respecting the desired structure. A comparison may instead be organized by a *span* of aligned descriptions $D \xleftarrow{p} C \xrightarrow{q} D'$, where C records admissible correspondences. Two invariant assignments on D and D' can then be compared after pulling them back along p and q to the common domain C . Such a span restricts the claim to the aligned pairs; it does not produce an equivalence relation across the disjoint union of domains unless additional relations and closure conditions are explicitly supplied. The type-preserving approach also avoids identifying the physical spacetime of a realization with its state manifold or with the geometric realization of an event–coupling complex by an undeclared map.

4.5 Incomparable criteria in a cellular example

The preorder of Equation 20 can contain genuinely incomparable criteria. Consider finite undirected simple graphs regarded as one-dimensional CW-complexes. Let $e(G)$ be the number of edges, $b_0(G)$ the rank of its zeroth integral homology, and $b_1(G)$ its first Betti number. These invariants have familiar graph-theoretic interpretations: b_0 is the number of connected components, while $b_1 = e - v + b_0$, where v is the number of vertices [14]. The three four-vertex graphs in Table 3 provide a finite, exact counterexample to a universal implication between equality of edge counts and equality of first Betti numbers.

Graphs A and B agree on edge count but differ in first Betti number. Graphs B and C agree on first Betti number but differ in edge count. Hence the edge-count and first-Betti criteria are incomparable on the class containing these graphs. In particular, edge-count equivalence relates A to B , and first-Betti equivalence relates B to C , while neither relates A to C . Their union is therefore nontransitive. Its equivalence closure identifies all three examples, as predicted by Proposition 4.1. The joint criterion (e, b_1) , in contrast, distinguishes every one of these three descriptions.

This counterexample also clarifies the limited reach of a topological criterion. In the specified convention for a relational one-complex, vertices can represent events and edges couplings. Equality of a Betti number then concerns a selected aspect of incidence topology. It does not imply equality of attaching or coupling semantics, a common transformation law, a shared history, or equivalence of phenomenology. Likewise, identical edge counts need not imply equal accessible dynamics on those graphs. Criteria can be compared rigorously without assigning them a universal ordering of scientific importance.

4.6 Approximate comparison and finite observations

Exact kernel equivalence gives a mathematically stable notion of classes but can be too restrictive for observations subject to measurement error, numerical approximation, or individual variability. Given a pseudometric d_U on an invariant codomain U , define the nonnegative discrepancy $\delta_I(A, B) = d_U(I(A), I(B))$. Equality at zero distance yields an equivalence relation even when the

pseudometric identifies different invariant values. By contrast, the threshold relation $\delta_I(A, B) \leq \varepsilon$ for $\varepsilon > 0$ generally lacks transitivity. For a real-valued invariant, values 0, 0.75ε , and 1.5ε provide an immediate counterexample: adjacent values are within tolerance while the endpoints are not. A tolerance therefore defines *proximity* rather than an equivalence class unless an additional clustering or closure convention is supplied.

When $J = g \circ I$ for a map g between metric targets, stability can be transferred under an explicit regularity assumption. The bound is stated in Equation 24.

$$d_V(J(A), J(B)) \leq L d_U(I(A), I(B)) \quad \text{if} \quad J = g \circ I \text{ and } g \text{ is } L\text{-Lipschitz on } I(D). \quad (24)$$

Equation 24 follows by applying the definition of a Lipschitz map to the two attained values $I(A)$ and $I(B)$. It quantifies how an approximation in one criterion constrains the difference in another when a stable factorization exists. Without a finite Lipschitz constant, exact factorization alone need not yield a useful finite-error bound. A converse quantitative estimate generally requires an inverse regularity condition; injectivity without a controlled inverse modulus need not suffice.

For manifestational comparisons involving uncertain outcomes, an invariant may be a probability law rather than a deterministic record. If two laws are declared equivalent because their complete distributions agree, the criterion is mathematically exact even though finite data may not determine which law generated the observations. Conversely, failure to reject a statistical null hypothesis does not establish equality of the source laws. A common stochastic measurement channel can preserve distributional equality while coarsening distinguishability, in accordance with the comparison of experiments studied by Blackwell [4]. Interpretation of an observed match therefore depends on the experiment class, sample size, dependence assumptions, and calibrated test or estimation procedure. These matters will be treated formally in the empirical comparison section; they are separate from the set-theoretic equivalence constructed here.

4.7 Implications for the four orientations and three registers

The indexed framework provides an operational vocabulary for later sections. In Variation, an invariant might retain the full transformation action, an equivariant quotient, or only the induced reachability relation. In Trajectory, it might retain parameterized histories, oriented orbit images, or based homotopy classes. In Generativity, it could retain indexed attaching maps, comparison morphisms, or selected invariants of the resulting complexes. In Repertoire, it might retain complete admissible path families, trace languages, or probability laws defined over observed histories. Each choice induces a kernel on an appropriately typed model class; comparison of different choices then depends on inclusion, factorization, a preservation square, or a carefully defined common correspondence domain.

The registers modify the domain and the meaning of these invariants. An ontological criterion concerns the realization stipulated by a physical model, a relational criterion concerns the selected organization, and a manifestational criterion concerns a specified observation protocol. The maps between them can preserve or erase distinctions according to Propositions 4.2 and 4.3. Their existence and suitability require an independent representational or experimental justification. In particular, equality of manifestational invariants may leave physically significant differences unresolved, while non-equality of relational invariants need not imply a corresponding difference in subjective experience.

A defensible equivalence claim should consequently record the following data in prose or in a formal specification: the common domain and its admissible descriptions; the orientation and register; the particular criterion and its equality convention; the comparison or translation maps; the scope of observations or interventions; and the mathematical or empirical argument warranting preservation or recovery. As the subsequent sections develop stronger and weaker criteria within each explanatory orientation, the implication order of Equation 20 will serve as a comparison instrument rather than as a predetermined ranking of theories of consciousness. The established lattice, factorization, and transport results are mathematical tools for locating explanatory differences. Whether those differences bear on physical implementation or phenomenal character remains a distinct question requiring suitable bridge assumptions and evidence.

5 Equivalence in Variation

Variation concerns the transformations admitted by a model and the organization of their composition, domains, constraints, and distinguishable effects. Its comparison therefore begins with transformation prescriptions rather than with one selected history or a family of observed endpoints. The purpose of this section is to distinguish equivalence of mechanism presentations, equivalence of effective actions, conjugacy of dynamical systems, agreement of reachability structures, and equivalence relative to an observation interface. We state preservation conditions for several of these comparisons and construct counterexamples where weaker criteria identify systems that stronger criteria separate. The analysis takes place in explicitly specified mathematical domains; a transformation parameter has no assumed identification with physical time, and equivalence of a variation model supplies no independent phenomenal-equivalence claim.

5.1 Action descriptions and the information carried by mechanisms

Let S be a nonempty set of configurations and let M be a monoid with identity e . A *unital action description* is a triple $A = (S, M, \alpha)$ in which $\alpha : M \times S \rightarrow S$ satisfies $\alpha(e, x) = x$ and $\alpha(mn, x) = \alpha(m, \alpha(n, x))$. Here mn denotes composition with n acting first. The monoid describes formal mechanism composition; S may be a state manifold, a finite configuration set, or an epistemically reconstructed carrier. When the intended system has an ordered family of enabled controls or partial transformations, these data must be supplied separately. The total-action model represents the simplest common comparison domain.

Each mechanism induces a transformation $\hat{\alpha}(m) : S \rightarrow S$ by $\hat{\alpha}(m)(x) = \alpha(m, x)$. The action law makes $\hat{\alpha} : M \rightarrow \text{End}(S)$ a unital monoid homomorphism. We distinguish the presented monoid M from the *effective transformation monoid* $\hat{\alpha}(M)$. A nonfaithful presentation can assign different mechanism symbols to the same transformation on every state. Consequently, comparison of mechanism names, comparison of their compositional laws, and comparison of their realized state transformations are separate criteria. This distinction is also relevant to state-based measurements that cannot directly interrogate mechanism identity.

For two action descriptions $A = (S, M, \alpha)$ and $B = (T, N, \beta)$, an *action isomorphism with mechanism transport* is a pair of bijections $h : S \rightarrow T$ and $\varphi : M \rightarrow N$, where φ is a monoid isomorphism, satisfying Equation 25.

$$h(\alpha(m, x)) = \beta(\varphi(m), h(x)) \quad (m \in M, x \in S). \quad (25)$$

This requirement fixes a strong comparison criterion: state descriptions are related bijectively, the mechanism algebra is preserved, and the actions commute with these correspondences. When S, T carry topology or differentiable structure, h may additionally be required to be a homeomorphism or diffeomorphism; a set-theoretic action isomorphism does not entail either regularity condition. An *action morphism* will mean a pair (h, φ) satisfying Equation 25 with h and φ merely maps and φ a unital homomorphism. Action isomorphism is an equivalence relation on a class of such descriptions, while a general morphism expresses directed preservation and may lose information.

The ability to change mechanism presentations while retaining effective transformations has a particularly simple algebraic characterization.

Proposition 5.1 (Effective-action quotient). *For a unital action description $A = (S, M, \alpha)$, define $m \approx_\alpha n$ if $\alpha(m, x) = \alpha(n, x)$ for all $x \in S$. Then $\approx_\alpha$ is a monoid congruence and the quotient monoid $M/\approx_\alpha$ is isomorphic to $\hat{\alpha}(M) \subseteq \text{End}(S)$. Hence the effective transformation monoid is determined by the action up to monoid isomorphism, even if the mechanism presentation is not faithful.*

Proof. The relation $\approx_\alpha$ is the kernel of the monoid homomorphism $\hat{\alpha}$, so it is an equivalence compatible with multiplication on either side. The map $[m] \mapsto \hat{\alpha}(m)$ is well-defined and multiplicative by the action law, is surjective onto $\hat{\alpha}(M)$ by construction, and is injective by the definition of the kernel. It therefore defines the asserted monoid isomorphism. $\square$

Proposition 5.1 licenses an equivalence criterion based on effective mechanisms, but its scope is precise. Quotienting can erase distinctions between physical implementations or control channels that happen to induce equal maps on the chosen S . Conversely, adding a redundant mechanism symbol can change the presentation without changing the effective monoid. Neither operation determines whether the presentation itself records experimentally meaningful differences in implementation, resource costs, or control accessibility.

5.2 Equivariant translations and reachability preservation

The comparison of actions can retain weaker information than Equation 25. Define the *reachability set* from x in a unital action description by $\mathcal{R}_A(x) = \{\alpha(m, x) : m \in M\}$. Because the action is unital, reachability is reflexive; because M is closed under multiplication, it is transitive. It need not be symmetric. A reachability criterion thus records which states can be obtained by at least one available mechanism while omitting the identity, labels, multiplicity, and composition history of those mechanisms. The corresponding relation is presented in Equation 26.

$$x \preceq_A x' \iff x' \in \mathcal{R}_A(x). \quad (26)$$

For a morphism $(h, \varphi) : A \rightarrow B$, equivariance ensures that every transformed source state has a transformed image in the target. If the mechanism translation fails to cover a target mechanism, additional target states can become reachable that have no antecedent through the chosen translation. This qualification gives a sharp preservation statement.

Proposition 5.2 (Reachability under equivariant translation). *Let $(h, \varphi) : A \rightarrow B$ be an action morphism between unital action descriptions. For every $x \in S$, the inclusion in Equation 27 holds. If $\varphi : M \rightarrow N$ is surjective, this inclusion is an equality. If in addition h is bijective, the induced reachability preorders are isomorphic via h .*

$$h(\mathcal{R}_A(x)) \subseteq \mathcal{R}_B(h(x)). \quad (27)$$

Proof. If $y = \alpha(m, x)$, equivariance gives $h(y) = \beta(\varphi(m), h(x))$, so $h(y)$ belongs to the target reachability set. If φ is surjective, each $n \in N$ equals $\varphi(m)$ for some $m \in M$; the corresponding target state $\beta(n, h(x))$ is therefore the image of $\alpha(m, x)$, which yields equality. Finally, for $u, v \in S$, equivariance and mechanism surjectivity give $u \preceq_A v$ if and only if $h(v) \in h(\mathcal{R}_A(u)) = \mathcal{R}_B(h(u))$ when h is injective. Surjectivity of h makes this an isomorphism of relations on the full state carriers. $\square$

The bijectivity and surjectivity conditions in Proposition 5.2 cannot be suppressed when a full reachability isomorphism is claimed. For example, a noninjective state quotient can map distinct reachable states to a single observed state even when every target mechanism has a source representative. The result concerns the reachability preorder determined by a selected action. Changing the admissible control set, mechanism domain, or state constraints can change that preorder even if the underlying carrier remains fixed.

5.3 Identical reachability with distinct mechanism algebra

Consider the carrier $S = \{0, 1\}$ and two monoids acting by ordinary function evaluation. Let A have transformation monoid $C_2 = \{\text{id}, \tau\}$, where $\tau(0) = 1$ and $\tau(1) = 0$. Let B have the full transformation monoid $T_2 = \{\text{id}, \tau, c_0, c_1\}$, where c_i sends every state to i . Each system reaches both states from every initial state. Their reachability relations therefore agree exactly, as recorded in Equation 28.

$$\mathcal{R}_A(0) = \mathcal{R}_A(1) = \mathcal{R}_B(0) = \mathcal{R}_B(1) = \{0, 1\}. \quad (28)$$

The effective monoids nevertheless differ. C_2 has two elements and is a group of bijections. T_2 has four elements, contains constant idempotent maps, and has transformations that erase the distinction

between its two states. For a distinguished state equivalence $\sim$ taken to be equality, the *distinction-collapse relation* of a mechanism m can be recorded as $\Delta_m = \{(x, x') : x \neq x', \alpha(m, x) = \alpha(m, x')\}$. Every Δ_m in A is empty, whereas Δ_{c_0} and Δ_{c_1} in B contain both ordered pairs of distinct states. Hence the two systems are inequivalent by effective-mechanism isomorphism and by the presence of distinction-erasing mechanisms, despite Equation 28.

This counterexample shows why an accessibility map by itself cannot serve as a complete invariant for Variation. It records outcomes reachable from a given state and omits how outcomes are reached and whether distinctions are preserved by the transformations involved. No irreversible physical process is inferred from a noninjective function alone: physical irreversibility depends on which states and environmental degrees of freedom are included in the model and on the interpretation of the action.

The failure of reconstruction can persist under a stronger observation protocol that records the first component of entire transformation sequences. Take $S' = \{0, 1\} \times \{0, 1\}$ with the observation $q(a, b) = a$. Define two transformations by $F(a, b) = (1 - a, b)$ and $G(a, b) = (1 - a, 0)$. For each $n \geq 0$, $q(F^n(a, b)) = q(G^n(a, b))$, since both transformations flip the first coordinate on every application. Yet F is bijective and G is not: G erases the hidden second coordinate. The generated effective monoids have respectively two and three maps, because $F^2 = \text{id}$ while $G^2(a, b) = (a, 0)$, $G^3 = G$. Thus complete output agreement under this fixed observation and common iteration protocol need not reconstruct the underlying transformation algebra. The examples remain finite mathematical constructions and establish only the specified relationships among their formal criteria.

5.4 Observed action equivalence and maximal stable distinctions

The preceding construction has a useful general form. Let $A = (S, M, \alpha)$ be a unital action description, and let $o : S \rightarrow Y$ be an observation map. For an initial state x , the *action-response assignment* $\Theta_o(x)$ maps each available mechanism m to the observation of its result. Two states are response-equivalent when every available mechanism produces identical observed output. This supplies a canonical observation-compatible action congruence, expressed in Equation 29.

$$\Theta_o(x)(m) := o(\alpha(m, x)), \quad x \equiv_o^M x' \iff \Theta_o(x) = \Theta_o(x'). \quad (29)$$

Unlike equality of $o(x)$ and $o(x')$ alone, Equation 29 includes the outputs of the full stipulated transformation set. It may distinguish states that are initially observationally identical but respond differently to an admissible intervention. The definition presupposes that the same mechanism label is meaningful for both initial states; comparisons of two different action descriptions need an explicit mechanism correspondence such as φ in Equation 25.

Proposition 5.3 (Largest observation-compatible action congruence). *For any unital action $A = (S, M, \alpha)$ and observation $o : S \rightarrow Y$, the relation $E_o = \ker \Theta_o$ is an equivalence relation on S , is stable under every action map, and is contained in $\ker o$. It contains every action-stable equivalence relation $E \subseteq \ker o$. Consequently, the observation of a transformation action descends to a well-defined deterministic action on the quotient S/E_o .*

Proof. Being the kernel of a function, E_o is an equivalence. If $x E_o x'$, then for $n, m \in M$ the action law gives $o(\alpha(m, \alpha(n, x))) = o(\alpha(mn, x)) = o(\alpha(mn, x')) = o(\alpha(m, \alpha(n, x')))$; hence $\alpha(n, x) E_o \alpha(n, x')$. Setting $m = e$ in the response equality and using unitality yields $o(x) = o(x')$, so $E_o \subseteq \ker o$. If E is action-stable and $E \subseteq \ker o$, then $x E x'$ implies $\alpha(m, x) E \alpha(m, x')$ for every m , and therefore equality of all their outputs; thus $E \subseteq E_o$. Stability makes the quotient action $[x] \mapsto [\alpha(m, x)]$ independent of the chosen representative. The inclusion $E_o \subseteq \ker o$ similarly makes the quotient observation well-defined. $\square$

Proposition 5.3 formalizes the strongest state identification allowed by a given observation and the chosen family of mechanisms, subject to invariance of that identification under the action. It is an elementary system-theoretic version of the factorization principle developed in Sections 3 and 4. Changing

the admissible mechanism set changes the response assignment and potentially changes its kernel. Such changes may follow a revision of experimental access, a representational refinement, or a constitutive modification; the mathematics alone does not decide which interpretation applies.

5.5 Smooth conjugacy and the geometry of generated flows

For smooth models, the strength of an equivalence also depends on whether the comparison preserves a flow parameter, only the geometric images of orbits, or merely accessible sets. Let P, Q be smooth manifolds and $X \in \mathfrak{X}(P)$, $Y \in \mathfrak{X}(Q)$ be C^1 vector fields with locally defined flows Φ_X^t and Φ_Y^t . A C^1 diffeomorphism $h : P \rightarrow Q$ is a *parameter-preserving smooth conjugacy* if $h(\Phi_X^t(x)) = \Phi_Y^t(h(x))$ whenever the compared trajectories and their images are defined. This condition implies that the tangent-level dynamics intertwine. With standard regularity and compatible maximal flow domains, the converse also holds.

Proposition 5.4 (Infinitesimal criterion for smooth flow conjugacy). *Let X, Y be C^1 vector fields on smooth manifolds P, Q , and let $h : P \rightarrow Q$ be a C^1 diffeomorphism. Their local flows are parameter-preservingly conjugate by h on common intervals of definition if and only if the differential identity in Equation 30 holds at every $x \in P$.*

$$Dh_x X(x) = Y(h(x)). \quad (30)$$

Proof. Differentiating the flow-conjugacy identity at the flow parameter $t = 0$ gives Equation 30. Conversely, fix x . The curve $t \mapsto h(\Phi_X^t(x))$ has derivative $Dh_{\Phi_X^t(x)} X(\Phi_X^t(x)) = Y(h(\Phi_X^t(x)))$ while it is defined. It starts at $h(x)$ and therefore agrees with $\Phi_Y^t(h(x))$ by local uniqueness of solutions of the C^1 vector field Y . Applying the same argument to h^{-1} identifies the corresponding maximal domains. $\square$

Parameter-preserving conjugacy is strong enough to preserve periods of periodic trajectories. In contrast, multiplying a nonvanishing vector field by a positive constant preserves its oriented unparameterized orbit images while changing its speed. On $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, the fields $X = \partial_\theta$ and $Y = 2\partial_\theta$ have the same oriented orbit images and the same complete reachability relation. Their least positive periods are 2π and π . They therefore admit no parameter-preserving topological conjugacy, since a conjugacy would transfer the least period of each orbit. This is a distinction between two specified dynamical criteria and does not presume a universal physical clock. Standard treatments of conjugacy and behavioral dynamics motivate this hierarchy [22].

A selected trajectory can provide even less information. On $\mathbb{R}^2$ consider the fields $X_-(x, y) = (1, -y)$ and $X_+(x, y) = (1, y)$. Both have the selected integral curve $\gamma(t) = (t, 0)$ through $(0, 0)$, while the transverse variational equations along this curve have respectively contracting and expanding solutions: $\eta_-(t) = e^{-t}\eta_-(0)$ and $\eta_+(t) = e^t\eta_+(0)$. Agreement of a selected path is compatible with different local organizations of nearby transformations. The example also indicates why a complete physical interpretation must declare what nearby states, perturbations, and parameter intervals are experimentally available.

5.6 Comparison criteria and their relative strengths

Table 4 collects the principal criteria for Variation. An implication holds only when its stated assumptions and common comparison domain are satisfied. An action isomorphism with mechanism transport implies equivalence of effective transformation monoids and an isomorphism of reachability preorders. The reverse directions fail in the finite constructions of Subsection 5.3. Parameter-preserving smooth conjugacy implies orbit correspondence and preservation of periods; correspondence of orbit images alone does not recover parameter-preserving conjugacy. Observation-response equivalence is indexed by a fixed observation and mechanism protocol, and the quotient of Proposition 5.3 cannot be interpreted as an equivalence of all physically available transformations without evidence that the tested family exhausts them.

Table 4: Criterion-indexed comparisons of Variation models. The table specifies preserved data rather than a universal order of explanatory adequacy.

Criterion	Data required for comparison	Distinctions that may remain unresolved
Presented-action isomorphism	Monoid isomorphism, state bijection, commuting action square	Different physical realizations of the same presentation
Effective-monoid isomorphism	Induced maps on the carrier and their composition	Redundant labels, implementation costs, enablement rules
Reachability isomorphism	Reachability relation on corresponding states	Labels, multiplicity, injectivity and mechanism composition
Smooth parameter-preserving conjugacy	Diffeomorphic state correspondence and intertwined flow parameters	Physical interpretation of the parameter and omitted variables
Orbit-image correspondence	Images of the selected orbits, with stated orientation convention	Speed, periods, and local transverse dynamics
Observation-response equivalence	Fixed observation, mechanism protocol and output assignments	Hidden states, distinctions erased by measurement

The table also clarifies why the four orientations remain useful as a working partition of research questions. Variation asks which transformations exist and how they compose. Trajectory selects paths that these transformations generate; Generativity investigates the conditions under which transformation descriptions and constraints themselves are constituted or revised; Repertoire concerns whole admissible families of paths. Their descriptions may overlap, and exact mathematical entailments across them require shared domains and explicit maps. For example, the full action determines a reachability relation on its carrier, but a reachability relation does not determine the action. That asymmetry is a property of the specified formal encodings, not a metaphysical ordering of the hypotheses.

5.7 Physical and phenomenal interpretation

In an empirical application to consciousness, a variation criterion might refer to perturbational responses, available neural control directions, transformation algebra under interventions, or symmetries of a reconstructed state manifold. A physically motivated action description must specify admissible perturbations, the preparation and measurement protocol, dynamical regularity, and the scope over which the represented state space is regarded as adequate. Distinct control protocols can induce different action monoids on the same observational carrier. Likewise, a reduced state description may transform a reversible full-system evolution into a noninjective effective map when hidden degrees of freedom are marginalized or identified.

The examples in this section therefore support a limited methodological conclusion: state accessibility, flow geometry, mechanism composition, and measured responses define different equivalence criteria. They specify different data that an empirical theory might try to reconstruct. The mathematical relations among those criteria are available without supposing that any one criterion is sufficient for consciousness. Connecting a specific equivalence of mechanisms with a physical account of an experiencing system, or with the structure of reported or unreported experience, requires additional empirical and philosophical bridge assumptions. The next section considers how such distinctions change when the object of comparison is a selected trajectory rather than the full organization of transformations.

6 Equivalence in Trajectory

Trajectory selects a path or historically interpreted course of configurations from a represented system. Its mathematical comparison requires an explicit account of the carrier, path parameter, admissible correspondence between carriers, and features retained by observation. This section develops five increasingly different questions: equality of parametrized paths, equivalence after an admissible reparametrization, coincidence of geometric images, equivalence of homotopy classes, and equivalence under a partial

record. We also examine the information needed to reconstruct a path from a measurement, distinguishing continuous observations from finite sampling. These comparisons concern selected histories rather than the total action of the transformations generating them, and the path parameter is an index supplied by the model rather than an ontologically privileged clock.

6.1 Typed paths and admissible reparametrization

Let $I = [0, 1]$ and let X be a Hausdorff topological space. A represented trajectory is a continuous map $\gamma : I \rightarrow X$. The choice of a compact interval simplifies comparison; circles, half-lines, partially ordered index domains, and paths defined only locally may require other constructions. In applications of the ontological, relational, and manifestational registers of Section 3, X may denote a hypothesized physical carrier, a space of relational configurations, or an epistemically reconstructed space. Those interpretations require separate arguments. A path on a configuration manifold is not automatically a curve in physical spacetime or a history of phenomenal experience.

For paths $\gamma : I \rightarrow X$ and $\eta : I \rightarrow Y$, even pointwise equality is undefined until a common comparison domain or a transport map is chosen. Fix a homeomorphism $h : X \rightarrow Y$. The strict transported comparison and the weaker reparametrization comparison are given in Equation 31.

$$\begin{aligned} \gamma \equiv_{h, \text{pt}} \eta &\iff h \circ \gamma = \eta, \\ \gamma \equiv_{h, \text{rep}} \eta &\iff \exists \varphi \in \text{Homeo}_+(I) : h \circ \gamma = \eta \circ \varphi. \end{aligned} \tag{31}$$

Here $\text{Homeo}_+(I)$ consists of increasing homeomorphisms of I , each fixing 0 and 1. For fixed h these formulae compare a pair of paths; to speak of an equivalence relation on one common path set, take $X = Y$ and $h = \text{id}_X$, or specify a coherent group of permissible carrier identifications. The endpoint conditions and increasing direction are part of the criterion. Allowing orientation reversal changes the comparison, while permitting arbitrary nondecreasing surjections also identifies trajectories with inserted stationary intervals and requires a separately defined quotient. The choice among these conventions should follow the experimental or interpretive question.

A nonlinear change of parameter can preserve the oriented traversal while changing its local pace. For example, on $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ the paths $e^{2\pi i t}$ and $e^{2\pi i \varphi(t)}$ are reparametrization-equivalent for any increasing homeomorphism $\varphi : I \rightarrow I$. They have distinct coordinate velocities wherever $\varphi'(t) \neq 1$. Consequently, even an exact reparametrization criterion discards information about speed relative to the chosen parameter. A comparison that treats intervals, dwell times, or physical durations as experimentally meaningful should retain or explicitly calibrate that additional structure.

6.2 Geometric images, traversal multiplicity, and differential invariants

For a path $\gamma : I \rightarrow X$, define its image $|\gamma| := \gamma(I)$. Equality of images under a chosen carrier map h is another criterion, stated in Equation 32.

$$\gamma \equiv_{h, \text{im}} \eta \iff h(|\gamma|) = |\eta|. \tag{32}$$

Reparametrization equivalence in Equation 31 implies the image criterion in Equation 32. The converse fails even for smooth based loops. Let $\gamma_n(t) = e^{2\pi i n t}$ on S^1 , with n a positive integer. Every γ_n has the same image, while γ_1 and γ_2 cannot be related by an increasing homeomorphism of I : their winding numbers are respectively one and two. Thus image equality forgets traversal multiplicity in addition to pace. Image equality may also forget ordering of visits and backtracking on more general carriers.

For piecewise C^1 paths on a smooth manifold X , integration against a smooth one-form provides an invariant retaining more information than image alone. The following elementary result specifies its parameter dependence.

Proposition 6.1 (Line integrals under an oriented reparametrization). *Let $\gamma : I \rightarrow X$ be piecewise C^1 , let ω be a smooth one-form on X , and let $\varphi : I \rightarrow I$ be an increasing piecewise C^1 diffeomorphism with*

piecewise C^1 inverse. Then the line integral is invariant under the changed parameter, as expressed in Equation 33.

$$\int_{\gamma \circ \varphi} \omega = \int_{\gamma} \omega. \quad (33)$$

Proof. On each smooth segment, the pullback of ω along $\gamma \circ \varphi$ evaluates on the parameter derivative as $\omega_{\gamma(\varphi(t))}(\gamma'(\varphi(t))\varphi'(t))$. Substitution $s = \varphi(t)$ preserves the order of the endpoints. Summing over a common refinement of the finitely many smooth pieces proves Equation 33. $\square$

Orientation reversal changes the sign of a line integral. A line integral is not in general invariant under homotopy: its value can vary as the path moves across a region with nonzero exterior derivative of the selected form. Closed one-forms provide a more specialized topological criterion.

Proposition 6.2 (Closed-form periods and endpoint-preserving homotopy). *Let X be a smooth manifold, let ω be a closed smooth one-form, and let $\gamma_0, \gamma_1 : I \rightarrow X$ be piecewise smooth paths connected by a smooth homotopy fixing both endpoints. Then their ω -integrals agree. In particular, the integral of ω over based loops depends only on the based homotopy class.*

Proof. Write $H : I \times I \rightarrow X$ for the endpoint-preserving homotopy. Apply Stokes' theorem to $H^*\omega$ over the oriented square. The integral of $d(H^*\omega) = H^*(d\omega)$ vanishes because $d\omega = 0$. The two boundary edges corresponding to fixed endpoints contribute zero; the other two edges are the original paths with opposite boundary orientations. Hence their integrals agree. Piecewise smooth boundaries follow by subdivision. $\square$

On $\mathbb{R}^2 \setminus \{0\}$ the globally defined closed one-form $\omega = (x dy - y dx)/(x^2 + y^2)$ has periods $2\pi n$ on the loops γ_n . Its period distinguishes the same geometric circle traversed different numbers of times. In a simply connected carrier every closed one-form is exact when the relevant smooth hypotheses hold, so these particular periods do not distinguish based loop classes; the availability of informative one-form invariants depends on the carrier and on the chosen collection of observables. The basic relation between homotopy and fundamental-group information is discussed in standard algebraic topology [14].

6.3 Based homotopy and observation of loop classes

A homotopy is a continuous deformation in a specified carrier, together with declared endpoint or boundary conditions. For based loops, the comparison domain is $\pi_1(X, x_0)$ rather than the raw path space. Homotopy equivalence of trajectories is different from equality of their images and from the reparametrization relation of Equation 31. For example, the constant loop at $1 \in S^1$ and the out-and-back loop $\beta(t) = e^{ia \sin(\pi t)}$, for $0 < a < \pi$, are based homotopic but have different images and are not reparametrizations of one another. Conversely, the positive loops γ_1 and γ_2 have the same image but represent different based homotopy classes.

Suppose a continuous observation $q : (X, x_0) \rightarrow (Y, y_0)$ is basepoint-preserving. The induced homomorphism $q_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ specifies exactly which loop distinctions survive observation.

Proposition 6.3 (Reconstruction of based loop classes). *Let X and Y be path-connected pointed spaces and $q : (X, x_0) \rightarrow (Y, y_0)$ be continuous. The equality of observed based homotopy classes determines equality of source based homotopy classes for every pair of source loops if and only if q_* is injective. Equivalently, Equation 34 holds.*

$$q_*([\gamma]) = q_*([\eta]) \implies [\gamma] = [\eta] \quad \text{for all based loops } \gamma, \eta \iff \ker q_* = \{1\}. \quad (34)$$

Proof. The first condition says directly that q_* reflects equality, which is the definition of injectivity. Since q_* is a group homomorphism, it is injective precisely when its kernel is trivial. For the converse direction when the kernel is nontrivial, choose $[\gamma] \neq 1$ in $\ker q_*$ and compare γ with the constant source loop. Their observed classes agree even though their source classes differ. $\square$

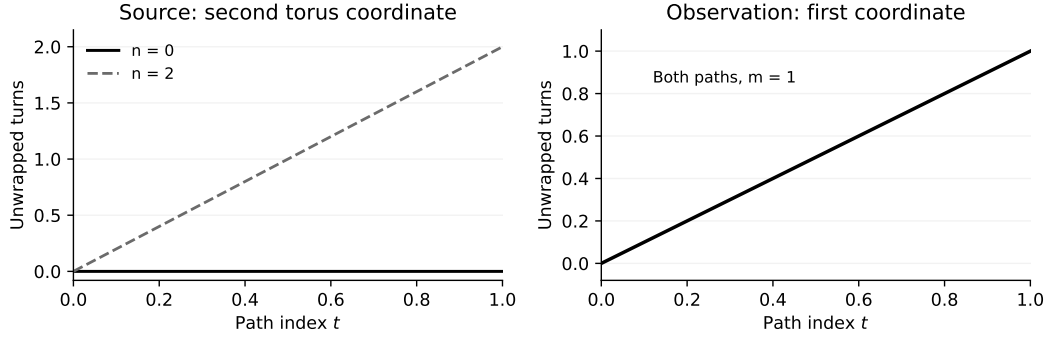


Figure 2: Coordinate lifts of two based torus loops. The second factor records different source winding numbers (left), while projection onto the first factor produces the same full observed path (right). The diagram uses an auxiliary loop parameter and makes no assumption about physical time.

Consider the torus $T^2 = S^1 \times S^1$ and the observation $q : T^2 \rightarrow S^1$ projecting onto the first factor. For integer pairs (m, n) , define the loops and their observations in Equation 35.

$$\gamma_{m,n}(t) = (e^{2\pi i m t}, e^{2\pi i n t}), \quad (q \circ \gamma_{m,n})(t) = e^{2\pi i m t}. \quad (35)$$

The source classes are indexed by $(m, n) \in \pi_1(T^2) \cong \mathbb{Z}^2$, while their observed classes are indexed only by m . The kernel of q_* is $\{0\} \times \mathbb{Z}$. Thus the curves $\gamma_{1,0}$ and $\gamma_{1,2}$ have exactly the same observed continuous path, yet their source homotopy classes differ. Figure 2 displays their coordinate lifts. The depicted coordinates are mathematical representations and do not assign a phenomenal dimension to either circle factor.

The group-theoretic test in Proposition 6.3 concerns recovery of *homotopy classes* from observed homotopy classes. It differs from reconstruction of each point of a parameterized path. These two tasks have different requirements, as the following example demonstrates.

6.4 Path lifting and the information supplied by initial conditions

An observation map with discrete local fibers can retain more information about a continuously observed path than one with extended fibers. A covering map is a useful limiting case. The standard path-lifting theorem gives the following uniqueness statement [14].

Proposition 6.4 (Uniqueness of a lifted observed path). *Let $p : E \rightarrow B$ be a covering map, $b : I \rightarrow B$ a continuous path, and $e_0 \in p^{-1}(b(0))$. If $\tilde{b}_1, \tilde{b}_2 : I \rightarrow E$ are lifts of b with $\tilde{b}_1(0) = \tilde{b}_2(0) = e_0$, then $\tilde{b}_1 = \tilde{b}_2$. Every such b admits a lift from e_0 .*

Proof. An evenly covered neighborhood of $b(t)$ separates the sheets over a sufficiently short parameter interval near t . Lifts agreeing at one point of that interval must remain in the same sheet and coincide there, because the sheet projection is a homeomorphism. Subdivide the compact parameter interval into finitely many subintervals subordinate to evenly covered neighborhoods along b . Starting at e_0 , choose the unique local inverse sheet at each step. This constructs a lift; the same local uniqueness applied successively proves uniqueness on the entire interval. $\square$

For example, $p : S^1 \rightarrow S^1, p(z) = z^2$, is a two-sheeted covering. The path $b(t) = e^{2\pi i t}$ has the unique lift starting at 1 given by $\tilde{b}(t) = e^{\pi i t}$, which ends at -1 . The two possible initial sheets yield different lifts, but fixing the initial point removes this ambiguity. By contrast, the torus projection in Equation 35 has circle fibers and admits multiple lifts of the same observed path even after fixing the initial source point: $\gamma_{1,0}$ and $\gamma_{1,2}$ both start at $(1, 1)$. Its extended fiber permits unobserved motion while the measured path remains fixed.

The uniqueness statement does not identify an unobserved initial sheet from its projected path. Moreover, a laboratory measurement map need not satisfy covering-space hypotheses or provide the continuous record assumed in Proposition 6.4. Path lifting illustrates a precise mathematical sufficiency condition rather than a general solution to historical or neural-state reconstruction.

6.5 Finite samples, regularity, and path reconstruction

Continuous path observation is stronger than observing a finite collection of samples. Let $y_\gamma, y_\eta : I \rightarrow (Z, d)$ be two measured paths in a metric space. Fix samples $0 = t_0 < \dots < t_N = 1$, with maximum gap $\Delta = \max_i(t_{i+1} - t_i)$. If the measured paths are known to be Lipschitz with constants L_γ and L_η , respectively, a finite-sample bound follows.

Proposition 6.5 (Uniform reconstruction bound from sampled discrepancy). *Suppose $y_\gamma, y_\eta : I \rightarrow (Z, d)$ are Lipschitz with constants L_γ, L_η and the discrepancy at every sampling point is at most ε . Then their uniform discrepancy satisfies Equation 36.*

$$\sup_{t \in I} d(y_\gamma(t), y_\eta(t)) \leq \varepsilon + \frac{L_\gamma + L_\eta}{2} \Delta. \quad (36)$$

Proof. For any t , choose a sample t_i at distance at most $\Delta/2$; both endpoints of each sampling interval are included. Apply the triangle inequality through $y_\gamma(t_i)$ and $y_\eta(t_i)$, bound their distance by ε , and bound the two remaining terms by the respective Lipschitz constants times $|t - t_i|$. Taking a supremum yields Equation 36. $\square$

The regularity assumptions in Proposition 6.5 are substantive. On a uniform grid $t_j = j/N$, the two smooth real-valued paths in Equation 37 agree at all sampled positions for any integer $N \geq 1$, despite having different inter-sample behavior.

$$y_0(t) = 0, \quad y_1(t) = a \sin(2\pi Nt), \quad y_0(t_j) = y_1(t_j) = 0 \quad (0 \leq j \leq N). \quad (37)$$

Their uniform distance is $|a|$, and the Lipschitz constant of y_1 increases with N . A sampling protocol without a justified regularity or bandwidth bound therefore cannot equate equality at sampled points with equality of complete trajectories. Even when the full measured path is recovered, reconstructing the underlying source path still requires an appropriate observation map, as shown in Subsections 6.3 and 6.4.

When comparing random measured trajectories, the relevant objects are probability measures on a specified path space with a declared measurable structure. Equality of one-point marginals supplies a weaker criterion than equality of the joint path law, and finite-dimensional distributions determine a process law only under a specified process space and suitable consistency and measurability conditions. Measurement noise, adaptation, synchronization, and dependence between records must be modeled separately. These issues connect trajectory equivalence to the statistical comparison developed in Section 2, while preserving the distinction between equality of a stochastic law and identity of an individual history.

