

Braid-Theoretic Trajectory Description of Historical Social Dynamics: A Preliminary Methodological Framework

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Abstract

This paper develops a preliminary methodological framework for braid-theoretic trajectory description in historical social inquiry. It begins from an epistemic asymmetry: a complete generative rule for a social process can remain unavailable or underdetermined while portions of the realized relational history remain reconstructible from documents, events, institutional transformations, and ordered interactions. The paper adds an explicit fact-regime layer between social reality, evidence, and historical trajectory reconstruction. Within GR's three-level reality architecture, ontological reality remains suspended, relational reality contains historically constituted and efficacious social objects, and manifestation reality contains the context-dependent forms through which those objects become accessible. Social inquiry is then represented through overlapping fact families, alternative fact taxonomies, compatible fact configurations, candidate histories, relational trajectories, and braid descriptions. The framework develops set-valued reconstruction, fact-conditioned research objects, braid admissibility, topological robustness under evidential, taxonomic, and geometric uncertainty, topology-changing boundaries, comparative trajectory descriptors, and inverse constraints on candidate generative explanations. Its central methodological claim is limited: non-identification of a complete generator need not imply non-identification of selected trajectory structure. Braid-theoretic description is treated as a conditional intermediate representation whose fact articulation, admissibility, resolution, robustness, semantic interpretation, and causal role must be stated separately. The framework complements historical, interpretive, process-tracing, path-dependence, and sequence-sensitive methods.

Keywords: historical social dynamics; braid theory; trajectory reconstruction; process tracing; path dependence

Discussion Paper Note

This manuscript is the methodological continuation of a preceding formal study of braid-theoretic trajectory description in Generative Relational Theory. The companion formal paper establishes the distinction among a generative specification, a realized history, and a trajectory-descriptive representation, and analyzes the domain, resolution, robustness, and generality profile of a braid descriptor. The present paper moves to historical social inquiry and asks what that descriptive layer can contribute when the generating mechanisms of a social process are only partially known.

The methodological object is therefore not an alleged universal law of social history. The paper studies reconstruction of relational histories from incomplete evidence, the stability of selected topological descriptions under uncertainty, comparison among reconstructed trajectories, and the extent to which recurrent trajectory structure can constrain rather than uniquely determine candidate explanations.

The framework is intended to remain methodologically plural. Process tracing, historical interpretation, archival reconstruction, sequence analysis, comparative-historical reasoning, and domain-specific substantive theory retain independent roles. Braid-theoretic description supplies one additional structural vocabulary when its assumptions are justified by the historical object and the evidence.

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Notices

Status. This manuscript is a substantially revised preprint and discussion paper. The current version contains fifteen sections and integrates the fact-regime and three-level-reality architecture into the historical reconstruction, robustness, ontology-change, example, scope, and reporting formalisms.

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Statement on the use of language models. OpenAI’s ChatGPT was used for exploratory dialogue, formal reconstruction, source discovery followed by source verification, argumentative criticism, and drafting in $\text{\LaTeX}$. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, verified the cited sources, and bears sole responsibility for the manuscript, including its definitions, formal constructions, arguments, conclusions, and errors. Verification records for cited external sources are maintained in the accompanying source ledger.

1 Introduction

This paper develops a methodological framework for using braid-theoretic trajectory description in historical social inquiry. Its role is downstream from the formal construction of braid-theoretic trajectory description in Generative Relational Theory (GR): the present task is to determine what that construction can contribute when historical evidence gives partial access to realized processes while the complete generative dynamics remain unknown or underdetermined. The argument is methodological rather than ontological. It does not treat braid structure as the substance of social reality, and it does not propose that historical explanation can be replaced by a topological descriptor.

Historical and qualitative social research already gives substantial analytical weight to temporal order, event sequence, process, and path dependence. Abbott emphasizes that social-scientific methods carry assumptions about time, events, actors, and causality, and develops process-sensitive alternatives to analyses that suppress temporal organization [1]. Collier’s account of process tracing similarly treats careful description and causal sequence as central to within-case analysis and to the evaluation of causal claims [2]. Historical-sociological work on path dependence identifies classes of sequences in which earlier events condition later possibilities and in which timing and ordering are analytically consequential [7]. The present paper enters this methodological field at a narrower point: it asks whether selected structural properties of a reconstructed relational trajectory can remain identifiable and comparable even when the complete rule that generated the history is not.

The motivation also has a mathematical precedent. Braid-based analysis has been used to extract global topological information from observed particle trajectories even when the underlying flow is poorly known [10]. That result cannot be transferred directly to social history. Social entities, identities, institutional positions, and evidential records require domain-specific interpretation, and the relevant configuration space need not be physical space. The precedent is methodological: observation of trajectories can sometimes support structural analysis that is weaker than full recovery of the generator but stronger than endpoint description alone.

1.1 Generator Opacity in Historical Social Systems

The central epistemic problem is the asymmetry between generative explanation and historical reconstruction. In a fully specified dynamical model, a rule and an initial condition determine or constrain a trajectory. Historical social inquiry commonly begins from a different position. Mechanisms can be heterogeneous across time, institutional rules can change during the process, actors can respond reflexively to prior events, and the evidential record can reveal only fragments of the underlying generative organization. A single stable function analogous to a classical vector field is therefore often unavailable as a starting object.

The present framework represents that asymmetry schematically rather than treating it as a universal property of social science. Let G denote whatever generative structure a particular inquiry would need to recover for a complete explanation, let h denote the realized history, and let Y denote the surviving or observed evidential record. The epistemic arrangement is summarized in Equation 1.

$$G \rightsquigarrow F_{0:T}^{(2)} \rightsquigarrow M_{0:T}^{(3)} \rightsquigarrow Y, \quad Y \longmapsto \mathcal{F}_{\tau,A}(Y) \longmapsto \mathcal{H}_{\tau,A}(Y) \longmapsto \mathcal{B}_{\tau,A}(Y), \quad (1)$$

where $F_{0:T}^{(2)}$ denotes relational-level fact configurations, $M_{0:T}^{(3)}$ their manifestation history, $\mathcal{F}_{\tau,A}(Y)$ the fact configurations compatible with the record under taxonomy τ and assumptions A , $\mathcal{H}_{\tau,A}(Y)$ the corresponding compatible histories, and $\mathcal{B}_{\tau,A}(Y)$ the compatible braid descriptions when the histories are braid-admissible. Equation 1 separates ontic production from epistemic reconstruction.

The distinction is essential. The reconstructive chain from Y through fact configurations and candidate histories is not an inverse of the historical process in general. Several histories can be compatible with one evidential record, and several generative structures can be compatible with one reconstructed history. Historical uncertainty therefore appears at more than one layer.

1.2 Trajectory Structure as an Intermediate Object of Analysis

The companion formal paper establishes a hierarchy in which generative specifications produce histories and trajectory descriptors subsequently classify selected structure in those histories. The present paper takes that hierarchy as its formal starting point [6]. Its methodological thesis is that a useful intermediate object can exist between raw historical evidence and a complete generative explanation.