6.6 Comparison hierarchy and translation between registers

Table 5 summarizes the information retained by the principal trajectory criteria. The first three comparisons are formulated relative to a common carrier identification h ; homotopy comparisons additionally require fixed endpoint or basepoint conventions. Observational comparisons require a declared record map q . These criteria form a partial hierarchy, with implications only under the common assumptions already specified. Neither homotopy nor image equality alone reconstructs the other; the constant and out-and-back example and the different-winding loops furnish explicit counterexamples.

Within the three analytical registers, trajectory equivalence can refer respectively to a model of the physical realization of a process, a historically constituted relational description, or a recorded sequence.

Table 5: Equivalence criteria for selected trajectories. Preserved information depends on the carrier identification, endpoint conventions, and observation protocol.

Criterion	Retained data	Distinctions potentially suppressed
Transported pointwise equality	Every indexed state after fixed carrier transport	Mechanism identity; unmodeled physical variables
Increasing reparametrization	Oriented ordering along the path and endpoints	Pace and parameter-specific dwell times
Geometric image equality	Subset of the compared carrier visited by the path	Traversal ordering, multiplicity, endpoints in general
Endpoint-preserving homotopy	Homotopy class with declared boundary conditions	Detailed geometric trace and path-wise kinematics
Closed-form period equality	Selected cohomological evaluations on paths	Homotopy distinctions invisible to chosen forms
Continuous observed-path equality	Complete path after a fixed observation map	Variation within observation fibers
Finite sampled equality	Values at specified indexed sample points	Inter-sample motion and hidden source coordinates

Moving among these registers requires interfaces like those in Section 3; an observed-path equality entails source-path equality only with additional injectivity or lifting assumptions. It is also important that a homeomorphism preserving homotopy need not preserve metric lengths or physical control costs, and a diffeomorphism transporting a smooth path need not preserve its generating vector field unless the stronger conjugacy conditions of Section 5 hold.

The formal findings thus give Trajectory its own comparison vocabulary without elevating one criterion to a universal measure of consciousness. A selected history can be considered at several resolutions, from its full indexed representation to its geometric and topological invariants and its recorded manifestations. Each resolution answers a different reconstruction question. The next section will examine equivalence of the historically indexed structures and conditions under which such histories and their admissibility are constituted.

7 Equivalence in Generativity

Generativity concerns the organization of conditions under which configurations, distinctions, constraints, and further transformations can be constituted. Comparing two generative descriptions therefore requires information about the connections among structural presentations, in addition to the structures occurring at individual positions of an index. The objective of this section is to formulate such comparisons in terms of diagrams of models, cellular attachments, rule admissibility, and observational mediation. We prove conditions for coherent diagram equivalence and for transporting constitutive cellular maps, and we construct examples in which agreement on objects or terminal structures conceals differences in the organization of their constitution. Our index categories are mathematical comparison devices. Interpreting their arrows as causal dependencies, physical evolution, or historically realized changes requires additional structure and evidence.

7.1 Constitutive descriptions and indexed model domains

Let J be a small category whose objects index structural presentations and whose arrows denote declared comparison or constitutive maps. No global order or privileged reference position is presumed. Let C be a category of presentations with morphisms that preserve a specified portion of their structural data. An *indexed constitutive description* is a functor $\mathcal{G} : J \rightarrow C$. For a simple cellular example, C can be the category of CW-complexes of dimension at most two and cellular maps; for rule-bearing systems, it can be a category of pairs (X, A) consisting of a set of states and an admissible-transition relation, with

relation-preserving maps. These are distinct categories, and their morphisms have different preservation obligations.

The data of a diagram and its compositional coherence are displayed in Equation 38.

$$\mathcal{G} : J \longrightarrow \mathbf{C}, \quad j \longmapsto G_j, \quad (a : j \rightarrow k) \longmapsto G_a : G_j \rightarrow G_k, \quad G_{b \circ a} = G_b \circ G_a. \quad (38)$$

A cellular G_j can represent a relational two-complex whose vertices denote events, edges couplings, and faces stipulated higher-order primitives. Its cell attachments and comparison maps form mathematical descriptions of relational constitution. Neither a CW-complex nor the assignment $j \mapsto G_j$ automatically determines a subject of experience, a physical spacetime, or a generative law. A complete physical account would identify the relevant carriers, interpreted arrows, energetic and causal constraints, and the mechanism producing the observed organization.

For two functors $\mathcal{G}, \mathcal{H} : J \rightarrow \mathbf{C}$ with the same index domain, a *coherent structural equivalence* is a natural isomorphism $\eta : \mathcal{G} \Rightarrow \mathcal{H}$. Its component $\eta_j : G_j \rightarrow H_j$ must be an isomorphism in the declared category, and it must respect every indexed comparison map. Equation 39 states the condition.

$$\eta_k \circ G_a = H_a \circ \eta_j \quad \text{for every arrow } a : j \rightarrow k \text{ in } J. \quad (39)$$

This is stronger than the mere existence of unrelated objectwise isomorphisms $G_j \cong H_j$. For descriptions indexed by different categories, an equivalence of index categories together with an explicitly coherent reindexing may be admitted, but choosing such a reindexing is itself part of the equivalence criterion. The distinction follows the standard mathematical meaning of natural transformations and diagram categories [23].

Proposition 7.1 (Objectwise equivalence without diagram equivalence). *There exist functors $\mathcal{G}, \mathcal{H} : J \rightarrow \mathbf{Set}$ for which $G_j \cong H_j$ at every object and every indexed arrow acts bijectively, yet no natural isomorphism $\mathcal{G} \Rightarrow \mathcal{H}$ exists.*

Proof. Let J be the free category on two parallel arrows $u, v : a \rightrightarrows b$. Set $G_a = G_b = H_a = H_b = \{0, 1\}$, and let $G_u = G_v = H_u = \text{id}$, while $H_v = \sigma$ interchanges 0 and 1. All four objects are isomorphic and all four designated structure maps are bijections. If a natural isomorphism η existed, naturality along u would give $\eta_b = \eta_a$, whereas naturality along v would give $\eta_b = \sigma \circ \eta_a$. Since η_a is surjective, this would imply $\sigma = \text{id}$, a contradiction. $\square$

This example concerns the coherence of constitutive descriptions rather than the mere magnitude of structural difference. An indexed system may contain several distinct ways to relate the same positions. Comparing only the positions suppresses those relations. In a physical interpretation, whether such multiplicity reflects distinct historical processes, simultaneous couplings, alternative modeling choices, or another organization must be separately determined.

7.2 Cellular attachments and compatible translations

A two-dimensional CW-complex can encode a rule of constitution through an attaching map, which may carry information invisible in a list of cells or in a homology rank. We use the standard degree- n attachment $K_n = S^1 \cup_{z \mapsto z^n} D^2$ for $n \geq 0$. The one-skeleton is the common circle and the two-cell is attached with specified degree; $n = 0$ denotes a constant attaching map. The cellular chain boundary and first homology are given in Equation 40.

$$C_2(K_n; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\partial_2=n} C_1(K_n; \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(K_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}, \quad (40)$$

where $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$. The cellular computation, together with the associated fundamental-group presentation, is a standard consequence of the attaching-map construction [14]. The following result specifies when a proposed translation of the common one-skeleton is compatible with the two-dimensional constitution.

Proposition 7.2 (Degree compatibility for a cellular attachment map). *Let n, m be nonnegative integers and let $h : S^1 \rightarrow S^1$ be a cellular map of degree $k \in \mathbb{Z}$. There exists a continuous extension $\bar{h} : K_n \rightarrow K_m$ of h if and only if an integer r satisfies the degree equation in Equation 41. Here $m = 0$ is interpreted literally, so that the condition is $kn = 0$.*

$$kn = mr, \quad r \in \mathbb{Z}. \quad (41)$$

Proof. The attaching loop of the source two-cell represents n times the generator of $\pi_1(S^1)$. Applying the degree- k map h produces a loop of winding kn on the target one-skeleton. An extension over the source disk exists precisely when this loop is null-homotopic in K_m . By the presentation $\pi_1(K_m) \cong \mathbb{Z}/m\mathbb{Z}$, this occurs exactly when kn belongs to $m\mathbb{Z}$, equivalently when an integer r satisfies Equation 41. If the condition holds, choose a null-homotopy of the boundary loop in K_m and use it as the image of the source disk. The maps on the circle and disk agree on the attaching boundary, so the CW adjunction property gives a continuous extension. Conversely, any such extension supplies the required null-homotopy. $\square$

For example, a degree-one map of the common one-skeleton extends from K_2 to K_1 because 1 divides 2, whereas it cannot extend from K_1 to K_2 . Thus preserving the one-skeleton and its apparent event-coupling carrier is not sufficient for preserving an admissible higher-cell attachment. Notice also that the integer r describes compatibility of the extension with cellular degrees; it is not a measure of physical generation or a chronological rate.

More general CW-attachments are described by pushout diagrams $X \leftarrow S^{d-1} \rightarrow D^d$. Maps between such presentations require compatible boundary data, or an explicitly weaker comparison such as a homotopy-commuting square with chosen coherence. Agreement only after applying cellular homology need not supply such a map: cellular chains can discard nonabelian fundamental-group relations, attaching homotopies, and the interpretation assigned to individual cells. A comparison may preserve some of those data deliberately, but its criterion must identify them.

7.3 Identical terminal complexes and distinct constitutive diagrams

Even when the resulting topological carrier is held fixed, a diagram that records intermediate subcomplexes can retain distinctions suppressed by comparison of the terminal complex alone. To construct an example, start with a circle $K_0 = S^1$ and attach two labeled two-cells of degrees 2 and 1. Let $K_{2,1}$ be the complex with both attached cells. Consider the two chains of inclusions in Equation 42, where each intermediate stage retains the same original circle but contains one of the two available cells.

$$\begin{aligned} \mathcal{G} : K_0 &\hookrightarrow K_2 \hookrightarrow K_{2,1}, \\ \mathcal{H} : K_0 &\hookrightarrow K_1 \hookrightarrow K_{2,1}. \end{aligned} \quad (42)$$

The displayed index is a selected three-object chain for mathematical comparison; it is not a claim that physical constitution necessarily occurs in a universal succession.

Proposition 7.3 (Endpoint identity does not imply diagram equivalence). *The two cellular diagrams in Equation 42 have identical initial and terminal complexes, but they are not naturally isomorphic as diagrams of CW-complexes. Their intermediate first-homology groups differ, while the common terminal complex has $H_1 = 0$ and $H_2 \cong \mathbb{Z}$.*

Proof. The boundary homomorphism at the first intermediate stage is multiplication by 2 in K_2 and multiplication by 1 in K_1 . Equation 40 therefore gives $H_1(K_2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(K_1; \mathbb{Z}) = 0$. A natural isomorphism would have an isomorphism $K_2 \rightarrow K_1$ at the intermediate object, and would consequently induce an isomorphism on H_1 , contradicting these computations. In the terminal complex the boundary map $C_2 \cong \mathbb{Z}^2 \rightarrow C_1 \cong \mathbb{Z}$ is the row matrix $(2 \ 1)$. It is surjective, and its kernel is generated by $(1, -2)$, so $H_1(K_{2,1}; \mathbb{Z}) = 0$ and $H_2(K_{2,1}; \mathbb{Z}) \cong \mathbb{Z}$, as claimed. $\square$

Two cellular diagrams with one terminal complex

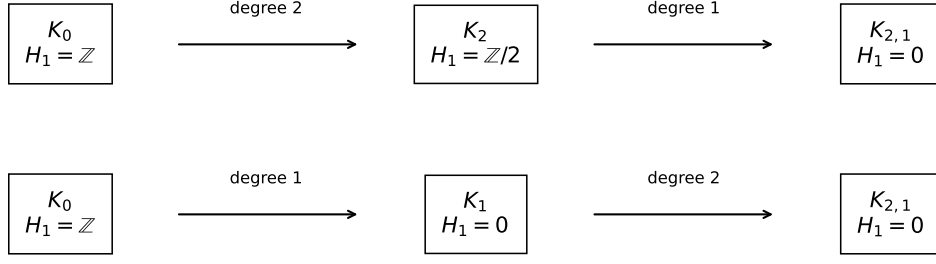


Figure 3: Two indexed cellular constructions with the same initial and terminal complexes. The first-homology group of the intermediate complex is $\mathbb{Z}/2\mathbb{Z}$ in the upper chain and zero in the lower chain. The index arrows represent declared inclusions, not an assumed physical clock.

Figure 3 illustrates the intermediate and terminal first-homology distinctions. The point is not that the index used here exhausts a system's history. Rather, once a set of attachment relations has been selected for representation, endpoint equivalence is strictly weaker than equivalence of the resulting diagrams. If a model permits erasure or rewriting of the intermediate records, whether such differences can be reconstructed becomes an additional epistemic question.

7.4 Changing admissibility and the equivalence of rule families

The topology of a relational carrier and the admissibility of further transformations are different pieces of information. A representation can hold the carrier fixed while modifying which edges of a transition relation are allowed; conversely, one can change cellular geometry without changing a chosen observable transition relation. To compare these aspects, let X_j be a set equipped with a relation $A_j \subseteq X_j \times X_j$ for each $j \in J$. A finite admissible word at j is a sequence $(x_0, \dots, x_N)$ satisfying $(x_i, x_{i+1}) \in A_j$ for all $i < N$. The family A_j may describe local permissions, available couplings, or constraints on a selected process model. It does not by itself identify a mechanism that changes those constraints.

For two rule families (X_j, A_j) and (Y_j, B_j) , fix bijections $\eta_j : X_j \rightarrow Y_j$. Their *exact rule correspondence* is stated in Equation 43.

$$(\eta_j \times \eta_j)(A_j) = B_j \quad \text{for every } j \in \text{Ob}(J). \quad (43)$$

Proposition 7.4 (Rule correspondence and finite admissible histories). *For bijections η_j , Equation 43 holds if and only if the componentwise images of admissible words of every finite length are exactly the admissible words of the corresponding target relation, at each j . It is sufficient to verify this correspondence for words with exactly one transition (two entries).*

Proof. If Equation 43 holds, every adjacent source pair belongs to A_j precisely when its image belongs to B_j . Applying the condition to every adjacent pair yields the equivalence for words of arbitrary finite length, in both directions. Conversely, every relation pair is a two-entry admissible word, and exact correspondence for these words identifies the target relation with the image of the source relation. This gives Equation 43. $\square$

If equality in Equation 43 is weakened to inclusion, the resulting comparison gives forward preservation of admissibility but can admit additional target histories. The similarity to simulation and trace comparison is intentional, but Generativity asks a further question: how are the rule-bearing objects and their correspondence maps constituted or modified across J ? An indexed relation family that reproduces every admissible word at a fixed j is not thereby equivalent to another family in its cross-index organization. To obtain such an equivalence, the maps η_j must additionally satisfy coherent conditions such as Equation 39 for the selected structural interfaces, and the changing constraint family must be interpreted.

For a concrete example with unchanged carrier $X = \{0, 1\}$, let A_a contain only the identity pairs and let A_b additionally contain $(0, 1)$. Although $X_a = X_b$ as sets, the repertoire of admissible words differs, and the index position $a \rightarrow b$ can record a relaxation of a rule. Describing the rule difference is possible without asserting that a transition from a to b has occurred physically. A law for endogenous modification would require a further operator acting on rule-bearing presentations, with declared conditions for its own admissibility and interpretation.

7.5 Observational factorization across a constitutive diagram

The interface between ontological, relational, and manifestational registers in Section 3 allows equivalence claims to depend on records produced from changing structural descriptions. It is useful to specify when an indexed structural property can be recovered consistently, rather than merely at each isolated position. Consider three functors $\mathcal{X}, \mathcal{O}, \mathcal{P} : J \rightarrow \text{Set}$, where X_j are represented source configurations, O_j are records and P_j are values of a stipulated property. Let $q : \mathcal{X} \Rightarrow \mathcal{O}$ and $p : \mathcal{X} \Rightarrow \mathcal{P}$ be natural transformations. Naturalness of q is an assumption about compatible observation interfaces across the index, not a generic fact about measurement devices.

Proposition 7.5 (Coherent recovery from a surjective observational diagram). *Suppose that each observation map $q_j : X_j \rightarrow O_j$ is surjective. There exists a unique natural transformation $r : \mathcal{O} \Rightarrow \mathcal{P}$ satisfying $p = r \circ q$ if and only if $\ker(q_j) \subseteq \ker(p_j)$ for every index object j . The componentwise factorization and the commuting observation square are expressed in Equation 44.*

$$p_j = r_j \circ q_j, \quad r_k \circ O_a = P_a \circ r_j \quad (a : j \rightarrow k). \quad (44)$$

Proof. If $p_j = r_j q_j$, equal q_j -outputs necessarily have equal p_j -outputs, proving the kernel condition. Conversely, if the kernel condition holds, define $r_j(o) = p_j(x)$ for any x with $q_j(x) = o$. Surjectivity supplies an x for every o , while the kernel condition makes the definition independent of the choice. For $a : j \rightarrow k$ and $x \in X_j$, naturality of p and q gives $P_a r_j q_j(x) = P_a p_j(x) = p_k X_a(x) = r_k q_k X_a(x) = r_k O_a q_j(x)$. Since q_j is surjective, $P_a r_j = r_k O_a$ on all of O_j . The family r is therefore natural. Componentwise surjectivity also makes each r_j unique. $\square$

The proposition strengthens the record-relative factorization criterion of Section 3 by requiring agreement across an entire selected diagram. Without surjectivity, the factorization may be determined only on the observed image, and extending it to unobserved record values can fail to be unique or fail to satisfy the naturality equations. Without naturality of the original interfaces, separately recovered properties need not be coherently comparable. Even when all hypotheses hold, r reconstructs only the stipulated property p ; this result supplies no inference from formal recovery to the constitution of phenomenal identity.

7.6 Criteria, physical interpretation, and explanatory limits

Table 6 summarizes several comparisons that should remain distinct when Generativity is investigated. An isomorphism of the terminal space is weaker than coherent isomorphism of a specified indexed diagram. An isomorphism of homology diagrams retains less than cellular or homotopical data. Equality of admissible words at one position describes a relation on local constraints but does not identify the procedures through which those constraints are reorganized. Likewise, observational equivalence can coexist with different internal attachments or rule families. These are relative comparisons with declared mathematical domains rather than levels of a single absolute hierarchy.

For consciousness, a possible research application is to compare reconstructed neural coupling complexes, rules inferred from perturbational responses, and the availability of particular dynamical regimes. Such a comparison would require specifying how events and couplings are measured, what the face primitives represent, whether the index arrows describe physical interventions or merely alternative models, and which observed signatures could distinguish the candidate constructions. The same diagram could

Table 6: Selected equivalence criteria for generative descriptions. Each criterion retains specified information and leaves other distinctions open.

Criterion	Information retained	Information not generally recoverable
Terminal-space homeomorphism	Topological type of a selected final carrier	Intermediate attachments and indexed comparisons
Objectwise cellular isomorphism	Each specified cellular presentation	Coherence of the maps between presentations
Natural isomorphism of cellular diagrams	Constitutive maps and cellular structure throughout one index	Additional physical mechanism, measured phenomena
Isomorphism of homology diagrams	Groups and connecting maps at a selected degree	Attaching words, nonabelian and finer homotopical data
Exact relation-family correspondence	All finite admissible words within each indexed rule relation	Endogenous changes of the rule system and its realization
Naturally recoverable observable property	A stipulated property factored through an observation diagram	Unmeasured structural properties, phenomenal identity

be equipped with different physical interpretations; equal diagram invariants alone do not adjudicate among them. In the ontological register, one would need a theory of realization; in the relational register, explicit constitutive organization; and in the manifestational register, an experimentally supported observation interface.

The mathematical conclusion is therefore modest but consequential. Generativity admits equivalence criteria that depend simultaneously on structures and the organization of their constitutive correspondences. The propositions in this section identify conditions for coherent structural comparison, cellular-map extension, preservation of admissibility and reconstruction through observations. Their counterexamples show why a static structure, an isolated rule set, or an observed record can underdetermine that organization. Whether such differences are important to consciousness remains a question for physically parameterized models and discriminatory empirical studies. Section 5 concerns transformation mechanisms, and Section 6 concerns selected paths; the next section considers equivalence of the admissible-history repertoires associated with those constructions.

8 Equivalence in Repertoire

Repertoire concerns a family of admissible histories relative to a specified system, representation, and notion of admissibility. It is therefore distinct from an individual Trajectory and from the transformation mechanisms considered in Variation. This section examines which aspects of such families survive different equivalence criteria. We first compare finite trace languages with branching organization, then analyze the lifting of abstract histories, the topology and geometry of continuous path families, and probability laws on admissible paths. We conclude with conditions for comparisons across the ontological, relational, and manifestational registers. The constructions use indexed paths and transition compositions as mathematical devices; an index becomes a physical time variable only through additional interpretation.

8.1 Finite-history languages and branching organization

Let $\mathcal{S} = (S, \Sigma, \longrightarrow, s_0)$ be a pointed, labeled transition system, where S is a set of states, Σ a finite alphabet, and $s \xrightarrow{a} s'$ a labeled transition. The *finite trace repertoire* consists of all words labeling a path from s_0 , including the empty word. For a finite horizon N , the truncated language and the induced equivalence criterion are defined in Equation 45.

$$\begin{aligned} \mathcal{R}_{\leq N}(\mathcal{S}) &= \{w \in \Sigma^* : |w| \leq N, s_0 \xrightarrow{w} s \text{ for some } s \in S\}, \\ \mathcal{S} \equiv_{\text{tr}, N} \mathcal{T} &\iff \mathcal{R}_{\leq N}(\mathcal{S}) = \mathcal{R}_{\leq N}(\mathcal{T}). \end{aligned} \tag{45}$$

Here $s_0 \xrightarrow{w} s$ abbreviates a sequence of transitions whose ordered labels form w . The full finite-trace criterion uses $\mathcal{R}_{<\infty} = \bigcup_{N \geq 0} \mathcal{R}_{\leq N}$. These are exact equivalence relations on pointed transition systems with the fixed common alphabet. The order of labels is a compositional feature of the model and does not, without interpretation, constitute a universal order of physical events.

Equality of the full trace repertoire retains possible finite label sequences. It does not by itself retain which intermediate state is responsible for a choice between different continuations. Proposition 2.1 established that strong bisimulation entails finite-trace equivalence, while the branching counterexample in Equation 6 shows that the converse fails. In particular, the state-level modal formula $\langle a \rangle (\langle b \rangle \top \wedge \langle c \rangle \top)$ distinguishes the roots of that example even though their trace languages agree. This is a comparison between predicates about the organization of alternatives and predicates about their possible sequential manifestations. It belongs to the established linear-time–branching-time distinction [13, 1].

A further limitation arises even if the branching structure itself is set aside: agreement on a finite observation horizon does not identify the full finite-trace repertoire.

Proposition 8.1 (Arbitrarily late disagreement of finite-trace repertoires). *For every integer $N \geq 0$, there exist finite, pointed, labeled transition systems $\mathcal{S}_N$ and $\mathcal{T}_N$ over the alphabet $\{a, c\}$ satisfying $\mathcal{S}_N \equiv_{\text{tr}, N} \mathcal{T}_N$ while their full finite-trace repertoires are different.*

Proof. Take two copies of a chain $s_0 \xrightarrow{a} s_1 \xrightarrow{a} \dots \xrightarrow{a} s_N$, with the evident single-state chain when $N = 0$. In $\mathcal{S}_N$ add one c -transition from s_N to a fresh terminal state. In $\mathcal{T}_N$ omit this additional transition. Both systems admit precisely the same words of length at most N , namely a^k for $0 \leq k \leq N$. However, $a^N c$ is a trace of $\mathcal{S}_N$ and is not a trace of $\mathcal{T}_N$. The systems are finite for every N , and the claim follows. $\square$

The preceding counterexample allows the number of states to grow with the observation horizon. A known size bound changes the logical problem. For an ϵ -free finite labeled transition system, associate with each symbol a the deterministic operation Δ_a that maps a set of source states to the union of its a -successors. A word belongs to the trace language exactly when the subset reached from the distinguished singleton is nonempty. This powerset presentation gives an explicit, though generally conservative, bound on the length of a distinguishing trace.

Proposition 8.2 (Size-bounded distinguishing horizon). *Let $\mathcal{S}$ and $\mathcal{T}$ be pointed finite labeled transition systems with respectively $m \geq 1$ and $n \geq 1$ states, over the same finite alphabet, and with no unobserved ϵ -transitions. If their finite trace languages differ, they admit a distinguishing word w satisfying the bound in Equation 46. Consequently, equality of their trace repertoires through that horizon implies equality of the full finite trace repertoires.*

$$|w| \leq 2^{m+n} - 1. \quad (46)$$

Proof. For each word v , form the ordered pair (U_v, V_v) of subsets of states reachable from the respective distinguished roots by a path labeled v . The empty word gives two singletons. Appending a symbol acts deterministically on the subset pair, regardless of nondeterministic branching in the original systems. There are at most $2^m 2^n = 2^{m+n}$ such pairs. A word is distinguishing precisely when exactly one of its two reached subsets is empty. Choose a shortest distinguishing word w of length L . Its $L+1$ prefix pairs must be distinct: if prefixes of lengths $i < j$ reached the same pair, deleting the intervening substring would produce the same final pair as w , contradicting minimality. Hence $L+1 \leq 2^{m+n}$, which proves Equation 46. If the trace repertoires agree for every word through this length, the existence of any longer distinguishing word would contradict the established bound. $\square$

Proposition 8.2 qualifies the finite-horizon nonidentification example under a strong model-class constraint. The exponential bound is a worst-case logical guarantee, rather than a claim of feasible experimental enumeration: the model sizes, complete label alphabet, absence of hidden transitions, and exact availability of each tested word are material assumptions. The bound can be reduced for more restricted model classes. Moreover, trace-language identification still leaves branching structure and

the physical interpretation of recorded processes unresolved. Its relevance to experiments is therefore conditional on both a justified system-size bound and the means to probe candidate traces.

Proposition 8.1 is a statement about the information supplied by a bounded protocol, rather than an impossibility of reconstructing finite-state languages under richer knowledge of the system. A known bound on model size supplies the finite separating horizon in Proposition 8.2; an informative set of experiments or a restricted automaton class may give substantially sharper identification criteria. Without those assumptions, increasing N merely refines the finite-horizon criterion and does not guarantee that the observed agreement extends to every longer admissible word. Likewise, an observed word may be compatible with more than one path of internal states, a multiplicity omitted by trace-language equivalence.

8.2 Observational quotients and the lifting of histories

Consider a transition relation $R \subseteq X \times X$ on a finite state space and a surjective state observation $q : X \rightarrow Y$. Define the *existential quotient* relation $\bar{R}$ on Y by declaring $y \bar{R} y'$ whenever at least one source transition links their observation fibers. This quotient always maps concrete finite paths to abstract paths. The converse is a separate property: consecutive quotient edges may depend on incompatible choices of representatives within an intermediate fiber.

For a fixed observation map, Equation 47 identifies an exact condition for lifting every abstract path from any prescribed representative of its initial observed state.

$$\forall x \in X \forall y' \in Y, \quad q(x) \bar{R} y' \implies \exists x' \in X : xRx' \text{ and } q(x') = y'. \quad (47)$$

Proposition 8.3 (Uniform finite-history lifting from observation fibers). *Let X and Y be finite sets, let R be a transition relation on X , and let $q : X \rightarrow Y$ be surjective. Every finite $\bar{R}$ -path beginning at $q(x)$ has a concrete R -path lift beginning at each specified $x \in X$ if and only if Equation 47 holds.*

Proof. If every abstract path lifts from every representative, apply the assertion to each single-edge path $q(x) \bar{R} y'$ to obtain the transition required in Equation 47. Conversely, assume that condition and fix an abstract path $y_0 \bar{R} \cdots \bar{R} y_n$ together with $x_0 \in q^{-1}(y_0)$. Recursively, Equation 47 supplies x_{i+1} such that $x_i R x_{i+1}$ and $q(x_{i+1}) = y_{i+1}$. This produces a lift of the entire finite path. The length-zero case is immediate. $\square$

For a concrete counterexample, take the deterministic three-cycle $0R1, 1R2, 2R0$ and the observation $q(0) = A, q(1) = q(2) = B$. Its existential quotient permits $A \bar{R} B, B \bar{R} B$, and $B \bar{R} A$. The abstract word of observed states ABA is therefore admitted in the quotient graph. Starting at source state 0, however, the only corresponding three-position word is ABB . The abstract $B \bar{R} A$ step is contributed by source state 2, while the unique concrete state reached immediately after $0R1$ is 1. Figure 4 includes this example. The existential quotient thus enlarges the repertoire of observed state words, without asserting that the extra abstract history has been physically realized.

For a particular initial distribution or a restricted subset of accessible states, the condition in Proposition 8.3 can be relaxed to the states actually relevant to that protocol. Its strength follows from requiring lifting from *each* representative of *every* fiber. The distinction matters when a model is reconstructed from coarse neural observables: a state-transition graph obtained by aggregating measurements can admit sequences that no underlying continuous or discrete process can generate. A fitted abstract repertoire therefore requires an explicit realization or lifting audit before its members are treated as candidates for physical histories.

8.3 Continuous path repertoires and geometric preservation

In a continuous model, a repertoire is naturally a subset $\mathcal{R} \subseteq C(I, X)$, where $I = [0, T]$ is a chosen compact parameter interval and (X, d_X) a metric state space. The uniform metric $d_\infty(\gamma, \eta) =$

$\sup_{t \in I} d_X(\gamma(t), \eta(t))$ turns the path set into a topological space. Comparing the two sets merely as unstructured collections discards the topology induced by their surrounding path spaces. For nonempty compact repertoires, the Hausdorff distance supplies one criterion for quantitative comparison.

Let $q : X \rightarrow Y$ be an L -Lipschitz observation, let $q^I(\gamma) = q \circ \gamma$, and let d_H^X, d_H^Y denote the Hausdorff metrics induced by the respective uniform path metrics. The estimate to be established is recorded in Equation 48.

$$d_H^Y(q^I(\mathcal{A}), q^I(\mathcal{B})) \leq L d_H^X(\mathcal{A}, \mathcal{B}). \quad (48)$$

Proposition 8.4 (Lipschitz preservation of repertoire proximity). *If $q : X \rightarrow Y$ is L -Lipschitz and $\mathcal{A}, \mathcal{B}$ are nonempty compact subsets of $C(I, X)$, then their observed images are compact and Equation 48 holds.*

Proof. The induced map q^I is L -Lipschitz for the uniform metrics, since the pointwise Lipschitz estimate holds at every $t \in I$. Continuous images of compact sets are compact. Given $\gamma \in \mathcal{A}$, its distance to $\mathcal{B}$ can be approximated arbitrarily closely by some $\eta \in \mathcal{B}$. The observed distance satisfies $d_\infty^Y(q^I\gamma, q^I\eta) \leq L d_\infty^X(\gamma, \eta)$. Taking the infimum in η and the supremum in γ gives one directed Hausdorff bound; reversing $\mathcal{A}$ and $\mathcal{B}$ gives the other. Their maximum yields Equation 48. $\square$

Proposition 8.4 concerns stability of comparison under a declared observation, and provides no inverse bound for a noninjective q . For example, a projection $T^2 \rightarrow S^1$ maps distinct source loop families to the same observed family when their differences occur exclusively in the second circle factor. Equal observed repertoires can thus coexist with different geometric or topological source organization. An injective observation with suitable quantitative inverse bounds can support stronger reconstruction, but injectivity and a stable inverse are independent requirements in more general domains. In particular, equality of Hausdorff distance zero is equality of *compact subsets in the specified metric space*; it is a stronger claim than having homeomorphic abstract repertoires.

For completeness, one can also compare repertoires using path homotopy, homology, connected components, or regularity-constrained subspaces. Each selects a different invariant. The topology of $C(I, X)$ depends on the chosen path topology; changing from uniform topology to a Sobolev or differentiable topology can change which sequences converge and which limits are admissible. Observational coordinates, alignment of path parameters, boundary restrictions, and admissibility constraints must therefore be included in any asserted repertoire equivalence.

8.4 Constraint-relative topology of based loop families

A circle illustrates how a fixed underlying carrier can support repertoires with different topological organization. Equip S^1 with a circle metric of radius $a > 0$, fix $T > 0$, and let $\mathcal{L}_a(E_0)$ consist of based H^1 loops whose angular lifts $\theta \in H^1([0, T], \mathbb{R})$ begin at 0, end at an integer multiple of 2π , and have energy at most E_0 . This definition depends on the metric, the chosen parameter interval, a specified basepoint, and the admissibility bound. The energy condition is displayed in Equation 49.

$$E_a(\gamma) = \frac{a^2}{2} \int_0^T |\dot{\theta}(t)|^2 dt \leq E_0, \quad \theta(T) = 2\pi k, \quad k \in \mathbb{Z}. \quad (49)$$

Proposition 8.5 (Components of an energy-bounded loop repertoire). *For $E_0 \geq 0$, the connected components of $\mathcal{L}_a(E_0)$, with the topology inherited from the uniform topology on based loops, are indexed exactly by integers k satisfying $2\pi^2 a^2 k^2 / T \leq E_0$. Every nonempty fixed-winding stratum is path-connected. Consequently its component count is given by Equation 50.*

$$\#\pi_0(\mathcal{L}_a(E_0)) = 2 \left\lfloor \sqrt{\frac{E_0 T}{2\pi^2 a^2}} \right\rfloor + 1. \quad (50)$$

Proof. For any lift with endpoints 0 and $2\pi k$, Cauchy–Schwarz gives $\int_0^T |\dot{\theta}|^2 dt \geq (2\pi k)^2 / T$. Equality holds for the affine lift $\theta_k(t) = 2\pi kt / T$, so the stated inequality is both necessary and sufficient for

winding k to be admissible. For fixed k , linear interpolation between two such H^1 lifts preserves the endpoints. The squared L^2 norm of the derivative is convex, so the interpolation remains within the energy bound. Projecting the interpolation to the circle yields a path of loops continuous in the uniform topology; the fixed-winding stratum is therefore path-connected. Since winding number is invariant under continuous based homotopy, no connected path in the uniform loop space can join two different winding strata. Each stratum is both open and closed relative to the loop repertoire because winding number is locally constant in the uniform topology. The number of admissible integers gives Equation 50. $\square$

When a is changed to a' , the exact path repertoire at the adjusted threshold $E'_0 = (a'/a)^2 E_0$ consists of the same angular loops. This is equality of subsets under the specified circle identification, even though the metric energy values differ. At an unchanged threshold, the admissible winding classes may change according to Equation 50. Conversely, equal component counts for two energy bounds need not give equality of the repertoires: fixed-winding loops with different nonminimal energies can belong to one family and fall outside another. Thus topology of the carrier, topology of the repertoire, and equality of the repertoire itself yield different equivalence criteria.

The construction is a mathematical constraint model. Assigning its radius parameter to a physical coupling strength, interpreting energy as neural expenditure, or reading its based loops as conscious episodes would require independent modeling and measurement. Its value here is the exact demonstration that a purported *Repertoire equivalence* must name which constraints and which invariants enter the comparison.

8.5 Selection laws and probabilistic repertoire equivalence

An admissible repertoire describes which paths are permitted by a model. A *selection law* additionally assigns probabilities to measurable path events. These are independent modeling choices: a fixed set of allowed trajectories can carry many different probability laws. Let S be finite, let μ be a chosen initial probability distribution, and let P be a stochastic transition matrix. The joint distribution of a word of states $(x_0, \dots, x_n)$ is given by Equation 51.

$$\mathbb{P}_{\mu, P}(X_0 = x_0, \dots, X_n = x_n) = \mu(x_0) \prod_{j=0}^{n-1} P(x_j, x_{j+1}). \quad (51)$$

This describes a conventional discrete-parameter Markov model. It does not posit that its step index is a physical time coordinate, or that randomness is an intrinsic property of the phenomena being represented. The corresponding measurable repertoire can be expressed through the support of admissible cylinder events. On a finite alphabet, path-law equality for all finite cylinder events determines the law on the product σ -algebra, whereas equality of the set of positive-probability cylinder words generally does not determine their probabilities [1, 15].

The two transition matrices in Equation 52 have the same strictly positive transition support and the same stationary distribution $\mu = (1/2, 1/2)$.

$$P = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}, \quad \mu = (1/2, 1/2). \quad (52)$$

Both models admit every finite word on $\{0, 1\}$ and have identical one-position marginal distributions at every index. Nevertheless, under P the probability of equal consecutive states is $3/4$, while under Q it is $1/4$. Their two-position joint laws have total-variation distance $1/2$. Figure 4 illustrates the different joint probabilities, alongside the spurious-path quotient example from Subsection 8.2. These are mathematically exact distinctions between *support equivalence*, *marginal-law equivalence*, and *complete path-law equivalence*.

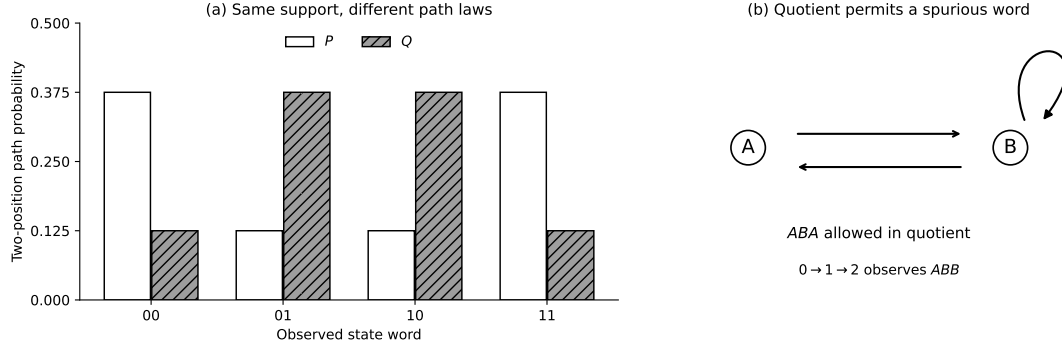


Figure 4: Two independent repertoire comparisons. Left: identical two-state transition supports and stationary one-point marginals give different two-position path probabilities under P and Q (exact rational probabilities). Right: the observed quotient of the deterministic cycle $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ admits an abstract A, B, A state word that cannot be lifted from source state 0. All data are synthetic mathematical constructions.

Proposition 8.6 (Reconstruction of a finite Markov kernel from two-position laws). *Let P and Q be stochastic transition matrices on the same finite state set S , with a common initial distribution μ satisfying $\mu(i) > 0$ for every $i \in S$. The two-position joint laws $\mathcal{L}_{\mu,P}(X_0, X_1)$ and $\mathcal{L}_{\mu,Q}(X_0, X_1)$ are equal if and only if $P = Q$.*

Proof. Equality of the joint laws implies $\mu(i)P(i, j) = \mu(i)Q(i, j)$ for every pair (i, j) . Since $\mu(i) > 0$, division yields $P(i, j) = Q(i, j)$ for all entries. The reverse implication follows directly from Equation 51 with $n = 1$. $\square$

If $\mu(i) = 0$ for some states, two-position observations impose no constraint on the corresponding rows. Observations at larger indices can identify such rows only if the states become accessible with positive probability, under an appropriate common-kernel protocol. Even complete finite-dimensional path laws do not automatically identify a unique underlying hidden-state representation: state splitting and measurement coarsening can preserve an observed process law while changing the latent organization. Probabilistic bisimulation and metrics for stochastic processes address different portions of these comparisons; their detailed definitions depend on the transition model and its observation structure [15, 10]. Statistical indistinguishability from finite samples is weaker again, depending on sample size, dependence, nuisance parameters, and measurement noise.

8.6 Criteria across the four orientations and three registers

A repertoire comparison can be formulated over an ontologically interpreted carrier, an intermediate relational representation, or a manifestational record space. The maps connecting these registers must be specified. For example, if $q : X \rightarrow Y$ is an observation map on states, then q^I maps a continuous path repertoire in X to a repertoire of observed paths in Y . Equality of the source repertoires implies equality of the observed images under the same map. Its converse requires additional reconstruction assumptions. Similarly, pushing two source probability laws forward through the same record map can make them equal while retaining differences in their source laws. This is a preservation statement about the specified interface rather than a claim of phenomenal identity.

Table 7 records a non-exhaustive taxonomy. The table distinguishes criteria according to their mathematical domains and preservation targets, rather than ranking them by a universal degree of explanatory value. An application may need several criteria simultaneously, such as equality of path supports plus a bound on distances between probability laws, or equivalence of cellular admissibility rules alongside topological preservation of the resulting path spaces.

Table 7: Representative equivalence criteria for admissible-history repertoires. The declared carrier, observation, and probability structures determine the comparison domain.

Criterion	Information retained	Information that can remain different
Finite trace language	All permitted finite labeled words	Internal branching, path multiplicities, selection probabilities
Strong bisimulation	Matched local branching transitions	Physical interpretation of labels and omitted internal variables
Uniform abstract path lifting	Concrete realization of each finite abstract path from each fiber point	Probabilities of lifts, phenomenal interpretation
Homeomorphism of path repertoires	Topological organization of specified path spaces	Embedding geometry, energetic costs, probability laws
Equality of compact path sets	Admissible continuous paths in a fixed carrier and topology	Differences in their selection laws
Equality of path probability laws	Probabilities of all specified measurable path events	Distinct latent realization systems under coarse observation
Finite-horizon statistical agreement	Joint laws available at the specified horizon	Longer-horizon dependencies and hidden state descriptions

The four orientations select different objects even when a model links them. Variation concerns transformations capable of generating admissible paths; Trajectory selects an individual path or history; Generativity concerns the constitution and modification of the conditions defining the family; Repertoire concerns the family and, when included, its measurable selection law. Equality of complete path sets does not generally determine the effective action monoid established in Section 5; equality of a single selected path determines still less. Coherent correspondence of changing constraint systems in Section 7 can yield compatible repertoire transport when the comparison maps preserve admissibility, but an equality of one terminal repertoire need not reconstruct the constitutive diagram.

For consciousness research, candidate repertoires could be constructed from neural state-estimation models, evoked responses, task-dependent dynamics, or other physically measured systems. The interpretation must identify the relevant variables, the spatial and temporal resolution, admissibility thresholds, observation maps, and statistical sampling design. The mathematical propositions in this section articulate what such comparisons can preserve and what alternative models may conceal. They provide neither a privileged definition of subjective experience nor an inference from repertoire equivalence to phenomenal identity. The next section compares the implications and incomparabilities among equivalence criteria across all four orientations.

9 Cross-Orientation Implications and Incomparabilities

The preceding sections introduced distinct equivalence criteria for transformation mechanisms, selected trajectories, constitutive descriptions, and admissible-history repertoires. Their proximity within one mathematical model invites the assumption that these criteria form a single hierarchy. This section instead asks which implications can be established on a *common and explicitly declared comparison domain*. We develop sufficient preservation conditions, examine the role of admissibility and selection, and construct counterexamples showing that local agreement can conceal differences in constitutive organization. Throughout, indices and compositional order are formal devices; their identification with physical time requires a separate modeling argument.

9.1 A common domain with explicitly typed comparison data

Consider a unital monoid action $\alpha : M \times S \rightarrow S$, together with a subset $G \subseteq M$ of mechanisms admitted as individual steps. A specified initial state $s_\bullet$ and an admissible selected path $\gamma_\bullet$ are optional additional data. Equation 53 defines the induced one-step relation and the corresponding repertoire of finite state

words, with unrestricted initial states.

$$x R_{\alpha,G} y \iff \exists g \in G : \alpha(g, x) = y, \\ \mathcal{P}_{\alpha,G} = \bigcup_{n \geq 0} \{ (x_0, \dots, x_n) \in S^{n+1} : x_i R_{\alpha,G} x_{i+1} \ (i < n) \}. \quad (53)$$

This construction exhibits a precise connection between Variation, expressed through the action, and Repertoire, expressed through admissible state words. An action does not determine $\mathcal{P}_{\alpha,G}$ without the extra choice of G . Conversely, $\mathcal{P}_{\alpha,G}$ determines the transition relation $R_{\alpha,G}$ by inspecting all two-entry words, but it need not recover the transformation monoid or distinguish two mechanisms producing the same transition. A repertoire starting at $s_\bullet$ contains less information than the unrestricted one; a labeled trace language can conceal yet more of the state sequence. Proposition 7.4 establishes the corresponding exact rule–word criterion for relational systems.

A comparison of Generativity additionally needs an index category J , carriers (S_j) , and maps $c_a : S_j \rightarrow S_k$ associated with arrows $a : j \rightarrow k$. Where appropriate, the indexed model is enriched with actions (M_j, α_j, G_j) , selected paths, or observation maps. Local comparisons at individual objects of J are not automatically comparisons of the indexed structure: the maps c_a retain constitutive information, and their interpretation as physical events, hypothetical alternatives, or epistemic translations is not determined by category theory. Selected-path evaluations likewise require an actual selection rule, which a set of permitted paths does not supply. These typed dependencies govern the implications below.

9.2 Action isomorphism and preservation of admissible histories

Suppose two action models (S, M, α, G) and (S', M', β, G') have a state bijection $h : S \rightarrow S'$ and a monoid isomorphism $\phi : M \rightarrow M'$. For a generator-compatible action isomorphism, the required identities are collected in Equation 54.

$$h(\alpha(m, x)) = \beta(\phi(m), h(x)), \quad \phi(G) = G'. \quad (54)$$

Proposition 9.1 (Generator-compatible action isomorphism preserves repertoire). *Under Equation 54, the componentwise map h^{n+1} induces a bijection of admissible state words of length $n + 1$, for every $n \geq 0$. It therefore identifies the complete finite state-word repertoires. If designated initial states are related by $h(s_\bullet) = s'_\bullet$, their pointed repertoires also correspond.*

Proof. Given an admissible word $(x_0, \dots, x_n)$, choose for every adjacent pair $g_i \in G$ with $\alpha(g_i, x_i) = x_{i+1}$. Equation 54 gives $\beta(\phi(g_i), h(x_i)) = h(x_{i+1})$, with $\phi(g_i) \in G'$. Thus the image word is admissible. The reverse direction follows by the same argument applied to h^{-1} and ϕ^{-1} , using $\phi(G) = G'$. The componentwise maps are mutually inverse. The pointed claim follows by restricting to corresponding initial states. $\square$

Generator compatibility is a substantive assumption. Take $S = \{0, 1\}$ with the same action of $M = \{e, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$ in both models, where e is the identity and σ swaps the two states. Let $G_1 = \{\sigma\}$ and $G_2 = \{e, \sigma\}$. The complete action systems are identical, but $(0, 0)$ is an admissible word only for G_2 . The first repertoire consists of alternating binary words; the second admits all binary words. Thus unqualified Variation equivalence does not imply Repertoire equivalence if admissibility data may differ.

The reverse implication fails even if every mechanism is permitted. Compare $M_{\text{rev}} = \{e, \sigma\}$ with the full transformation monoid $M_{\text{all}} = \{e, \sigma, c_0, c_1\}$ on $\{0, 1\}$, where c_i denotes the constant map to i . Both produce the complete relation $S \times S$ and hence identical unrestricted state-word repertoires. Their effective transformation monoids have respectively two and four elements; one is a group, the other contains noninvertible maps. Thus exact equality of all *unlabeled* state words does not reconstruct the internal mechanism algebra. This distinction is compatible with established linear-time and branching-time semantics [13, 1].

9.3 Selected trajectories and the absence of automatic selection

Trajectory equivalence concerns a selected member of a family, with specified parameterization, end-points, and comparison map. Repertoire equivalence concerns the admissible family itself. Neither determines the other on a domain where the selected member is an independent component of a model. For instance, the complete binary transition relation admits both the constant sequence $(0, 0, 0, \dots)$ and the alternating sequence $(0, 1, 0, 1, \dots)$. Two systems with that same relation may designate these different selected trajectories, thereby agreeing in Repertoire but differing in exact parameterized Trajectory. Conversely, the identity-only relation $R_0 = \{(0, 0), (1, 1)\}$ and the relation $R_1 = R_0 \cup \{(0, 1)\}$ admit the same selected constant path at 0, although their admissible repertoires differ.

A *transported selection* can be made compatible with repertoire correspondence, but must be stipulated. For paths on the same parameter domain I , Equation 55 expresses the exact requirement.

$$\gamma'_\bullet = h^I(\gamma_\bullet), \quad h^I(\gamma) = h \circ \gamma. \quad (55)$$

If Equation 55 holds, the selected paths are exactly related by h . Weaker notions such as equivalence under reparameterization, equality of images, or homotopy require the distinct assumptions developed in Section 6. Moreover, a state path need not determine the mechanisms that produced each step. Recording triples (x_i, g_i, x_{i+1}) adds mechanism labels and thereby forms an enriched path representation containing information belonging also to Variation. Without such a record, separate generators realizing the same transition are merged.

Consequently, the four orientations should not be pictured as successively stronger statements about a single unspecified object. The comparison must say whether a path *exists*, whether its complete trajectory is *selected*, whether its *mechanism labels* are retained, and whether it is observed through a quotient. Each question has a different answer in the counterexamples above.

9.4 Constitutive coherence beyond local mechanisms and trajectories

An indexed constitutive description retains maps among local structures as well as the local structures themselves. The following example isolates that additional information. The local Variation actions, selected Trajectories, and Repertoires coincide at every index, even though the constitutive diagrams are not equivalent by natural isomorphism.

Proposition 9.2 (Pointwise agreement does not establish indexed constitutive equivalence). *There are two J -diagrams of finite action-equipped sets, where J consists of one arrow $0 \rightarrow 1$, for which the local Variation descriptions, selected Trajectories, and complete finite Repertoires agree at each object, but whose underlying set-valued diagrams are not naturally isomorphic.*

Proof. Take $S = \{0, 1\}$ at both index objects. At each object, let the mechanism monoid consist only of the identity, and permit this mechanism at every step; all local admissible paths are constant state words. Select the constant path at state 0 at both indices. The first diagram has connecting map $c = \text{id}_S$, while the second has $d = c_0$, the constant map with value 0. All pointwise data in the other three orientations agree exactly. A natural isomorphism would consist of bijections $\eta_0, \eta_1 : S \rightarrow S$ satisfying $\eta_1 \circ c = d \circ \eta_0$. The left-hand side is bijective, whereas the right-hand side is constant. Such an isomorphism cannot exist. $\square$

The example is a statement about information retained by diagrams. It does not imply that an arbitrary index arrow constitutes a physically realized change, or that every distinction among diagrams is relevant to consciousness. Its value is that even comprehensive *local* observations leave the relation among structural descriptions open. A naturality condition can constrain that relation, whereas object-wise agreement cannot recover it. Figure 5 exhibits the two distinct connecting maps.

A positive implication is available if an indexed description includes all the relevant action and admissibility data *and* compares them coherently. Suppose two J -indexed families of unital monoid actions possess state bijections h_j , monoid isomorphisms ϕ_j , and admissible subsets G_j, G'_j , satisfying Equation 54 at every object. Suppose also that the h_j form a natural isomorphism of the carrier diagrams.

Proposition 9.3 (Local implications of enriched constitutive equivalence). *Under the indexed hypotheses just stated, Variation equivalence holds pointwise, and the complete local finite state-word Repertoires correspond through the state bijections h_j . If selected paths are additionally matched by Equation 55, their exact Trajectory equivalence follows. Naturality of (h_j) establishes coherence with the specified constitutive connecting maps.*

Proof. Each (h_j, ϕ_j) is by assumption an action isomorphism respecting admissible generators, so the Variation claim holds. Proposition 9.1 applies at every j and establishes the Repertoire correspondence. Equation 55 supplies the optional selected-path claim. Naturality supplies the commuting equations for the carrier maps and is logically distinct from the pointwise conclusions. $\square$

This conditional result does not create an unconditional hierarchy. A weaker Generativity criterion, such as the homeomorphism type of a terminal complex, the isomorphism type of homology groups, or objectwise equivalence of unadorned carrier diagrams, need not preserve mechanism or admissibility data. Conversely, Proposition 9.2 shows that simultaneous local agreement in the other orientations may still leave a constitutive connection undetermined. To compare changing neural coupling structures, for example, one would have to state what constitutes a coupling, which relationships are represented by the index arrows, and whether the comparison is intended to preserve an empirically measurable feature or only a mathematical representation.

9.5 Manifestational agreement and different source topology

Comparisons across the three research registers raise a further question. Can the full collection of continuously observed trajectories determine the topology of the state carrier? Even in elementary smooth dynamics, the answer depends on a restrictive observation hypothesis. Consider the circle $X = S^1$ with a rotation flow and the torus $X' = S^1 \times S^1$ with a corresponding rotation of its first coordinate. Define both systems and their observations in Equation 56.

$$\begin{aligned} \Phi_t(\theta) &= \theta + \omega t \pmod{2\pi}, & \Psi_t(\theta, \psi) &= (\theta + \omega t, \psi) \pmod{2\pi}, \\ q(\theta) &= \theta, & q'(\theta, \psi) &= \theta, & \omega &\neq 0. \end{aligned} \tag{56}$$

Proposition 9.4 (Equality of continuously observed flows does not identify carrier homotopy type). *For the systems in Equation 56, the complete sets of observed parameterized flow trajectories from all initial states are equal. However X and X' are not homotopy equivalent; their integral first homology groups are respectively $\mathbb{Z}$ and $\mathbb{Z}^2$.*

Proof. Every observed trajectory in either system has the form $t \mapsto \theta_0 + \omega t \pmod{2\pi}$ for some θ_0 , and all such initial angles occur in both systems. Thus their observed trajectory repertoires agree, including complete parameterization on any common interval. The standard homology calculations give $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ and $H_1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2$ [14]. Since homotopy equivalence preserves integral homology, the two carriers cannot be homotopy equivalent. $\square$

The observation in this example is continuous, rather than a finite sampling scheme. The hidden angular coordinate does not affect the selected record map, so additional sampling precision of that same observable cannot resolve the concealed circle. The proposition concerns the topology of mathematical state carriers, which must be distinguished from physical spacetime and from any hypothesized geometry of phenomenal experience. It also demonstrates why matching Manifestation-level trajectory data cannot be promoted automatically into Relational or Ontological equivalence. The converse fails without further assumptions as well: equal carrier topology does not require equal flow generators or equal admissible repertoires.

9.6 A conditional implication matrix

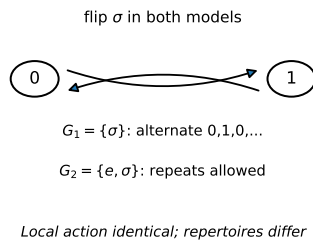
Table 8 collects implications and failures on the particular domains established above. A valid arrow between two criteria must state the comparison maps and preserved structures. A counterexample to an implication means that no implication holds throughout the domain containing that example; it does not preclude stronger results on a restricted class of systems.

Table 8: Cross-orientation implications and counterexamples under explicitly stated comparison conditions. Every entry is mathematical and does not by itself establish phenomenal equivalence.

Proposed relation	Comparison conditions	Finding
Variation $\Rightarrow$ Repertoire	Action isomorphism preserving permitted generators	Holds (Proposition 9.1)
Unqualified Variation $\Rightarrow$ Repertoire	Same action but different permitted subsets	Fails: flip only versus flip and identity
Repertoire $\Rightarrow$ Variation	All state words matched; mechanism labels omitted	Fails: two- and four-element transformation monoids
Trajectory $\Rightarrow$ Repertoire	Selected path only, other transitions free	Fails: extra unselected edge
Repertoire $\Rightarrow$ Trajectory	Complete path set matched, selection unrestricted	Fails: distinct distinguished paths
Local V, T, R $\Rightarrow$ Generativity	Indexed connecting maps retained	Fails: identity versus constant connector
Enriched Generativity $\Rightarrow$ local V, R	Natural carrier maps and generator-compatible action isomorphisms	Holds (Proposition 9.3)
Observed Repertoire $\Rightarrow$ source topology	Complete continuous observation, noninjective map	Fails: circle versus torus

Figure 5 pairs two finite examples to show why an implication may fail for different reasons. In the first, an unchanged action is accompanied by different admissibility choices. In the second, every local action and its admissible paths agree, while constitutive connectors are incompatible with natural isomorphism. These are *different kinds of information loss*. Their distinction would be obscured by treating equivalence as a yes-or-no property of models independently of the data and context selected for comparison.

(a) Same action, different permissions



(b) Same local systems, different connectors

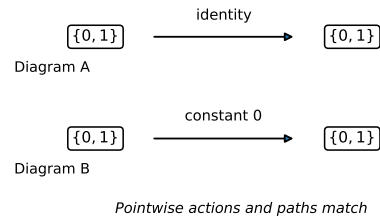


Figure 5: Two finite cross-orientation countermodels. Left: one action, different sets of admissible steps, and hence different state-word repertoires. Right: two diagrams with identical local identity actions but incompatible identity and constant connecting maps. All displayed structures are mathematical examples and do not represent measurements of consciousness.

9.7 Comparative obligations for consciousness research

The cross-orientation analysis suggests a discipline for comparing neural and physical theories. A Variation comparison should declare its action or transition mechanism and whether admissibility is part of that description. A Trajectory comparison should identify path carriers, parameter conventions, initial conditions, selection rules and observation maps. A Generativity comparison should identify the constitutive structure and the maps connecting its indexed descriptions, explaining whether those maps describe realized processes or counterfactual model relations. A Repertoire comparison should state its path or trace space, admissibility constraints, topology, and, if modeled, probability selection law. A claim spanning two orientations is supported only when the required bridge conditions are exhibited.

The same care applies when moving between Ontological, Relational and Manifestational registers. A shared record map can send distinct source systems to equal observations; a topology-preserving structural comparison does not automatically preserve a selected measurement model; and a statistical matching procedure cannot establish that underlying mechanisms or subjectively lived experience coincide. In the language of Section 4, a comparison criterion reflects another only when the relevant kernel inclusion or factorization requirements are satisfied on the declared domain. Section 3 supplied the interfaces needed to express that dependency; Section 10 will examine cross-register preservation in more detail.

A promising empirical comparison would therefore specify an action model for perturbed neural states, a measured trajectory under a defined intervention, a family of admissible responses and an indexed representation of changes in effective coupling. Its comparative value would derive from experimentally accessible *divergences* between candidate models after controlling observation resolution and model complexity, rather than simply demonstrating that several invariants can be computed from the same data. Such work would remain an investigation of putatively consciousness-relevant physical organization. Additional substantive theory and evidence are necessary to associate these formal objects with consciousness, recognize the constitutive boundaries of a subject, or infer phenomenal equivalence.

The mathematical contribution of this section is a typed, conditional structure of implications together with concrete witnesses for nonimplication. Several positive arrows arise only when one orientation has explicitly been *enriched* with information usually treated by another. That dependence is an analytical finding: it shows where translation requires extra commitments and why the four orientations cannot be ranked by a universal strength ordering without declaring those commitments first.

10 Ontological, Relational, and Manifestational Comparison

This section examines equivalence claims that connect the ontological, relational, and manifestational registers within an explicitly specified modeling context. Its objective is to identify the data and assumptions required to transport an equivalence between registers, to characterize information that may be suppressed by a representational interface, and to clarify the conditions under which additional observations recover formerly inaccessible distinctions. The analysis proceeds through comparisons of typed interfaces, property-specific reconstruction, combined observational protocols, dynamical compatibility, probabilistic measurements, and a smooth quotient with nontrivial topology. Throughout, *ontological* designates a model's candidate realization domain, *relational* a mathematical description of constituted organization, and *manifestational* an accessible record or report domain. These roles remain context-dependent and do not supply an independent ontology of experience.

10.1 Typed comparisons between representational contexts

Recall from Section 3 the contextual interface $X_c \xrightarrow{\rho_c} R_c \xrightarrow{\omega_c} Y_c$, with composite record map $m_c = \omega_c \rho_c$. When comparing two contexts c and d , equality of records has mathematical meaning only after specifying a correspondence between their outcome types. For example, EEG feature vectors produced by different filtering and sensor configurations may have superficially similar coordinates while differing in their calibration, statistical dependencies, and physical referents. Likewise, two graph reconstructions

may have isomorphic underlying graphs while encoding different meanings for their vertices and higher cells. The comparison must therefore include morphisms that convey the intended types of structure and records.

Let $f : X_c \rightarrow X_d$, $g : R_c \rightarrow R_d$, and $h : Y_c \rightarrow Y_d$ be comparison maps. Their vertical interface compatibility is expressed in Equation 57.

$$\rho_d \circ f = g \circ \rho_c, \quad \omega_d \circ g = h \circ \omega_c, \quad m_d \circ f = h \circ m_c. \quad (57)$$

The third equality in Equation 57 follows from the first two, and can hold even when no intermediate comparison map g is available. The two stronger equations record how an ontological comparison is mediated through the proposed relational model. A map f need not be physically implementable merely because it exists mathematically; a map g need not preserve cellular attaching data merely because it sends vertices and edges to vertices and edges. Each comparison must specify its type and admissibility assumptions separately.

Proposition 10.1 (Preservation and reflection of manifestational equivalence). *Suppose the three maps in Equation 57 are functions and its composite equality holds. For $x, x' \in X_c$, equality $m_c(x) = m_c(x')$ implies $m_d(f(x)) = m_d(f(x'))$. The converse holds for all x, x' if h is injective on $m_c(X_c)$. More generally, the converse holds precisely when $\ker(h|_{m_c(X_c)})$ is the equality relation on the image $m_c(X_c)$.*

Proof. Applying h to the equality $m_c(x) = m_c(x')$ and using $m_d f = h m_c$ gives the first implication. If h is injective on $m_c(X_c)$, then $h(m_c(x)) = h(m_c(x'))$ implies equality of the two m_c -values. Conversely, if h is noninjective on $m_c(X_c)$, choose distinct y, y' in that image with $h(y) = h(y')$. By definition of image there exist x, x' with $m_c(x) = y$ and $m_c(x') = y'$. Their images under $m_d f$ agree, while their m_c -records differ. This proves the criterion. $\square$

Proposition 10.1 identifies two distinct preservation obligations. A commuting map preserves record equality in the forward direction. Reflecting record equality requires the additional injectivity condition on the records actually attained. The distinction is important when a comparison modifies observation resolution, collapses channels, or changes a reporting vocabulary. If f is not injective, the comparison may identify source realizations even while h is injective: reflection in the proposition concerns the records of source states, rather than reconstruction of the states themselves. Similarly, a bijection of record alphabets has no necessary implication for the equivalence of the constitutive relational descriptions. Interpretive correspondence between reports and experiences is a further bridge assumption rather than a property of Equation 57.

10.2 Property-specific reconstruction and the unobserved remainder

An interface can preserve a particular relational property while leaving many other relations unidentified. Fix c and restrict the relational domain to the image $R_c^{\text{att}} = \rho_c(X_c)$, the descriptions attained under the stipulated modeling context. Let $u : R_c^{\text{att}} \rightarrow U$ be a relational invariant or property, for example an effective coupling rank, a specified homology class, a transformation semigroup invariant, or an admissibility signature. We ask whether u can be computed exactly from the manifestational record $\omega_c(r)$ on every attained relational description. The answer, a restricted form of Proposition 3.1, is expressed in Equation 58.

$$\exists \hat{u} : \omega_c(R_c^{\text{att}}) \rightarrow U, \quad u = \hat{u} \circ \omega_c|_{R_c^{\text{att}}} \iff \ker(\omega_c|_{R_c^{\text{att}}}) \subseteq \ker u. \quad (58)$$

Equation 58 explicitly restricts the recovery claim to attained descriptions. A property may be recoverable on this restricted domain and fail to be recoverable on the larger space of conceivable descriptions. Experimental extrapolation therefore depends on whether the class of attained configurations adequately represents the interventions and conditions to which the claim is applied.

There is no unique quantity called the *unobserved remainder* independently of the selected property family. For a declared collection $\mathcal{U}$ of properties on R_c^{att} , the distinctions jointly retained by that collection

are described by $E_{\mathcal{U}} = \bigcap_{u \in \mathcal{U}} \ker u$. The manifestational interface retains $E_{\omega} = \ker(\omega_c|_{R_c^{\text{att}}})$. Pairs belonging to E_{ω} but not to $E_{\mathcal{U}}$ are record-indistinguishable yet distinguished by at least one property in $\mathcal{U}$. Equivalently, the failure of the kernel inclusion in Equation 58 for some u is a witness of property-specific inaccessible information. Such a witness need not have phenomenological significance. Its relevance depends on whether the chosen property is physically instantiated and connected to the explanatory task.

The principle is particularly important when comparing the four orientations. Equality of a measured Trajectory criterion may coexist with variation in the effective action, in an indexed constitutive map, or in the unobserved Repertoire. Even within one orientation, recovery is criterion-relative: an observed path can determine an endpoint while concealing a winding number, and a degree summary can determine the total number of graph edges while concealing connected components. The purpose of the kernel formulation is to identify the lost distinctions precisely, rather than assign a generic hierarchy of epistemic value to the three registers.

10.3 Combining observational protocols and recovering distinctions

Observation depends on the questions and interventions made available. Suppose a fixed candidate source domain X supports a family $\{m_p : X \rightarrow Y_p\}_{p \in P}$ of deterministic record maps, where each p is a declared observational protocol. The index p may specify a sensor configuration, intervention, reporting instruction, or reconstruction algorithm; its meaning must be stated locally. For a finite family $P_0 \subset P$, the joint protocol M_{P_0} records the tuple of outcomes. Its induced equivalence is displayed in Equation 59.

$$M_{P_0}(x) := (m_p(x))_{p \in P_0}, \quad \ker M_{P_0} = \bigcap_{p \in P_0} \ker m_p. \quad (59)$$

Each additional protocol can refine the induced partition of X . The refinement can be strict, redundant, or insufficient to recover the property of interest. When the protocols themselves intervene on and alter the source, the common-domain construction of Equation 59 requires an explicit counterfactual or controlled-response model; recorded outcomes from different realized conditions cannot automatically be identified as simultaneous measurements of an unchanged state.

Proposition 10.2 (Joint-protocol separation of a property). *Let X and P_0 be sets, with P_0 finite, and $m_p : X \rightarrow Y_p$ for each $p \in P_0$. A property $q : X \rightarrow Q$ is exactly recoverable from joint outcomes if and only if, for every $x, x' \in X$ with $q(x) \neq q(x')$, at least one $p \in P_0$ satisfies $m_p(x) \neq m_p(x')$. If $P_0 \subseteq P_1$, every property recoverable under P_0 remains recoverable under P_1 .*

Proof. The separation condition is the contrapositive of $\ker M_{P_0} \subseteq \ker q$. The equivalence with recoverability follows from Proposition 3.1, applied to M_{P_0} . For the second claim, Equation 59 gives $\ker M_{P_1} \subseteq \ker M_{P_0}$, and hence inclusion in $\ker q$ is preserved. $\square$

An elementary example gives the criterion operational content. Let $X = \{(a, b) : a, b \in \{0, 1\}\}$, and let $m_1(a, b) = a$, $m_2(a, b) = b$. Neither protocol alone reconstructs the pair (a, b) , while (m_1, m_2) does. If the target property is $q(a, b) = a$, the first protocol alone suffices. If the target is $q(a, b) = a \oplus b$, both are needed. Adding a third copy of m_1 changes no observational equivalence class. These cases distinguish additional *measurement volume* from additional *distinguishing power*.

The formulation admits a limitation relevant to consciousness research. If the source domain X contains phenomenally distinct states only through a stipulation or hypothesized bridge map $\varphi : X \rightarrow \mathcal{P}$, neither separability of X nor recovery of a neural property supplies empirical access to φ unless the experimental protocol provides an independent anchor for its interpretation. A statistically powerful instrument may support discrimination of clinical or reported conditions while leaving the equivalence of subjective experiences unsettled.

10.4 Dynamical compatibility across the registers

The maps ρ_c and ω_c must preserve more than instantaneous states when the equivalence under comparison concerns transformations or trajectories. Fix one context and a shared monoid M of stipulated mechanisms. Suppose M acts unitaly on X , R , and Y , with actions α , β , and γ , respectively. A dynamically compatible interface makes $\rho : X \rightarrow R$ and $\omega : R \rightarrow Y$ equivariant. Its compatibility is given by Equation 60.

$$\rho(\alpha(a, x)) = \beta(a, \rho(x)), \quad \omega(\beta(a, r)) = \gamma(a, \omega(r)) \quad (a \in M). \quad (60)$$

Under Equation 60, the manifest record map $m = \omega\rho$ is equivariant. As a consequence, for a shared mechanism word $w = a_k \cdots a_1$ and any x , the record of its source evolution coincides with the evolution of its initial record under the corresponding observed action. The statement concerns the admissible words of the chosen mechanism protocol. Source systems can have additional untested mechanisms and paths, and a quotient may conceal which states in a fiber can actually realize a concatenation of observed transitions.

Proposition 10.3 (Equivariant interfaces preserve fixed-label responses). *Assume the unital actions and equivariant interfaces of Equation 60. If $m(x) = m(x')$, then $m(\alpha(a, x)) = m(\alpha(a, x'))$ for every $a \in M$, and the complete manifest responses along every finite mechanism word agree. Conversely, equality of the complete manifest responses, including the empty word, implies $m(x) = m(x')$. Therefore the manifest kernel is an action congruence under these assumptions.*

Proof. Equivariance of the composite gives $m(\alpha(a, x)) = \gamma(a, m(x))$. Equality of initial records therefore implies equality of the records after any a , and repeated application establishes the finite-word statement. The empty word acts by the identity, so equality of all responses entails equality of initial records. Finally, $m(x) = m(x')$ implies $m(\alpha(a, x)) = m(\alpha(a, x'))$ for each a , which is the congruence property. $\square$

The converse inference to underlying dynamical equivalence would be considerably stronger. Consider $X = \{0, 1\}^2$ and the observation $m(a, b) = a$. Two single-generator systems can act by $F(a, b) = (1 - a, b)$ and $G(a, b) = (1 - a, 0)$. Their observed responses agree at every finite iterate, while F is bijective and G collapses the hidden coordinate. This finite construction, already motivated in Section 5, makes the failure of source-mechanism recovery explicit. Proposition 10.3 does not turn equality of recorded histories into equivalence of the complete physical transformation systems.

A further distinction concerns dynamic reconstruction itself. If the quotient record m fails to respect the action, an autonomous observed evolution γ satisfying Equation 60 may not exist. In that case the observed path can depend on inaccessible source information, and adding an observational memory state or a richer relational representation may be necessary to achieve closure. Which augmentation is appropriate depends on the physically justified class of hidden processes, noise sources, and admissible interventions.

10.5 Stochastic manifestations and exact single-observation recovery

Real observation protocols often yield probability distributions rather than deterministic outcomes. For finite source and record domains, let $K(y \mid x)$ be a stochastic matrix, with $\sum_y K(y \mid x) = 1$. It specifies the law of one manifest record Y conditional on a given source configuration x . The law-level equivalence criterion $K(\cdot \mid x) = K(\cdot \mid x')$ is different from exact reconstruction of a source property from *one realized draw*. The latter requires a deterministic decision map $d : Y \rightarrow Q$ that succeeds with probability one for every possible source configuration. The following criterion makes the difference explicit.

Proposition 10.4 (Perfect single-draw property recovery). *Let X, Y, Q be finite sets, K a stochastic channel from X to Y , and $q : X \rightarrow Q$. A decoder $d : Y \rightarrow Q$ with $K(\{y : d(y) = q(x)\} \mid x) = 1$ for every $x \in X$ exists if and only if $\text{supp } K(\cdot \mid x)$ and $\text{supp } K(\cdot \mid x')$ are disjoint whenever $q(x) \neq q(x')$.*

Proof. If a perfect decoder exists, every y in the support of $K(\cdot | x)$ must satisfy $d(y) = q(x)$, since all outcome probabilities are nonnegative and the correct-decoding probability is one. An outcome shared by the supports of two source configurations with different q -values would require two different values of $d(y)$, a contradiction. Conversely, suppose the stated support-disjointness condition holds. For each outcome y lying in the support of some source state, assign $d(y)$ the common value of $q(x)$ among all source states with $K(y | x) > 0$. Such a common value exists by the assumption. Define d arbitrarily on outcomes never produced by the channel. It succeeds with probability one for each x . $\square$

To compare this with recoverability from idealized probability laws, take $X = Y = \{0, 1\}$ and the binary symmetric channel in Equation 61, with $0 < \varepsilon < 1/2$.

$$K = \begin{pmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 1 - \varepsilon \end{pmatrix}, \quad q(x) = x. \quad (61)$$

The two rows in Equation 61 are different, so the complete conditional observation laws distinguish the source states. Each row, however, has full support, so Proposition 10.4 rules out perfect recovery from any one draw. With repeated independent observations, inference error can decline, but the repetition assumption and estimation procedure are substantive additional conditions. For correlated neural recordings, one must instead model the appropriate joint channel. Classical comparisons of statistical experiments treat precisely the information carried by different observation procedures [4]; the present proposition is a finite elementary criterion focused on recovering a declared property in one observation.

A record of verbal report adds another interpretation layer. A probabilistic map from modeled experience to report categories, if introduced, must be identified or constrained by independent hypotheses; report agreement can arise from shared language, task instruction, and response policies even when the sought phenomenal property remains unanchored. The support criterion therefore states a property of an assumed measurement channel and its declared source labels. It supplies no independent warrant for identifying those labels with phenomenal character.

10.6 A smooth observation quotient with different topology

A smooth example shows that a manifestational dynamics can be exactly closed while concealing topological information about its source. Let the relational source carrier be the circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ with angular coordinate θ , and let the record carrier be the closed interval $[-1, 1]$. Consider the source vector field $\dot{\theta} = -\sin \theta$, its observation $q(\theta) = \cos \theta$, and the candidate induced observed vector field $\dot{z} = 1 - z^2$. These objects satisfy the compatibility identity in Equation 62.

$$\frac{d}{dt}q(\theta(t)) = -\sin \theta \dot{\theta} = \sin^2 \theta = 1 - q(\theta)^2. \quad (62)$$

The parameter t in Equation 62 is the stipulated flow parameter. It is neither a universal temporal coordinate nor a model-independent assertion about the constitution of consciousness.

Proposition 10.5 (Closed manifestational flow with topologically distinct source). *The map $q : S^1 \rightarrow [-1, 1]$, $q(\theta) = \cos \theta$, semiconjugates the flow of $\dot{\theta} = -\sin \theta$ to the flow of $\dot{z} = 1 - z^2$ on $[-1, 1]$. The two systems have the same collection of $[-1, 1]$ -valued observed parameterized trajectories when all initial conditions are admitted. Nevertheless, the state carriers are not homotopy equivalent, and q is noninjective on every generic fiber $q^{-1}(z)$ with $-1 < z < 1$.*

Proof. Both vector fields are smooth on their stipulated carriers, and their flows exist globally: smoothness on the compact circle gives completeness, while $[-1, 1]$ is invariant for $\dot{z} = 1 - z^2$ and its vector field is Lipschitz there. The chain rule in Equation 62 and uniqueness for the observed ODE establish the semiconjugacy for all flow parameters. Since q is surjective, every observed initial state has a source preimage and hence every observed trajectory occurs as the record of a source trajectory; the reverse inclusion follows from semiconjugacy. For $-1 < z < 1$, the angles $\theta = \pm \arccos z$ are distinct points of S^1 and give the same record. Finally, $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$, while $H_1([-1, 1]; \mathbb{Z}) = 0$. Homotopy equivalence preserves homology, so the two carriers cannot be homotopy equivalent [14]. $\square$

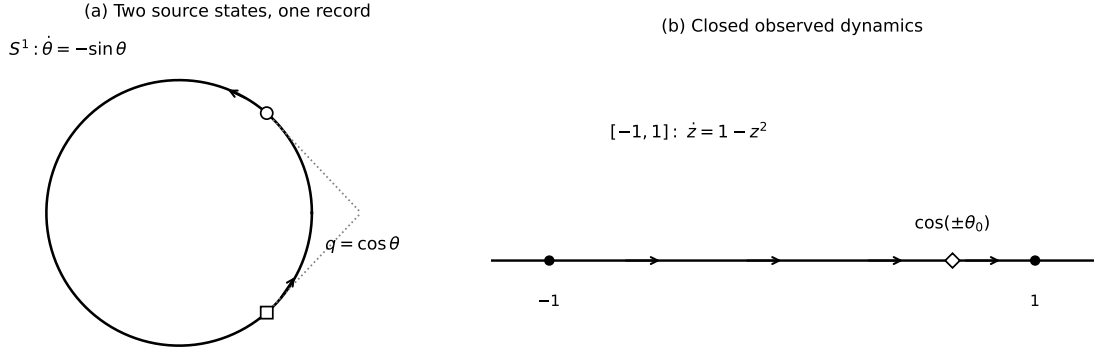


Figure 6: Exact observation quotient of a smooth source flow. Left: two generic points on the source circle have the same cosine record. Right: their common manifestational coordinate evolves autonomously on the interval under $\dot{z} = 1 - z^2$. The figure concerns synthetic mathematical dynamics.

Figure 6 depicts the noninjective fibers and the resulting interval dynamics. The source circle has a nontrivial loop class, while the recorded carrier is contractible. Equality of complete observed trajectories therefore does not reconstruct the source's first homology, even though the observation possesses an autonomous dynamical evolution. The example is stronger in a specific respect than a comparison of isolated measured values: it preserves every observed flow trajectory under the declared dynamics. Its explanatory reach remains deliberately limited. The circle can serve as a state or relational carrier under a stipulated model; it is not asserted to represent physical spacetime, a complete neural implementation, or phenomenal topology.