The strongest form of the thesis can be stated without assuming that the exact history is reconstructible. Suppose the evidence supports several candidate histories. If all braid-admissible reconstructions compatible with the evidence yield the same braid class, then the trajectory descriptor is identifiable even though the geometric or event-level reconstruction remains nonunique. In that situation the evidence supports a structural statement about the historical trajectory without supporting a unique realized history.

Conversely, failure of braid identifiability does not imply failure of historical inquiry. It means only that the selected braid-level description is not determined by the available evidence under the declared reconstruction assumptions. Other historical, interpretive, causal, or semantic analyses can remain available. Braid-theoretic description is therefore one conditional descriptive layer rather than a universal test of historical knowledge.

The methodological claim can be summarized in Equation 2.

$$\text{non-identification of } G \not\Rightarrow \text{non-identification of selected structure in } h. \quad (2)$$

Equation 2 is a statement about logical possibility within the layered architecture, not an empirical claim that useful trajectory structure will always be identifiable. Whether it is identifiable depends on the evidence, the reconstruction rule, the choice of historical entities, the configuration space, and the descriptive resolution.

1.3 Interface with Historical Qualitative Methodologies

The framework is designed to complement methods that already analyze process and sequence. Process tracing evaluates diagnostic evidence in relation to descriptive and causal claims and pays close attention to the temporal sequence in which observations occur [2]. Braid-theoretic trajectory description does not supply the causal-process tests used in process tracing. Its narrower contribution is a formal representation of selected relational ordering and intertwining within a reconstructed trajectory.

Likewise, path-dependence analysis studies historically consequential sequences and mechanisms of reproduction or reaction [7]. A braid class does not by itself establish path dependence in that substantive sense. Two histories can be topologically distinct without one displaying a path-dependent causal mechanism, and two path-dependent histories can share a braid class. The possible contribution lies in adding a trajectory-level structural descriptor to a broader historical explanation.

The relation to sequence-sensitive social analysis is therefore one of partial complementarity. Temporal ordering, event semantics, institutional interpretation, and causal mechanisms remain domain-specific. Braid theory becomes relevant only when the inquiry can justify persistent or typed relational traces, a suitable configuration space, and an equivalence relation under which topological deformation corresponds to analytically negligible variation.

1.4 Methodological Commitments

Six commitments organize the framework.

First, *reconstruction is explicit*. A historical trajectory is not silently equated with the evidential record. The map from records to candidate histories must state its assumptions, admissible uncertainty, and identity criteria.

Second, *fact articulation is explicit*. Historical records constrain fact configurations only under a declared taxonomy or fact regime. Fact families, categories, research objects, and their continuity conditions are therefore part of the reconstructive commitments rather than invisible preprocessing choices.

Third, *configuration spaces are domain-specific*. The “space” in which trajectories move need not be geographic. It can encode relational positions, institutional roles, diplomatic ordering, authority and dependence, network positions, or another domain-specific structure. The interpretation must be supplied by the substantive inquiry rather than imported from braid mathematics.

Fourth, *robustness is conditional*. Topological stability can make selected descriptors insensitive to some reconstruction perturbations, but only within an admissible deformation class that avoids topology-changing boundaries. Robustness is therefore evidence- and constraint-relative.

Fourth, *invariants are descriptive before they are explanatory*. Recurrent braid classes or derived invariants can support comparison among histories. They do not identify the generating mechanism without additional assumptions.

Fifth, *inverse use is constraining rather than uniquely reconstructive*. Observed trajectory structure can eliminate candidate explanations that cannot generate the observed class, but several generators may remain compatible with the same trajectory description.

These commitments preserve the methodological pluralism required by the wider GR programme and by historical inquiry more generally. The higher GR metamodel supplies abstract roles for constraint, change, structure, and difference without fixing a universal metric or state geometry [5]. The present paper adds only those lower-level commitments needed for a historical reconstruction and its braid-theoretic description.

1.5 Methodological Research Questions

The paper addresses six connected questions. The first concerns reconstruction: how should incomplete historical evidence define a set of candidate relational histories rather than a single silently assumed trajectory? The second concerns admissibility: under what conditions can historical entities, roles, or institutions be represented as persistent strands in a relational configuration space? The third concerns robustness: which braid-level conclusions survive plausible perturbations of timing, position, and reconstruction? The fourth concerns topology-changing boundaries: when can small evidential changes alter the inferred topological class, and how should those boundary regions be interpreted? The fifth concerns comparison: which braid classes or derived invariants can support comparison across cases without replacing domain semantics? The sixth concerns explanation: how can observed trajectory structure constrain a class of candidate generators without being mistaken for a recovered causal law?

The principal contribution is a methodological architecture that makes these questions formally separable. It extends the companion braid formalism by inserting an explicit evidence-and-reconstruction layer between realized history and trajectory description. That extension is particularly important in historical social research because the exact realized path is often itself an inferential object.

The paper proceeds in thirteen further sections. Section 2 formalizes generator opacity, historical reconstruction, and the distinction between explanatory and descriptive identifiability. Section 3 develops evidence objects, candidate-history sets, and reconstruction constraints. Section 4 defines domain-specific relational configuration spaces and historical strand semantics. Section 5 derives the set-valued trajectory and braid reconstruction procedure. Sections 6 and 7 study robustness under evidential uncertainty and topology-changing boundaries. Section 8 develops comparative descriptors and invariant families. Section 9 formalizes inverse constraints on candidate generative explanations. Sections 10 and 11 position the framework relative to established methods and state semantic and evidential obligations. Sections 12 and 13 give worked examples and identify failure modes and scope limits. Section 14 concludes.

2 Fact Structures and Reality Levels in Historical Social Inquiry

This section identifies the social-scientific objects that later enter historical reconstruction and trajectory description. Its role is foundational: an evidential record does not directly yield a historical trajectory, because the inquiry first articulates what kinds of facts, entities, relations, events, states, and processes

count as research objects. The objective is to make that articulation explicit without treating any one disciplinary taxonomy as ontologically final. The section first locates fact articulation within the three GR reality levels. It then distinguishes fact families, fact categories, fact tokens, and fact configurations; introduces a preliminary fact-regime representation; relates alternative fact taxonomies to the four GR meta-primitives; and finally separates the ontic direction of social-historical production from the epistemic direction of historical reconstruction. The discussion is compatible with the long-standing importance of social facts in sociology and with contemporary social ontology's plural treatment of social facts, institutional facts, groups, practices, properties, processes, law, language, and other social entities [3, 4, 9].

2.1 Three-Level Reality in Social Inquiry

GR distinguishes three analytical levels of reality while suspending claims about ultimate ontology. The levels are stated in Equation 3.

$$\mathcal{R}^{(1)} := \text{ontological reality}, \quad \mathcal{R}^{(2)} := \text{relational reality}, \quad \mathcal{R}^{(3)} := \text{manifestation reality}. \quad (3)$$

The notation in Equation 3 is classificatory rather than a claim that the three levels form three disjoint mathematical universes.