10.7 A register-sensitive comparison protocol

The constructions of this section motivate an explicit protocol for cross-register comparison. First, specify the candidate realization domain and the experimental or theoretical context that gives its variables physical meaning. Second, declare the relational carrier and its mathematical type, together with any state-to-structure reconstruction map; a graph, manifold, group action, and cellular two-complex have different structural preservation obligations. Third, specify the manifest record map or stochastic kernel, its calibration and admissible interventions. Fourth, name the actual criterion of equivalence within each register, and state any commuting or factorization condition that transports it. Finally, evaluate which distinctions are inaccessible to the stipulated protocol and which can be recovered by justified additional observations. Table 9 organizes these obligations by comparison type.

The three registers are therefore most usefully understood as linked but separately typed explanatory commitments. Their interfaces can be inspected, refined, and challenged. In some model classes, carefully chosen observations recover the specific relational properties relevant to an explanatory task. In others, a persistent observational equivalence class contains materially different mechanisms, topologies, or constitutive descriptions. The results in Propositions 10.1–10.5 identify sufficient criteria, necessary conditions, and explicit counterexamples for such comparisons. They establish mathematical boundaries on inference, while leaving the physical constitution of subjects and the phenomenology of experience to additional substantive theories and evidence. Section 11 will develop stronger algebraic and categorical preservation mechanisms for translations among the resulting typed model structures.

11 Algebraic and Categorical Preservation

Equivalence of modeled structures can be compared at several levels: configurations, transformations, morphisms between systems, and coherent families of such morphisms. The objective of this section is to specify what algebraic and categorical translations preserve at each level, and which distinctions

Table 9: Register-sensitive equivalence claims, required interfaces and interpretive limits. Each row concerns a declared criterion rather than a universal hierarchy among registers.

Comparison	Required mathematical data	Limitation retained
Context transport	Maps f, g, h ; commuting squares; type-preservation declarations	Record preservation may fail to reflect source distinctions
Relational property from record	Attained relational domain, u, ω , and kernel inclusion	Other invariants may remain inaccessible
Joint experimental access	Protocol-indexed response maps and joint-response model	Independence and invariance under intervention require justification
Dynamical record equivalence	Common mechanism labels, unital actions, equivariant record maps	Mechanism identity and hidden admissible paths may remain unidentified
Stochastic manifestation	Conditional laws, chosen property, noise and sampling protocol	Law-level identification is weaker than certain single-draw decoding
Topology from observed trajectories	Source and observed carriers, observation map, invariant of interest	Closed observed dynamics may hide source homology
Phenomenal comparison	Proposed bridge between physical/relational model and experience, independent anchors	Formal equivalence alone gives no phenomenal identity claim

their targets may conceal. We first construct quotients of monoid actions under observation-stable congruences. We then distinguish faithful, full, and essentially surjective functors; derive a hierarchy of object-isomorphism criteria under successive translations; analyze naturality and colimit information loss; and compare cellular chain representations with the topology of their sources. Finally, we give a precise treatment of translations defined only on selected portions of an indexed model. These are standard categories and elementary constructions whose relevance to consciousness depends on additional physical and epistemic interpretation [23, 2].

11.1 Congruences and the descent of transformation actions

Let M be a monoid with identity e , and let (X, α) be a unital left M -set: $\alpha(e, x) = x$ and $\alpha(mn, x) = \alpha(m, \alpha(n, x))$. A comparison of configurations through a map $q : X \rightarrow Y$ generally identifies points of X . Such identification supports an autonomous M -action on the record carrier only when the identified configurations remain indistinguishable after every stipulated mechanism. We state this as a condition on $\ker q$, with the orientation of composition fixed by the left-action convention.

The action-stability requirement is given by Equation 63.

$$x \sim_q x' \implies \alpha(m, x) \sim_q \alpha(m, x') \quad (m \in M), \quad x \sim_q x' \iff q(x) = q(x'). \quad (63)$$

Proposition 11.1 (Exact descent of a unital action). *Suppose $q : X \rightarrow Y$ is a surjection and (X, α) a unital left M -set. There exists a unital action $\bar{\alpha} : M \times Y \rightarrow Y$ such that $q(\alpha(m, x)) = \bar{\alpha}(m, q(x))$ for all m, x if and only if $\ker q$ satisfies Equation 63. The resulting action is unique.*

Proof. If $\bar{\alpha}$ exists, equality $q(x) = q(x')$ implies equality of $\bar{\alpha}(m, q(x))$ and $\bar{\alpha}(m, q(x'))$, which gives action stability. Conversely define $\bar{\alpha}(m, q(x)) := q(\alpha(m, x))$. Condition 63 makes this well-defined. The unit and composition laws follow from those of α and surjectivity of q ; for example, $\bar{\alpha}(mn, q(x)) = q(\alpha(mn, \alpha(n, x))) = \bar{\alpha}(m, \bar{\alpha}(n, q(x)))$. Every element of Y has the form $q(x)$, so the compatibility identity also establishes uniqueness. $\square$

A small failure case helps identify the necessity of Equation 63. Let $X = \{a, b, c\}$, with observation $q(a) = q(b) = 0, q(c) = 1$, and let a generating transformation send a to a and b to c . The generator can be extended to a unital action of the free monoid $\mathbb{N}$ by iteration, but the quotient has no deterministic induced action for that generator: the record 0 has two possible successors. Replacing the projected

behavior with a nondeterministic transition relation is a legitimate alternative model with a different type and different preservation obligations. Section 8 examines how this change of type can introduce abstract histories without compatible concrete lifts.

A useful categorical domain for these quotients has objects (X, α, E) , where E is an M -stable equivalence relation. Its morphisms $f : (X, E) \rightarrow (X', E')$ are equivariant functions such that xEx' implies $f(x)E'f(x')$. Equation 64 defines an induced morphism of the quotient actions.

$$Q(f) : X/E \longrightarrow X'/E', \quad Q(f)([x]_E) = [f(x)]_{E'}. \quad (64)$$

The compatibility assumptions make Equation 64 well-defined, equivariant, and compatible with identities and composition. Hence Q is a functor from the category of equipped actions to the category of unital M -sets. Choosing arbitrary observational partitions without action stability or choosing maps that fail to respect the partitions would remove the data required for this functor. An observational representation is therefore a functorial quotient only relative to a suitable category of admissible models and arrows.

11.2 Faithfulness, fullness, and categorical equivalence

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between locally small categories. For objects A, B , its hom-set function is $F_{A,B} : \mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB)$. Faithfulness means that every $F_{A,B}$ is injective; fullness means that every $F_{A,B}$ is surjective. Essential surjectivity means that each target object is isomorphic to an object in the image. The standard full-faithful-essentially-surjective characterization of an equivalence of categories must be distinguished from literal isomorphism of categories [23]. The definitions are collected in Equation 65.

$$\begin{aligned} F \text{ faithful} &\iff F_{A,B} \text{ injective for all } A, B, \\ F \text{ full} &\iff F_{A,B} \text{ surjective for all } A, B, \\ F \text{ essentially surjective} &\iff (\forall D \in \mathcal{D})(\exists A \in \mathcal{C}) D \cong FA. \end{aligned} \quad (65)$$

Faithfulness concerns arrows between *fixed source objects*; it need not distinguish nonisomorphic objects after translation. Fullness addresses the availability of target arrows between objects already in the image. Essential surjectivity concerns the reach of the target object domain. None of these conditions by itself determines whether the functor preserves a stipulated metric, topology, experimental protocol, physical implementation, or phenomenal classification that has not been included in the categorical data.

Proposition 11.2 (Reflection of isomorphisms by a fully faithful functor). *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is full and faithful, then every morphism $f : A \rightarrow B$ whose image Ff is invertible is invertible in $\mathcal{C}$. Moreover, $FA \cong FB$ in $\mathcal{D}$ implies $A \cong B$ in $\mathcal{C}$.*

Proof. Let $g : FB \rightarrow FA$ be the inverse of Ff . Fullness supplies $\tilde{g} : B \rightarrow A$ with $F\tilde{g} = g$. Functoriality gives $F(\tilde{g}f) = gFf = 1_{FA} = F(1_A)$ and $F(f\tilde{g}) = Ff g = 1_{FB} = F(1_B)$. Faithfulness implies $\tilde{g}f = 1_A$ and $f\tilde{g} = 1_B$. For the second statement, lift a chosen isomorphism $u : FA \rightarrow FB$ and its inverse by fullness, and apply the same argument; its lift is an isomorphism. $\square$

A faithful functor alone need not reflect invertibility. Regard the additive monoid $(\mathbb{N}, +)$ and the additive group $(\mathbb{Z}, +)$ as one-object categories. Inclusion of $\mathbb{N}$ into $\mathbb{Z}$ is a faithful functor, yet the arrow 1 becomes invertible in the target while having no inverse in the source. The target has added a morphism that acts as the inverse. This counterexample identifies why faithful observational encoding, by itself, may fail to preserve distinctions concerning *which transformations can be undone* within the source category. The example is purely algebraic; treating inverse morphisms as physically realizable inverse processes requires an independent assumption.

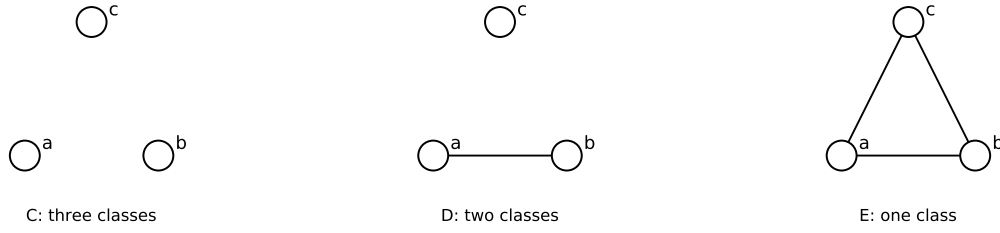


Figure 7: Three categories with successive faithful functors and strictly coarsening object-isomorphism partitions. Nodes a, b, c represent mathematical objects; connecting lines denote pairs admitting inverse isomorphisms. The diagram is an exact finite categorical construction, without an assigned physical or temporal interpretation.

11.3 Successive translations and coarsening of object equivalence

Every functor preserves isomorphisms in the forward direction. Accordingly, if two objects of C are isomorphic, their images under F are isomorphic in D . An even weaker comparison can occur when the functor identifies previously nonisomorphic objects. For a chosen set S of objects of C , the equivalence relations induced by source isomorphism, target isomorphism after F , and target isomorphism after a second functor G obey Equation 66.

$$\begin{aligned}
 A E_C B &\iff A \cong_C B, \\
 A E_F B &\iff F A \cong_D F B, \\
 A E_{GF} B &\iff G F A \cong_E G F B, \\
 E_C &\subseteq E_F \subseteq E_{GF} \quad \text{on } S.
 \end{aligned} \tag{66}$$

Proposition 11.3 (Strict coarsening under successive faithful functors). *Both inclusions in Equation 66 can be strict even when F and G are faithful functors between finite groupoids or categories.*

Proof. Let C be the discrete category on objects a, b, c . Let D be the groupoid with exactly one morphism between each ordered pair of objects in the component $\{a, b\}$, and only the identity morphism on c . Let E be the indiscrete connected groupoid on a, b, c , with exactly one morphism between every ordered pair. Inclusion-on-objects functors $F : C \rightarrow D$ and $G : D \rightarrow E$ send the unique arrows in each existing hom-set to the corresponding unique arrows. Each hom-set function is injective, so both functors are faithful. In C the three isomorphism classes are singletons. In D the two classes are $\{a, b\}$ and $\{c\}$; in E there is a single class $\{a, b, c\}$. Thus both inclusions are strict. $\square$

Figure 7 exhibits the three finite groupoids. An edge joins objects only when they are isomorphic; the identity arrows and inverse arrows are suppressed from the drawing. In the relation order of Section 4, the successive functors make object-isomorphism criteria less discriminating. The functors remain faithful on their existing hom-sets, yet neither is full: each target adds arrows between selected object pairs that the source lacks. This illustrates the difference between recovering a morphism *if it came from the source* and knowing whether a morphism present only in the target was available in the source.

11.4 Naturality, diagram categories, and constitutive coherence

Section 7 used natural isomorphisms to compare indexed generative descriptions. The category $[J, C]$ of functors $J \rightarrow C$ and natural transformations makes this comparison itself categorical. For $F : C \rightarrow D$, postcomposition induces a functor $F^J : [J, C] \rightarrow [J, D]$. Its action on a natural transformation $\eta : A \Rightarrow B$ is componentwise, and its naturality follows from that of η . Equation 67 states this construction.

$$F^J(A) = F \circ A, \quad (F^J \eta)_j = F(\eta_j), \quad F(\eta_k \circ A_u) = F(B_u \circ \eta_j) \quad (u : j \rightarrow k). \tag{67}$$

Proposition 11.4 (Full faithfulness extends to indexed diagrams). *Let J be a small category and let $F : \mathbf{C} \rightarrow \mathbf{D}$ be full and faithful. Then $F^J : [J, \mathbf{C}] \rightarrow [J, \mathbf{D}]$ is full and faithful, and it reflects natural isomorphisms. If F is an equivalence of categories, then F^J is an equivalence of diagram categories.*

Proof. Suppose $\eta, \theta : A \Rightarrow B$ satisfy $F^J \eta = F^J \theta$. Faithfulness of F gives $\eta_j = \theta_j$ at every index, so the transformations agree. For fullness, take a natural transformation $\sigma : F \circ A \Rightarrow F \circ B$. Fullness of F provides a component $\eta_j : A_j \rightarrow B_j$ with $F\eta_j = \sigma_j$, and faithfulness makes this component unique. The naturality equations for σ are images under F of the corresponding equations for η . By faithfulness they hold in $\mathbf{C}$, so the components assemble to a natural transformation lifting σ . Reflection of natural isomorphisms follows from Proposition 11.2, applied to the fully faithful functor F^J . If F is an equivalence with quasi-inverse R , whiskering its unit and counit natural isomorphisms with J supplies quasi-inverse functors F^J, R^J and the required natural isomorphisms. $\square$

The sufficient conditions of Proposition 11.4 provide a precise standard for *coherent translation* of relationally indexed structures. A natural transformation between translated diagrams may then be lifted, and translated diagram equivalence reflects source diagram equivalence. For a faithful functor lacking fullness, natural components or their inverses may occur only in the target and thus need not lift. For a transformation of index domains $u : J' \rightarrow J$, the restriction $A \mapsto A \circ u$ likewise preserves existing natural isomorphisms but can discard comparisons not seen in J' . For example, restriction to a one-object subcategory of the two-object arrow category remembers the selected object while forgetting the map joining the two positions. Such restriction expresses a declared scope of comparison, not an assertion that the missing constitutive map is unimportant.

11.5 Colimits and cellular chains as information-reducing translations

A colimit combines an indexed diagram through a universal construction. Its universal property provides a principled way to retain the result of assembly, but it need not retain enough information to reconstruct the original diagram. This limitation can be exhibited without imposing a chronology on the indexing arrows. Let J be the free category on two parallel arrows $u, v : 0 \rightrightarrows 1$. For two diagrams $A, B : J \rightarrow \mathbf{FinSet}$, use two-element sets at both indices. Write σ for the swap of 0 and 1, and c_i for the constant map with value i .

Proposition 11.5 (Identical colimits with nonisomorphic diagrams). *Let $A_u = \text{id}$, $A_v = \sigma$, $B_u = c_0$, and $B_v = c_1$ on the two-element set. The colimits $\text{colim}_J A$ and $\text{colim}_J B$ in $\mathbf{FinSet}$ are isomorphic one-element sets, while A and B are not naturally isomorphic diagrams.*

Proof. The colimit of two parallel arrows in $\mathbf{FinSet}$ is their coequalizer. For A , the relations $\text{id}(0) \sim \sigma(0)$ and $\text{id}(1) \sim \sigma(1)$ identify the two target elements, so the quotient has one class. For B , $c_0(x) \sim c_1(x)$ likewise identifies the two target elements, giving the same one-class quotient. Suppose that a natural isomorphism $\eta : A \Rightarrow B$ existed. Both η_0 and η_1 would be bijections, and naturality on u would give $\eta_1 \circ \text{id} = c_0 \circ \eta_0$. The left side is a bijection while the right side is constant, a contradiction. $\square$

The coequalizer example displays a loss of *constitutive mapping information*: the colimit remembers the assembled object, while the diagrams differ even in the ranks of corresponding maps. It complements the intermediate-CW-complex construction of Section 7, using a categorical universal construction rather than a selected cellular attachment order. It also shows why agreement on a reconstructed terminal relational carrier does not automatically yield a coherent comparison of the maps used in a description of its constitution.

A second reduction applies a functor to cellular structures. For CW-complexes with cellular maps, the cellular chain construction gives a functor to chain complexes of abelian groups, and cellular maps induce chain maps. This functor preserves the composition of cellular maps and provides computable homology; it discards attaching-map distinctions that can affect homotopy groups. Consider two CW-complexes with one zero-cell, two one-cells representing a, b , and one two-cell. Attach the two-cell

of K_{com} along the commutator $aba^{-1}b^{-1}$ and that of K_{triv} by the constant loop. Both cellular chain complexes have $C_2 \cong \mathbb{Z}$, $C_1 \cong \mathbb{Z}^2$, $C_0 \cong \mathbb{Z}$, with zero boundary maps. Their fundamental groups, however, are displayed in Equation 68.

$$\begin{aligned} C_*(K_{\text{com}}; \mathbb{Z}) &\cong C_*(K_{\text{triv}}; \mathbb{Z}), \\ \pi_1(K_{\text{com}}) &\cong \langle a, b \mid [a, b] \rangle \cong \mathbb{Z}^2, \\ \pi_1(K_{\text{triv}}) &\cong \langle a, b \mid \rangle \cong F_2. \end{aligned} \tag{68}$$

The groups in Equation 68 are different: the free group on two generators is nonabelian. Since homotopy equivalences induce isomorphisms on fundamental groups, the two complexes cannot be homotopy equivalent. Their identical cellular chains give a specific counterexample to reflection of homotopy type by the cellular chain functor, despite exact preservation of the corresponding cellular homology [14]. Under the paper's relational convention, zero-cells may be interpreted as events and one-cells as couplings; the meaning of two-cells remains model-dependent. This convention by itself does not turn either complex into a biological network or a subject of experience.

11.6 Partial translation domains and a typed preservation ledger

Mathematical translations in an applied investigation are frequently defined on selected objects, morphisms, or protocols. If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a full subcategory of admissible models and $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}$ is fully faithful, Proposition 11.2 applies on $\mathcal{C}_0$. The domain restriction is part of the conclusion: a result proved for models in $\mathcal{C}_0$ does not compare objects excluded from that domain. Similarly, a reindexing $u : J' \rightarrow J$ retains only the maps accessible to J' . When a proposed translation acts on selected arrows without respecting their identities and compositions, it is a partial mapping of data rather than a functor. It can still serve a modeling purpose, provided its preservation claims are restricted accordingly.

One can compare the four consciousness orientations through a typed diagram of transformations, trajectories, rule-bearing structures, or path repertoires. A claimed equivalence must identify the relevant category and the data whose preservation is required. For example, a faithful recording of specified response arrows need not reflect object-isomorphism classes; a fully faithful translation of indexed diagrams can reflect their natural isomorphisms; and a cellular chain representation can calculate homology while hiding fundamental-group relations. Table 10 records these distinctions and their additional requirements.

The comparison machinery identifies conditions under which representation changes preserve selected algebraic and categorical equivalences. It also explains why preservation claims can fail when arrow structure, coherences, or source-domain distinctions are omitted. These are properties of formal model categories. Whether a representation preserves features pertinent to conscious experience requires a theory connecting model morphisms and invariants to physical organization, experimental access, and candidate phenomenal properties. Section 10 supplies the relevant register distinctions; Section 12 will develop the geometric and topological obligations of translations for which smooth, cellular, and path-space structure is explicitly part of the criterion.

12 Geometric and Topological Preservation

Geometric and topological equivalence criteria apply to different mathematical aspects of a represented system. A homeomorphism preserves the topology of a carrier; a diffeomorphism preserves its smooth structure; an isometry additionally preserves a stipulated Riemannian metric. Maps between CW-complexes may preserve cellular homology while obscuring attaching maps, and transformations between path spaces can preserve continuous trajectories without preserving the conditions under which those trajectories are admissible. This section develops explicit preservation obligations for these comparisons. The argument proceeds from smooth transformations and transported path spaces to metric constraints, coefficient-dependent homology, indexed cellular attachments, and families whose

Table 10: Algebraic and categorical preservation claims. Each row stipulates the compared type and identifies the information whose recovery is or is not supplied by the stated criterion.

Translation or comparison	Preservation condition	Remaining obligation or information loss
Action quotient (Variation)	Stable congruence; surjective equivariant projection	Mechanism or source-state distinctions within fibers may be lost
Faithful functor (morphisms)	Injective map on every source hom-set	Object isomorphism and invertibility need not be reflected
Fully faithful functor	Bijjective maps on all source hom-sets	Objects outside the essential image remain unmatched
Diagram postcomposition (Generativity)	Fully faithful F gives fully faithful F^J	Indexing type and physical interpretation must also correspond
Colimit of constitutive diagram	Universal cocone on specified index category	Distinct maps and diagrams may have identical colimits
Cellular chain representation	Functorial chain maps and homology	Fundamental-group and attaching-word information may be lost
Observational record (Trajectory, Repertoire)	Equivariant measurement map or compatible stochastic kernel	Continuous history, branching, and probability laws need separate tests
Partial translation	Declared subcategory or subindex domain	Conclusions are local to that domain

fibers change topological type. Throughout, a modeled state manifold, a geometric realization of an event-coupling two-complex, and a physical spacetime remain distinct representational types unless a particular application supplies an identification [16, 17, 14].

12.1 Smooth structure and transported path spaces

Let M, N be finite-dimensional smooth manifolds and let $I = [0, T]$ for a fixed $T > 0$. Write $C(I, M)$ for the continuous path space with the compact-open topology. Given a continuous map $h : M \rightarrow N$, denote its pointwise action on paths by $h^I(\gamma) = h \circ \gamma$. This map describes a change of representation for each selected path. It does not specify which paths may arise through a physical or formal rule. For smooth dynamics, an additional relation between vector fields is needed: if X on M and Y on N satisfy $Dh_x X(x) = Y(h(x))$, the corresponding local flows intertwine wherever both are defined, as discussed in Section 5.

Proposition 12.1 (Path-space equivalence under a homeomorphism). *If $h : M \rightarrow N$ is a homeomorphism, then $h^I : C(I, M) \rightarrow C(I, N)$ is a homeomorphism. It induces a homeomorphism between based loop spaces at $x \in M$ and $h(x) \in N$, preserves path components of these spaces, and transports endpoint conditions. If h is a diffeomorphism, its restriction also maps piecewise C^1 paths bijectively to piecewise C^1 paths.*

Proof. For a compact domain I , postcomposition by a continuous map is continuous in the compact-open topology. Since $(h^{-1})^I$ is continuous, and the compositions $(h^{-1})^I \circ h^I$ and $h^I \circ (h^{-1})^I$ are identity maps, h^I is a homeomorphism. Based loops and prescribed-endpoint paths are defined by endpoint equalities, which are preserved and reflected by h . Homeomorphisms preserve path components. For the final assertion, both h and h^{-1} are smooth, so their compositions with piecewise C^1 paths remain piecewise C^1 . $\square$

Proposition 12.1 concerns the *unconstrained* space of paths. A selected repertoire $\mathcal{A} \subseteq C(I, M)$ is transported exactly only when the comparison also declares the target repertoire to be $h^I(\mathcal{A})$, or demonstrates this identity from the relevant constitutive rules. In particular, a homeomorphism of carriers need not carry an energy-limited or control-limited repertoire to the repertoire obtained by independently applying the same numerical constraint on the target. Thus carrier equivalence, trajectory equivalence, and

repertoire equivalence have different mathematical obligations, even when they are compared through the same map.

12.2 Metric preservation and admissible-energy repertoires

Equip M and N with Riemannian metrics g and g' . For a piecewise C^1 path γ , define its kinetic energy by Equation 69. This definition uses an abstract path parameter on I ; treating that parameter as physical time is an additional modeling decision.

$$\mathcal{E}_g(\gamma) = \frac{1}{2} \int_0^T g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s)) ds. \quad (69)$$

The metric relation controlling this quantity is the pullback h^*g' , defined through $h^*g'(v, w) = g'(Dh(v), Dh(w))$. An isometry satisfies $h^*g' = g$ and hence preserves both lengths and the energy of every parameterized piecewise C^1 path. More general diffeomorphisms distort these quantities. The distinction is quantitative when the distortion is uniformly bounded.

Proposition 12.2 (Energy distortion and repertoire inclusions). *Let $h : M \rightarrow N$ be a diffeomorphism and let $0 < c \leq C < \infty$ satisfy $c^2g \leq h^*g' \leq C^2g$ as quadratic forms on TM . For any piecewise C^1 path, its energy obeys the bounds in Equation 70. If $\mathcal{A}_g(E)$ denotes the paths with fixed transported endpoint conditions and $\mathcal{E}_g \leq E$, the inclusions of Equation 71 hold.*

$$c^2\mathcal{E}_g(\gamma) \leq \mathcal{E}_{g'}(h \circ \gamma) \leq C^2\mathcal{E}_g(\gamma). \quad (70)$$

$$h^I \mathcal{A}_g(E/C^2) \subseteq \mathcal{A}_{g'}(E) \subseteq h^I \mathcal{A}_g(E/c^2). \quad (71)$$

Proof. The quadratic-form bounds apply to $\dot{\gamma}(s)$ at every point of differentiability. Using the chain rule $\frac{d}{ds}(h \circ \gamma) = Dh(\dot{\gamma})$ and integrating proves Equation 70. A source path of energy at most E/C^2 maps to a target path of energy at most E . Conversely, each target path β is $h \circ \gamma$ for $\gamma = h^{-1} \circ \beta$, and the lower energy bound implies $\mathcal{E}_g(\gamma) \leq E/c^2$ whenever $\mathcal{E}_{g'}(\beta) \leq E$. These give Equation 71. $\square$

The inclusions require matching endpoint conditions; they make no assertion about control laws or physically available interventions. Moreover, the constants c, C are data of a chosen comparison and are not determined by the mere existence of a diffeomorphism. A compact source manifold and a fixed smooth diffeomorphism between smooth Riemannian metrics ensure suitable positive global bounds, while on noncompact domains these bounds may fail.

For an explicit geometric example, let S_a^1 be a circle with angular coordinate $\theta \pmod{2\pi}$ and metric $a^2 d\theta^2$. Consider absolutely continuous based loops of duration T with square-integrable derivative. The minimum energy in winding class $n \in \mathbb{Z}$ is attained by the constant-angular-speed lift. Equation 72 gives its value, followed by the number of components in the energy-sublevel repertoire.

$$\mathcal{E}_{a,\min}(n) = \frac{2\pi^2 a^2 n^2}{T}, \quad \#\pi_0(\mathcal{L}_a(E)) = 2 \left\lfloor \sqrt{\frac{ET}{2\pi^2 a^2}} \right\rfloor + 1. \quad (72)$$

The formula in Equation 72 follows from the Cauchy–Schwarz inequality for the lifted angle, $\int_0^T \dot{\theta}^2 ds \geq (2\pi n)^2/T$. Within each integer winding class, the lifted paths with fixed endpoints and bounded quadratic energy form a convex set; distinct winding classes cannot be joined by a continuous family of based loops. Hence each nonempty winding class contributes exactly one component. This is a metric-dependent change in repertoire topology on a carrier that remains topologically a circle. Figure 8 displays the contrast between two circle metrics, alongside a separate topology-changing family discussed below.

12.3 Topological maps and coefficient-dependent information

A homotopy equivalence between suitable path-connected spaces induces isomorphisms on fundamental groups and homology groups for each coefficient group. A continuous map can also induce an isomorphism on a selected homology theory without being a homotopy equivalence. The chosen coefficient ring matters, especially when torsion is present. For the circle map $p_k : S^1 \rightarrow S^1$ given by $p_k(z) = z^k$ with nonzero integer k , the induced map on first integral homology is multiplication by k . Equation 73 states the difference between rational and integral coefficients for $|k| > 1$.

$$(p_k)_* : H_1(S^1; \mathbb{Z}) \rightarrow H_1(S^1; \mathbb{Z}), \quad n \mapsto kn; \quad (p_k)_* \otimes \mathbb{Q} : \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}. \quad (73)$$

The rational map in Equation 73 is invertible, whereas the integral homology map is not surjective for $|k| > 1$. In addition, the map p_k is a nontrivial covering and is not a homeomorphism for such k . Equality of the topological types of its domain and codomain does not alter these properties of the specified comparison map. This example distinguishes a property of spaces from a preservation requirement on arrows between spaces [14].

A related difference arises for a two-dimensional CW-complex. Attach one two-cell to the circle by a map of degree $n \geq 2$, obtaining the Moore complex $K_n = S^1 \cup_{p_n} D^2$. It has one cell in each dimension zero, one and two; its cellular differential from degree two to degree one is multiplication by n . The resulting integral and field-coefficient homology is explicit.

Proposition 12.3 (Torsion concealed by real-coefficient homology). *For K_n with $n \geq 2$, one has the groups shown in Equation 74. If a prime p divides n , both $H_1(K_n; \mathbb{F}_p)$ and $H_2(K_n; \mathbb{F}_p)$ are one-dimensional over $\mathbb{F}_p$. Consequently, vanishing $H_1(K_n; \mathbb{R})$ does not entail vanishing integral first homology.*

$$H_1(K_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}, \quad H_2(K_n; \mathbb{Z}) = 0, \quad H_1(K_n; \mathbb{R}) = 0. \quad (74)$$

Proof. The cellular chain complex in degrees two through zero is $\mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$. Its first homology is the quotient $\mathbb{Z}/n\mathbb{Z}$ and its second homology is the kernel of multiplication by n , which vanishes. Tensoring the cellular chain complex with $\mathbb{R}$ makes multiplication by n invertible, giving zero first and second real homology. Over $\mathbb{F}_p$, the differential $\times n$ is zero precisely when p divides n , yielding one-dimensional homology in both degrees one and two. $\square$

Proposition 12.3 gives a concrete limitation of replacing discrete cellular invariants by real-valued linear features. A real cohomological or spectral description may be well suited to a chosen observable, while exact integral attaching data retains additional torsion information. The model-dependent faces in a relational two-complex can encode attaching constraints, but identifying those cells with physical interactions or phenomenal distinctions requires a separate semantic model. In particular, the coefficient-sensitive result is an algebraic-topological fact about the stipulated CW-complex.

12.4 Indexed cellular constructions and mapping cones

Generativity comparisons require retaining maps between differently indexed structural descriptions when such maps are part of the formal object. A useful test case consists of two diagrams of circles indexed by the arrow category $0 \rightarrow 1$. Let A assign the identity circle map to the indexing arrow, and let B assign the degree-two map p_2 . At each index, their objects are homeomorphic. Their connecting maps, however, represent different classes in the homotopy category. The distinction survives attempts to identify the two diagrams coherently.

Proposition 12.4 (Degree obstruction to indexed homotopy equivalence). *For integers k with $|k| > 1$, the two diagrams $S^1 \xrightarrow{\text{id}} S^1$ and $S^1 \xrightarrow{p_k} S^1$ admit no natural transformation whose two components are homotopy equivalences. Their mapping cones also have nonisomorphic first integral homology groups: respectively 0 and $\mathbb{Z}/|k|\mathbb{Z}$.*

Proof. Each homotopy equivalence $S^1 \rightarrow S^1$ has degree $+1$ or -1 . Naturality of components η_0, η_1 would imply $\eta_1 \circ \text{id} = p_k \circ \eta_0$, and taking degrees would give $\deg \eta_1 = k \deg \eta_0$. The two sides cannot have equal absolute value when $|k| > 1$. The mapping cone of a degree- m circle map has a CW decomposition with boundary homomorphism $C_2 \rightarrow C_1$ equal to multiplication by m . Therefore its first integral homology is $\mathbb{Z}/|m|\mathbb{Z}$ for $m \neq 0$. For $m = 1$ this is zero, while for $m = k$ it is nontrivial. $\square$

The obstruction of Proposition 12.4 is stronger than the observation that different intermediate complexes may share a final homology rank. Both diagrams begin and end with circles, and the two components are individually of the same homeomorphism type; nevertheless the specified *connecting structures* are not naturally equivalent by homotopy equivalences. Moreover, the mapping cone translates the connecting map into an attached two-dimensional complex with an observable integral torsion invariant. This is a precise bridge between diagrammatic Generativity and cellular topology, without equating a formal index arrow to physical causal succession.

12.5 Changing carriers and locally trivial families

Topology-changing families are often represented by a map $\pi : E \rightarrow B$ whose fibers $E_b = \pi^{-1}(b)$ supply distinct carrier spaces. The map may be a locally trivial bundle, a more general continuous family, or a singular quotient representation. Each choice carries different preservation theorems. For example, consider the subset $E \subseteq [0, 1] \times \mathbb{R}^2$ whose fiber at t is the circle of radius t for $t > 0$, and the single origin for $t = 0$. The family is expressed in Equation 75.

$$E = \{(t, u, v) : 0 \leq t \leq 1, u^2 + v^2 = t^2\}, \quad \pi(t, u, v) = t. \quad (75)$$

The total space in Equation 75 is a cone over S^1 , topologically a closed disk. Over $(0, 1]$ it is a trivial circle family, but its fiber over 0 is a point. A local trivialization at 0 would identify that point fiber with neighboring circle fibers, which is impossible. The following elementary implication identifies the role of the bundle assumption.

Proposition 12.5 (Fiber topology in locally trivial families). *If $\pi : E \rightarrow B$ is a locally trivial topological fiber bundle and b_0, b_1 lie in the same path component of B , then E_{b_0} and E_{b_1} are homeomorphic. Consequently, the family of Equation 75 cannot be locally trivial at $t = 0$.*

Proof. Choose a path from b_0 to b_1 . Its compact image admits a finite sequence of local trivialization neighborhoods ordered along the path with consecutive overlaps. Within each neighborhood all fibers are homeomorphic to the local model fiber; on overlaps the homeomorphism types agree. Composing these comparisons establishes a homeomorphism between the endpoint fibers. In the cone family, the fiber at zero is a singleton and every positive fiber is a circle, so no such local trivialization exists near zero. $\square$

The parameter t in Equation 75 indexes fibers. It need not denote a physical clock or an ontologically privileged transition from one actuality to another. A claim about physical topology change requires a dynamically interpreted structure on the total representation, including a rule governing admissibility of the fibers or their embeddings. Figure 8 presents the cone example as an explicit mathematical family rather than a simulation of neural geometry.

12.6 Typed preservation criteria for the four orientations

The mathematical invariants above are useful only with a stated domain of comparison. Table 11 summarizes the principal criteria and their limits. For Variation, a diffeomorphism preserving the chosen vector fields is sufficient for parameter-preserving flow conjugacy, and a metric-preserving diffeomorphism is stronger still. For Trajectory, a carrier homeomorphism transports entire path spaces, while maps with nontrivial fibers may conceal source homotopy. For Generativity, the connecting maps of an indexed

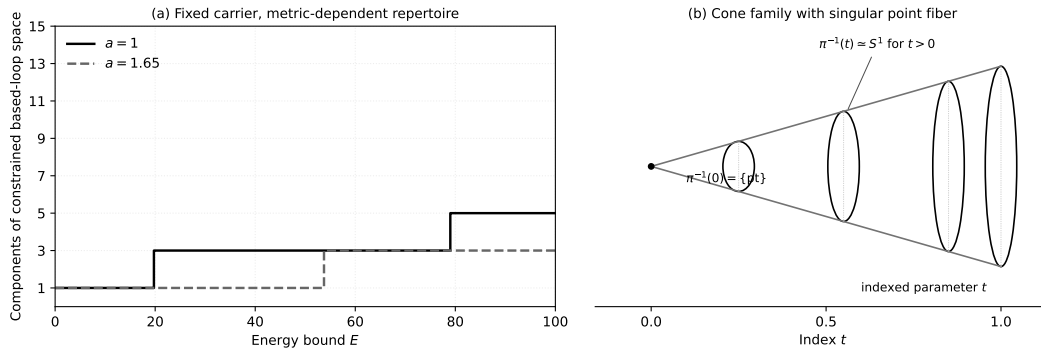


Figure 8: Two mathematically distinct preservation questions. Left: the component count of energy-constrained based-loop repertoires on circles with metrics $a^2 d\theta^2$ for two values of a , for $T = 1$. Right: sample fibers of the cone family in Equation 75; the point fiber at $t = 0$ obstructs local triviality even though positive fibers are circles. Both are formal constructions, not empirical models of consciousness.

Table 11: Geometric and topological preservation obligations, indexed by explanatory orientation. The relevant space or family and the criterion must be stipulated before equivalence is asserted.

Orientation	Stipulated preservation criterion	Distinction requiring additional data
Variation	Diffeomorphism intertwining vector fields; isometry when metric dynamics matter	Orbit images alone do not fix generators, parameterization or metric distortion
Trajectory	Homeomorphism on carriers and induced path-space map; induced π_1 map for based classes	Projection may merge winding classes, path lifts, or path-speed profiles
Generativity	Natural or homotopy-coherent comparison of indexed connecting maps	Objectwise equal homology can conceal mapping degree and attachment relations
Repertoire	Transport of admissibility rules, energy budgets and the topology of constrained path families	Equal carrier type need not fix allowed winding classes or control constraints
Cross-register comparison	Declared interface preserving a specified geometric or topological invariant	Observational closure does not determine source topology or phenomenal identity

family supply information that objectwise homeomorphism leaves out. For Repertoire, admissibility predicates and metrics require explicit transport; carrier equivalence alone leaves these properties open.

Sections 6, 7, and 8 analyze the four orientations at the level of their respective formal objects. The results here show how the preservation criteria themselves depend on a geometric or topological enrichment of those objects. Metric structure, degree, attaching maps, and local triviality are additional data; a translation preserving a weaker invariant may omit them while remaining mathematically legitimate for its declared purpose. Whether any of these distinctions is explanatorily relevant to consciousness remains a separate physical and epistemological question. The next section considers how such model distinctions can, under appropriate experimental assumptions, be made empirically identifiable.

13 Empirical Identifiability and Experimental Discrimination

A mathematical equivalence criterion acquires empirical significance only relative to a physical model, an experimental protocol, and a specified record-generating mechanism. This section determines what distinct empirical designs can establish about such criteria and which distinctions may remain unrecoverable. We develop a common experiment-indexed framework, examine passive observations and interventions, derive finite-sample discrimination limits, and analyze the identification of latent phenomenal proxies under imperfect measurement. A final comparative protocol identifies operational contrasts for

Variation, Trajectory, Generativity, and Repertoire. The focus is on how to compare precisely formulated models of putatively consciousness-related processes; neither equality of records nor an observed treatment effect alone supplies a theorem concerning subjective experience [24, 4].

13.1 Experiments, response laws, and equivalence classes

Let $\mathfrak{M}$ be a declared class of candidate models, and let $\mathcal{E}$ be a family of experimentally implementable protocols. Each protocol e specifies an admissible preparation, assignment or manipulation, measurement apparatus, observation schedule, inclusion criteria, and a common measurable record space $(\mathcal{Y}_e, \mathcal{B}_e)$. A model $m \in \mathfrak{M}$ induces a probability law P_m^e on that record space. We take the law to include all records needed for the stated comparison: trajectories rather than single-time marginals when temporal dependence matters; intervention assignments when randomization is used; and measurement metadata when the apparatus varies. Equality of such laws is an exact mathematical idealization distinct from equality of finite datasets.

For a set $E \subseteq \mathcal{E}$ of declared protocols, define *experiment-relative equivalence* by Equation 76. The parameter or model property of interest is a map $q : \mathfrak{M} \rightarrow \mathcal{Q}$, such as a class of effective action monoids, a homotopy invariant, an isomorphism type of constitutive diagram, or a law on admissible paths.

$$m \equiv_E m' \iff P_m^e = P_{m'}^e \text{ for every } e \in E. \quad (76)$$

The relation in Equation 76 is an equivalence relation because equality of measures is reflexive, symmetric and transitive, and intersections of equivalence relations are equivalence relations. A property q is *identifiable relative to E and $\mathfrak{M}$* when every observational equivalence class carries a single value of q . This recovers the kernel-factorization idea from Section 3, now with probability laws and feasible experimental protocols as its specified interface. Enlarging E refines the equivalence classes, provided both models can be represented under the additional protocols and the response laws refer to their respective controlled experimental conditions. An arbitrary hypothetical intervention outside a model's intended domain does not establish empirical identifiability within the original domain.

Proposition 13.1 (Experimental refinement and property identification). *Let $E \subseteq E' \subseteq \mathcal{E}$ and $q : \mathfrak{M} \rightarrow \mathcal{Q}$. Then $\equiv_{E'} \subseteq \equiv_E$. Moreover, q is identifiable relative to E if and only if there is a function $\tilde{q}$ on the image of the joint law map $P_E : m \mapsto (P_m^e)_{e \in E}$ such that $q = \tilde{q} \circ P_E$.*

Proof. Equality of all E' -indexed response laws entails equality of the subfamily indexed by E , proving inclusion. If $q = \tilde{q} \circ P_E$, equality of joint law images forces equality of q . Conversely, if q is constant on each fiber of P_E , define $\tilde{q}(P_E(m)) = q(m)$; the fiber assumption makes this definition independent of the representative. No continuity or computability of $\tilde{q}$ follows from this set-theoretic construction. $\square$

Proposition 13.1 identifies an exact reconstructibility condition. Practical reconstruction additionally depends on sensitivity to perturbations in the law, finite data, computational feasibility, and the calibration of the observation channel. For two fixed candidate models, the existence of an experiment with distinct predicted laws is a requirement for *ideal* experimental separation. A test with acceptable power at a stated sample size requires further conditions on the magnitude of that difference and the sampling design.

13.2 Controlled access to hidden dynamical variation

Controlled interventions can turn an inaccessible direction of dynamical variation into an observed response. This is an experimental design principle rather than an implication that arbitrary manipulation reveals a unique physical mechanism. Consider a two-dimensional linear system with one observable coordinate x and a controllable, initially held coordinate y . For a parameter $a \in \mathbb{R}$, take $x(0) = 0$, $y(0) = u$, and use the model in Equation 77. The parameter s is an evolution parameter in the stipulated differential equation; correspondence with physical elapsed time would require calibration in an application.

$$\frac{dx}{ds} = -x + ay, \quad \frac{dy}{ds} = 0, \quad x_a(s; u) = au(1 - e^{-s}). \quad (77)$$

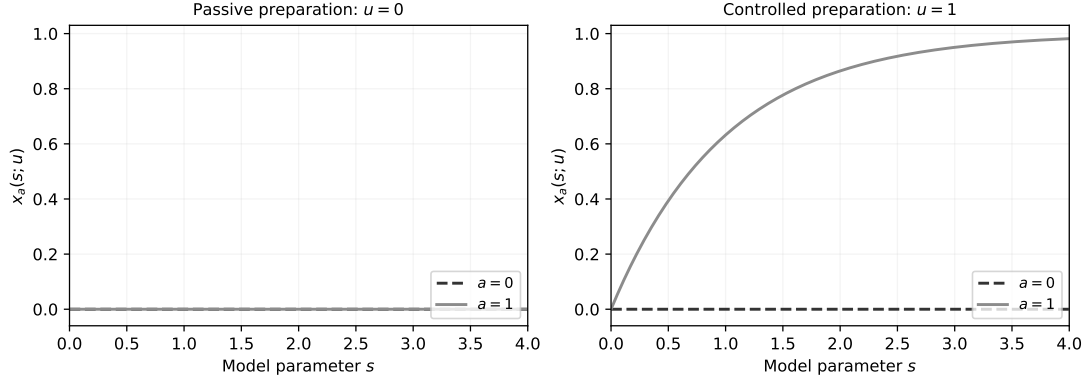


Figure 9: Synthetic intervention-sensitive identifiability in Equation 77. Left: two coupling parameters produce identical passive trajectories for $u = 0$. Right: controlled preparation with $u = 1$ distinguishes their predicted response curves. Curves are analytical model outputs, not recorded neural signals.

Under the passive preparation $u = 0$, every a gives the identical continuous record $x \equiv 0$. Preparing $u = 1$ makes the response parameter-dependent. Figure 9 presents the analytically computed trajectories for two candidate values; it is a mathematical construction with no neuroscientific data.

Proposition 13.2 (Identification of a hidden coupling by controlled preparation). *In the model of Equation 77, suppose u is known, $s_* > 0$ is calibrated, and the noise-free observation $x_a(s_*, u)$ is available. The coupling parameter a is identifiable from that observation when $u \neq 0$. Under $u = 0$, it is unidentifiable for every observation schedule based solely on x . If independent repetitions at fixed s_* are measured with additive Gaussian noise of known variance $\sigma^2 > 0$, the two-model Kullback–Leibler divergence per repetition is given by Equation 78.*

$$D_{\text{KL}}(P_a^{u, s_*} \| P_b^{u, s_*}) = \frac{(a - b)^2 u^2 (1 - e^{-s_*})^2}{2\sigma^2}. \quad (78)$$

Proof. Solving the scalar inhomogeneous differential equation gives the expression for x_a in Equation 77. For $u \neq 0$ and $s_* > 0$, inversion yields $a = x_a(s_*, u)/[u(1 - e^{-s_*})]$; for $u = 0$ every output equals zero. Under the specified Gaussian measurement model the distributions are normal with common variance and respective means $au(1 - e^{-s_*})$ and $bu(1 - e^{-s_*})$. The standard Gaussian density calculation yields Equation 78. Independence yields an n -fold divergence equal to n times this value; the assertion concerns repetitions from the specified preparation, not successive correlated samples from one physical run. $\square$

The result illustrates how additional experimental access refines an observational equivalence class. It also isolates a crucial assumption: the intervention changes the declared preparation $y(0)$ without altering the parameter a , sensor gain, or equations of motion. Violations of modularity or measurement invariance can invalidate this comparison. In a biological application, preparatory manipulation may affect several coupled processes, and a response divergence may support discrimination between two fully specified models without uniquely identifying an underlying neural or phenomenal mechanism. The experiment’s causal interpretation requires an explicit account of treatment assignment, intervention fidelity and unmeasured pathways [9].

13.3 Randomization, measurement outcomes, and causal scope

Let $D \in \{0, 1\}$ denote a randomized assignment and let $Y(0), Y(1)$ be potential outcomes for a measured response, under a well-specified treatment definition. Suppose consistency holds, assignment is independent of the potential outcomes, and both assignments have positive probability. The observed record is $Y = Y(D)$. Under these assumptions, randomization identifies an average treatment effect on

the measured variable. The distinction between response-level causal identification and a claim about phenomenal experience is essential.

Proposition 13.3 (Randomized identification of a measured effect). *Suppose $\mathbb{E}|Y(d)| < \infty$ for $d = 0, 1$, consistency $Y = Y(D)$ holds, $D \perp (Y(0), Y(1))$, and $0 < P(D = d) < 1$ for both assignments. Then the average causal effect on the measured response equals Equation 79.*

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}[Y \mid D = 1] - \mathbb{E}[Y \mid D = 0]. \quad (79)$$

Proof. Consistency gives $\mathbb{E}[Y \mid D = d] = \mathbb{E}[Y(d) \mid D = d]$. Independence replaces the right side by $\mathbb{E}[Y(d)]$, and positivity ensures both conditional expectations are defined from the experimental assignment. Subtracting the cases $d = 1$ and $d = 0$ gives Equation 79. $\square$

Proposition 13.3 specifies what a randomized experiment can establish under its model. When Y is a neural signal, classifier output, behavior, or verbal report, the identified effect concerns that variable. Interpreting Y as a calibrated indicator of conscious content or phenomenal organization requires further assumptions about the observation interface. The target may also be mediation through a proposed geometric invariant, in which case random assignment to a treatment alone is insufficient to identify the mediator’s causal contribution. This is a source of underdetermination shared by many physical models, including models that borrow language from dynamical systems and algebraic topology.

Consciousness research supplies instructive examples of these distinctions. The perturbational complexity index introduced by Casali and colleagues uses responses to transcranial magnetic stimulation and compression of evoked electrophysiological patterns as a theory-motivated clinical measure; an observed index is a physiological summary with an empirically studied association to states of consciousness [6]. No-report paradigms attempt to reduce the measurement contributions of overt report processes, while their interpretation continues to require assumptions about surrogate behavioral and neural indicators [25]. The preregistered adversarial collaboration reported by the Cogitate Consortium in 2025 juxtaposed predictions of integrated information theory and global neuronal workspace theory through multiple neural modalities and found results partly compatible with, and partly challenging to, specified predictions of both theories [8]. These studies illustrate disciplined operational tests; none supplies a general equivalence theorem across the four orientations considered here.

13.4 Finite samples and probabilistic discrimination

Even distinct probability laws can be difficult to discriminate at finite sample size. For two candidate models under a fixed protocol, let P and Q be probability laws of one full experimental record, including whatever correlated observations belong to a *single* run. Under equal prior probabilities and optimal binary testing, the minimal error is $(1 - \text{TV}(P, Q))/2$, as established in the statistical comparison of Section 8. The data-processing inequality shows that a shared stochastic measurement channel cannot increase the total-variation separation of the candidate source laws [4].

Ideal-law identifiability further differs from the stability of statistical reconstruction: an invertible observation channel may have a nearly singular inverse that amplifies estimation variance. Appendix E gives an explicit variance formula for the binary channel and the conditions of its applicability. A finite-sample bound makes the difference between ideal-law identifiability and operational separation quantitative. The bound assumes independent repetitions of the *entire* record, a condition more restrictive than merely observing numerous correlated time points within one recording.

Proposition 13.4 (Lower bound on repeated-experiment testing error). *Let P, Q be probability laws on a finite record space, let $\delta = \text{TV}(P, Q)$, and suppose $n \geq 1$ conditionally independent and identically distributed repetitions of one protocol are observed. Under equal priors, the optimal discrimination error obeys Equation 80.*

$$P_{\text{err}}^{*(n)} = \frac{1 - \text{TV}(P^{\otimes n}, Q^{\otimes n})}{2} \geq \frac{(1 - \delta)^n}{2}. \quad (80)$$

Proof. For finite spaces, a maximal coupling of P, Q agrees on the two coupled records with probability $1 - \delta$. Coupling n such pairs independently yields a coupling of $P^{\otimes n}, Q^{\otimes n}$ that agrees in all coordinates with probability $(1 - \delta)^n$. By the coupling characterization of total variation, $\text{TV}(P^{\otimes n}, Q^{\otimes n}) \leq 1 - (1 - \delta)^n$. Substitution into the equal-prior Bayes error formula gives Equation 80. $\square$

For example, if $P = \text{Bernoulli}(0.52)$ and $Q = \text{Bernoulli}(0.48)$, then $\delta = 0.04$ and Equation 80 yields a positive lower bound at every finite n . Distinct laws make ideal identification possible in a sufficiently rich asymptotic setting, but the achievable finite-sample accuracy depends on effect size, dependence structure, model complexity and selection of a test. This elementary bound is deliberately conservative; an actual sample-size analysis should also specify the rejection rule, type-I error constraint, and the design-specific alternative. In neural recordings, hierarchical dependence among participants, trials and repeated samples further limits naive reuse of independent-repetition formulae.

13.5 Reports, latent proxies, and measurement invariance

Reports and behavioral proxies are often treated as access paths to features of experience. Their epistemic status is made explicit by a latent-variable measurement model. Let $C \in \{0, 1\}$ be a stipulated latent classification whose interpretation must be supplied independently, and let $R \in \{0, 1\}$ be a recorded report. Write $p = P(C = 1)$, sensitivity $\alpha = P(R = 1 \mid C = 1)$, and specificity $\beta = P(R = 0 \mid C = 0)$. Equation 81 gives the observed positive-report rate r .

$$r = \alpha p + (1 - \beta)(1 - p), \quad p = \frac{r + \beta - 1}{\alpha + \beta - 1} \quad (\alpha + \beta \neq 1). \quad (81)$$

Proposition 13.5 (Recovery and nonidentification under binary misclassification). *For the binary measurement model in Equation 81, a known observation channel recovers p uniquely if and only if $\alpha + \beta \neq 1$. If $\alpha + \beta = 1$, the distribution of R is independent of p . When the channel parameters are unspecified, observing r alone does not generally identify p .*

Proof. The first expression in Equation 81 is affine in p with slope $\alpha + \beta - 1$. A nonzero slope gives the stated inverse; zero slope makes $r = 1 - \beta$ for every p . For nonidentification with unknown parameters, take $r = 1/2$. Both the perfect channel $\alpha = \beta = 1$ with $p = 1/2$ and the uninformative channel $\alpha = \beta = 1/2$ with any $p \in [0, 1]$ yield the same observed report rate. Therefore one recorded marginal admits multiple latent explanations. $\square$

Equation 81 supports a narrow methodological lesson. A report criterion can be calibrated relative to a specified latent classification only if that classification and the observation channel are themselves meaningfully defined and appropriately constrained. Even a mathematically identified binary latent prevalence says nothing by itself about the full structure of subjective experience. Across anesthetic levels, tasks, sensory modalities, or populations, changes in the reporting channel can imitate changes in the latent target unless measurement invariance is examined. Conversely, a common measurement channel may erase differences between source laws, a fact formalized by stochastic data processing. The ontological, relational, and manifestational registers are useful here as declarations of what is modeled, not as mutually guaranteed descriptions of the same phenomenal object.

13.6 Adaptive protocols and persistent experimental equivalence

Repeated experiments can select later protocols using earlier records. Such adaptivity increases the accessible experiment family, yet it cannot overcome equivalence when the candidate systems induce identical conditional observation laws for every allowed action and attainable record history. This gives an informative boundary condition for active attempts to recognize another system's generative organization.

Table 12: Candidate experimental comparisons for four equivalence orientations. Predicted contrasts are conditional on fully specified physical instantiations and shared measurement protocols.

Orientation	Candidate observable and contrast	Principal underdetermination or confound
Variation	Perturbation-response families, transfer functions, intervention-sensitive reachable sets under common controlled inputs	Unmeasured states, off-target stimulation, source-action reparameterization, sensor compression
Trajectory	Continuously sampled state estimates, path order, temporal correlations, possible winding or path-class proxies	Sampling aliasing, noninjective projection, uncertain physical-time calibration and trajectory selection
Generativity	Changes in coupled effective rules or network organization across matched contexts, with maps between inferred descriptions	Constitutive-map ambiguity, nonstationarity, model-selection dependence, contextual and developmental effects
Repertoire	Full trial-level path laws, branching under repeated controlled conditions, accessible behavioral or neural histories	Rare-path under-sampling, latent branching, selection bias, altered admissibility and correlated repetitions

Proposition 13.6 (Adaptive non-discrimination under equal response kernels). *Let m, m' be two models with a common finite action set $\mathcal{A}$ and finite record set $\mathcal{Y}$. Suppose their conditional one-step record kernels agree for every admissible action and history, and their initial observed record laws agree. For any common, possibly randomized history-dependent policy selecting actions, the joint law of observed histories and selected actions up to each finite stage is identical under m and m' .*

Proof. The initial joint distributions coincide by assumption. Suppose the distributions of the observed histories and selected actions up to stage k agree. Conditional on a fixed history, the common policy selects the next action according to the same distribution under both models. Conditional on the chosen action, the next record has the same kernel under both models by hypothesis. Integrating these two kernels against the identical distribution of previous histories gives equality of the joint law through stage $k + 1$. Induction proves the claim at each finite stage. $\square$

The hypothesis of Proposition 13.6 is intentionally strong. A new intervention can separate models when it reaches conditions under which their response kernels differ, as in Proposition 13.2. Persistent equality over the entire experimentally accessible policy class yields an operational equivalence relation for that access class; it does not imply identity of the underlying physical realization, constituted relational substructure, or a phenomenal assignment. A scientific programme can therefore improve its observational scope while explicitly recording which distinctions remain inaccessible.

13.7 Operational comparison across the four orientations

A practical equivalence study should formulate at least two fully specified competing models, identify a common observation interface, and declare the criterion whose preservation is being tested. It should derive predicted joint response laws and, where permissible, responses under controlled perturbation. Disagreement between models on a measurable record is informative only after accounting for protocol fidelity, nuisance parameters, measurement changes and the resolution limits of the experiment. The inferential target must also be fixed: identifying a mathematical transformation law, an observed trajectory class, a constitutive correspondence, or an admissible repertoire requires different data. Table 12 provides candidate operational comparisons rather than claims of established consciousness-specific biomarkers.

The practical investigation should preregister competing response predictions when confirmatory tests are intended, separate exploration from evaluation, retain uncertainty estimates, document observation maps, and report negative as well as positive comparisons. Such practices were central to the adversarial experimental strategy of the Cogitate collaboration [8]. The distinction between a model's mathematical equivalence and its empirical explanatory sufficiency remains fundamental. Even when

an experiment identifies a response law and rejects one candidate orientation-specific instantiation, the relation between the retained model, the constitution of an observer, and subjective experience requires further examination. Section 13 therefore provides a disciplined interface between our formal equivalence taxonomy and possible evidence, while leaving phenomenological and normative interpretation for the following section.

14 Limits of Phenomenal Inference

The preceding sections identify mathematical equivalence criteria within four candidate explanatory orientations, describe how these criteria vary across ontological, relational, and manifestational registers, and establish experimental conditions for recovering selected model properties. A further question concerns their bearing on subjective experience. Two mathematical models can agree on every response available to a specified protocol, yet the claim that they instantiate equivalent experiences depends on assumptions additional to their response laws. Conversely, a disagreement between formal invariants provides limited grounds for inferring a difference in experience until the invariants receive an appropriate physical and phenomenal interpretation. This section makes those inferential obligations explicit. Its purpose is to delineate conditional paths from formally defined equivalences to candidate phenomenal claims, while preserving the possibility that future theory and evidence may justify stronger conclusions.

Philosophical work distinguishes the subjective character of experience from an observer's third-person description [21], the explanatory gap associated with some psychophysical identities [18], and the question of how physical or functional organization relates to the presence of experience [7]. These are disputed philosophical analyses, with different implications under physicalist, functionalist, representationalist, and other approaches. They motivate a formal accounting of inferential commitments rather than a universal impossibility theorem. In particular, we distinguish the representational equivalence already examined in Sections 13 and 3, a *phenomenal bridge hypothesis* relating represented features to a stipulated target, and the epistemic and normative use of conclusions drawn from that bridge.

14.1 Phenomenal targets and declared bridge hypotheses

Let $\mathfrak{M}$ denote a fixed, typed class of candidate descriptions, and let $P_E : \mathfrak{M} \rightarrow \prod_{e \in E} \mathcal{P}(\mathcal{Y}_e)$ be the protocol-indexed law map of Equation 76. In the present section, the members of $\mathfrak{M}$ may already contain specific constitutive and dynamical models. They are descriptions to be compared, not stipulated independently existing subjects. Let $\mathcal{B}$ be a nonempty family of proposed *bridge hypotheses*. Each $b \in \mathcal{B}$ assigns to a model m a target $b(m)$ in a specified comparison space $\mathcal{F}_b$. The proposed targets could range from a binary claim that experience occurs, through a dimension of experienced organization, to a phenomenally interpreted relation between modeled subjects. Any such reading requires a separately justified interpretation of $\mathcal{F}_b$ and a defensible application of b to the represented physical systems.

The notation $b(m)$ is a formal placeholder for a theoretical commitment. It does not make phenomenal properties directly observable or presuppose that they can be uniquely encoded. Different hypotheses may even use target types that lack a canonical comparison map. Within each declared target type, we can nevertheless ask whether equality of the available response laws suffices to recover the target. The uniform recovery condition for the entire bridge family is given in Equation 82.

$$\ker(P_E) \subseteq \bigcap_{b \in \mathcal{B}} \ker(b). \quad (82)$$

The intersection in Equation 82 is an equivalence relation on $\mathfrak{M}$, even when the codomains $\mathcal{F}_b$ differ: each $\ker(b)$ simply compares the values assigned by the *same* bridge. It expresses agreement across the specified family of interpretive hypotheses. It is stronger than empirical equivalence under a selected bridge and does not impose a prior ranking on the hypotheses.

Proposition 14.1 (Uniform bridge-relative recoverability). *For a fixed set of protocols E and a nonempty family $\mathcal{B}$ of functions $b : \mathfrak{M} \rightarrow \mathcal{F}_b$, the following are equivalent: (i) the inclusion in Equation 82 holds;*

(ii) for every $b \in \mathcal{B}$ there exists a function $\tilde{b} : \text{im}(P_E) \rightarrow \mathcal{F}_b$ with $b = \tilde{b} \circ P_E$; and (iii) for every pair m, m' with identical E -indexed response laws, all bridges in $\mathcal{B}$ assign equal target values to m and m' .

Proof. The equivalence of (i) and (iii) follows by expanding the definitions of the kernels and their intersection. If (i) holds, each b is constant on every fiber of P_E , so defining $\tilde{b}(P_E(m)) = b(m)$ is well defined and gives (ii). Conversely, (ii) implies that equality of P_E -images entails equality of the corresponding b -values for each b , yielding (iii). $\square$

Proposition 14.1 gives an exact, set-theoretic criterion; it does not determine whether the postulated bridges are true. A physical theory might propose that a specific invariant determines a phenomenal property. Under that added thesis, the proposition describes when the property's value survives observation. It neither derives the thesis from the invariant nor shows that a rival bridge family is empty. Allowing every conceivable mapping from $\mathfrak{M}$ to a two-element target makes the uniform condition fail whenever the observation map has a nontrivial fiber. Conversely, restricting $\mathcal{B}$ to a single untested bridge can make a formal reconstruction claim appear stronger than its empirical warrant. The scholarly obligation is therefore to state both the observation protocol and the grounds for admitting particular bridges into $\mathcal{B}$.

An equivalence criterion within Variation, Trajectory, Generativity, or Repertoire may refine the empirical fibers and thereby contribute to bridge testing. It may also conceal distinctions valued by another bridge. For example, equivalence of measured path repertoires does not fix a constitutive diagram, as established in Section 9, while a proposed phenomenal bridge could refer precisely to the latter. A successful theory needs to make this discrepancy scientifically consequential through explicit predictions or independent arguments about physical realization.

14.2 Latent descriptions and calibration symmetries

Suppose a reported experience is modeled through a finite latent alphabet $\mathcal{Z}$, an observed alphabet $\mathcal{R}$, a latent distribution $\mu \in \mathcal{P}(\mathcal{Z})$, and a stochastic reporting channel $K(r | z)$. The observed report law is obtained by marginalization, just as in the calibrated measurement models of Section 13. A latent label may initially identify a state of a theoretical coding system without constituting an independent name for an experienced quality. The distinction matters because changes of latent coordinates can leave the probability of every report unchanged.

For a bijection $g : \mathcal{Z} \rightarrow \mathcal{Z}$, define $\mu_g(z') = \mu(g^{-1}z')$ and $K_g(r | z') = K(r | g^{-1}z')$. Equation 83 states their effect on the observed law.

$$\sum_{z' \in \mathcal{Z}} K_g(r | z') \mu_g(z') = \sum_{z \in \mathcal{Z}} K(r | z) \mu(z). \quad (83)$$

Proposition 14.2 (Relabeling invariance of latent reporting models). *For finite $\mathcal{Z}$ and $\mathcal{R}$, every bijection g of $\mathcal{Z}$ leaves the report distribution invariant under the paired transformation $(\mu, K) \mapsto (\mu_g, K_g)$. A latent target that changes under this transformation is not identifiable as a property of the report distribution without additional calibration or restrictions on the admitted reporting channels.*

Proof. The change of variables $z' = g(z)$ is a bijection of $\mathcal{Z}$. Substituting the definitions of μ_g and K_g into the left-hand side of Equation 83 gives the right-hand side for every r . If a proposed target assigns different values to two parameterizations yielding the same report law, the target fails the fiber-constancy criterion in Proposition 14.1 for this observation map. $\square$

Proposition 14.2 establishes an identifiability symmetry in a statistical representation. A pure relabeling of latent coordinates is ordinarily a *gauge change in description*, rather than an alteration in the modeled subject's experience. It therefore supplies no demonstration that two worlds with inverted phenomenal qualities are empirically possible. A genuine claim about distinct experiences would require a target interpretation that remains meaningful across coordinate changes and a physical account identifying distinct realizations. The mathematical symmetry demonstrates why uncalibrated reports cannot, by themselves, secure an interpretation of latent labels.

Additional measurements or interventions may remove an identification ambiguity if they constrain the permissible transformations of the latent model. For example, pairing reports with a stable stimulus protocol, neural recordings, or independent behavioral tasks can restrict the compatible reporting channels. Such calibration still calls for an account of why a measured relation is evidence for a proposed phenomenal category rather than merely a reliable task-dependent reporting disposition. Block's distinction between access-consciousness and phenomenal consciousness is relevant precisely because a functional route to reportability and the subjective character of experience need distinct argumentative treatment [5]. Empirical improvements in access measures may strengthen a theory while leaving its proposed phenomenal bridge open to contestation.

14.3 Recognition of another subject and relational constitution

Inference about another subject requires identification of the subject whose experience is being compared. In a relational two-complex model, vertices represent events, edges represent couplings, and faces represent locally declared higher-order primitives. A subject is therefore a candidate historically constituted organization, represented by a selected and appropriately interpreted subcomplex or related structured object. Its persistence conditions, boundaries, internal organization, and possible overlap with other subjects are further modeling assumptions. A vertex-level identity label would bypass those assumptions rather than solve them.

To expose the relevant dependency, let $\Sigma(m)$ denote the candidate subject description associated with a model m . A comparison across models requires a rule for relating values of Σ , as well as a phenomenal bridge defined on the corresponding subjects. If two descriptions possess observably matching responses, their subject selections may still differ. For instance, identical degree statistics of event–coupling graphs can coexist with different connected-component structures, while a subject-identification rule sensitive to connectedness may distinguish the two. That formal counterexample does not imply that graph components are conscious beings; it demonstrates why a chosen representation of observer constitution belongs among the premises of an inference concerning another system.

This problem differs from skeptical dismissal of first-person testimony. A subject's report of pain or visual experience can be strong practical evidence under many ordinary communicative and clinical conditions. The evidential value of that report is mediated by language, task design, memory, capacity to respond, and conditions of interpretation. These mediations can be investigated, calibrated, and sometimes bypassed with auxiliary physiological measures. Yet access to an external record remains an interpretive relation, even when the record is generated by a highly reliable measurement process. In populations or contexts with limited reportability, a missing or ambiguous report has a different epistemic meaning from a well-calibrated report of absence.

A related distinction concerns the phenomenal *character* of experience and the operational classification of global states. Bayne, Hohwy, and Owen criticize a simplistic one-dimensional scale of consciousness and examine a multidimensional account of global states [3]. Within the present framework, two individuals may be equivalent under a chosen arousal measure and differ under content, access, or other measures. The absence of a universal comparison metric is therefore an explicit scope condition for the target space of any bridge. A mathematical equivalence statement concerning one operational dimension does not determine which dimensions are relevant for a particular scientific or clinical question.

14.4 Statistical limits on phenomenal classification

The difference between law-level and finite-data discrimination is established in Section 13. Here we examine the corresponding epistemic cost when the target is supplied by a proposed phenomenal bridge. Consider two candidate descriptions m_0, m_1 whose bridge values fall in disjoint stipulated classes, designated 0 and 1. Fix an experimental design, including any permitted interventions and its actual joint distribution of records. Denote its two joint data laws by P_0 and P_1 . The design may include repeated or sequential measurements; independence is not assumed. A classification rule δ maps the recorded data to a possibly randomized choice in $\{0, 1\}$.

Proposition 14.3 (Discrimination bound for a bridge-assigned binary target). *For any two candidate models with different stipulated binary bridge values and joint record laws P_0, P_1 , every decision rule δ satisfies the bound in Equation 84. If $P_0 = P_1$, the worst-case probability of erroneous classification is at least one half, irrespective of the size of the record when the complete joint laws remain equal.*

$$\max_{i \in \{0,1\}} P_i(\delta(Y) \neq i) \geq \frac{1 - \text{TV}(P_0, P_1)}{2}. \quad (84)$$

Proof. The maximum of the two conditional errors is at least their arithmetic mean. Under equal prior weights, the minimum mean error over all binary tests equals $(1 - \text{TV}(P_0, P_1))/2$, as follows by choosing the more likely hypothesis on a measurable decision region and applying the variational characterization of total-variation distance. Hence every test satisfies Equation 84. Equal laws have zero total-variation distance. Because P_0 and P_1 denote the entire joint record laws, the conclusion needs no assumption about independent samples. $\square$

The target classification in Proposition 14.3 is provided by the bridge as an assumption. The proposition bounds a statistical decision task; it does not establish that there exist two physically realizable systems with different experiences and identical laws under every conceivable physical intervention. When a broader protocol class separates the candidate models, its joint record distributions can have positive total variation, and the bound changes. The experimenter's admissible access and the model's physical scope must therefore accompany the discrimination claim. Even a low classification error for the bridge-assigned label does not verify the bridge's substantive interpretation without an independently justified target or convergent theoretical evidence.

These limitations are particularly important for attempts to infer equivalence from complex physiological indicators. An estimated complexity score, for instance, is a function of a measured signal and its processing pipeline. It may discriminate some clinically characterized states under validated conditions [6]. It remains necessary to examine calibration across tasks and populations, confounding mechanisms, the dependence of the score on data processing, and the theoretical reasons for connecting the score to the intended phenomenal classification. A formal map from signals to a scalar summary can be reproducible while preserving too little information for a proposed bridge.

14.5 Obligations for the four explanatory orientations

Each of the four orientations supplies potential physical or relational properties that might enter a phenomenal bridge. The equivalences developed in Sections 5, 6, 7, and 8 identify what particular transformations preserve. Their phenomenal interpretation depends on a theory selecting which preserved features are relevant. Table 13 summarizes the inferential step that remains to be justified for each orientation. The entries are methodological questions rather than predictions already supported by the manuscript.

Several nonimplications follow from the typing of these candidate objects. An invariant of a state manifold is not automatically an invariant of the physical substrate or of an admissible repertoire. Identical selected paths may conceal different accessible alternatives. Identical aggregate homology may conceal different attaching maps and fundamental groups. These distinctions matter when an independently motivated bridge singles out the suppressed feature. Their existence alone provides no reason to assign a specific phenomenal difference to the pair being compared.

A phenomenally informative programme can therefore use the four orientations comparatively. It can first specify alternative bridges on one physical model, then identify controlled protocols under which those bridges produce different response predictions, and finally examine whether surviving models account for the target experiences and their relational constitution. This is a stronger test than simply asking whether a selected invariant correlates with a preassigned diagnostic category, because the competing theoretical interpretations and their empirical implications remain explicit. The four orientations need not be exhaustive or exclusive for this comparison to be meaningful.

Table 13: Phenomenal-bridge obligations for the four explanatory orientations. Formal equivalence specifies preserved data; the candidate phenomenal connection requires an additional justified premise.

Orientation	Possible preserved data	Additional explanatory and empirical obligation
Variation	Effective actions, generators, accessible transformations	Explain which physical transformations are relevant to experience and why equivalence at the selected action level predicts a phenomenal invariant.
Trajectory	Selected paths, orbit types, homotopy or geometric features	Justify the relation between an observed or reconstructed path and a subject's experience, including the status of the path parameter.
Generativity	Constitutive diagrams, incidence changes, evolving constraints	Specify how a subject and the relevant phenomenal property are constituted, and test the significance of alternative constitutive mappings.
Repertoire	Admissible path families, branching, constrained path topology, path laws	Explain how modal availability or probability of histories bears on experience beyond realized and recorded responses.

14.6 Interpretive and normative consequences

The interpretation of another system's experience can affect medical care, animal research, human-machine interactions, and the treatment of individuals whose capacity to report is restricted. The mathematical account developed here has a limited but relevant normative consequence: a declaration of observational or structural equivalence establishes only the property named by its criterion. It does not, without additional premises, entail equality of interests, entitlements, vulnerability, dignity, or the appropriate form of care. Likewise, formal distinctions between architectures or dynamic organizations cannot directly settle claims of moral status. The norms relevant to these questions draw upon ethical reasoning, legal and institutional obligations, evidence of harm, and the affected parties' circumstances, in addition to models of consciousness.

There is an important asymmetry between an inference used to understand a system and a decision whose consequences affect that system. An investigator may withhold a phenomenal attribution because current evidence is inconclusive, while an institution may still adopt protective procedures in the face of uncertainty. A decision rule depends on the relative consequences of false positive and false negative classifications, which are not specified by observational equivalence or a homology group. Recording this dependence guards against treating the absence of a decisive physiological marker as a demonstrated absence of morally relevant experience.

The three-register framework provides an audit trail for such reasoning. The ontological register states what sort of physical subject and experience a model assumes; the relational register states the structures and constitutive interpretations used to characterize it; the manifestational register states how observations and testimony were obtained. An additional normative argument determines the decision to be made under residual uncertainty. Ethical and legal considerations remain partly independent of whether a formal equivalence has been established. These limits should be recorded as substantive uncertainty rather than hidden by a more convenient mathematical description.

14.7 Conditions for a defensible phenomenal inference

The analysis suggests a sequence of research obligations. A proposed phenomenal claim should specify its target and subject-identification rule; identify the physical realization and the relevant relational description; declare the bridge hypotheses relating those structures to the target; determine the observational

and experimental protocols under which the bridge-relevant properties can be recovered; and examine the effects of reporting channels, latent-description symmetries, and finite data. Finally, it should state the scope of any comparative phenomenal conclusion and the alternative bridges that remain compatible with the evidence. This sequence is methodological rather than temporal or ontological: it is a collection of inferential dependencies that can be investigated in different orders.

The central formal distinction is thus among *equivalence relative to a mathematical criterion*, *equivalence relative to a family of physically interpreted bridges*, and *equivalence of measured manifestations*. Proposition 14.1 gives an exact condition for transferring equality of records to the values of the declared bridges; Proposition 14.2 shows that uncalibrated latent labels can resist identification; and Proposition 14.3 quantifies a limitation of statistical classification even when the target labels are stipulated. None of these propositions rules out a future satisfactory physical explanation of experience. Their contribution is to expose the premises such an explanation must provide and to maintain a disciplined distinction between formal preservation, empirical support, and phenomenal interpretation. Section 14 thereby prepares the comparative evaluation of the paper's equivalence framework in the following Discussion.

15 Comparative Discussion of Equivalence Criteria

The equivalence relations examined in this paper delimit distinct kinds of information that a mathematical comparison can preserve. The present section synthesizes their consequences for model selection and consciousness research. Its aim is to specify when equivalence provides a useful explanatory constraint, when a distinction warrants an experiment, and when a comparison remains sensitive to unexamined choices of representation or phenomenal interpretation. The argument proceeds through three levels: comparisons within the four explanatory orientations, comparison across ontological, relational, and manifestational registers, and the requirements for an empirical research programme. The synthesis draws on the formal results established in Sections 4–14; it introduces no assumption that the four orientations exhaust possible explanations of consciousness.

15.1 The explanatory scope of an equivalence criterion

An equivalence relation partitions a declared class of descriptions. Its explanatory use depends on why the distinctions suppressed by that partition can be set aside for the scientific question being posed. Equivalence under a particular observation map may be suitable for predicting a defined record, while equivalence under action conjugacy can support a substantially stronger comparison of dynamical organization. In the first case, the target is the distribution or trajectory of records under a stated protocol. In the second, the relevant commitment includes a correspondence between the transformations acting on the modeled states. The same pair of systems can be classified differently by these two relations, consistently and without logical conflict.

Section 4 formalized the relative discrimination of criteria by kernel inclusion. This order yields an exact statement about recoverable invariants, since a coarser invariant factors through a finer one when its kernel contains the finer kernel. It is a comparison of partitions, not an absolute ranking of theoretical merit. A criterion that retains every microscopic degree of freedom may discriminate a larger class of systems while making a clinical or explanatory question needlessly dependent on nuisance variation. Conversely, a tractable scalar index can give reproducible classification while concealing the distinctions needed to test a proposed mechanism. Criteria must therefore be evaluated simultaneously for their mathematical strength and their scientific relevance to a specified target.

This distinction supplies a practical rule for comparing consciousness theories. A theoretical claim should identify its target property, the physical or relational domain on which the property is defined, the equivalence criterion proposed to preserve it, and the admissible changes of representation under which the claim is expected to remain stable. A comparison of scalar markers should not be advertised as a comparison of the underlying constitutive organizations unless an additional reconstruction argument

is supplied. Similarly, identifying an isomorphism between formal descriptions is informative only after the objects and structure-preserving maps have acquired a warranted physical interpretation. These conditions are especially important in an area where theoretical frameworks frequently employ different kinds of explanatory primitives [24, 11].

15.2 Four orientations and their comparative dependencies

The results for Variation, Trajectory, Generativity, and Repertoire distinguish four families of mathematical questions. Variation concerns transformations, generators, and action structures. Trajectory concerns a selected path with its parameterization, geometry, and possible homotopy data. Generativity concerns the constitution and transformation of relational structures and of their admissibility conditions. Repertoire concerns families of admissible histories, their branching or path-space topology, and, when modeled, their probability laws. A given physical description may carry data belonging to all four families. The orientations distinguish the explanatory claims made about that data rather than furnishing mutually exclusive types of physical system.