At $\mathcal{R}^{(1)}$, the ultimate ontological status of the objects represented by the theory remains suspended. A state, institution, legal order, organization, identity, role, or social relation can be modeled without the paper claiming that the model establishes what such an object ultimately is.

At $\mathcal{R}^{(2)}$, the analysis concerns historically constituted relational beings and structures that are sufficiently stable, persistent, and efficacious to matter within the domain of inquiry. Examples can include institutions, offices, currencies, corporations, legal statuses, organizations, roles, norms, identities, or relations of authority. Relational reality is therefore the primary level at which many social-scientific research objects are treated as effective participants in historical processes.

At $\mathcal{R}^{(3)}$, relational reality becomes accessible through context-dependent presentations. These can include recognition, measurement, records, interpretation, symbols, or legal manifestations. Documents, utterances, registrations, judgments, archives, institutional decisions, public symbols, and statistics can therefore belong to manifestation reality without exhausting the relational objects to which they refer.

The three levels should not be confused with fact taxonomies. A legal fact, political fact, social fact, institutional fact, or economic fact is not thereby a fourth reality level. Such fact categories are inquiry-relative articulations of structures, events, properties, states, relations, or processes that can concern relational reality, manifestation reality, or the relations between them.

2.2 Fact Families, Categories, Tokens, and Configurations

Social inquiry uses many major fact families. A representative, nonexhaustive collection is denoted in Equation 4.

$$\mathcal{K}_F := \left\{ \begin{array}{l} \text{social, institutional, legal, political, economic, organizational,} \\ \text{cultural-symbolic, demographic, linguistic-discursive, historical, } \dots \end{array} \right\}. \quad (4)$$

Equation 4 is illustrative rather than exhaustive. Contemporary social ontology itself contains multiple competing proposals about which categories of social entities or phenomena should be treated as basic [4].

Fact-family membership need not form a partition. Let $\mathcal{T}_F$ be a set of fact categories and define the category-to-family relation in Equation 5.

$$R_{\text{fam}} \subseteq \mathcal{T}_F \times \mathcal{K}_F. \quad (5)$$

A category can be related to several fact families through R_{fam} . For example, a treaty-entry-into-force category can simultaneously participate in legal, political, institutional, social, and historical fact families.

A second classificatory axis concerns the formal shape of a fact. One possible taxonomy is stated in Equation 6.

$$\mathcal{T}_F^{(6)} := \{\text{entity, property, state, event, process, relation}\}. \quad (6)$$

Equation 6 is only one possible taxonomy. GR does not promote the six categories to a privileged ontology, and another discipline can use a different category system when the research object requires it.

Let $\mathcal{X}_F$ be the set of particular fact tokens admitted by an inquiry. Define the token-to-category typing relation in Equation 7.

$$R_{\text{type}} \subseteq \mathcal{X}_F \times \mathcal{T}_F. \quad (7)$$

A token can have several types when the taxonomy permits overlapping classification.

The distinction among family, category, and token is therefore stated schematically in Equation 8.

$$\text{fact family} \neq \text{fact category} \neq \text{fact token}. \quad (8)$$

For example, legal facts form a family, legal event can be a category, and a particular treaty's entry into force at a specified time can be a fact token.

Historical analysis rarely operates on isolated fact tokens. It usually requires mutually related facts concerning actors, roles, temporal relations, statuses, events, and processes. Let $\mathfrak{C}_\tau$ denote the set of fact configurations admissible under a declared taxonomy τ . A fact configuration $F \in \mathfrak{C}_\tau$ is understood as a typed relational organization of fact tokens rather than an unordered bag of claims.

2.3 Fact Regimes and the GR Four-Primitive Meta-Language

A fixed fact taxonomy is insufficient for representing how fact categories, family memberships, admissibility rules, and recognition mechanisms themselves change historically. The paper therefore introduces a preliminary fact-regime representation.

Definition 2.1 (Fact regime). A *fact regime* is a higher-order representational object of the form stated in Equation 9.

$$\mathfrak{G}_F := (\mathcal{X}_F, \mathcal{T}_F, \mathcal{K}_F, R_{\text{type}}, R_{\text{fam}}, \mathcal{C}_F, \mathcal{M}_F), \quad (9)$$

where $\mathcal{C}_F$ collects the constitutive, evidential, or admissibility constraints governing fact formation and $\mathcal{M}_F$ collects the declared mechanisms capable of transforming fact tokens, category assignments, family memberships, or other regime components.

Definition 2.1 does not introduce seven new GR primitives. It is a higher-order organization built over the four primitive roles inherited from the GR metamodel: constraint, change, structure, and difference [5].

The correspondence is summarized in Table 1.

Table 1 treats familiar fact forms as higher-order organizations rather than primitive ontology. An entity can be represented through persistence of a structured configuration under declared constraints; a state through a locally stabilized configuration; an event through a localized relevant change; a process through organized composition of changes; a property through a stable or discriminating structural feature; and a relation through structured conditioning among distinguishable terms. These are representational reconstructions, not claims that the six forms are ontologically reducible to the four primitives.

The category set itself can change. Let $\mathcal{T}_F(t)$ be the category system available at historical position t . Fact-taxonomy change is permitted when Equation 10 holds.

$$\mathcal{T}_F(t_1) \neq \mathcal{T}_F(t_2). \quad (10)$$

Equation 10 covers emergence, splitting, merging, disappearance, or inadequacy of categories. It should be distinguished from change in the fact configuration while the taxonomy remains fixed.

Table 1: One preliminary GR representation of fact-regime components through the four meta-primitives.

GR primitive	Fact-regime role	Illustrative questions
Constraint	Admissibility, constitutive rules, evidential thresholds, category-membership requirements	What must hold for a token to count as an instance of a category or for an attribution to be admissible?
Change	Transformations of tokens, statuses, categories, recognitions, or regime rules	Which acts or processes create, revise, stabilize, or dissolve fact status?
Structure	Relations among tokens, categories, families, institutions, and temporal positions	Which facts depend on, support, contradict, or condition other facts?
Difference	Distinctions the regime preserves, collapses, or makes newly available	Which differences remain trackable, and which become indistinguishable under the current regime?

2.4 Relational Facts, Manifestation Facts, and Records

The three-level reality architecture permits two different fact articulations to refer to one historical situation without identifying them.

Consider a relational-level fact token $f^{(2)}$ such as: an actor holds an office, an organization possesses a legal status, or two institutions stand in a declared relation. A manifestation-level fact token $f^{(3)}$ can concern a document, judgment, announcement, registry entry, public recognition, or other presentation through which the relational fact becomes accessible.

The distinction is summarized in Equation 11.

$$\text{relational fact} \neq \text{manifestation fact} \neq \text{evidential record}. \quad (11)$$

The final distinction matters because an evidential record is a surviving object used by an inquiry, whereas manifestation reality includes more presentations than the surviving archive contains.