Variation equivalence can preserve an action's structural organization where equality of reachable states cannot. The reversible and distinction-erasing mechanisms compared in Section 5 illustrate this gap on a finite carrier. Under the generator correspondence required by Proposition 9.1, a sufficiently strong action isomorphism also transports the specified history repertoire. This implication is conditional on a common admissibility scheme. Removing generator correspondence can preserve a transformation monoid while changing the repertoire generated by its permitted moves. The scientific significance is the need to distinguish the system's available transformations from the experimenter's stipulated opportunities to activate them.

Trajectory equivalence is sensitive to the chosen identification of path data. Equality of geometric images can suppress multiplicity of traversal, orientation, and parameter-dependent behavior. Homotopy class can survive changes of parameterization while distinguishing loops that project onto an identical observed path. Section 6 supplied explicit lifting and recoverability criteria. Those criteria matter whenever a measured neural time series is treated as a reconstruction of an underlying state-space orbit: the shape of a projected record constrains the source path only through the declared observation map and its fibers. A trajectory's mathematical parameter must also be related to an experimental clock or an ordering structure before it can support a physically temporal interpretation.

Generativity equivalence adds coherence requirements to comparisons made pointwise across indexed structures. The examples in Section 7 show how a terminal CW-complex or a list of componentwise isomorphism types can omit the correspondence maps that constitute the modeled development. This observation is not a theorem that physical systems possess an irreducible historical essence. It is a property of the chosen diagrammatic descriptions: when a theoretical property depends on connecting maps, a comparison that discards those maps can cease to determine the property. A physical application must justify which attachments, incidence changes, and admissibility revisions represent identifiable processes and which are artifacts of a modeler's chosen discretization.

Repertoire equivalence concerns possibility relative to constraints. Equal supports of admissible histories need not determine branching structures or their probability laws. Section 8 supplied finite-state, metric, and probabilistic examples, including equal stationary marginals with different path distributions. These results allow a meaningful distinction between properties of a realized record and properties of alternative histories admitted by the model. Their use in consciousness research requires an additional empirical and interpretive step: the scientifically relevant repertoire must be related to a physical preparation or intervention scheme, and a phenomenal interpretation of unreached alternatives requires a separate bridge hypothesis.

The comparison across orientations is therefore governed by conditional implication, partial recoverability, and genuine incomparability. A common synthetic example can carry an action, an observed path, an indexed constitutive diagram, and a set of admissible histories. Establishing equivalence in one projection of that example yields exactly the invariants specified for that projection. The implications

assembled in Table 8 give the relevant formal qualifications. Treating one orientation as automatically superior to the others would confuse information retention with explanatory adequacy.

15.3 Equivalence across the three analytical registers

The ontological, relational, and manifestational registers provide another, independently indexed comparison. Ontological claims concern a proposed physical realization and, where introduced, its relation to experience; relational claims concern the structures used to characterize that realization; manifestational claims concern accessible records and reporting procedures. The interfaces linking these registers are conditional modeling commitments. A physical carrier, a neural state manifold, and a cellular realization of a relational two-complex may each enter a study, but their identification with one another demands explicit maps and interpretation.

The central mathematical danger is to reverse an information-losing map. A relational model can produce identical manifestational outputs from different source structures. Section 10 provided a smooth example in which an autonomous observed flow arises from a source with different topology. Section 12 showed separately that a homeomorphism of carriers can preserve continuous path-space topology without preserving Riemannian energy or a corresponding energy-limited repertoire. Together, these constructions demonstrate that preservation of a target invariant and reflection of its equality are different properties. A translated description can faithfully support predictions at its own level while leaving source-level alternatives unresolved.

The relational register also exposes the status of subjects and boundaries within a proposed model. In a relational two-complex, vertices represent events, edges represent couplings, and faces represent locally stipulated higher-order primitives. Neither a vertex nor an arbitrary connected subcomplex is automatically an experiencing subject. A subject-identification rule must specify a relational organization and the conditions under which its purported persistence, unity, and boundaries are compared. The choice of this rule can affect whether two putative subjects are classified as equivalent even when some neural signals, graph summaries, or homology groups agree.

Manifestational equivalence is nevertheless scientifically useful. It specifies exactly which claims are supported by records obtained under compatible instruments and protocols. Additional experiments can refine an observational partition; Proposition 13.1 establishes this at the level of ideal response laws. The possibility of refinement does not imply that every source distinction will become experimentally accessible. An instrument may be incapable of measuring the relevant degrees of freedom, a proposed manipulation may alter the model's structure, or a theoretical property may be absent from the specified response laws altogether. Experimental access is part of the comparison's domain, not an afterthought.

Finally, a phenomenal interpretation places an additional premise on any register comparison. The bridge-family criterion of Proposition 14.1 describes the precise condition under which observational equality determines the values assigned by every bridge in a stipulated family. The mathematical result neither selects that family nor validates the bridges empirically. The latent-report symmetry of Proposition 14.2 likewise identifies a concrete ambiguity in the reconstruction of subjective categories. A defensible phenomenal claim must therefore report the structure preserved, the subject-identification convention, the candidate bridge, and the evidence for the bridge's interpretation.

15.4 Representation choice, invariance, and robustness

A comparison between models can change when the representations used to construct its invariants change. Coordinate changes, coarse-graining, observation projection, cellular subdivision, discretization, and statistical aggregation are distinct operations; each has its own preservation obligations. Category-theoretic equivalences, homotopy equivalences, and equality of observed probability laws do not supply universal interchangeability among these operations. Sections 11 and 12 supplied formal examples of information lost by functors and geometric reductions. The resulting lesson is methodological: every asserted equivalence should be accompanied by a declaration of which admissible translations preserve it and which transformations fall outside that guarantee.

There are at least three levels of robustness to examine. *Representational robustness* concerns preservation under changes in mathematical encoding that are justified as describing the same target. *Model robustness* concerns persistence when uncertain mechanisms or nuisance parameters vary within an explicitly declared family. *Empirical robustness* concerns recovery under finite observations, measurement error, cohort variation, and changes in experimental access. These levels are logically distinct. A property may be a topological invariant of an assumed carrier and nonetheless be poorly identifiable from EEG, or be accurately measurable while remaining highly sensitive to a model's choice of state variables. Robustness at one level should be reported with its quantifier and scope.

A useful example concerns aggregation of dynamical records. Suppose two models have equal distributions of a scalar summary produced by a fixed processing pipeline. The inference licensed by this equality is equality at the summary-law level. A new observation protocol retaining temporal dependencies may separate the models, as in the Markov constructions of Section 8. Conversely, recording a longer time series can fail to separate models whose entire measured joint laws agree under every available protocol. These cases distinguish removable representational loss from persistent protocol-relative underdetermination. The distinction is relevant to comparisons among contemporary consciousness theories, whose experimental implementations can differ in the anatomical targets, tasks, reporting assumptions, and measures used to operationalize their predictions [11, 8].

Robustness also has a resource dimension. A mathematically stronger criterion may require inaccessible states, exponentially many histories, unreliable inverse maps, or measurements that a clinical context cannot supply. Finite-sample inference places another cost on increasingly discriminating criteria. The lower bound in Proposition 13.4 makes clear that exact separation of predicted laws and reliable statistical discrimination are different achievements. Computational or experimental budgets should be treated as explicit constraints on proposed comparison protocols, rather than as reasons to elevate a convenient coarse criterion into an ontological identity claim.

15.5 Experimental contrasts and evidential dependencies

The formal taxonomy can inform experimental design when candidate physical models are sufficiently specified to yield competing predictions. The operative unit of comparison should be a pair or family of explicit models under a shared, interpretable protocol, rather than abstract labels such as Variation and Repertoire taken alone. To test a difference in transformation structure, an experiment requires a manipulable condition predicted to reveal that difference. To test a distinction in path geometry or topology, a measurement requires access to suitable state-space information and an assessment of projection ambiguities. To test a difference in constitutive diagrams, the protocol requires a defensible correspondence between observed interventions or developmental conditions and the modeled structural maps. To test a distinction in repertoires, the sampling and perturbation regime must access informative alternatives or their probability laws.

Table 14 synthesizes these requirements as a comparison design rather than a set of predictions already confirmed. Each row begins with a difference between specified candidate models. It then states the additional access that could make the difference empirically relevant and the strongest conclusion that would be supported by successful discrimination. A response difference can validate a proposed prediction under the test conditions without establishing the necessity or sufficiency of the selected mathematical feature for subjective experience.

One appropriate use of this table is to design adversarial comparisons. Investigators can agree on a common target, predeclare the physical implementations, identify a predicted response discrepancy, and establish analysis and exclusion rules before measuring outcomes. The recent adversarial evaluation of selected predictions from integrated information theory and global neuronal workspace theory provides a relevant precedent for testing specified implementations; it does not settle the adequacy of their broader philosophical formulations [8]. Our framework can contribute by recording, alongside the prediction, exactly which equivalence criterion it tests and which mathematical distinctions its operationalization discards.

Table 14: Evidence dependencies for testing equivalence criteria across four explanatory orientations. Each proposed contrast presupposes fully specified candidate physical models and a common measurement interpretation; entries are research designs rather than empirical findings.

Orientation	Target contrast between candidate models	Required experimental or reconstructive access	Licensed claim from successful discrimination
Variation	Different effective transformations despite matching reachable endpoints	Controlled initial conditions or perturbations with comparable readout dynamics	Difference in a specified intervention–response law; mechanistic interpretation requires further constraints
Trajectory	Different source paths or topological classes despite equal coarse records	Calibrated observation map, sufficient temporal resolution, and identifiable path invariants	Difference in the reconstructed path criterion under the model assumptions
Generativity	Different connecting maps, attachments, or admissibility updates despite matching terminal summaries	Longitudinal or comparative access to constituent changes and justified cross-state correspondence	Difference in a declared constitutive model; endogenous physical generation remains to be explained
Repertoire	Different branching, constraints, or path laws despite matching selected histories	Access to alternative preparations, perturbations, or joint-history statistics	Difference in admissible or probabilistic response families under the tested protocol

A further concern is asymmetry in measurement accessibility. Different model families may have different parameters, latent-state dimensions, and calibration requirements. A protocol that can recover one model’s preferred invariant may remain insensitive to an alternative theory’s target. Shared datasets can therefore support asymmetric evidential claims. The appropriate remedy is an explicit mapping of each model’s predictions into the same record domain, accompanied by uncertainty intervals, nuisance-model sensitivity analysis, and an account of whether the relevant intervention is physically realizable. The possibility of unobserved confounding or interference should be assessed separately from statistical power. Randomization can identify an effect on a measured endpoint under the assumptions of Proposition 13.3; it does not itself identify the intermediate causal mechanism or a phenomenal state.

15.6 Research priorities and cumulative model comparison

A productive research programme can proceed by progressively tightening the scientific commitments associated with a particular equivalence claim. The initial task is to identify a candidate pair of descriptions and the target property for which equality is proposed. The second is to specify how the descriptions are physically instantiated and how their representations relate to a common observable domain. The third is to exhibit a feasible discrimination protocol or to show that the proposed property is unrecoverable under the available experimental class. The fourth is to test the stability of that assessment across plausible observation channels, nuisance conditions, and changes of model representation. A possible phenomenal interpretation then requires an independently motivated bridge and evidence supporting its use. These are logical and methodological dependencies; they do not prescribe a universal chronological sequence for scientific discovery.

Three priorities follow from the manuscript’s concrete counterexamples. First, the field needs *typed cross-theory translation*, because apparent agreement can reflect the use of different invariants or incompatible mathematical carriers. The categorical preservation results of Section 11 clarify why a faithful translation may still erase object-level distinctions and why coherence of indexed structures matters. Second, it needs *counterexample-guided operationalization*: before treating a marker as evidence of a mechanism, investigators can construct models that agree on that marker while differing on the mechanism. Such countermodels determine what additional measurement would have discriminating value.

3133 Third, it needs *bridge-explicit phenomenology*, in which the premises linking mathematical properties to
3134 phenomenal targets are made available for critique, calibration, and empirical comparison. Each priority
3135 concerns an identifiable deficiency in inference rather than a claim that the general research problem
3136 originated here.

3137 The theory of equivalence also provides a way to record unresolved disagreements without forcing
3138 premature consensus. If two proposals are equivalent under a limited observation protocol, their relation
3139 can be registered at that level while their untested constitutive claims remain separate. If their criteria are
3140 incomparable, the mathematical nonimplication can motivate a search for a more informative common
3141 target. If no physically interpretable map between descriptions is currently available, the comparison
3142 can remain partial. This explicit accounting is preferable to manufacturing a single universal similarity
3143 score that silently mixes formal, empirical, and phenomenal commitments.

3144 There are practical limits to this programme. Many neural systems cannot be experimentally reset to
3145 precisely controlled microscopic states; anatomical and behavioral observations provide partial proxies
3146 for underlying processes; high-dimensional dynamics may defeat stable reconstruction; and a proposed
3147 relational two-complex may be one of several useful encodings of the same physical substrate. The use
3148 of exhaustive finite counterexamples or reproducible synthetic models can clarify logical possibilities,
3149 but cannot measure the prevalence of those possibilities in biological cognition. A broader account will
3150 also require deliberate treatment of development, learning, bodily and environmental couplings, and the
3151 social conditions in which reports and interpretive categories are formed. The mathematical vocabulary
3152 supplied by the four orientations can accommodate several of these concerns only after the corresponding
3153 physical variables and maps are specified.

3154 15.7 Interpretive scope of the completed formal comparison

3155 The paper's established formal contribution is a set of typed equivalence criteria and conditional relations
3156 among them. It identifies information preserved by action isomorphisms, selected-path comparisons, in-
3157 dexed constitutive equivalences, path-language and path-law comparisons, and algebraic or geometric
3158 translations. The counterexamples establish that natural-seeming implications often fail when their extra
3159 hypotheses are omitted. None of these results singles out a mathematical invariant as necessary or suf-
3160 ficient for consciousness. They delimit the inferential commitments that a future candidate theory must
3161 make if it attributes phenomenal significance to that invariant.

3162 This limitation does not render mathematical equivalence irrelevant to the science of consciousness.
3163 The value of a well-specified relation lies in determining exactly which observations and transformations
3164 can bear on a theory's claims, how proposed similarities depend on the observer's access, and where al-
3165 ternative explanations remain compatible with the evidence. The formal results permit a more exact
3166 research conversation between dynamical neuroscience, topology, philosophy of science, and phenome-
3167 nal inquiry. They also caution against identifying the successful measurement of a clinically informative
3168 variable with the complete recovery of another subject's experience, a distinction that matters particularly
3169 in contexts of impaired reporting [3, 6, 25].

3170 The central synthesis is therefore conditional and plural. Equivalence is a relation between specified
3171 descriptions under specified criteria. Its preservation under translation is a further mathematical propo-
3172 sition. Its empirical accessibility depends on calibrated protocols and model assumptions. Its relevance
3173 to subjective experience depends on an argued and testable, where possible, phenomenal interpretation.
3174 Section 15 brings these dependencies together without treating any one of them as a substitute for the
3175 others. Section 16 consolidates these results and states the outstanding mathematical and scientific obli-
3176 gations.

3177 16 Conclusion

3178 The present investigation has developed a conditional mathematical framework for comparing equiva-
3179 lence claims in models relevant to consciousness. Its starting point is that an assertion of equivalence

acquires determinate content through a declared comparison domain, a criterion of preservation, and a mapping between the descriptions under consideration. The same pair of candidate systems may accordingly be equivalent with respect to their measured responses and distinguishable with respect to their transformation mechanisms, selected paths, constitutive correspondences, or admissible histories. These judgments can coexist without inconsistency because they refer to different mathematical properties. The principal contribution of the paper is to make their dependencies and limits of transfer explicit across four provisional explanatory orientations and three analytical registers.

Variation identifies equivalences of transformations and their organization. The analysis of monoid actions, effective actions, equivariant maps, and smooth conjugacies established conditions under which transformation structure, reachability, and parameter-preserving flow behavior are transported between descriptions. The counterexamples demonstrated that equality of reachable states or observed responses can leave the action generating those responses underdetermined. These distinctions matter whenever an experiment is interpreted as evidence about a mechanism rather than only about an observed outcome. They also show why a change of permitted generators may alter the experimentally accessible histories even when the full transformation monoid is unchanged.

Trajectory identifies equivalences of selected histories under criteria that include parameterization, reparameterization, geometric image, path homotopy, and observational projection. The formal results demonstrated that these criteria retain different information: a path image does not determine traversal multiplicity, an observed loop may conceal source winding, and finite sampling does not identify an unrestricted continuous path. Covering-space lifting and regularity assumptions supply explicit circumstances in which particular path properties become reconstructible. The use of a path parameter remains a modeling commitment whose physical meaning depends on an interpretation of ordering, clocks, or experimentally defined events. A mathematical history does not acquire a universal temporal direction merely from its notation.

Generativity identifies equivalence of modeled constitutive organization, including indexed relational structures, connecting maps, cellular attachments, and changing admissibility conditions. The examples of nonisomorphic diagrams with objectwise isomorphic constituents and of identical terminal complexes reached through different cellular constructions make a specific information-loss point: the structure obtained after a constitutive operation may omit distinctions encoded in its connecting maps. The categorical and cellular extension criteria establish when such mappings can be translated coherently. Within this framework, an indexed diagram records a proposed pattern of structural differences and correspondences. An account of how the relevant rules, constraints, and structures arise in a physical system remains a further explanatory task. Indexing alone does not establish endogenous generation or a physically realized succession.

Repertoire identifies equivalences among admissible-history families, including trace languages, branching structures, constrained path spaces, and probability laws. The finite-state constructions distinguished trace agreement from bisimulation and showed that bounded observations can leave longer admissible histories unresolved. The geometric results demonstrated that a homeomorphism of carrier spaces can preserve continuous path-space topology while failing to preserve an energy-constrained repertoire, unless the metrics and admissibility rules satisfy additional conditions. Probabilistic examples showed that equality of path support and stationary one-point marginals can coexist with different joint-history distributions. These distinctions are relevant when a theory concerns possible responses under a protocol, rather than the one history selected in a particular trial.

The comparisons across orientations provide a second result. Some implications can be proved under matched carriers, compatible generator sets, and stipulated admissibility schemes; other implications fail through explicit counterexamples. In particular, preservation of an action can transport a specified repertoire when its admissible generators are preserved, while agreement on a repertoire generally leaves distinctions among available mechanisms or selection laws. A selected trajectory supplies less information than a fully specified repertoire; terminal relational structures can retain less information than the diagrams defining their constitution. These results establish a partially ordered family of criteria rather than a universal ranking of the four orientations. The criteria are mathematical tools for separating ex-

planatory commitments. Their relative scientific importance depends on the question, model class, and evidence available.

The analysis across *ontological*, *relational*, and *manifestational* registers gives those distinctions an epistemic interpretation. Proposed physical realizations, their relational representations, and accessible records or reports can be connected by explicit, context-dependent maps. A map can preserve a target observation while identifying distinct source descriptions; recovering a property from that observation additionally requires a factorization or injectivity condition appropriate to the property. The smooth-flow and graph examples exhibited complete agreement of selected manifestations alongside different source topology or relational organization. Accordingly, a statement about observational equivalence has the scope of the stipulated observation protocol. Extending it to physical realization, relational constitution, or a phenomenological target requires further premises and supporting evidence. The registers specify distinct questions of access and interpretation; they imply no intrinsic temporal hierarchy or ordering of epistemic authority.

The algebraic, categorical, and geometric preservation results sharpen this methodological conclusion. Quotient congruences determine which actions descend to an observation space; fully faithful functors provide a stronger guarantee for reflecting isomorphisms than faithful functors alone; natural transformations test coherence across indexed descriptions; homotopy and homology preserve selected topological properties while omitting other attaching information. A diffeomorphism preserves smooth carrier structure and induces a correspondence between continuous paths, yet preservation of their energy and admissibility requires metric bounds or stronger geometric structure. These are established properties of the mathematical representations considered here. They neither designate a privileged representation of a neural system nor establish that any one of these invariants is a necessary or sufficient condition of consciousness.

The empirical consequences of an equivalence claim arise only after the formal criterion is coupled to an operational model. Sections 13 and 15 distinguished identification of a property from ideal response laws, statistical discrimination under finite observations, and the separate demands of causal attribution. A controlled preparation can reveal a parameter that remains unidentifiable under passive observation, while stochastic measurement and nuisance variation can reduce the available discriminating information. Even a successful randomized comparison identifies an effect on its specified outcome under the assumptions of the experimental design. Additional modeling and evidence are needed to attribute that effect to an intervening generative mechanism or to a phenomenal difference. Table 14 records the corresponding evidential obligations for each of the four orientations; its proposed contrasts require physically specified candidate models before they become testable hypotheses.

The paper has also delimited the reach of formal equivalence in phenomenal inference. A proposed mapping from physical or relational descriptions to phenomenal properties can make recovery or preservation questions mathematically precise, provided the mapping and its admissible alternatives are declared. The bridge-family and latent-report results of Section 14 show how different phenomenal assignments or latent descriptions may remain consistent with the same observational record under the modeled assumptions. This underdetermination identifies a burden of calibration and explanation. It does not prove that satisfactory phenomenal bridges are impossible, nor that a particular mathematical classification determines what another subject experiences. For systems with limited reporting capacity, explicit recognition of this distinction is especially important. Moral and legal judgments about a candidate subject likewise require their own normative arguments; mathematical equivalence relative to a convenient marker does not settle those judgments.

Several limitations determine the status of the present results. The finite automata, cellular complexes, smooth flows, stochastic channels, and path spaces provide proofs and counterexamples in stipulated mathematical domains. Their synthetic character limits their immediate empirical interpretation. The ontological and relational carriers considered here have not been identified with a validated neurobiological model or a warranted space of phenomenal states. Some constructed examples concern idealized exact observations or unconstrained interventions that may not be achievable in neural experiments. The framework also leaves open how bodily, developmental, environmental, and interpersonal

relations should enter a physically grounded constitutive description. Mathematical exactness within the chosen domains therefore coexists with substantial uncertainty concerning explanatory adequacy for consciousness.

The resulting research agenda is precise in its dependencies. One task is to construct physically interpretable translations between competing model families, specifying which invariants survive a change of state coordinates, coarse-graining, discretization, or observation protocol. A second is to develop counterexample-guided experiments: for each proposed marker, one can seek candidate systems that agree on the marker yet diverge in a purported mechanism, and identify an intervention or richer observation that separates their predictions. A third is to assess phenomenal bridge proposals using declared access conditions, report calibration, nuisance assumptions, and evidence capable of distinguishing alternatives. These tasks can be pursued together without requiring the four orientations to be treated as exclusive theories or the three registers to be treated as levels of a single predetermined ontology.

The completed formal comparison establishes the scope of several equivalence criteria, the assumptions under which they imply one another, and explicit circumstances in which they remain incomparable. It provides a structured account of the information lost when transformation mechanisms are replaced by reachable outcomes, selected histories by coarse traces, constitutive diagrams by terminal summaries, or complete repertoires by limited statistical records. Its contribution to consciousness research is a set of checkable comparison obligations for future physical models and experimental designs. Determining whether any of the preserved distinctions explains the existence, organization, or character of subjective experience remains an open scientific and philosophical problem, to be approached through models whose mathematics, measurements, causal interpretations, and phenomenal premises can be examined independently.

A Formal Definitions, Equivalence Axioms, and the Implication Pre-order

This appendix records the mathematical types and logical qualifications required by the equivalence comparisons of Sections 4–10. Its purpose is to make the domain restrictions, factorization conventions, and comparison operations independently inspectable. The results are elementary consequences of set-theoretic equivalence, with supplementary counterexamples chosen to expose conditions that can be concealed by shorthand. They concern mathematical descriptions and declared observations; none is an axiom of phenomenal identity.

A.1 Typed descriptions and equality conventions

A comparison begins with a context c , an orientation $h \in \{V, T, G, R\}$, a register $\ell \in \{O, L_r, M\}$, and a criterion label k . The index selects a set $D = \mathcal{D}_{h,\ell}^c$ of admissible, well-typed descriptions, and a total function $I : D \rightarrow U$ whose values are the features to be retained. If a property is defined only for a subset $D_0 \subseteq D$, then D_0 is the comparison domain; assertions about descriptions outside it do not follow. Two invariant assignments that use different targets or differently typed source descriptions require an explicit translation before their equalities can be compared.

Definition A.1 (Typed equivalence criterion). A *typed criterion record* is a tuple $C = (c, h, \ell, k; D, U, I)$, where $I : D \rightarrow U$ is a total function into a target supplied with a specified equality relation. The induced equivalence is equality of invariant values, as defined in Equation 85.

$$E_C = \ker I := \{(x, x') \in D \times D : I(x) = I(x')\}. \quad (85)$$

Equation 85 defines an equivalence relation because equality in U is reflexive, symmetric, and transitive. When U consists of objects compared up to isomorphism, one can instead take the set of isomorphism classes as target; an arbitrary choice of object labels is not an intrinsic invariant. In category-valued

comparisons, an isomorphism criterion must similarly declare the relevant category. This distinction matters for cellular complexes, dynamical systems with chosen coordinates, and historically indexed diagrams.

The register label is an analytical role, not a proof that the corresponding mathematical carrier is an independently established ontology. In the relational two-complex convention, vertices represent events, edges couplings, and faces locally specified higher-order primitives. The emergence of subjects or other persistent entities requires a separate constitutive interpretation. An index of diagram objects, the parameter of a path, and a physical clock are not identified without a further assumption.

A.2 Canonical quotients and the comparison preorder

Fix a common set D . For two criteria $I : D \rightarrow U$ and $J : D \rightarrow V$, write $I \succeq_D J$ when $\ker I \subseteq \ker J$. This expresses the implication from equality under I to equality under J ; it does not order the scientific importance of either criterion. The preorder identifies different encodings that retain the same partition of D . The quotient map $q_I : D \rightarrow D/\ker I$ sends a description to its invariant-equivalence class.

Proposition A.2 (Canonical representation and factorization). *Let $I : D \rightarrow U$ and $J : D \rightarrow V$ be total functions. There is a canonical bijection $\bar{I} : D/\ker I \rightarrow I(D)$ taking $[x]$ to $I(x)$. Moreover, $\ker I \subseteq \ker J$ holds if and only if there exists a unique function $g : I(D) \rightarrow J(D)$ for which $J = g \circ I$. The induced relation $\succeq_D$ is a preorder; after quotienting criteria by equality of kernels it is a partial order.*

Proof. If $[x] = [x']$, then $I(x) = I(x')$, proving that $\bar{I}$ is well defined. The same equivalence proves injectivity, and the definition of $I(D)$ gives surjectivity. Suppose $\ker I \subseteq \ker J$. Define $g(I(x)) = J(x)$. Kernel inclusion makes this assignment independent of the choice of x ; it is a map into $J(D)$ and satisfies the displayed factorization. Every element in $I(D)$ equals $I(x)$ for some x , giving uniqueness. Conversely, $J = g \circ I$ implies that equality of I -values entails equality of J -values. Reflexivity and transitivity of $\succeq_D$ follow from inclusion; antisymmetry holds once criteria with identical kernels are identified. $\square$

The factorization in Proposition A.2 applies to the *realized image* $I(D)$. It does not guarantee a canonical extension of g to unused values of U , let alone a smooth, measurable, causal, or physically implementable extension. The quotient-order interpretation is summarized in Equation 86.

$$[I] \succeq_D [J] \iff E_I \subseteq E_J, \quad [I] := \{J' : \ker J' = \ker I\}. \quad (86)$$

The brackets in Equation 86 designate equivalence classes of criterion *descriptions*; they do not denote equivalence classes of physical systems or subjects. In a finite domain, the criterion order can be computed from partitions. In an unrestricted domain, its existence is a mathematical statement independent of the possibility of enumerating or measuring all comparisons.

A.3 Combination, restriction, and hidden intermediates

For equivalence relations $E, F \subseteq D \times D$, their meet is intersection, and their join is the least equivalence relation containing their union. For functions I and J on D , recording both values gives $\ker(I, J) = \ker I \cap \ker J$. By contrast, joining kernels allows finite chains of alternating identifications. The join need not equal the union because the latter may fail transitivity.

A subtlety arises when only a restricted domain is observed. For a map $t : X \rightarrow D$, define the pullback $t^{-1}E$ by $(x, x') \in t^{-1}E$ precisely when $(t(x), t(x')) \in E$. Pullback respects intersections but need not respect the equivalence closure of unions.

Proposition A.3 (Restriction of combined equivalences). *For every map $t : X \rightarrow D$ and equivalence relations E, F on D , the relations in Equation 87 hold. The second inclusion can be strict even when t is an inclusion of a subset.*

$$\begin{aligned} t^{-1}(E \wedge F) &= (t^{-1}E) \wedge (t^{-1}F), \\ (t^{-1}E) \vee (t^{-1}F) &\subseteq t^{-1}(E \vee F). \end{aligned} \tag{87}$$

Proof. Membership in the pullback of an intersection means membership in both pullbacks, proving the equality. Each of $t^{-1}E$ and $t^{-1}F$ is contained in $t^{-1}(E \vee F)$; the latter is an equivalence relation. Their least common upper bound is therefore contained in that pullback. For strictness take $D = \{a, b, c\}$, $X = \{a, c\}$, and let t be inclusion. Let E have classes $\{a, b\}, \{c\}$ and let F have classes $\{a\}, \{b, c\}$. The restrictions of E and F to X are both equality, so their join on X is equality. In D , however, $a E b F c$; hence $E \vee F$ identifies a with c . Its pullback to X is the universal equivalence, proving strictness. $\square$

The counterexample shows that a comparison made only among sampled models can miss an intermediate description needed to witness an equivalence chain in a larger domain. Conversely, a join computed in the larger domain may identify observed descriptions by way of an inaccessible intermediate. Neither convention is intrinsically preferred. The scope of the join and the available intermediates must be declared when comparing an experimental or historically reconstructed repertoire of descriptions.

A.4 Families of protocols and finite separation

Let P be an index set of observational protocols on a common description set D . Protocol p produces an ideal record $I_p : D \rightarrow Y_p$, and their joint record is the product map into $\prod_{p \in P} Y_p$. For nonempty P , the distinction retained by the joint family is stated in Equation 88.

$$\ker((I_p)_{p \in P}) = \bigcap_{p \in P} \ker I_p. \tag{88}$$

Equation 88 refers to ideal protocol outputs, not to a finite data set drawn under experimental noise. Recovery of a property $q : D \rightarrow Q$ is possible at the level of exact records when the left-hand kernel is contained in $\ker q$, by Proposition A.2.

Proposition A.4 (Finite subfamily for finite-domain property recovery). *Suppose D is finite and $q : D \rightarrow Q$ is a property such that $\bigcap_{p \in P} \ker I_p \subseteq \ker q$. Then there exists a finite set of protocols $P_0 \subseteq P$ with $\bigcap_{p \in P_0} \ker I_p \subseteq \ker q$. If N_q is the number of unordered pairs of descriptions with different q -values, the family can be chosen with at most N_q members.*

Proof. For every unordered pair $\{x, x'\}$ with $q(x) \neq q(x')$, the assumed kernel inclusion guarantees a protocol $p_{x, x'} \in P$ for which $I_{p_{x, x'}}(x) \neq I_{p_{x, x'}}(x')$. Choose one witness for each such pair and collect the chosen protocols as P_0 . There are N_q eligible pairs, hence at most N_q chosen protocols. If all protocols in P_0 agree on a pair, that pair cannot have different q -values, giving the required inclusion. If $N_q = 0$, the empty family suffices because q is constant. $\square$

The finite-domain hypothesis matters. Let $D = \mathbb{N}$ and let protocol $I_n : D \rightarrow \{0, 1\}$ test whether $x = n$. All protocols jointly distinguish any two distinct natural numbers, but any finite subfamily leaves infinitely many natural numbers with the same all-zero record. Thus an infinite family may yield exact model-level identification without a finite separating subfamily. This is a claim about ideal distinguishability; neither the cost of a protocol nor its noisy statistical power is addressed by Proposition A.4.

A.5 Translation squares and preservation versus reflection

Translations between contexts may change both the model domain and the invariant target. Let $t : D \rightarrow D'$ be a description translation, let $I : D \rightarrow U$ and $I' : D' \rightarrow U'$ be criteria, and suppose $u : U \rightarrow U'$ satisfies the commuting condition in Equation 89.

$$I' \circ t = u \circ I. \tag{89}$$

Under Equation 89, source criterion equivalence always implies equivalence of the translated descriptions. The converse is sensitive to the values discarded by u on the image actually realized by I .

Proposition A.5 (Exact reflection criterion for a commuting comparison). *Under Equation 89, the implication $I(x) = I(y) \Rightarrow I'(t(x)) = I'(t(y))$ holds for all $x, y \in D$. The reverse implication holds for all $x, y \in D$ if and only if u is injective on $I(D)$.*

Proof. The forward implication follows by applying u to equal values. If $u|_{I(D)}$ is injective, equality of translated values yields $u(I(x)) = u(I(y))$ and hence $I(x) = I(y)$. Conversely, if u is not injective on $I(D)$, choose distinct realized values $a = I(x)$ and $b = I(y)$ such that $u(a) = u(b)$. The commuting square gives $I'(t(x)) = I'(t(y))$ while $I(x) \neq I(y)$, contradicting reflection. $\square$

Proposition A.5 is a statement about the selected *criterion*; it does not assert that t is injective or that every source mechanism, path, or cellular attachment can be reconstructed. A separate typed comparison is required when the data retained by I' belong to a different mathematical category. In particular, post-composing indexed diagrams objectwise is weaker than preserving their natural transformations unless compatibility with the connecting maps is also established, as developed in Section 11.

A.6 Approximate agreement and stochastic observations

Suppose $I : D \rightarrow (U, d)$ takes values in a metric space. A tolerance test may declare two descriptions close whenever $d(I(x), I(y)) \leq \varepsilon$. Reflexivity and symmetry follow, but transitivity generally fails. On the real line, for instance, the three values $0, \varepsilon, 2\varepsilon$ show failure when $\varepsilon > 0$. There is a mathematical circumstance in which a fixed positive tolerance *does* define an equivalence: the strong triangle inequality of an ultrametric.

Proposition A.6 (Tolerance equivalence for ultrametric invariants). *If d is an ultrametric on U , then for every $\varepsilon \geq 0$ the relation in Equation 90 is an equivalence relation on D . For an ordinary metric, it need not be transitive.*

$$x \sim_{I, \varepsilon} y \iff d(I(x), I(y)) \leq \varepsilon. \quad (90)$$

Proof. Reflexivity and symmetry follow from the corresponding metric axioms. For an ultrametric, $d(a, c) \leq \max\{d(a, b), d(b, c)\}$. If $x \sim_{I, \varepsilon} y$ and $y \sim_{I, \varepsilon} z$, both distances on the right are at most ε , so $x \sim_{I, \varepsilon} z$. In a general metric the triangle inequality provides only an upper bound 2ε , and the real-line example exhibits nontransitivity for positive tolerance. $\square$

The exactness of this mathematical relation is determined by the codomain metric, not by a presumption that two nearby neural measurements correspond to phenomenally equivalent states. For a noisy protocol represented by a Markov kernel $K : D \rightsquigarrow Y$, distinct comparisons are available: equality of the complete laws $K(\cdot \mid x)$ and $K(\cdot \mid y)$; approximate closeness of those laws under a specified statistical distance; and failure to reject a test of equality from finitely many observed draws. Only the first directly induces the kernel equivalence of a law-valued invariant. The last describes evidence under a testing rule, sampling design, nuisance model, and error criterion. The statistical and causal obligations of these comparisons are discussed in Sections 13 and 14.

A.7 Assumption register for the twelve comparison cells

The four orientations and three analytical registers produce twelve indexing cells, but an entry specifies a *question and admissible type*, not a universal natural classification. Table 15 records the information that must accompany an equivalence assertion before the implications developed in the main text can be applied.

Table 15: Formal assumption register for criterion-indexed comparisons. Each row lists the mathematical object, the required explicit data, and a conclusion excluded without further premises.

Comparison focus	Required declaration	Additional conclusion requiring independent premises
Variation	Carrier, action law, generators, congruence and action translation	Identity of mechanisms from reachability alone
Trajectory	Path domain, parameter convention, endpoints, topology, allowed reparametrizations	Identity of source homotopy classes from coarse projections
Generativity	Indexed objects, connecting maps, coherence and admissibility of constitutive changes	Equivalence of constitutive diagrams from terminal objects
Repertoire	Admissible path family, path topology, branching or probability law	Equality of mechanisms or selection laws from trace support
Ontological register	Candidate realization domain and physically interpreted comparison map	Phenomenal identity from a formal realization isomorphism
Relational register	Event/coupling/face semantics and selected structural invariants	Complete reconstruction from homology or local statistics
Manifestational register	Protocols, record types, sampling and calibration assumptions	Source-level recovery or phenomenal identity from observed equality
Cross-register transfer	Typed commuting interfaces and realized invariant images	Reflection of source distinctions without image-level injectivity

The conjunction of a declared orientation and register does not by itself fix the mathematical invariant: two models can be equivalent relative to one criterion in a cell and inequivalent relative to another. Proposition A.5 states the extra condition for reflecting an exact criterion through a specified commuting translation; Proposition A.4 clarifies what finite collections of ideal protocols can recover on finite domains. The relations remain formal tools for comparison. Establishing their explanatory relevance to consciousness requires a defensible physical model, an explicit observational and causal design, and, where phenomenal claims are made, a separately argued bridge between the formal structures and experience.

B Homotopy, Torus Examples, and Cellular Invariants

This appendix consolidates explicit calculations used when an equivalence criterion is defined on based paths, fundamental groups, cellular complexes, or maps between constitutive descriptions. Its objective is to distinguish three mathematical tasks: recovery of a based homotopy class from an observed class; unique reconstruction of a parameterized path after supplying an initial lift; and preservation of attaching information by a cellular or chain-level translation. These tasks use different assumptions, even when they are expressed through the same carrier. All paths below are mathematical maps from a declared parameter interval; the interval is not taken to constitute physical time. The topological constructions supplement Sections 6, 7, and 12.

B.1 Integer torus maps and the recovery of loop classes

Write $T^d = \mathbb{R}^d / \mathbb{Z}^d$, with basepoint $[0]$. For an integer matrix $A \in M_{r \times d}(\mathbb{Z})$ define the continuous homomorphism $q_A : T^d \rightarrow T^r$ by Equation 91. Integer entries are necessary for the formula to be independent of the representative of a point in the source quotient.

$$q_A([x]) = [Ax], \quad (q_A)_* : \pi_1(T^d, [0]) \cong \mathbb{Z}^d \longrightarrow \pi_1(T^r, [0]) \cong \mathbb{Z}^r, \quad w \longmapsto Aw. \quad (91)$$

An integer winding vector w is represented by $t \mapsto [tw]$. Consequently, comparison of observed based loop classes is the linear-algebraic problem of deciding whether A retains distinct integral vectors.