Scientific or conceptual priority illustrates the point. The following fact tokens are analytically distinct:

1. a historical-event fact that person A publicly formulated method M at time t ;
2. an evidential fact that document d records the formulation;
3. an attribution fact that A is credited with authorship or priority;
4. a social-recognition fact that a community recognizes A as a founder or originator;
5. where applicable, a legal fact concerning an enforceable legal status or entitlement.

Historical priority therefore need not entail social recognition of priority. Mechanisms of indexing, citation, institutional recognition, education, archives, search, metadata, or professional convention can affect whether a recognition fact stabilizes without changing the underlying historical event.

This distinction also matters in law and governance. Searle’s treatment of institutional facts provides a canonical example of facts whose status depends on constitutive social arrangements rather than on physical description alone [9]. Legal and institutional acts can recognize pre-existing facts, constitute new statuses, or alter the consequences attached to a fact configuration.

2.5 Research Objects as Fact Configurations

A social-scientific research object is not assumed to be a biologically fixed individual or a naturally given unit. It can be an institution, office, legal person, organization, coalition, role-bearing entity, relation, process, or another historically constituted object selected by the inquiry.

Let τ denote a declared fact taxonomy and let $F \in \mathfrak{C}_\tau$ be an admissible fact configuration. A research-object construction is represented by the map in Equation 12.

$$\text{Obj}_\tau : \mathfrak{C}_\tau \longrightarrow \mathcal{O}_\tau, \quad (12)$$

where $\mathcal{O}_\tau$ is the set of research objects available under the declared taxonomy.

Equation 12 does not state that an object is nothing more than its fact configuration. It records how the inquiry operationally identifies the object to be tracked.

This representation clarifies later strand identity. A strand label in the braid analysis can represent a research object constructed from a compatible fact configuration rather than being restricted to one biological person. If an organization splits into two institutions, if one office is replaced by several offices, or if a legal person ceases and successor entities emerge, the change can alter the research-object structure itself. Such cases should be analyzed as potential changes in fact configuration or taxonomy before being treated as failures of geometric strand tracking.

2.6 Ontic Production and Epistemic Reconstruction

The fact layer allows the paper to distinguish the direction in which social-historical reality is produced from the direction in which historical inquiry reconstructs it.

The ontic or generative direction is represented schematically in Equation 13.

$$G \rightsquigarrow F_{0:T}^{(2)} \rightsquigarrow M_{0:T}^{(3)} \rightsquigarrow Y, \quad (13)$$

where G denotes the generative structure sought by the inquiry, $F_{0:T}^{(2)}$ denotes the evolving relational-level fact configuration, $M_{0:T}^{(3)}$ denotes its historically manifested forms, and Y denotes the surviving evidential record. The symbol $\rightsquigarrow$ in Equation 13 indicates generative or manifestation dependence rather than a claim that each transition is a deterministic function.

Historical inquiry proceeds in the opposite epistemic direction. For a declared taxonomy τ and assumption set A , the reconstruction chain is stated in Equation 14.

$$Y \longmapsto \mathcal{F}_{\tau,A}(Y) \longmapsto \mathcal{H}_{\tau,A}(Y) \longmapsto \mathcal{Q}_{\tau,A}(Y) \longmapsto \mathcal{B}_{\tau,A}(Y), \quad (14)$$

where the sets contain respectively compatible fact configurations, compatible histories, compatible relational trajectories, and compatible braid descriptions.

Figure 1 places the two directions side by side.

Equation 14 and Figure 1 give the revised methodological spine of the paper. Historical evidence does not directly reveal the generator, the complete relational reality, or one uniquely privileged history. It supports sets of fact configurations under a declared taxonomy; those configurations support sets of historical objects and trajectories; and only then can a braid descriptor be applied.

This architecture also introduces a new robustness question: whether a trajectory-level conclusion survives reasonable alternative fact taxonomies. Later sections incorporate the taxonomy τ into the reconstruction regime and define taxonomic robustness alongside evidential and geometric robustness.

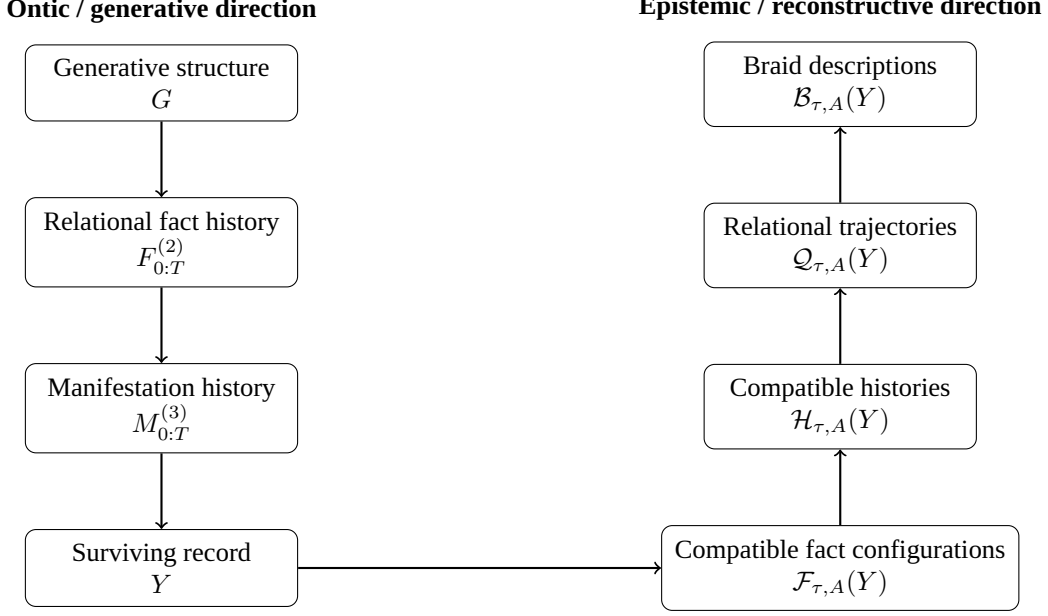


Figure 1: Ontic production and epistemic reconstruction are distinct directions. The left side represents generative and manifestation dependence; the right side represents set-valued reconstruction under a declared fact taxonomy and assumption regime.

3 Epistemic Architecture of Historical Trajectory Analysis

This section formalizes the epistemic architecture that follows from the fact-regime analysis of Section 2. Its role is to distinguish five objects that historical trajectory analysis can otherwise conflate: a candidate generative specification, a realized history, a fact configuration articulated under a declared taxonomy, an evidential record, and a trajectory description. The objective is to derive conditions under which a trajectory-level description can be identified even when the compatible fact configuration, complete realized history, and generating specification remain nonunique. The argument retains the generator–history architecture of the companion formal paper while inserting explicit fact-articulation and evidential-compatibility relations between history and record [6]. Section 3.1 defines the layered architecture. Section 3.2 derives generator underdetermination relative to a candidate class. Section 3.3 defines fact and history reconstruction as set-valued maps. Section 3.4 then derives descriptive identifiability under generator and reconstruction underdetermination.

3.1 Generative, Fact-Articulation, and Evidential Layers

Let $\mathfrak{D} = (S, M, \alpha, \sim)$ be a GR dynamical instantiation. For each positive integer k , the companion formal paper defines a generation map $\text{Gen}_k : S \times M^k \rightarrow \Gamma_k(\mathfrak{D})$. Because the relevant historical length can itself be uncertain, define the global generative domain in Equation 15.

$$\mathcal{U}(\mathfrak{D}) := \coprod_{k \geq 1} (S \times M^k). \quad (15)$$

The corresponding piecewise generation map is defined in Equation 16.

$$\text{Gen} : \mathcal{U}(\mathfrak{D}) \longrightarrow \Gamma(\mathfrak{D}), \quad \text{Gen}|_{S \times M^k} := \text{Gen}_k. \quad (16)$$

Fix a declared fact taxonomy τ with admissible fact-configuration space $\mathfrak{C}_\tau$ as introduced in Section 2.2. A realized history need not determine one fact configuration independently of analytical articulation. Represent the relation between realized histories and fact configurations by Equation 17.

$$\mathbf{A}_\tau \subseteq \Gamma(\mathfrak{D}) \times \mathfrak{C}_\tau. \quad (17)$$

The statement $(h, F) \in A_\tau$ means that F is an admissible fact configuration through which the inquiry articulates history h under taxonomy τ . The relation can be one-to-many when the same history supports several admissible fact articulations.

Let $\mathcal{Y}$ be the set of possible evidential records. The relation between a fact configuration and an evidential record is defined in Equation 18.

$$E_\tau \subseteq \mathfrak{C}_\tau \times \mathcal{Y}. \quad (18)$$

For $(F, Y) \in E_\tau$, the fact configuration F is compatible with the record Y under the currently admitted interpretation rules. Equation 18 does not assume that the record is a lossless manifestation of the fact configuration.

The resulting architecture is summarized in Equation 19.

$$u \xrightarrow{\text{Gen}} h, \quad (h, F) \in A_\tau, \quad (F, Y) \in E_\tau, \quad D(h) \in \Omega_D. \quad (19)$$

Equation 19 separates generation, fact articulation, evidence, and description. The generative specification u contains mechanism-bearing information; the realized history h records the generated state sequence; the fact configuration F states which social-scientific facts are articulated under τ ; the record Y contains surviving manifestation-level evidence; and the descriptor D retains selected structure from an admitted history.

3.2 Generator Underdetermination Relative to a Candidate Class

Statements about generator identification are always relative to a declared candidate class. Let $\mathcal{U}^* \subseteq \mathcal{U}(\mathfrak{D})$ be the candidate generative class admitted by the inquiry. For a realized history h , define the unrestricted and candidate-restricted generation fibers in Equation 20.

$$\mathcal{G}(h) := \text{Gen}^{-1}(\{h\}), \quad \mathcal{G}^*(h) := \mathcal{G}(h) \cap \mathcal{U}^*. \quad (20)$$

Generator underdetermination for a known history is present when $|\mathcal{G}^*(h)| > 1$. A singleton fiber gives identification only relative to $\mathcal{U}^*$.

The fact layer introduces another source of nonuniqueness before generator inversion is attempted. The same evidential record can support several fact configurations, and one fact configuration can be compatible with several histories. Generator opacity can therefore arise through at least three distinct routes:

1. several fact configurations are compatible with the record;
2. several realized histories are compatible with one or more of those fact configurations;
3. one realized history has several admitted generators.

The formal decomposition of these sources follows after the set-valued reconstruction maps are defined.

3.3 Fact and History Reconstruction as Set-Valued Maps

For $Y \in \mathcal{Y}$, define the compatible fact-configuration set in Equation 21.

$$\mathcal{F}_{\tau, E}(Y) := \{F \in \mathfrak{C}_\tau : (F, Y) \in E_\tau\}. \quad (21)$$

Equation 21 is the first epistemic inversion of the record. It states which fact configurations survive the declared evidential and taxonomic constraints.

The compatible history set is then induced through the history–fact articulation relation. Define Equation 22.

$$\mathcal{R}_{\tau, E}(Y) := \{h \in \Gamma(\mathfrak{D}) : \exists F \in \mathcal{F}_{\tau, E}(Y) \text{ such that } (h, F) \in A_\tau\}. \quad (22)$$

Equation 22 makes history reconstruction explicitly conditional on fact articulation. For a fixed taxonomy and evidential regime, the shorter notation $\mathcal{R}_E(Y)$ will be used when no ambiguity results.

Definition 3.1 (Fact-configuration identifiability). A fact configuration is *identified from Y relative to (τ, E_τ)* when Equation 23 holds.

$$|\mathcal{F}_{\tau,E}(Y)| = 1. \quad (23)$$

Definition 3.2 (History identifiability). A realized history is *identified from Y relative to (τ, E_τ, A_τ)* when Equation 24 holds.

$$|\mathcal{R}_{\tau,E}(Y)| = 1. \quad (24)$$

Neither Definition 3.1 nor Definition 3.2 is a claim of metaphysical uniqueness. Each is relative to the declared taxonomy, evidence interpretation, and articulation relation.

Define the evidence-compatible candidate-generator set in Equation 25.

$$\mathcal{G}_\tau^*(Y) := \{u \in \mathcal{U}^* : \text{Gen}(u) \in \mathcal{R}_{\tau,E}(Y)\}. \quad (25)$$

Proposition 3.3 (Generator decomposition over reconstructed histories). *For every evidential record Y , Equation 26 holds.*

$$\mathcal{G}_\tau^*(Y) = \bigcup_{h \in \mathcal{R}_{\tau,E}(Y)} \mathcal{G}^*(h). \quad (26)$$

Proof. If $u \in \mathcal{G}_\tau^*(Y)$, then $h := \text{Gen}(u)$ belongs to $\mathcal{R}_{\tau,E}(Y)$ and $u \in \mathcal{G}^*(h)$. This gives containment in the right-hand side. Conversely, if $u \in \mathcal{G}^*(h)$ for some $h \in \mathcal{R}_{\tau,E}(Y)$, then $u \in \mathcal{U}^*$ and $\text{Gen}(u) = h$, so $u \in \mathcal{G}_\tau^*(Y)$. $\square$

Proposition 3.3 separates generator underdetermination from fact-configuration and history underdetermination. Several fact configurations can map to overlapping history sets, and several histories can each have several admitted generators.

3.4 Descriptive Identifiability under Generator Underdetermination

Let $D : \Gamma_D \rightarrow \Omega_D$ be a trajectory descriptor. For record Y , define the evidence-compatible description set in Equation 27.

$$\mathcal{D}_{\tau,E}(Y) := \{D(h) : h \in \mathcal{R}_{\tau,E}(Y) \cap \Gamma_D\}. \quad (27)$$

The descriptor is completely admissible for the reconstructed history set when Equation 28 holds.

$$\mathcal{R}_{\tau,E}(Y) \neq \emptyset, \quad \mathcal{R}_{\tau,E}(Y) \subseteq \Gamma_D. \quad (28)$$

Definition 3.4 (Descriptive identifiability). The descriptor value is *identified from Y relative to the declared reconstruction regime* when Equation 28 holds and Equation 29 is satisfied.

$$|\mathcal{D}_{\tau,E}(Y)| = 1. \quad (29)$$

For $\omega \in \Omega_D$, define the descriptor fiber $\mathcal{F}_D(\omega) := D^{-1}(\{\omega\})$.