Proposition B.1 (Integer rank and based-loop reconstruction). *For the map in Equation 91, observed based homotopy classes determine source based homotopy classes if and only if A has rank d over $\mathbb{Q}$. If A has rank less than d , there are distinct based source loops with the same complete observed path, not merely with the same observed homotopy class.*

Proof. The induced map on fundamental groups is the restriction of the linear map $A : \mathbb{Q}^d \rightarrow \mathbb{Q}^r$ to the integer lattice. If A has rational rank d , its kernel is zero and the restriction is injective. If it has smaller rank, Gaussian elimination over $\mathbb{Q}$ provides a nonzero rational kernel vector; multiplying by common denominators gives $w \in \mathbb{Z}^d \setminus \{0\}$ with $Aw = 0$. The source loops $\gamma_w(t) = [tw]$ and $\gamma_0(t) = [0]$ are in distinct based homotopy classes, yet both have the constant observed path because $q_A(\gamma_w(t)) = [tAw] = [0]$. $\square$

For projection $q_{(1\ 0)} : T^2 \rightarrow S^1$, the kernel is $\{0\} \times \mathbb{Z}$. More generally, all loops $\gamma_{u,v}(t) = ([ut], [vt])$ with fixed u have the same observed path under this projection. The observation may therefore preserve a first-factor winding while losing the entire second-factor winding. By contrast, an integral 2×2 matrix of nonzero determinant reflects based homotopy equivalence, even when its observation is not a homeomorphism. The distinction between injectivity and invertibility on the integral lattice will matter for path lifting.

B.2 Covering maps, initial sheets, and lifted histories

Let $A \in M_{d \times d}(\mathbb{Z})$. If $\det A \neq 0$, the induced torus map is a finite-sheeted covering; its kernel as a homomorphism of tori is the finite group $A^{-1}\mathbb{Z}^d / \mathbb{Z}^d$. Its number of sheets equals the index of the image lattice. The formula is given in Equation 92.

$$\# \ker(q_A) = [\mathbb{Z}^d : A\mathbb{Z}^d] = |\det A|. \quad (92)$$

Proposition B.2 (Closing criterion for an observed loop lift). *Let $A \in M_{d \times d}(\mathbb{Z})$ have nonzero determinant. An observed based loop b in T^d admits a unique lift under q_A starting at a prescribed point in the fiber above $b(0)$. This lifted path is closed if and only if $[b] \in A\mathbb{Z}^d \subseteq \pi_1(T^d) \cong \mathbb{Z}^d$.*

Proof. Nonsingularity of A over $\mathbb{R}$ gives local inverses for $[x] \mapsto [Ax]$, and its finite quotient kernel provides a finite evenly covered neighborhood around every target point. Existence and uniqueness of the path lift from a specified initial sheet follow from the covering-space lifting theorem. Write the winding vector of b as $v \in \mathbb{Z}^d$ and lift b to a real path $B : [0, 1] \rightarrow \mathbb{R}^d$ with $B(1) - B(0) = v$. A lift through q_A may be written locally, and hence globally with a consistent initial choice, using $A^{-1}B(t)$. Its endpoint difference in the covering vector space is $A^{-1}v$. The lifted torus path closes precisely when this difference belongs to $\mathbb{Z}^d$, equivalently when $v \in A\mathbb{Z}^d$. $\square$

Consider $A = \text{diag}(2, 1)$. It is injective on $\pi_1(T^2)$, but has two initial sheets over each target point. The target loop of winding $(1, 0)$ lifts from either sheet to a path joining distinct sheets and does not lift to a closed loop. The target loop of winding $(2, 0)$ lifts to a closed loop. Thus, *class recoverability for loops which arise from source loops, unique lifting after initial-sheet specification, and surjectivity onto all target loop classes* are distinct properties. Figure 10 compares the image of this covering with the kernel of a rank-deficient observation.

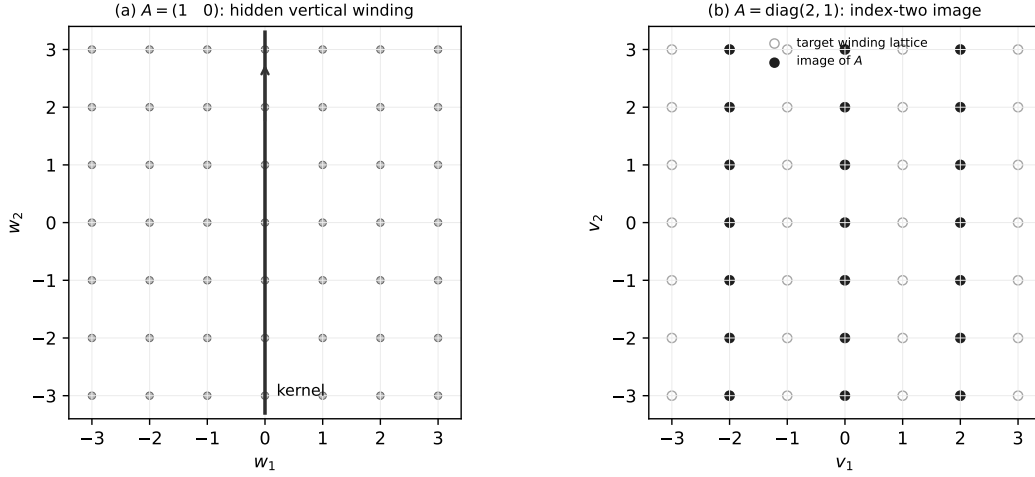


Figure 10: Exact integer-lattice representations of two observation maps. Left: projection $A = (1 \ 0)$ forgets the entire vertical winding direction. Right: $A = \text{diag}(2, 1)$ injects source winding vectors into the even-first-coordinate sublattice, preserving source loop distinctions but not covering every target winding class. The lattice depicts algebraic homotopy information rather than physical spatial coordinates.

B.3 Cellular attaching words and integral homology

Let $W = S_a^1 \vee S_b^1$ be a bouquet of two oriented circles. For a word w in the free group $F(a, b)$, let $K_w = W \cup_w D^2$ be obtained by attaching one closed two-cell according to a based loop representing w . Define $n_a(w)$ and $n_b(w)$ as the total signed exponent sums of a and b in w . Cellular homology detects these exponent sums but does not reconstruct the entire attaching word. The relevant integral boundary map is Equation 93.

$$0 \longrightarrow C_2(K_w; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\partial_2} C_1(K_w; \mathbb{Z}) \cong \mathbb{Z}^2 \xrightarrow{\partial_1=0} \mathbb{Z} \longrightarrow 0, \quad \partial_2(1) = \begin{pmatrix} n_a(w) \\ n_b(w) \end{pmatrix}. \quad (93)$$

Proposition B.3 (Smith invariant of a one-relator cellular attachment). *Let $n = n_a(w)$ and $m = n_b(w)$. If $(n, m) \neq (0, 0)$ and $g = \gcd(|n|, |m|)$, then $H_1(K_w; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/g\mathbb{Z}$ and $H_2(K_w; \mathbb{Z}) = 0$, with $\mathbb{Z}/1\mathbb{Z}$ understood as trivial. If $(n, m) = (0, 0)$, then $H_1(K_w; \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(K_w; \mathbb{Z}) \cong \mathbb{Z}$. In both cases $\pi_1(K_w)$ has the presentation $\langle a, b \mid w \rangle$.*

Proof. The cellular boundary in Equation 93 has rank one if $(n, m) \neq (0, 0)$ and rank zero otherwise. In the first case elementary unimodular integer row operations transform the column $(n, m)^T$ to $(g, 0)^T$, so its cokernel is $\mathbb{Z}/g\mathbb{Z} \oplus \mathbb{Z}$ and its kernel vanishes. In the second case the boundary vanishes and the asserted homology follows immediately. The fundamental-group presentation is supplied by the attaching-map form of the Seifert–van Kampen theorem. $\square$

The case $w = [a, b] = aba^{-1}b^{-1}$ and the case $w = 1$ have the same exponent-sum vector $(0, 0)$. Their cellular chain complexes and integral homology groups agree, yet their fundamental groups are different. In Equation 94 the first complex is the standard two-dimensional torus CW structure, whereas the second is $S^1 \vee S^1 \vee S^2$.

$$\begin{aligned} H_1(K_{[a,b]}; \mathbb{Z}) &\cong H_1(K_1; \mathbb{Z}) \cong \mathbb{Z}^2, \\ H_2(K_{[a,b]}; \mathbb{Z}) &\cong H_2(K_1; \mathbb{Z}) \cong \mathbb{Z}, \\ \pi_1(K_{[a,b]}) &\cong \mathbb{Z}^2, \quad \pi_1(K_1) \cong F(a, b). \end{aligned} \quad (94)$$

Since the free group on two generators is nonabelian, these complexes are not homotopy equivalent. In particular, an identity chain isomorphism between their cellular chain complexes does not imply the

existence of a topological equivalence preserving the respective attaching maps. Homology is a controlled invariant of a specified comparison; it is not a complete coding of relational incidence and higher attaching structure.

B.4 Relative homology and the preservation of attaching degrees

Write $K_n = S^1 \cup_{p_n} D^2$ for the result of attaching a two-cell by a degree- n circle map, and let $i : S^1 \hookrightarrow K_n$ be the distinguished one-skeleton. Regardless of n , the pair (K_n, S^1) has $H_2(K_n, S^1; \mathbb{Z}) \cong \mathbb{Z}$. The boundary homomorphism in the exact sequence of this pair remembers the attaching degree, as stated in Equation 95.

$$\partial_n^{\text{rel}} : H_2(K_n, S^1; \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}, \quad z \longmapsto nz. \quad (95)$$

Proposition B.4 (Naturality obstruction for degree-preserving cellular translations). *Let n, m be nonzero integers. If a map of pairs $f : (K_n, S^1) \rightarrow (K_m, S^1)$ has degree k on the distinguished one-skeleton and degree r on relative H_2 , then $kn = mr$. Conversely, for any integers k, r satisfying this equality, a cellular map of pairs exists with those induced degrees. In particular, a prescribed degree- k map on the one-skeleton extends across the attached two-cell if and only if m divides kn .*

Proof. Naturality of the long exact sequences of pairs gives the commuting square $\partial_m^{\text{rel}} \circ f_{*,2}^{\text{rel}} = f_{*,1} \circ \partial_n^{\text{rel}}$. Substituting the multiplication homomorphisms of Equation 95 yields $mr = kn$. Conversely, a degree- k circle map sends the degree- n attaching loop to a loop of degree kn in the target one-skeleton. Its class in $\pi_1(K_m) \cong \mathbb{Z}/m\mathbb{Z}$ vanishes precisely when m divides kn , allowing extension over the disk. Cellular approximation relative to the one-skeleton yields a cellular extension. Naturality then forces its relative degree to equal $kn/m = r$. $\square$

An exact square of relative homology groups therefore retains a degree constraint that cannot be inferred from the homeomorphism type of the one-skeleton alone. The argument also illustrates the necessity of declaring which subcomplex and which comparison map count as part of the model: the same unmarked terminal complex may admit inequivalent marked descriptions. The parameter n is attaching data within a mathematical construction and does not by itself specify a physically realized transition in a neural system.

B.5 Coefficient fields and torsion orders

Real- and field-coefficient homology can conceal torsion orders. A particularly transparent comparison is K_2 versus K_4 . Both have trivial H_1 with rational coefficients, and for each field $\mathbb{F}_p$ they have isomorphic homology groups in all degrees. Indeed, multiplication by either 2 or 4 is invertible modulo every odd prime and zero modulo 2. The complete coefficient-sensitive description is stated in Equation 96.

$$\begin{aligned} H_1(K_2; \mathbb{Z}) &\cong \mathbb{Z}/2\mathbb{Z}, & H_1(K_4; \mathbb{Z}) &\cong \mathbb{Z}/4\mathbb{Z}, \\ H_j(K_2; \mathbb{F}_p) &\cong H_j(K_4; \mathbb{F}_p) \quad \text{for every prime } p \text{ and every } j \geq 0. \end{aligned} \quad (96)$$

For $p = 2$, both complexes have one-dimensional first and second homology over $\mathbb{F}_2$; for odd p , those groups are zero. Their degree-zero homology is one-dimensional over every field. Nevertheless, the two integral first-homology groups in Equation 96 have different exponents. Thus, agreement of Betti numbers over every prime field does not determine the isomorphism type of integral homology. The example is especially relevant when an observational translation retains ranks of matrices while omitting their Smith normal forms.

B.6 Topological preservation ledger and interpretive scope

Table 16 records the specific information retained by the constructions above. Its columns distinguish a mathematical criterion from the extra premises necessary for reconstruction of source data. Each comparison must also specify the marked subcomplex, basepoint and coefficient system where applicable.

Table 16: Selected preservation and recovery tests for topological comparisons. These statements concern mathematical carriers and maps, not presumed phenomenal invariants.

Information retained	Sufficient or exact test in the examples	Source information still potentially lost
Based torus loop class	$\text{rank}_{\mathbb{Q}} A = d$	Detailed path geometry and unmodeled source coordinates
Initial-point path lift	Finite torus covering and prescribed initial sheet	Initial-sheet identity if it is not supplied
Closed lift of observed loop	Observed winding belongs to $A\mathbb{Z}^d$	Nonclosed source lifts of other target loops
Cellular integral homology	Smith invariant factors of boundary maps	Nonabelian attaching words and fundamental group
Relative attaching data	Naturality of relative boundary and marked one-skeleton	Unrecorded intermediate structures or constitutive semantics
Field-coefficient homology	Rank after reduction in chosen fields	Exponents of integral torsion summands

The relations among these criteria are partial. Fundamental-group injectivity under observation may permit reconstruction of source loop classes but says nothing about source path speed or the organization of transformations generating those loops. A covering observation may permit unique path lifting once an initial sheet is declared while failing to cover all target homotopy classes by source loops. Equality of cellular chain complexes may fail to reconstruct attaching words, and equality of homology over every field may fail to reconstruct integral torsion. Moreover, passing from one complex to another through an indexed diagram includes connecting maps as mathematical data; comparison of isolated objects cannot substitute for naturality of those maps.

In the relational two-complex convention used here, vertices represent events, edges represent couplings, and faces represent locally specified higher-order primitives. The present CW calculations concern possible mathematical presentations of such structures; they do not identify a vertex with a conscious subject or assign a physical temporal direction to an attachment. The relevance of any preserved invariant to Variation, Trajectory, Generativity, Repertoire, or subjective experience must be separately justified by a physical interpretation, an observation protocol, and a model of the constitutive relations at issue. In this restricted role, the examples provide a reproducible collection of tests for statements of topological equivalence across the three analytical registers.

C Dynamical and Process Equivalence Counterexamples

This appendix develops finite and continuous constructions that distinguish the equivalence of transformation prescriptions, sampled transitions, selected histories, branching possibilities, observational quotients, and probabilistic selection laws. The objective is to supply verifiable countermodels for the conditional implications discussed in Sections 5, 6, 8, and 9. The constructions use explicit mathematical clocks, sampling intervals, or word positions; interpreting these parameters as physical or experiential time requires an additional modeling commitment. They rely on established distinctions between linear-time and branching-time semantics [13, 1], while the examples below are chosen to make their preservation obligations inspectable.

C.1 Sampling a continuous flow and losing its generator

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, fix a sampling interval $\Delta > 0$, and for any $\omega \in \mathbb{R}$ write $\Phi_s^\omega([\theta]) = [\theta + \omega s]$. Here $s \in \mathbb{R}$ is the parameter of the mathematical flow. The sampled map is the single transformation $P_\omega = \Phi_\Delta^\omega$. Sampling therefore retains one element of the one-parameter transformation group, not necessarily the

full group representation or its infinitesimal generator. The aliasing condition is given by Equation 97.

$$P_\omega = P_\nu \iff (\omega - \nu)\Delta \in 2\pi\mathbb{Z}. \quad (97)$$

Proposition C.1 (Sampled equivalence with inequivalent continuous flows). *For each $\Delta > 0$ there are continuous smooth flows on S^1 whose sampled maps at interval Δ agree at every state, but whose full flows are not orbit equivalent by any homeomorphism carrying flow orbits onto flow orbits. Among nonstationary constant-speed rotations, preservation of the full parameterized flow by a topological conjugacy requires equality of the absolute speeds.*

Proof. Take $\omega = 0$ and $\nu = 2\pi/\Delta$. Both time- Δ maps are the identity by Equation 97; their every integral iterate agrees. For $\omega = 0$, each flow orbit is a singleton; for $\nu \neq 0$, every full real-parameter orbit is S^1 . A homeomorphism maps singleton sets to singleton sets and cannot carry the first orbit partition to the second. For nonzero speeds, the minimal positive return parameters are $2\pi/|\omega|$ and $2\pi/|\nu|$. A parameter-preserving topological conjugacy preserves those return parameters, so it requires $|\omega| = |\nu|$. Conversely, equal speeds admit the identity conjugacy and opposite equal-magnitude speeds admit the circle reflection $[\theta] \mapsto [-\theta]$. $\square$

This construction yields an exact identifiability failure even when the entire sampled map is known. It is stronger than equality of a few sampled paths, and it does not depend on measurement noise. Additional inter-sample observations, generator restrictions, or alternative sampling intervals can sometimes distinguish the models; those resources must be specified rather than inferred from the discrete map. The observation neither establishes nor refutes any claim about physical or phenomenal identity.

C.2 Equal visible traces and inequivalent branching structures

A labelled transition system is a set S with relations $\xrightarrow{a} \subseteq S \times S$ for labels a in a specified alphabet Σ . Its finite trace language from state s comprises all finite words labeling paths from s , including the empty word ϵ . A *strong bisimulation* R between two such systems matches every transition from either related state by a transition carrying the same label into another related pair. This back-and-forth requirement retains information concerning where alternatives are available, beyond the existence of their concatenated labels.

Consider the following two pointed finite systems, with 0 and 0' terminal states. The left root p branches under label a , while the right root q makes one a -transition to a state retaining both possible continuations. Their transitions are listed in Equation 98.

$$\begin{array}{llll} p \xrightarrow{a} p_b, & p \xrightarrow{a} p_c, & p_b \xrightarrow{b} 0, & p_c \xrightarrow{c} 0, \\ q \xrightarrow{a} q_{bc}, & & q_{bc} \xrightarrow{b} 0', & q_{bc} \xrightarrow{c} 0'. \end{array} \quad (98)$$

Proposition C.2 (Trace agreement and modal separation). *The pointed systems in Equation 98 have the same finite trace language, but their roots are not strongly bisimilar. They are distinguished by the elementary modal formula $\langle a \rangle (\langle b \rangle \top \wedge \langle c \rangle \top)$.*

Proof. Every nonempty path from either root has label a , ab , or ac ; thus both finite trace languages are $\{\epsilon, a, ab, ac\}$. In any proposed bisimulation relating p and q , the a -transition from p to p_b must be matched by the sole a -transition from q to q_{bc} , requiring (p_b, q_{bc}) to be related. The state q_{bc} has a c -transition, whereas p_b has none; the back condition fails. Moreover q satisfies the displayed modal formula by choosing q_{bc} , where both b and c are available. The state p does not satisfy it, since neither of its a -successors enables both labels. $\square$

The example makes explicit why trace equivalence and strong bisimulation occupy different positions in the linear-time–branching-time spectrum. The modality $\langle a \rangle \varphi$ asks for an immediately accessible a -transition leading to a state satisfying φ ; it is not reducible to asserting the existence of isolated traces ab and ac . For image-finite systems a much broader modal characterization of bisimulation is available in established process semantics, but the explicit separating formula is sufficient for the present finite pair [20, 13].

C.3 Unobservable transitions and the scope of process comparison

A further choice concerns the treatment of unobservable labels. Let τ be an internal label. Consider a system with $u \xrightarrow{\tau} v \xrightarrow{a} t$ and another system with $u' \xrightarrow{a} t'$, the target states being terminal. These roots are not strongly bisimilar because u has a τ -transition that u' cannot match strongly. They are weakly bisimilar under the standard convention that a visible a -step may be matched by a finite τ -sequence, followed by an a -step and another finite τ -sequence. An explicit weak bisimulation relates u and v to u' , and relates t to t' .

This comparison is sensitive to the observational protocol. In particular, if the duration, count, or physical cost of an internal transition matters, concealing τ can remove information relevant to a chosen Variation or Trajectory criterion. Divergence-sensitive equivalence may also distinguish behaviors identified by a divergence-insensitive weak relation. The formal decision to hide a label therefore represents a specified abstraction rather than an ontological statement about whether the corresponding process occurs.

C.4 A quotient graph admitting unrealizable histories

Let $X = \{0, 1, 2, 3\}$ carry the deterministic cycle $F(i) = i + 1 \pmod{4}$. Define the observation $q : X \rightarrow Y = \{A, B, C\}$ by $q(0) = q(2) = A$, $q(1) = B$, and $q(3) = C$. The abstract edge relation on Y is constructed existentially: $y \Rightarrow y'$ whenever some source edge $x \rightarrow F(x)$ has $q(x) = y$ and $q(F(x)) = y'$. This gives exactly $A \Rightarrow B$, $A \Rightarrow C$, $B \Rightarrow A$, and $C \Rightarrow A$. Figure 11 depicts both graphs.

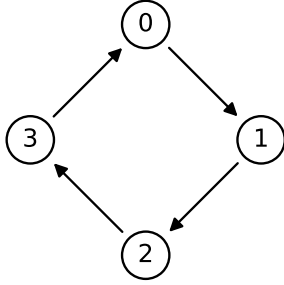
Proposition C.3 (Exact growth of spurious abstract histories). *Let N_n be the number of distinct length- n symbol words admitted by the abstract relation, starting at A , where $n \geq 1$ counts symbols including the initial A . Let R_n be the number of such words realizable by a source trajectory starting at either representative of A . Then Equation 99 holds.*

$$N_n = 2^{\lfloor n/2 \rfloor}, \quad R_1 = 1, \quad R_n = 2 \quad (n \geq 2). \quad (99)$$

In particular, $N_n - R_n > 0$ for $n \geq 4$, even though every abstract edge is the image of a source edge.

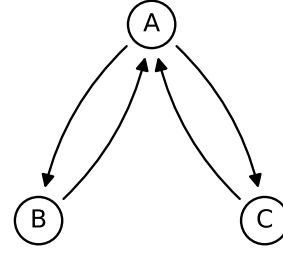
Proof. Every abstract path starting at A alternates between A and a freely chosen symbol in $\{B, C\}$. Among n positions, $\lfloor n/2 \rfloor$ positions permit independent choices, giving N_n . The two possible source initial states 0 and 2 produce respectively the two alternating words $ABAC \dots$ and $ACAB \dots$, which are distinct whenever $n \geq 2$. The one-symbol word is A in either case. These are the only source trajectories from the prescribed initial fiber because F is deterministic. For $n \geq 4$, the abstract word $ABAB$ is admitted, but the source path whose first symbols are ABA must next generate C rather than B . $\square$

(a) Source: deterministic four-cycle



$$q(0) = q(2) = A$$

(b) Existential observation quotient



ABAB: abstractly allowed, not realizable

Figure 11: A deterministic four-state cycle (left) and its three-symbol existential observation quotient (right). The quotient contains both outgoing choices at A , although each individual source representative of A permits only one. The abstract cycle $A B A B$ is consequently not the observation of any corresponding four-position source trajectory. Arrows describe a mathematical transition relation; no physical temporal orientation is inferred.

The essential loss occurs when the quotient combines different source representatives into one abstract state and subsequently permits choices belonging to different representatives. Existence of a concrete preimage of *each abstract edge* does not guarantee existence of a single concrete preimage of an *abstract path*. This is the same logical difference between locally matching transitions and a globally coherent lift discussed in Section 8. The count in Equation 99 makes the magnitude of this discrepancy explicit.

C.5 Finite observation memory and reconstruction of a hidden cycle

For a deterministic map $F : X \rightarrow X$ and observation $q : X \rightarrow Y$, define its length- k observation code $H_k : X \rightarrow Y^k$ from k consecutive observations of the same mathematical trajectory. Its expression is Equation 100.

$$H_k(x) = (q(x), q(Fx), \dots, q(F^{k-1}x)). \quad (100)$$

Proposition C.4 (Exact conjugacy from an injective finite observation code). *Let X be any set, $F : X \rightarrow X$ a map, and $q : X \rightarrow Y$ an observation. If H_k in Equation 100 is injective, there exists a unique map $\hat{F} : H_k(X) \rightarrow H_k(X)$ for which $H_k \circ F = \hat{F} \circ H_k$, and H_k is an isomorphism of the two deterministic transition systems. For the four-state model in Proposition C.3, H_2 is injective whereas $H_1 = q$ is not.*

Proof. Because H_k is one-to-one and onto its image, define $\hat{F} = H_k \circ F \circ (H_k|_X)^{-1}$ on $H_k(X)$. This definition is single-valued, satisfies the commuting identity, and is unique because each code has a unique source representative. In the four-state model the four length-two codes are (A, B) , (B, A) , (A, C) , and (C, A) , hence distinct; the one-position observation identifies states 0 and 2. Their deterministic successor sequence, formed by shifting the observation window, is therefore exactly conjugate to the original four-state cycle. $\square$

This reconstruction is conditional on access to the stipulated consecutive records, a stable deterministic transition rule, and adequate observation resolution. If H_k is not injective, its failure alone does not rule out every weaker property reconstruction; it prevents full state reconstruction by that specific code. Nor does injectivity supply evidence for phenomenal identity. In a relational model, the codes may reconstruct a represented transition mechanism while preserving uncertainty about the physical meaning of its states or the formation of its observer.

C.6 Path support, selection probabilities, and observational statistics

Trace languages record admissibility but need not specify how histories are selected. To separate these tasks, take binary states $\{0, 1\}$ and the family of transition matrices in Equation 101 with $0 < p < 1$. Start every chain with the uniform initial distribution.

$$K_p = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}, \quad \pi = (\tfrac{1}{2}, \tfrac{1}{2}), \quad \pi K_p = \pi. \quad (101)$$

Proposition C.5 (Equal supports and marginals with different path laws). *For any $p, r \in (0, 1)$, the two stationary Markov chains with kernels K_p and K_r have identical finite path supports and identical one-position laws. The total-variation distance between their two-position joint laws is $|p - r|$.*

Proof. Every matrix entry is strictly positive, so every finite binary word has positive probability under both chains. Both kernels preserve the uniform distribution π from Equation 101, giving identical marginals at every position. Their two-position joint laws assign mass $p/2$ to each of $(0, 0)$, $(1, 1)$ and mass $(1 - p)/2$ to each of $(0, 1)$, $(1, 0)$, with the corresponding expressions for r . The sum of four absolute differences is $2|p - r|$; division by two yields the claimed total-variation distance. $\square$

A difference in two-position laws can in principle be used to distinguish selection mechanisms when the observation retains both positions. If only the state at a single position is recorded, the models remain indistinguishable under the stated stationary preparation. If a further observation map sends both binary states to one symbol, their entire observed path laws coincide. Consequently, one-point equality, support equivalence, path-law equivalence, and source-mechanism equivalence are separately indexed criteria; their relationships depend on the available observation and preparation protocols [15, 10].

C.7 Cross-orientation comparison and reproducibility obligations

The examples support a common comparative lesson without collapsing the four orientations into a hierarchy. A sampled flow preserves a discrete transformation but can lose its continuous generator. A trace language retains available words but not the arrangement of choices. An existential quotient can create paths absent from the source; observation memory can restore an exact deterministic transition representation when its code separates states. A common path support can conceal the probabilities assigned to individual histories. These are concrete mathematical claims about specified model classes, not general theories of consciousness.

Table 17: Preservation obligations made explicit by the process counterexamples. Each row concerns the comparison domain and protocol stated in the relevant construction.

Comparison	What agrees	What may differ or require further access
Sampled smooth flows	Maps at sampling interval Δ	Continuous generators, fixed-point sets of the full flow, minimal periods
Finite trace semantics	Admissible words from selected roots	Branching location and modal availability of joint alternatives
Weak versus strong process semantics	Visible responses with internal steps hidden	Number, duration, and divergence of internal transitions
Existential state quotient	Every abstract edge has a source witness	Joint realizability of a multi-edge history
Finite observation memory	Encoded deterministic states when H_k is injective	Unmodeled physical and phenomenal interpretation
Stationary Markov repertoire	Path support and one-position statistics	Multi-position law and probabilistic selection mechanism

A reproducible companion script, `review/appendix_C_process_audit.py`, checks the state graphs, trace/bisimulation distinction, word-count formula, memory-code injectivity, and probability-law calculation. The figure is generated from the same state and observation definitions. These tests are

finite audits of the chosen examples rather than a replacement for the general mathematical arguments. None of the equivalence relations considered here by itself identifies a phenomenal experience, and no result assumes a privileged present or a universal physical ordering of the indices used to represent a path.

D Categorical Translations and Coherence Obligations

An equivalence asserted separately for every member of an indexed family need not extend to an equivalence of the family. This appendix develops the corresponding coherence obligations for translations of transformation systems and relational descriptions. It supplements the categorical results in Sections 7 and 11 by distinguishing the existence of quotient diagrams, the descent of translations, preservation under reindexing, and the information carried by connecting maps. The categorical constructions are standard applications of functoriality and naturality [23]. Their interpretation as histories of physical or conscious systems requires independently specified constitutive and observational assumptions. In particular, an index morphism represents a stipulated comparison or composition, without imposing a universal direction of physical time.

D.1 Indexed diagrams and coherence on generators

Let I be a small category and let $X, Y : I \rightarrow \text{Set}$ be covariant diagrams. For each object i , the sets X_i and Y_i are carriers of representations of the same declared mathematical type. For a morphism $a : i \rightarrow j$, the maps $X_a : X_i \rightarrow X_j$ and $Y_a : Y_i \rightarrow Y_j$ represent the prescribed comparisons. A family of maps $\eta_i : X_i \rightarrow Y_i$ defines a natural transformation exactly when Equation 102 holds for every index morphism.

$$Y_a \circ \eta_i = \eta_j \circ X_a \quad (a : i \rightarrow j \text{ in } I). \quad (102)$$

An objectwise bijection may fail this condition. The missing information is the compatibility of the proposed identification with the connecting transformations, which is part of the description of Generativity in the present formal programme.

Proposition D.1 (Generator test for naturality). *Suppose I is generated as a category by a specified directed graph, possibly subject to relations between composable paths. A family of functions $\eta_i : X_i \rightarrow Y_i$ is natural if Equation 102 holds for each generating edge. If all its components are bijections, the resulting natural transformation is a natural isomorphism.*

Proof. The equality is automatic for identity arrows. If it holds for arrows $a : i \rightarrow j$ and $b : j \rightarrow k$, then $Y_b Y_a \eta_i = Y_b \eta_j X_a = \eta_k X_b X_a$. Induction proves the equality for each composable word in the generators. Every arrow has a generating expression, and functoriality makes its source and target maps independent of that expression; hence the relations impose no additional condition once X and Y are genuine functors. If each η_i is bijective, composing the naturality square with its inverses gives $X_a \eta_i^{-1} = \eta_j^{-1} Y_a$, so the componentwise inverse family is natural. $\square$

This gives a finite verification procedure when the indexing category has finitely many objects and generators and the component functions are explicitly represented. It does not reconstruct the full indexing category from a finite observational record. When the diagrams represent event/coupling two-complexes, the maps in Equation 102 must be specified at the appropriate cellular or topological level; equality of their induced homology maps alone supplies a different, weaker test.

D.2 Existence of coherent observational quotients

An observational reduction may be chosen separately for each carrier without ensuring that the reduced carriers form a diagram. Let $X : I \rightarrow \text{Set}$ and, for each i , let $q_i : X_i \twoheadrightarrow Q_i$ be a surjective function

onto a prescribed set. The problem is to construct connecting maps Q_a for which the q_i become a natural transformation. The exact condition is recorded in Equation 103.

$$\ker q_i \subseteq \ker(q_j \circ X_a) \quad \text{for every } a : i \rightarrow j. \quad (103)$$

Proposition D.2 (Coherent descent of an indexed quotient). *There exists a unique functor $Q : \mathbb{I} \rightarrow \mathbf{Set}$ with the prescribed object sets Q_i such that the maps q_i constitute a natural transformation $q : X \Rightarrow Q$ if and only if Equation 103 holds. The connecting maps are necessarily given by Equation 104.*

$$Q_a(q_i(x)) = q_j(X_a(x)) \quad (x \in X_i). \quad (104)$$

Proof. If Q and q exist, naturality implies that equal q_i -values have equal $q_j X_a$ -values, proving the kernel inclusion. Conversely, the inclusion makes Equation 104 independent of the chosen representative x ; surjectivity defines Q_a on every element of Q_i . For identity arrows, $Q_{1_i}(q_i(x)) = q_i(x)$; for composable $a : i \rightarrow j$ and $b : j \rightarrow k$, $Q_b Q_a(q_i(x)) = q_k(X_b X_a(x)) = Q_{ba}(q_i(x))$. Surjectivity therefore establishes the identity and composition laws everywhere. The same formula proves naturality and uniqueness. $\square$

For a concrete failure, take the indexing category generated by $a : 0 \rightarrow 1$. Let $X_0 = \{u, v\}$, $X_1 = \{r, s\}$, and $X_a(u) = r$, $X_a(v) = s$. A record map that identifies u and v at index 0 but distinguishes r and s at index 1 violates Equation 103. The image of one reduced value would have to be both $q_1(r)$ and $q_1(s)$, so no deterministic connecting arrow Q_a exists. Replacing the quotient arrow by a relation is possible, but changes the model category and the equivalence criterion; it may introduce spurious concatenated histories of the kind exhibited in Appendix C.

D.3 Descent of translations between quotient diagrams

Even if two observational quotient diagrams exist, a translation between their source diagrams need not induce a translation between the quotients. Let $X, Y : \mathbb{I} \rightarrow \mathbf{Set}$ carry pointwise-surjective natural transformations $q^X : X \Rightarrow \bar{X}$ and $q^Y : Y \Rightarrow \bar{Y}$, and suppose $\eta : X \Rightarrow Y$ is natural. A translated comparison should act on a record $q_i^X(x)$ without depending on its unobserved representative. This requires the condition in Equation 105.

$$\ker q_i^X \subseteq \ker(q_i^Y \circ \eta_i) \quad (i \in \mathbf{Ob}(\mathbb{I})). \quad (105)$$

Proposition D.3 (Descent of a coherent translation). *A unique natural transformation $\bar{\eta} : \bar{X} \Rightarrow \bar{Y}$ satisfying $q^Y \eta = \bar{\eta} q^X$ exists if and only if Equation 105 holds. The induced $\bar{\eta}$ can be a natural isomorphism even when η is not componentwise bijective.*

Proof. The necessary kernel condition follows by applying $\bar{\eta}_i$ to equal q_i^X -values. Under this condition, define $\bar{\eta}_i(q_i^X(x)) = q_i^Y(\eta_i(x))$. Surjectivity and the kernel condition make the definition total and unambiguous. For an arrow $a : i \rightarrow j$, naturality of q^X , q^Y , and η implies that both $\bar{Y}_a \bar{\eta}_i q_i^X$ and $\bar{\eta}_j \bar{X}_a q_i^X$ equal $q_j^Y \eta_j X_a$. Because q_i^X is surjective, $\bar{Y}_a \bar{\eta}_i = \bar{\eta}_j \bar{X}_a$. Uniqueness follows from the same surjectivity. For the final assertion, take the terminal indexing category, $X = \{0, 1\}$, $Y = \{*\}$, and let both quotient diagrams have the one-element carrier. The unique source map $\eta : X \rightarrow Y$ is not bijective, while its descended map on singletons is an isomorphism. $\square$

The proof uses the commuting quotient squares in addition to Equation 105. Proposition D.3 shows that an isomorphism established only after quotienting cannot by itself be promoted to an isomorphism of source representations. Its relevance to comparisons between relational and manifestational registers depends on the type and interpretation of the quotient maps.

D.4 Monodromy, constitutive cycles, and lost coherence

A compact counterexample is furnished by an indexing category with one object whose endomorphism group is $\mathbb{Z}$. A functor from this category to finite sets assigns a finite set S and a permutation $T : S \rightarrow S$, with integer n acting by T^n . A natural isomorphism between two such functors is a bijection h intertwining their permutations. The condition appears in Equation 106.

$$h \circ T = U \circ h, \quad h : S \xrightarrow{\cong} S'. \quad (106)$$

Proposition D.4 (Cycle-type criterion and orbit-colimit information loss). *For finite sets, the two indexed permutation diagrams are naturally isomorphic if and only if their permutations have the same cycle-type partition. Their colimits in Set are isomorphic if and only if they have the same number of permutation orbits. Therefore, isomorphic colimits do not in general imply equivalent indexed diagrams.*

Proof. If Equation 106 holds, then $h(T^n x) = U^n h(x)$ for every integer n . The restriction of h to a cycle of length ℓ must therefore map it bijectively to a cycle of the same length. Conversely, a pairing of cycles of matching lengths permits a bijection h defined by corresponding positions within each cycle, and this bijection satisfies the intertwining equation. For the colimit, the coequalizer of the identity and T identifies x with $T(x)$, and its equivalence classes are precisely the permutation orbits. Consequently its underlying set has one element per cycle, independently of the lengths of those cycles. $\square$

Consider the permutations of $S = \{1, \dots, 6\}$ with disjoint-cycle notation $T = (1\ 2\ 3)(4\ 5\ 6)$ and $U = (1\ 2)(3\ 4\ 5\ 6)$. Both have exactly two orbits, hence both orbit-colimits are two-element sets. Their cycle types $(3, 3)$ and $(2, 4)$ differ, so Proposition D.4 excludes a natural isomorphism. Figure 12 displays these two indexed transformations. The index $\mathbb{Z}$ here encodes iterated reversible comparison operations; it is not an assumption that the physical world or experience possesses a global integer-valued temporal axis.

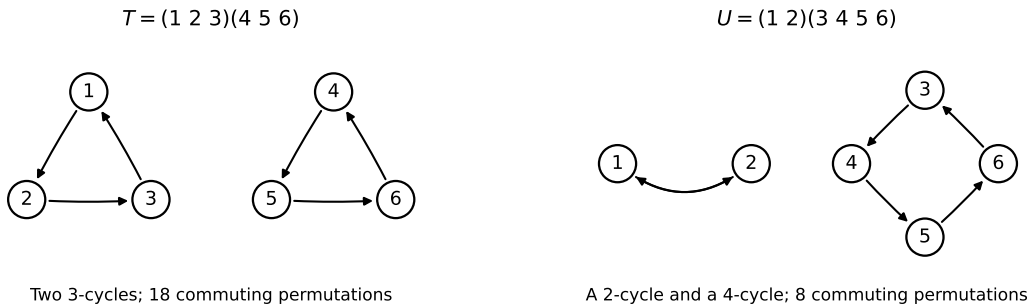


Figure 12: Two permutations on six points with equal numbers of orbits and therefore isomorphic orbit-colimits, but different cycle decompositions. The group-action diagrams are not naturally isomorphic, because the two connecting transformations cannot be intertwined by a bijection.