Proposition 3.5 (Fiber criterion for descriptive identifiability). *Assume $\mathcal{R}_{\tau,E}(Y) \neq \emptyset$. The descriptor is identified with value ω if and only if Equation 30 holds.*

$$\mathcal{R}_{\tau,E}(Y) \subseteq \mathcal{F}_D(\omega). \quad (30)$$

Proof. If Equation 30 holds, every reconstructed history is in the descriptor domain and receives value ω , so the compatible description set is $\{\omega\}$. Conversely, complete admissibility and singleton descriptive value imply that every reconstructed history maps to that same value and therefore lies in the corresponding fiber. $\square$

Proposition 3.5 shows that descriptive identification is a property of the entire reconstructed-history set lying inside one descriptor fiber; it does not require the histories themselves to be identical.

The central methodological possibility now follows.

Proposition 3.6 (Trajectory identification under generator underdetermination). *Suppose a record Y satisfies Equation 31.*

$$|\mathcal{D}_{\tau, \mathbf{E}}(Y)| = 1, \quad |\mathcal{G}_{\tau}^*(Y)| > 1, \quad (31)$$

together with complete descriptor admissibility. Then the selected trajectory structure is identified at resolution D while the candidate generator remains nonidentified.

Proof. The first condition in Equation 31 gives descriptive identifiability by Definition 3.4. The second leaves more than one admitted generator compatible with the reconstructed history set. The two claims concern different layers and can therefore hold simultaneously. $\square$

Proposition 3.6 does not require fact-configuration or full history identification. Several fact configurations and several histories can all remain compatible with the record while lying inside one descriptor fiber. This is the formal version of the paper’s central epistemic thesis.

The next section constructs the compatibility relations and shared witnesses through which record items, temporal order, identity continuity, roles, fact configurations, and auxiliary assumptions determine the reconstruction sets used here.

4 Constraint-Based Reconstruction of Fact Configurations and Histories

This section constructs the fact- and history-level compatibility relations introduced abstractly in Section 3. Its role is to make visible the constraints by which heterogeneous records are translated first into admissible fact configurations and then into candidate histories. The objective is not to impose one universal fact ontology on historical inquiry. Instead, record interpretation is indexed by a declared taxonomy, while temporal order, identity persistence, role continuity, and auxiliary assumptions are given separate formal roles. This constraint-based approach is compatible with sequence-sensitive historical analysis and process-tracing practice while leaving substantive interpretation to the historical inquiry itself [1, 2]. Section 4.1 defines record-level fact constraints and their history projection. Section 4.2 formalizes partial temporal order. Section 4.3 distinguishes identity persistence from role continuity. Section 4.4 combines the fact and history constraints through shared witnesses. Section 4.5 then derives assumption monotonicity and a leave-one-assumption-out audit criterion.

4.1 Record-Level Fact Constraints

Section 3.1 leaves the evidential record $Y \in \mathcal{Y}$ without a universal internal data structure. The present subsection assumes only that the record can be decomposed into a finite set of items $\mathcal{J}(Y)$ for the purpose of the analysis. An item can be a document, dated event statement, institutional act, material trace, coded interaction, registry entry, testimony, or another manifestation-level object.

Fix a declared fact taxonomy τ with admissible configuration space $\mathfrak{C}_{\tau}$. Each record item $j \in \mathcal{J}(Y)$ is interpreted first through a fact-configuration compatibility set $\mathcal{C}_{j, \tau}^F(Y) \subseteq \mathfrak{C}_{\tau}$. The record-level compatible fact configurations are defined in Equation 32.

$$\mathcal{F}_{\text{rec}, \tau}(Y) := \bigcap_{j \in \mathcal{J}(Y)} \mathcal{C}_{j, \tau}^F(Y). \quad (32)$$

Equation 32 does not state that each record item is complete or correct. It states how the current interpretation of the record constrains the fact configurations admissible under τ .

The corresponding history-level candidate set is induced through the history–fact articulation relation A_{τ} from Equation 17. Define Equation 33.

$$\mathcal{C}_{\text{rec}}(Y) := \{h \in \Gamma(\mathfrak{D}) : \exists F \in \mathcal{F}_{\text{rec}, \tau}(Y) \text{ such that } (h, F) \in A_{\tau}\}. \quad (33)$$

Equation 33 is the first explicit bridge from fact reconstruction to historical reconstruction.

Some record items contain event-level information that can be localized inside an ordered history. Let $\mathcal{E}(Y) \subseteq \mathcal{J}(Y)$ be the subset of event-bearing items. For a history $h \in \Gamma_k(\mathfrak{D})$, define its transition length in Equation 34.

$$\ell(h) := k. \quad (34)$$

Event localization is represented by the relation in Equation 35.

$$\mathsf{L}_Y \subseteq \{(e, h, j) : e \in \mathcal{E}(Y), h \in \mathcal{C}_{\text{rec}}(Y), 1 \leq j \leq \ell(h)\}. \quad (35)$$

The statement $(e, h, j) \in \mathsf{L}_Y$ means that transition j of history h is an admissible realization of event item e under the current fact interpretation and event-localization rule.

Relational observations are handled similarly. A record stating that two historical actors interacted, one office succeeded another, an institution entered a legal relation, or an organization changed status first constrains the compatible fact configurations and only then the histories capable of realizing them. Translation into a braid crossing occurs later and requires additional relational-position and strand assumptions.

4.2 Temporal Order as a Partial Constraint

Historical records often determine some temporal relations while leaving others unresolved. A total ordering can therefore add information not contained in the evidence. The present framework records only the precedence relations that the reconstruction regime is prepared to assert.

Let $\prec_Y$ be a strict partial order on $\mathcal{E}(Y)$. Thus $\prec_Y$ is irreflexive and transitive. The interpretation $e_1 \prec_Y e_2$ is that the evidence and admitted chronology require event item e_1 to occur before event item e_2 . If neither $e_1 \prec_Y e_2$ nor $e_2 \prec_Y e_1$ holds, the formalism does not infer an order between them.

For a candidate history h , a *temporal realization* is a map $\tau : \mathcal{E}(Y) \rightarrow \{1, \dots, \ell(h)\}$ that satisfies both event localization and recorded precedence. These conditions are stated together in Equation 36.

$$\begin{aligned} (e, h, \tau(e)) &\in \mathsf{L}_Y && \text{for every } e \in \mathcal{E}(Y), \\ e_1 \prec_Y e_2 &\implies \tau(e_1) < \tau(e_2) && \text{for every } e_1, e_2 \in \mathcal{E}(Y). \end{aligned} \quad (36)$$

Let $\mathsf{T}_Y(h)$ denote the set of temporal realizations satisfying Equation 36. The temporal-compatibility set is defined in Equation 37.

$$\mathcal{C}_{\text{time}}(Y) := \{h \in \Gamma(\mathfrak{D}) : \mathsf{T}_Y(h) \neq \emptyset\}. \quad (37)$$

Equation 37 preserves chronology without silently completing it. Two reconstructed histories can both satisfy the same partial order while assigning different positions to incomparable events. This residual ambiguity is part of the reconstruction set rather than an error to be removed automatically.