D.5 Reindexing and the recovery of coherent comparisons

An observational study may compare an indexed family only along a selected subcategory. For a functor $u : \mathbf{J} \rightarrow \mathbf{I}$, restriction takes $X : \mathbf{I} \rightarrow \mathbf{Set}$ to $X \circ u : \mathbf{J} \rightarrow \mathbf{Set}$. This is a mathematical model of limiting which connecting comparisons are available. If u is the inclusion of the terminal category into the one-object category of $\mathbb{Z}$, then restriction retains the underlying set while forgetting its permutation. Consequently the diagrams T and U above become isomorphic after restriction, although they were not isomorphic before it.

Proposition D.5 (Reflection of natural isomorphism under equivalence of index categories). *Let $u : \mathbf{J} \rightarrow \mathbf{I}$ be fully faithful and essentially surjective. If the restrictions $X \circ u$ and $Y \circ u$ of two functors $X, Y : \mathbf{I} \rightarrow \mathbf{Set}$ are naturally isomorphic, then X and Y are naturally isomorphic.*

Proof. Let $\theta : Xu \Rightarrow Yu$ be a natural isomorphism. For every $i \in I$, choose an object $j_i \in J$ and an isomorphism $s_i : u(j_i) \rightarrow i$. Define $\eta_i = Y(s_i)\theta_{j_i}X(s_i)^{-1}$. Each η_i is a bijection. To verify naturality for $a : i \rightarrow k$, full faithfulness supplies a unique $b : j_i \rightarrow j_k$ such that $u(b) = s_k^{-1}as_i$. Naturality of θ gives $Y(u(b))\theta_{j_i} = \theta_{j_k}X(u(b))$. Composing with $Y(s_k)$ and $X(s_i)^{-1}$ yields $Y(a)\eta_i = \eta_kX(a)$. Thus the family η is a natural isomorphism. No ordering on the objects of I or J is used. $\square$

Full faithfulness and essential surjectivity are sufficient here because they prevent the selected indexing domain from losing the comparison arrows needed to transport a proposed equivalence. They do not imply that a particular physical observation protocol covers all mechanisms of a system. Applied to the four orientations, Proposition D.5 is a criterion for retaining coherent *diagrammatic* equivalence under a declared reindexing, with any stronger ontological or phenomenal conclusion requiring separate argument.

D.6 Symmetries retained by coherent indexed representations

Beyond mere existence of an isomorphism, the group of natural automorphisms of an indexed diagram can provide a finer invariant of its internal organization. For a diagram associated with a permutation T of a finite set, a natural automorphism is precisely a permutation h commuting with T . Let m_ℓ be the number of cycles of length ℓ in T . The number of natural automorphisms is given by Equation 107.

$$|\text{Aut}(X_T)| = |\{h \in \text{Sym}(S) : hT = Th\}| = \prod_{\ell \geq 1} \ell^{m_\ell} m_\ell!. \quad (107)$$

Proposition D.6 (Symmetry count of a finite indexed permutation diagram). *For a finite permutation diagram X_T , Equation 107 holds. In particular, the diagrams X_T and X_U above possess respectively 18 and 8 natural automorphisms even though their orbit-colimits have the same cardinality.*

Proof. A permutation commuting with T must send each T -orbit to another orbit of the same length. Among the m_ℓ orbits of length ℓ it may choose any permutation of those orbits, giving $m_\ell!$ possibilities. Once the target orbit is chosen, the image of one point determines the rest by commutation, giving ℓ choices of cyclic phase per orbit. Choices on cycles of different lengths are independent, proving the product formula. The cycle types $(3, 3)$ and $(2, 4)$ yield respectively $3^2 \cdot 2! = 18$ and $2 \cdot 4 = 8$. $\square$

These automorphism counts are invariants of the *specified indexed action*. They do not assign probabilities to trajectories or quantify phenomenal complexity. Their significance is diagnostic: an aggregation retaining only orbit classes conceals distinctions not only in connecting operations but also in the coherent self-symmetries of the source descriptions.

D.7 Preservation obligations across representations

Table 18 consolidates the categorical obligations most directly relevant to cross-register comparisons. A translation may preserve a selected quotient criterion while failing to reflect it; a coherent family may cease to exist under incompatible pointwise identifications; and aggregation may erase constitutive arrows despite preserving a terminal set. These are failures relative to declared mathematical information. Further assumptions would be needed to determine whether any erased distinctions bear on physical organization or experience.

Table 18: Categorical preservation obligations for equivalence claims across indexed representations. The conditions refer to specified diagrams and arrows; the last column gives information that a weaker translation may conceal.

Comparison		Formal obligation	Information not retained by a weaker test
Objectwise correspondence	correspondence	Compatible naturality squares on the generating arrows	Constitutive compatibility and coherent diagram equivalence
Pointwise quotient	observation	Kernel stability for every index arrow	Existence of a deterministic reduced diagram
Quotient-level translation		Kernel stability of the translated observation	Whether the source translation was invertible
Orbit-colimit comparison		Retain connecting permutation or comparable action data	Cycle lengths, monodromy and coherent symmetries
Restriction of index category		Full faithfulness and essential surjectivity suffice to reflect natural isomorphism	Comparisons along unobserved indexing arrows
Cross-register application		Typed observational interface and a property-specific bridge	Source-level and phenomenal equivalence

The results give a finite hierarchy of checks rather than a universal ranking of mathematical models. For Variation, coherent translation requires compatibility with the selected mechanisms; for Trajectory, it requires compatibility with the chosen paths and parameter conventions; for Generativity, it includes compatibility across constitutive connecting morphisms; and for Repertoire, it must address the selected families of admissible histories and possible aggregation-induced paths. A categorical equivalence is strong only relative to the structure explicitly encoded in its objects and arrows. Equivalence of consciousness-relevant systems remains an additional scientific question rather than a consequence of categorical coherence alone.

E Observation, Identification, and Finite-Sample Illustrations

This appendix develops exact calculations for the observation- and experiment-indexed equivalence criteria introduced in Sections 13 and 10. Its objective is to specify what is identified from an individual record, from an ideal response distribution, or from a finite sequence of experimentally generated records. These information regimes have different mathematical requirements even for finite carriers and fully known observation channels. The analysis separates distributional reconstruction, statistical decision risk, intervention-dependent observability, and causal mediation. All examples are synthetic; any application to a neural system would additionally require a justified physical model, calibrated apparatus and a specified connection to the phenomenal properties under discussion.

E.1 Finite stochastic channels and identification from complete response laws

Let X, Y, Z be finite nonempty sets. A channel K from X to Y is an $|X| \times |Y|$ row-stochastic matrix with entries $K_{xy} = P(Y = y \mid X = x)$. A target property $q : X \rightarrow Z$ determines the deterministic row-stochastic matrix $Q_{xz} = \mathbf{1}\{q(x) = z\}$. A column probability vector $\pi \in \Delta(X)$ induces the observed law $K^\top \pi$ and the target-property law $Q^\top \pi$. The question here concerns recovery of the *property distribution* rather than decoding the value of the property on an individual trial.

Proposition E.1 (Recovery of a target distribution from an ideal observed law). *For finite X, Y, Z and known row-stochastic K, Q , the following are equivalent: (i) every two source laws $\pi, \pi' \in \Delta(X)$ with $K^\top \pi = K^\top \pi'$ satisfy $Q^\top \pi = Q^\top \pi'$; (ii) the real linear kernels obey Equation 108; and (iii) there exists a linear map B on the image of $K^\top$ with $Q^\top = BK^\top$. The factorization in (iii) is defined on the image and is not necessarily a stochastic decoding channel.*

$$\ker K^\top \subseteq \ker Q^\top. \quad (108)$$

Proof. Under (ii), equality of observed laws implies $\pi - \pi' \in \ker K^\top \subseteq \ker Q^\top$, proving (i). For the converse, let $v \in \ker K^\top$ and let u be the uniform strictly positive law on X . Row stochasticity implies $\sum_x v_x = \sum_y (K^\top v)_y = 0$. For sufficiently small $t > 0$, both $u + tv$ and $u - tv$ are probability vectors and have the same observed law. Condition (i) then gives $2tQ^\top v = 0$, establishing (ii). Finally, (ii) makes $B(K^\top v) := Q^\top v$ well defined on the linear image of $K^\top$, which yields (iii). The factorization itself immediately implies (ii). $\square$

A particularly instructive example is the binary symmetric observation channel shown in Equation 109, with input and output both in $\{0, 1\}$. For $0 < \varepsilon < 1/2$, its rows have overlapping supports, so one noisy outcome cannot reveal the input state with certainty, although the matrix is invertible and the full source distribution is identifiable from the exact observed distribution. At $\varepsilon = 1/2$, the rows coincide and even ideal-law identification of the input distribution fails.

$$K_\varepsilon = \begin{pmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 1 - \varepsilon \end{pmatrix}, \quad \det K_\varepsilon = 1 - 2\varepsilon. \quad (109)$$

E.2 Certain decoding of individual properties

A different recovery problem asks for a measurable decoder $d : Y \rightarrow Z$ whose output equals $q(x)$ with probability one for each input x . The criterion refers to support overlap and is stronger than recovering $q_\# \pi$ from a complete population law. Equation 110 states the condition.

$$\text{supp } K(\cdot | x) \cap \text{supp } K(\cdot | x') = \emptyset \quad \text{whenever } q(x) \neq q(x'). \quad (110)$$

Proposition E.2 (Single-record exact property decoding). *For a known finite channel K and target map $q : X \rightarrow Z$, a deterministic decoder $d : Y \rightarrow Z$ satisfying $P(d(Y) = q(x) | X = x) = 1$ for each x exists if and only if Equation 110 holds.*

Proof. If a common outcome y has positive probability under inputs x, x' with different q -values, then $d(y)$ would have to equal two distinct targets, contradicting certain decoding. Conversely, under Equation 110, every y in the union of channel supports belongs only to supports of inputs sharing one q -value. Assign this common value to $d(y)$, and assign arbitrary values to outcomes outside the union. The resulting decoder succeeds on every input with probability one. $\square$

For the binary symmetric channel in Equation 109, taking q to be the identity yields certain one-record decoding only at $\varepsilon = 0$ within $0 \leq \varepsilon \leq 1/2$. By contrast, Proposition E.1 yields ideal-law distribution identification whenever $\varepsilon < 1/2$. These claims assume that the observation channel is fully known and remains invariant across the observations supplying its law. An estimated or changing report channel requires additional uncertainty analysis.

E.3 Garbling, experimental informativeness, and decision risk

Suppose one experiment produces a finite record Y through $K : X \rightsquigarrow Y$ and a second produces W by applying a state-independent randomization $L : Y \rightsquigarrow W$. Its channel is $K' = KL$. A decision rule takes a record to an action in a finite action set A , and its loss $\ell(x, a)$ is assumed bounded. An experiment is evaluated relative to a prior π over the stipulated candidate states and this loss. The ordering below is a standard consequence of statistical comparison of experiments [4]; it is stated explicitly here because it provides a concrete preservation condition for manifestational equivalences.

Proposition E.3 (Decision-risk monotonicity under stochastic garbling). *For every finite prior π , bounded loss function ℓ , and finite action set, the minimum Bayes risk available through K is no larger than the minimum Bayes risk through KL . For any two source laws π_0, π_1 , Equation 111 holds.*

$$\text{TV}(L^\top K^\top \pi_0, L^\top K^\top \pi_1) \leq \text{TV}(K^\top \pi_0, K^\top \pi_1). \quad (111)$$

Proof. A possibly randomized decision rule $\delta : W \rightsquigarrow A$ following KL can be simulated with the richer record Y by first sampling $W \sim L(\cdot | Y)$ and then applying δ . This reproduces its joint law of source state and action, including its expected loss. Taking the infimum over rules establishes the Bayes-risk inequality. For Equation 111, write the signed difference of observed laws as $v = K^\top(\pi_0 - \pi_1)$. Because L is row-stochastic, $\sum_w |\sum_y L_{yw} v_y| \leq \sum_y |v_y|$ by the triangle inequality and row sums. Half of the respective sums gives the total-variation inequality. $\square$

The qualification that L be a common state-independent garbling matters. A new intervention may alter the source response law itself and reveal distinctions suppressed by the first protocol; such an experiment is not merely a post-processing of a fixed source record. A protocol judged less informative for one property may also be more practical or less intrusive, questions left outside this mathematical ordering.

E.4 Exact finite-sample discrimination under binary noise

We next compute achievable discrimination for a completely specified pair of hypotheses. Let H_0 and H_1 yield independent Bernoulli source records with parameters $p_0 = 2/5$ and $p_1 = 3/5$, respectively. Passing each source record independently through the binary channel K_ε in Equation 109 gives Bernoulli observed parameters $r_0(\varepsilon) = \varepsilon + (1 - 2\varepsilon)(2/5)$ and $r_1(\varepsilon) = 1 - r_0(\varepsilon)$. Each of the n complete trials has the same response law conditional on the hypothesis, and trials are mutually independent; this assumption is part of the present synthetic experiment.

Proposition E.4 (Exact equal-prior error for a symmetric Bernoulli comparison). *Let $0 \leq \varepsilon < 1/2$, $r = r_0(\varepsilon) < 1/2$, and let n be odd. Under equal hypothesis priors, the likelihood-ratio-optimal decision chooses H_1 when the number of observed ones exceeds $n/2$. Its minimum average error is given by Equation 112.*

$$P_{\text{err}}^*(n, \varepsilon) = \sum_{k=(n+1)/2}^n \binom{n}{k} r^k (1 - r)^{n-k}. \quad (112)$$

Proof. For an observed word containing k ones, the likelihood ratio of H_1 against H_0 is $[(1 - r)/r]^{2k-n}$. Since $0 < r < 1/2$ for the stated parameters, this is greater than one exactly when $k > n/2$. Odd n rules out ties. The error under H_0 is the binomial upper tail in Equation 112; by the symmetry $r_1 = 1 - r_0$, the error under H_1 has the same value. Their equal-prior average therefore equals this tail. $\square$

At $\varepsilon = 1/2$, the two observed laws coincide and the optimal error is $1/2$ for every finite sample size. The determinant in Equation 109 can be arbitrarily small yet nonzero: ideal-law identifiability then persists, although many repetitions may be needed to obtain useful discrimination. Figure 13 displays the exact binomial errors computed with rational arithmetic for multiple noise rates, alongside the channel's smallest singular value. The calculations are illustrative properties of a specified stochastic model, without an interpretation as empirical neural measurements.

E.5 Identification stability under a noisy measurement channel

The invertibility of a measurement channel and the stability of inference through that channel are different mathematical requirements. In the binary example of Equation 109, the two rows of K_ε become arbitrarily similar as ε approaches $1/2$. Although the channel remains invertible at every $\varepsilon < 1/2$, small errors in an estimated response frequency are amplified by the inverse map. This matters when an ideal response law is represented by finitely many experimental trials.

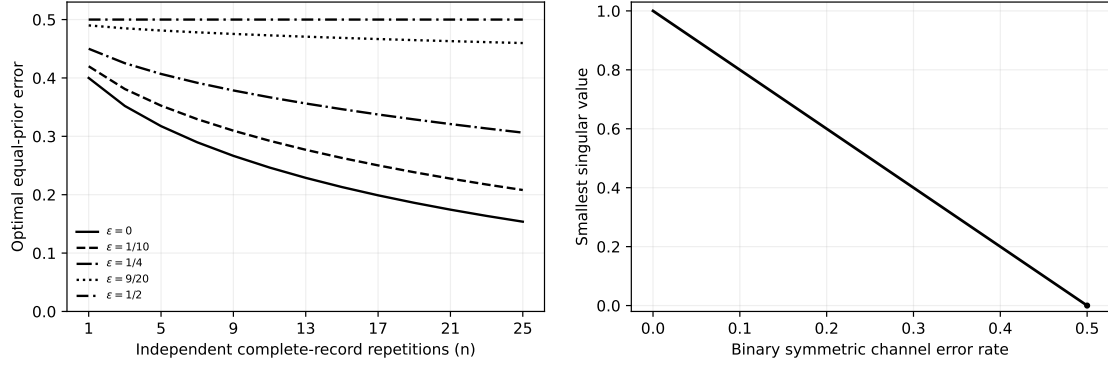


Figure 13: Exact finite-sample comparison under binary symmetric measurement noise. Left: the optimal equal-prior error for the two specified Bernoulli hypotheses as a function of the number of conditionally independent whole-record repetitions, evaluated by exact rational binomial sums. Right: the smallest singular value of the channel K_ε , showing the transition between ideal-law identifiability and distributional collapse at $\varepsilon = 1/2$. Values are synthetic and computed by the reproducible script in the editable archive.

Suppose $X_1, \dots, X_n$ are independent source Bernoulli records with success probability p , and Y_i is obtained by passing X_i independently through the known, fixed binary symmetric channel K_ε . Denote $r = P(Y_i = 1) = \varepsilon + (1 - 2\varepsilon)p$ and $\hat{r} = n^{-1} \sum_i Y_i$. We consider the unprojected linear inverse estimator $\hat{p} = (\hat{r} - \varepsilon)/(1 - 2\varepsilon)$; the word “unprojected” is material because its finite-sample values need not lie in $[0, 1]$.

Proposition E.5 (Exact inverse-channel variance amplification). *Fix $n \geq 1$, $0 \leq \varepsilon < 1/2$, and $p \in [0, 1]$. Under the independent measurement model just specified, $\hat{p}$ is unbiased for p , with exact mean-square error given by Equation 113. At $\varepsilon = 1/2$, the complete observed record law is independent of p , so no estimator based only on these records can be unbiased simultaneously for every $p \in [0, 1]$.*

$$\mathbb{E}_p[\hat{p}] = p, \quad \mathbb{E}_p[(\hat{p} - p)^2] = \text{Var}_p(\hat{p}) = \frac{r(1-r)}{n(1-2\varepsilon)^2}, \quad r = \varepsilon + (1-2\varepsilon)p. \quad (113)$$

Proof. Conditional independence and the specified binary channel give independent Bernoulli observations with common parameter r . Hence $\mathbb{E}[\hat{r}] = r$ and $\text{Var}(\hat{r}) = r(1-r)/n$. The affine inversion defining $\hat{p}$ implies $\mathbb{E}[\hat{p}] = (r - \varepsilon)/(1 - 2\varepsilon) = p$ and scales the variance by $(1 - 2\varepsilon)^{-2}$, giving Equation 113. If $\varepsilon = 1/2$, every observed record is Bernoulli with probability $1/2$, independently of the source parameter p . Every statistic of the observed records then has a distribution and expectation independent of p , so its expectation cannot equal each distinct value of p . $\square$

For a fixed interior source parameter p , the variance in Equation 113 diverges as $\varepsilon \uparrow 1/2$ at any fixed sample size; its numerator tends to $1/4$ while its denominator vanishes. In particular, exact invertibility gives no noise-independent, finite-sample accuracy guarantee. Projecting $\hat{p}$ onto $[0, 1]$ may improve squared-error risk for some parameters, but generally changes its unbiasedness. An unknown or drifting ε introduces a further identification problem, because the observed frequency r can correspond to multiple source probabilities paired with different channel parameters. These conclusions concern the specified model and do not determine how a phenomenal property should be represented or measured.

E.6 Adaptive protocols and the persistence of equivalence

One may choose the next experimental protocol after examining earlier records. For a finite set of fixed candidate models $\mathfrak{M}$, let $K_e(\cdot \mid m)$ be the distribution of a complete record under protocol e when model m is true. Consider a finite-horizon policy choosing each next protocol using only the previous

protocols and records and fresh randomization whose distribution is independent of m . Conditional on m and the selected protocol, the record of each round follows $K_e(\cdot \mid m)$ independently of previous records. This explicitly excludes unmodeled interventions that permanently modify the source dynamics across rounds.

Proposition E.6 (Adaptive preservation of pointwise experimental equivalence). *If two candidate models m_0, m_1 satisfy $K_e(\cdot \mid m_0) = K_e(\cdot \mid m_1)$ for every available protocol e , then every finite-horizon randomized adaptive policy of the preceding type induces identical laws of the complete protocol–record transcript under m_0 and m_1 .*

Proof. The policy’s first protocol distribution is independent of the candidate model. Given a selected first protocol, the conditional distributions of its record coincide by hypothesis. Thus the law of the first transcript segment coincides. Assume equality of the transcript laws through round k . The policy’s conditional law of the next selected protocol given a past transcript is the same under either model, and the record distribution conditional on that protocol again coincides. Multiplication of these conditional probabilities establishes equality at round $k + 1$. Induction proves the claim for each finite horizon, including policies with stopping encoded as a designated terminal action. $\square$

This result complements Proposition 13.6 by specifying the repeated-trial and policy assumptions for the transcript factorization. An adaptive strategy can improve finite-sample performance by concentrating measurements where already distinct kernels differ; it cannot separate models whose entire accessible experiment-indexed response families coincide under these assumptions.

E.7 Causal mediation and observationally equivalent pathways

The identification of a randomized treatment effect on a measured response does not automatically determine which internal variable mediates that effect. Consider three binary variables: a randomized assignment A , a candidate intermediate state M , and an observed response Y . Two deterministic structural causal models may agree on the complete joint law of (A, M, Y) under randomization of A , yet respond differently to a separate surgical intervention on M . Their structural assignments are listed in Table 19.

Table 19: Two synthetic causal structures with the same recorded joint response under randomized assignment and different predictions when M is intervened upon. In the counterfactual comparison $\text{do}(M = m)$, the assignment A is held fixed.

Model	Structural assignments	Under randomized A	Under $\text{do}(M = m)$
Mediated	$M := A, Y := M$	$(A, M, Y) = (A, A, A)$	$Y = m$
Direct	$M := A, Y := A$	$(A, M, Y) = (A, A, A)$	$Y = A$

The equality of the randomized observational laws holds for any specified assignment probability $0 < P(A = 1) < 1$. Nevertheless, the models disagree, for example, when $A = 0$ and $\text{do}(M = 1)$. This difference concerns a claimed internal mechanism and requires the additional intervention to be well defined and to preserve the other assignments. When M denotes a reconstructed geometric or topological statistic, treating $\text{do}(M = m)$ as a physically implementable operation involves a further realizability problem. Identifying an effect of A on a neural or behavioral variable Y is logically separate from identifying a phenomenal property, whose interpretation requires an explicit bridge hypothesis.

E.8 Assumption register and consequences for equivalence criteria

The preceding results supply a set of mutually distinct reconstruction claims. Proposition E.1 concerns properties of source distributions; Proposition E.2 concerns exact recovery on individual records; Proposition E.3 concerns decision informativeness; Proposition E.4 evaluates finite-sample discrimination; and

Proposition E.5 quantifies finite-sample instability despite ideal-law invertibility; and Proposition E.6 specifies a persistent equivalence class under adaptive experiment selection. Table 20 records the minimal distinctions required when any of these results is used to interpret a claim about Variation, Trajectory, Generativity, or Repertoire.

Table 20: Information regimes and the assumptions needed for interpreting a model equivalence claim. A row identifies what the corresponding calculation can establish within its declared model domain.

Regime	Required information and assumptions	Established target
Single-record decoding	Known fixed channel; disjoint supports across property classes	Property value per individual record
Ideal response laws	Complete law, known channel, kernel containment	Distribution of a stated property
Finite-sample testing	Specified model pair, priors, independent whole trials, rule	Achievable discrimination error
Inverse-channel estimation	Known stationary channel, independent draws, invertibility	Exact variance penalty and noise sensitivity
Decision informativeness	Common state-independent channel garbling and loss	Ordering of optimal decision risks
Adaptive experiments	Protocol-indexed response kernels, independent resets, history-based policy	Equality of full transcript laws
Causal mediation	Structural assignments and physically defined interventions	Distinguish pathways beyond assignment effects

The comparison applies separately to the source model type and the property of interest. A Variation criterion may concern an effective transformation action, a Trajectory criterion a path invariant, a Generativity criterion a coherent constitutive diagram, and a Repertoire criterion a family or probability law of admissible histories. Observing one of these objects through a statistical channel can introduce an additional equivalence relation induced by its response law. Across the ontological, relational and manifestational registers, these conditions define what has been identified within the chosen representation. They do not provide a general inference from observable agreement to the identity of subjective experience.

F Reproducibility, Proof Dependencies, and Notation Crosswalk

This appendix specifies how the formal comparisons and synthetic experiments in the present investigation can be reproduced and critically reviewed. It has three related objectives. First, it identifies the modeling objects inherited from the preceding paper, *The Geometry, Topology, and Dynamics of Consciousness: Four Preliminary Formal Hypotheses*, and separates their definitions from the equivalence criteria introduced here. Second, it establishes a reproducibility contract for the finite computations: source files, parameters, exact and floating-point arithmetic, executable checks, and the domain of each test. Third, it records proof dependencies and interpretive assumptions requiring assessment independently of a successful computational run. The preceding manuscript is identified by its versioned source-archive digest in `reproducibility/TWO_MANUSCRIPT_CROSSWALK.json`; references to it below are cross-document locators rather than active LaTeX references. The present appendix makes no claim that syntactic correspondence of symbols establishes a semantic identity between the two theoretical models.

F.1 Object types and the four-orientation correspondence

The same descriptive names recur in the two papers, while their formal roles differ. The preceding paper investigates candidate structures for consciousness-related explanations: transformation actions, selected paths, historically constituted diagrams, and families of admissible paths. The present paper asks which

information is retained when two descriptions of one such type are compared. Table 21 lists the precise section-level correspondence and the mathematical information introduced by the second inquiry. The earlier paper’s sections are cited by their written section numbers and validated source labels, and the current paper’s sections by LaTeX cross-reference.

Table 21: Typed comparison between the preceding four-hypothesis manuscript (v2.9) and the present equivalence investigation. The correspondence concerns the object studied; each equivalence criterion has additional hypotheses.

Orientation	Preceding manuscript: source model	Present manuscript: added comparison obligations
Variation	Section 7 (<code>sec:variation</code>): actions, generators, constraints, vector fields	Section 5: action intertwining, effective monoids, observational congruence, generator sensitivity
Trajectory	Section 8 (<code>sec:trajectory</code>): parameterized and selected paths, homotopy classes	Section 6: reparameterization, homotopy recovery, sampled-path uncertainty, covering lifts
Generativity	Section 9 (<code>sec:generativity</code>): constitutive diagrams, cellular attachments, changing carriers	Section 7: naturality, diagram equivalence, degree compatibility, rule-family transport
Repertoire	Section 10 (<code>sec:repertoire</code>): admissible histories, geometry and topology of path families	Section 8: trace and branching distinctions, constrained paths, complete trajectory laws

The four correspondences preserve a shared research question without identifying different mathematical types. In particular, a transformation law on a carrier X is not automatically a selected trajectory γ in X , an indexed constitutive diagram $\mathcal{D}$, or a repertoire $\mathcal{R}$ of candidate histories. The present paper introduces a further parameter: an equivalence criterion whose domain, invariant, comparison morphism and observational protocol are declared. The distinction is especially important for any translation between the four orientations. Section 9 gives the corresponding conditional implications, while Appendix C supplies finite counterexamples to several unqualified implications.

F.2 Registers, carriers, and the limits of a common notation

Notation becomes misleading when the same symbol denotes a physical substrate, a mathematical state space, a cellular realization, and a space of subjective experiences. In the two manuscripts, X can name a generic model carrier, while M often denotes a smooth manifold and K a CW-complex or finite measurement channel depending on context. Consequently, a valid crosswalk records the *type* of an expression together with its symbol. The convention for this appendix is the typed assignment displayed in Equation 114.

$$\begin{aligned} (X, M \curvearrowright X) \in \text{Act}, \quad \gamma : J \longrightarrow M, \quad \mathcal{D} : \mathbf{I} \longrightarrow \mathbf{C}, \\ \mathcal{R} \subseteq \text{Path}(M), \quad \rho : X \longrightarrow R, \quad \omega : R \longrightarrow Y, \quad m = \omega \circ \rho. \end{aligned} \tag{114}$$

Here M is a specified acting monoid, J a chosen parameter space, $\mathbf{I}$ an indexing category, and $\mathbf{C}$ a stipulated model category. The symbol M represents a smooth carrier when that type is required, and R denotes an explicitly chosen relational representation. The composite m is a *deterministic* manifestation map in Equation 114; probabilistic observations are represented separately by a stochastic kernel. The M notation prevents confusion with a manifold M . Equation 114 is a convention for this appendix, not a claim that every section uses exactly these letters or that the spaces are canonically related.

Table 22 records common mistakes that typed notation is intended to prevent. Its entries are obligations on the analyst, rather than universally valid identifications.

Table 22: Notation and interpretation safeguards for cross-manuscript comparisons.

Object	Type and admissible operation	Additional interpretation required
Smooth manifold M	Charts, smooth maps, tensor and vector-field data	Neural or physical realization, chosen observable coordinates
Cellular complex K	Cells, attaching maps, cellular chains and homotopy data	Meaning of vertices as events, edges as couplings, and faces as declared primitives
History γ	Path with domain J and prescribed regularity	Physical clock or historical ordering, if asserted
Indexed system $\mathcal{D}$	Functor with specified index morphisms	What an index arrow represents; physical time is not assumed
Repertoire $\mathcal{R}$	Set, topological space, or law of admissible paths	How admissibility is physically constrained or empirically tested
Record Y	Measurable observations through map or kernel	Calibration, report reliability and phenomenal bridge hypotheses

The term *ontological register* is conditional on an assumed realization model, *relational register* on the specified constitutive organization, and *manifestational register* on a record interface. Their distinct roles are developed in Sections 3 and 10. For a relational two-complex, vertices represent events and edges couplings; a vertex has no default identification with a subject. The comparison of two such complexes therefore requires both a stated cellular morphism and a separate interpretive account of the represented system.

F.3 Invariants, comparison maps, and a transport certificate

A reported computational equivalence should specify the map that translates between the two compared carriers. Let $I : X \rightarrow A$ and $J : X' \rightarrow A'$ be the invariants used to classify candidate models, and let $f : X \rightarrow X'$ and $u : A \rightarrow A'$ be explicitly typed comparison maps. When $uI = Jf$, the preservation of distinctions depends on the restriction of u to the realized image of I . Equation 115 states the basic commuting requirement.

$$J \circ f = u \circ I. \quad (115)$$

Proposition F.1 (Typed certificate for comparison on a tested domain). *Let $S \subseteq X$ be nonempty, and suppose Equation 115 holds on S . If u is injective on $I(S)$, then for $x, x' \in S$ one has $I(x) = I(x')$ if and only if $J(f(x)) = J(f(x'))$. In contrast, the commutation condition alone guarantees only the forward implication. Every counterexample to the converse is witnessed by a pair in S on which I differs while uI agrees.*

Proof. Equality under I implies equality under uI , and Equation 115 identifies uI with Jf . Conversely, equality under Jf implies $u(I(x)) = u(I(x'))$; injectivity on $I(S)$ gives equality under I . If injectivity fails on the realized image, two distinct values $a, a' \in I(S)$ with $u(a) = u(a')$ have preimages $x, x' \in S$, which witness failure of the converse. This last statement characterizes the existence of a counterexample to reflection when such a pair lies in the tested image. $\square$

Proposition F.1 supplies an explicit three-part certificate: the compared source values, the declared comparison maps, and the preserved or lost invariant equality. It also indicates a limit of computational checks. A script that tests Equation 115 for a finite subset S establishes the corresponding commutation only on S , unless a separately proved reduction or an exhaustive finite-domain enumeration extends its scope. Appendix A develops the general equivalence-preorder theory; the present statement reformulates a small portion for executable provenance.

F.4 Finite enumeration, regression tests, and logical scope

The executable source distinguishes exact finite arithmetic, numerical approximations, symbolic formula evaluations, and compilation tests. For an explicit finite universe U and a decidable predicate $P : U \rightarrow$

$\{\text{true}, \text{false}\}$, exhaustive evaluation checks the finite universal statement shown in Equation 116.

$$\forall x \in U, \quad P(x) = \text{true}. \quad (116)$$

Proposition F.2 (Scope of a finite exhaustive audit). *Suppose a program enumerates each element of a finite, explicitly defined set U exactly once or more, evaluates $P(x)$ by an exact implementation whose semantics agrees with the mathematical definition of P , and terminates without a failed assertion. Then Equation 116 holds. If the intended claim instead concerns a larger set $V \supsetneq U$, the computation alone establishes no statement about $V \setminus U$.*

Proof. For every $x \in U$, exhaustive enumeration provides at least one tested occurrence, and the assumed sound predicate implementation supplies the mathematical truth of $P(x)$. The second assertion follows because the program contains no evaluation for the untested elements; one may define a predicate on V agreeing with all test values on U and false at a selected point of $V \setminus U$. $\square$

The assumptions in Proposition F.2 are substantial. A passing test script is evidence that code ran consistently with its assertions; verification of the enumerator’s coverage, the implementation’s correspondence to the formal definition, and any dependency or arithmetic assumptions requires code inspection and, in critical cases, an independent implementation. Conversely, a fully specified exact counterexample can refute a universal assertion even when it was discovered through a finite computation. Simulation with floating-point error and random sampling has a further approximation layer and must record tolerances, numerical methods, and the sampling design.

Table 23 maps major formal constructions to the available scripts and to the type of verification actually performed. The names are executable paths in the source archive’s `review/` directory. They are not substitutes for the complete detailed inventory in the machine-readable manifest.

Table 23: Selected audit families, reproducible scripts, and the boundary of each computational claim. Other manuscript-specific scripts are recorded in the released audit manifest.

Object or criterion	Audit script in review/	Verification type and scope
Criterion refinement	<code>criterion_lattice_audit.py</code>	Exact finite equivalence and partition tests
Transformation actions	<code>variation_action_audit.py</code>	Explicit finite actions and congruences
Homotopy/cellular cases	<code>appendix_B_cellular_torus_audit.py</code>	Integer matrices, attaching compatibility, bounded lattice cases
Process abstraction	<code>appendix_C_process_audit.py</code>	Finite traces, bisimulation witnesses, Markov examples
Bounded-state trace separation	<code>finite_trace_horizon_audit.py</code>	Exhaustive one- and two-state systems, selected three-state systems
Coherent diagrams	<code>appendix_D_coherence_audit.py</code>	Finite naturality, quotient stability and centralizer counts
Observational tests	<code>appendix_E_observation_audit.py</code>	Exact stochastic channels and binomial-error cases
Phenomenal bridges	<code>phenomenal_inference_audit.py</code>	Finite latent-label examples; no phenomenal validation

A machine-readable audit manifest in `reproducibility/AUDIT_MANIFEST.json` includes a SHA-256 digest for each audited script, the invocation command, test scope, and outcome of the release run. The manifest describes the released sources and makes accidental source drift detectable. A checksum verifies byte identity to a named release; it does not certify a theorem, the correct semantics of the code, or empirical applicability. The previous paper’s experiments, including its numerical convergence and finite abstraction examples, remain reproducible from the separate v2.9 archive and are not silently included as part of the present paper’s audit suite.

F.5 Reproduction protocol and environment boundaries

The canonical reproduction unit is the editable source ZIP. Its `manuscript.tex` compiles under XeLaTeX with the bibliography processed by Biber. On a system with these programs, one may execute `make`

pdf from the extracted project root. The compilation produces `build/manuscript.pdf`. Python computational audits can be rerun from the same project root using the commands in `reproducibility/AU-DIT-MANIFEST.json`; a separate convenience runner `review/run_release_audits.py` executes the registered scripts and writes a machine-readable report. The reference environment and script digests are recorded in the release manifest. Where scripts require scientific Python packages, the installed versions are recorded rather than asserted to be universally interchangeable.

For each run, a reproducible report should identify the release archive, source digests, Python version, operating environment, invoked commands, exit codes and any output artifacts. Exact rational and integer calculations can be compared directly, whereas numerical results should report precision and tolerances. Graphical renderings serve as visualizations of defined mathematical cases. The diagrams of quotient histories, cellular structures, torus maps, and experimental discrimination are synthetic and carry no observational neuroscience data. Retaining both the source code and the numerical values is useful for auditing plotting errors or mismatched panel captions.

This protocol permits technical reproduction within the declared computational environment. It does not by itself demonstrate cross-platform determinism for every library version, semantic soundness of all assertions, or the relevance of selected mathematical toy systems to consciousness. A source archive that compiles successfully demonstrates that a document can be generated; the truth of the mathematics and the scientific support for its interpretation require different evaluation procedures.

F.6 Proof dependencies and unresolved research obligations

The main text and appendices contain three different sorts of formal claims. A *deductive* claim follows from its explicitly stated mathematical hypotheses; a *verified finite example* is calculated on a declared finite model; and a *provisional explanatory proposal* interprets those objects in a physical or phenomenal context. The corresponding verification obligations differ. Table 24 identifies dependencies of particular importance to external review.

Table 24: Independent review obligations for the main families of mathematical and interpretive claims. The third column specifies work that compilation and internal regression tests cannot replace.

Claim family	Mathematical or computational dependencies	Additional review obligation
Action and process equivalence	Monoids, equivariance, trace and branching semantics; Sections 5, 8	Check morphism types, hidden-transition conventions, and universal quantifiers
Geometry and topology	Regularity, basepoints, attaching maps, coefficient rings; Section 12, Appendix B	Independent topological proof check and preservation of declared carrier type
Categorical translation	Functoriality, quotient stability and naturality; Section 11, Appendix D	Validate chosen category, generators and coherence of physical interpretations
Experiment and statistics	Known kernels, sampling laws, resets and decision losses; Section 13, Appendix E	Check protocol realizability, nuisance parameters and correlated observations
Phenomenal inference	Candidate bridge family, latent reporting channel; Section 14	Distinguish stipulated bridges from independently supported phenomenology

The preceding manuscript’s geometric and numerical hypotheses serve as motivation and a source of candidate mathematical objects, with no theorem imported solely by a matching name. In the present manuscript, explanatory adequacy remains separate from preservation of a mathematical invariant. An experimental programme would therefore require explicit selection of the physical system, the observation and perturbation interfaces, the empirical estimand, and the proposed phenomenal bridge, followed

by genuinely discriminating predictions. Historical indices, continuation operations, and modal possibility are not treated as a pre-existing universal ordering. These conditions also govern any future extension to heterogeneous relational systems or to the recognition of another subject's reported experience.

The immediate next stage is a manuscript-wide scholarly review: independently check the strongest propositions, seek counterexamples to informal interpretations, validate external sources against their exact claims, and examine numerical and experimental assumptions. The crosswalk and test manifests make that review traceable. They do not resolve the ontological and epistemic questions on which the broader consciousness programme remains open.

Research provenance

The four-orientation taxonomy inherits research questions from the author's prior independent manuscript on the geometry, topology, and dynamics of consciousness. Its prior formal results are motivation rather than transferred theorems; any result needed here will be restated with its assumptions and independently checked. External references used in the completed sections were verified online before insertion and entered in the bibliographic ledger. Publication-level evaluation remains pending.

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