Interval-dated evidence can be incorporated through the localization relation L_Y . For example, if an event is known only to have occurred within a historical interval, the admissible transition positions can be restricted to those compatible with that interval without converting the entire evidential chronology into a total order. This keeps temporal precision proportional to the evidence actually used by the reconstruction.

4.3 Continuity Constraints on Historical Identity and Roles

Temporal placement alone does not determine which mentions in the record refer to the same continuing historical entity. Identity persistence and role continuity are separate reconstruction problems. The distinction is important because one entity can change role while remaining historically continuous, and one institutional role can persist while being occupied by different entities.

Let $\mathcal{M}(Y)$ be the set of entity mentions used by the evidential record. Examples can include personal names, institutional labels, offices, organizations, or other domain-specific references. A reconstruction

scheme that supports identity analysis must provide, for each candidate history h , a set $\text{Ref}(h)$ of candidate persistent referents available in that history. The construction of $\text{Ref}(h)$ is domain-specific and is not part of the highest GR metamodel.

The evidential record can impose two kinds of identity constraint. Let $\equiv_Y^{\text{id}}$ be an equivalence relation on $\mathcal{M}(Y)$ representing mention pairs required to denote the same historical referent. Let $\#_Y^{\text{id}}$ be a symmetric, irreflexive relation on $\mathcal{M}(Y)$ representing mention pairs required to denote distinct referents. A candidate identity assignment is a map $\iota : \mathcal{M}(Y) \rightarrow \text{Ref}(h)$. It is identity-compatible when Equation 38 holds.

$$\begin{aligned} m_1 \equiv_Y^{\text{id}} m_2 &\implies \iota(m_1) = \iota(m_2), \\ m_1 \#_Y^{\text{id}} m_2 &\implies \iota(m_1) \neq \iota(m_2). \end{aligned} \quad (38)$$

Role information requires an additional domain vocabulary. Let $\mathcal{Q}(Y)$ be the set of role labels used by the study. These labels are local analytical categories rather than a universal ontology of social roles. For a candidate history h , let the role-admissibility relation be defined in Equation 39.

$$\text{Role}_h \subseteq \text{Ref}(h) \times \{1, \dots, \ell(h)\} \times \mathcal{Q}(Y). \quad (39)$$

Membership $(a, j, q) \in \text{Role}_h$ means that referent a can occupy role q at transition position j in history h under the declared reconstruction model.

Let $\mathcal{O}_{\text{role}}(Y) \subseteq \mathcal{M}(Y) \times \mathcal{E}(Y) \times \mathcal{Q}(Y)$ be the set of role observations in the record. A pair (τ, ι) of temporal and identity assignments satisfies the observed-role constraints when Equation 40 holds.

$$(m, e, q) \in \mathcal{O}_{\text{role}}(Y) \implies (\iota(m), \tau(e), q) \in \text{Role}_h. \quad (40)$$

A claim of role continuity is stronger than a pair of isolated role observations. Let $\mathcal{K}_{\text{role}}(Y)$ be a set of quadruples (m, e_-, e_+, q) asserting that the referent denoted by m continuously occupies role q between two evidenced events. Such a continuity claim is satisfied when Equation 41 holds.

$$\begin{aligned} (m, e_-, e_+, q) \in \mathcal{K}_{\text{role}}(Y), \quad \tau(e_-) \leq j \leq \tau(e_+) \\ \implies (\iota(m), j, q) \in \text{Role}_h. \end{aligned} \quad (41)$$

Equation 41 makes continuity an explicit reconstruction commitment. The formalism does not infer continuous role occupancy merely because the same role is observed at two separated points.

4.4 Fact-History Compatibility from Shared Reconstruction Witnesses

The record-level fact constraints, history articulation, temporal order, identity assignments, and role claims should be satisfied by one coherent reconstruction. The reconstruction therefore combines a compatible fact configuration with one shared temporal and identity witness for its candidate history.

For a history h , define a *reconstruction witness* $w = (\tau_Y, \iota)$, where $\tau_Y \in \mathbb{T}_Y(h)$ satisfies Equation 36, ι satisfies Equation 38, and the pair satisfies the role conditions in Equations 40 and 41. Let $\mathcal{W}_Y(h)$ be the set of all such witnesses.

The witness-compatible history set is defined in Equation 42.

$$\mathcal{C}_{\text{wit}}(Y) := \{h \in \Gamma(\mathfrak{D}) : \mathcal{W}_Y(h) \neq \emptyset\}. \quad (42)$$

Define the base compatible fact-history pairs in Equation 43.

$$\mathcal{P}_0(Y; \tau) := \left\{ (F, h) : \begin{array}{l} F \in \mathcal{F}_{\text{rec}, \tau}(Y), \\ h \in \mathcal{C}_{\text{wit}}(Y), \\ (h, F) \in \mathbf{A}_\tau \end{array} \right\}. \quad (43)$$

The history projection of Equation 43 is the base reconstruction set in Equation 44.

$$\mathcal{R}_0(Y) := \{h : \exists F \text{ with } (F, h) \in \mathcal{P}_0(Y; \tau)\}. \quad (44)$$

The corresponding surviving fact configurations are defined in Equation 45.

$$\mathcal{F}_0(Y; \tau) := \{F : \exists h \text{ with } (F, h) \in \mathcal{P}_0(Y; \tau)\}. \quad (45)$$

Equations 44 and 45 make fact and history reconstruction mutually constrained. A record-level fact configuration that cannot be embedded in any witness-compatible history is removed; a history unsupported by any record-compatible fact configuration is likewise removed.

The resulting historical evidential relation can be written as Equation 46.

$$E_0^H := \{(h, Y) : h \in \mathcal{R}_0(Y)\}. \quad (46)$$

The construction remains relative to τ , the record interpretation sets, L_Y , identity assignments, role vocabulary, and the articulation relation A_τ . The subscript 0 therefore means only that no additional auxiliary assumptions have yet been imposed.

4.5 Auditability of Reconstruction Assumptions

Historical reconstruction commonly requires assumptions not directly encoded in the record items themselves. Examples can include continuity through an undocumented interval, a decision to treat two institutional labels as one continuing organization, a rule for resolving an ambiguous date, or a boundary convention for what counts as the same event. Such assumptions should remain visible because they can determine whether a trajectory-level conclusion is identified.

Let $\mathcal{A}(Y)$ be a finite set of auxiliary reconstruction assumptions. Each assumption $a \in \mathcal{A}(Y)$ induces a history constraint $\mathcal{C}_a(Y) \subseteq \Gamma(\mathfrak{D})$. For any subset $A \subseteq \mathcal{A}(Y)$, define the assumption-indexed reconstruction set in Equation 47.

$$\mathcal{R}_A(Y) := \mathcal{R}_0(Y) \cap \bigcap_{a \in A} \mathcal{C}_a(Y). \quad (47)$$

Equation 47 makes the inferential effect of the auxiliary assumptions explicit.

The corresponding surviving fact configurations are defined in Equation 48.

$$\mathcal{F}_A(Y; \tau) := \{F \in \mathcal{F}_0(Y; \tau) : \exists h \in \mathcal{R}_A(Y) \text{ such that } (h, F) \in A_\tau\}. \quad (48)$$

Equation 48 keeps the fact layer synchronized with auxiliary assumptions imposed at the history level.

Adding assumptions can only remove candidate histories.

Proposition 4.1 (Monotonicity under additional reconstruction assumptions). *If $A_1 \subseteq A_2 \subseteq \mathcal{A}(Y)$, then Equation 49 holds.*

$$\mathcal{R}_{A_2}(Y) \subseteq \mathcal{R}_{A_1}(Y). \quad (49)$$

Proof. Every constraint indexed by A_1 also appears among the constraints indexed by A_2 . A history satisfying all constraints in A_2 therefore satisfies all constraints in A_1 . Intersecting the larger constraint family can only preserve or reduce the reconstruction set. $\square$

Proposition 4.1 has a direct methodological consequence. A highly specific reconstruction can result from strong assumptions rather than from highly discriminating evidence. Cardinality reduction should therefore be reported together with the assumptions that caused it.

For a trajectory descriptor $D : \Gamma_D \rightarrow \Omega_D$, define the assumption-indexed description set by Equation 50.

$$\mathcal{D}_A(Y) := \{D(h) : h \in \mathcal{R}_A(Y) \cap \Gamma_D\}. \quad (50)$$

For $a \in A$, the leave-one-assumption-out reconstruction is $\mathcal{R}_{A \setminus \{a\}}(Y)$. The assumption is *D-material* for the current conclusion when removing it changes descriptor admissibility or the compatible description set.

Table 2: Minimum audit record for a constraint-based historical reconstruction.

Reconstruction component	Formal object	Audit question
Record interpretation	$\{\mathcal{C}_{j,\tau}^F(Y)\}_{j \in \mathcal{J}(Y)}$	Which fact configurations are admitted by each interpreted record item?
Fact taxonomy	$\tau, \mathcal{T}_F$, and R_{fam}	Which fact categories and family relations are available to the reconstruction?
Event localization	L_Y	Which history transitions can realize each event item?
Temporal order	$\prec_Y$ and $T_Y(h)$	Which precedence claims are asserted, and which orders remain unresolved?
Identity persistence	$\equiv_Y^{\text{id}}, \#_Y^{\text{id}}$, and ι	Which mentions are required to denote the same or different referents?
Role interpretation	$\text{Role}_h, \mathcal{O}_{\text{role}}(Y)$, and $\mathcal{K}_{\text{role}}(Y)$	Which role observations and continuity claims constrain the reconstruction?
Auxiliary assumptions	$\mathcal{A}(Y)$ and $\{\mathcal{C}_a(Y)\}$	Which constraints are introduced beyond the interpreted record items?
Sensitivity	$\mathcal{R}_{A \setminus \{a\}}(Y)$ and $\mathcal{D}_{A \setminus \{a\}}(Y)$	Which trajectory conclusions change when an assumption is removed?

Definition 4.2 (Descriptor-material reconstruction assumption). Let $a \in A$. The assumption a is *D-material* for (Y, A) when at least one condition in Equation 51 holds.

$$\begin{aligned} \mathcal{R}_{A \setminus \{a\}}(Y) &\not\subseteq \Gamma_D, \\ \mathcal{D}_{A \setminus \{a\}}(Y) &\neq \mathcal{D}_A(Y). \end{aligned} \tag{51}$$

The first line of Equation 51 detects whether the removed assumption had been necessary to keep every compatible history inside the descriptor domain. The second detects whether it changes the set of descriptive conclusions.

A descriptor identified under the full assumption set can therefore be audited by testing the assumptions individually. If $\mathcal{D}_A(Y) = \{\omega\}$ and $\mathcal{R}_A(Y) \subseteq \Gamma_D$, but removing a produces several descriptor values or introduces histories outside the domain, then the identification of ω depends materially on that assumption.

An empty reconstruction set requires separate treatment. If $\mathcal{R}_A(Y) = \emptyset$, the admitted record interpretations and assumptions are jointly inconsistent within the current historical model. The empty set is not a successful point identification. It is a diagnostic result requiring examination of the record coding, identity assignments, chronology, auxiliary assumptions, or the historical model itself.

Table 2 summarizes the minimum information needed to make the reconstruction auditable.

Table 2 turns reconstruction from an implicit preprocessing step into an inspectable component of the analysis. The framework does not require one historical method for constructing the listed objects. It requires that the objects carrying evidential and interpretive commitments be stated clearly enough that later trajectory-level conclusions can be traced back to them.

The next section supplies the domain-specific representational layer needed to turn fact-conditioned historical reconstructions into trackable relational research objects before braid theory is applied. It asks what counts as a component, what configuration space represents relational position, how historical

identity is connected to a strand, and which conditions make the resulting trajectory braid-admissible.

5 Relational Research Objects and Configuration Spaces

This section supplies the representational layer required to convert fact-conditioned historical reconstructions into braid-admissible relational trajectories. Its role is to state what research object is being tracked, what counts as the object’s relational position, and which additional commitments turn witness-compatible histories into strands in a configuration space. The construction is deliberately non-geographic: the braid plane represents a declared relational coordinate system unless an application explicitly chooses physical geography. Section 5.1 defines a planar relational representation scheme. Section 5.2 connects fact configurations, research objects, persistent referents, and strand labels. Section 5.3 states configuration-level and standard braid admissibility. Section 5.4 then distinguishes diagrammatic crossings from domain-level historical meanings. The construction inherits the configuration-space logic of the companion braid paper while making the social-historical extraction assumptions explicit [6].

5.1 Nongeographic Relational Position Spaces

A braid descriptor requires trajectories of several persistent components in a collision-free configuration space. Historical inquiry, however, does not begin with physical particles moving in Euclidean space. A social application must first define what a component’s position means. The present framework therefore separates a domain-level relational position domain from the planar chart used by the braid construction.

Let $\mathcal{Z}_Y$ be a relational position domain chosen for the historical study associated with evidential record Y . Its elements can encode institutional, political, economic, organizational, discursive, or other domain-specific positions. No universal metric, ordering, or arithmetic on $\mathcal{Z}_Y$ is assumed at this stage.

A braid-theoretic analysis additionally requires a declared planar representation. Define a *braid chart* by Equation 52.

$$\chi_Y : \mathcal{Z}_Y \longrightarrow \mathbb{R}^2. \quad (52)$$

Equation 52 is a modeling map. Its two coordinates need not be geographic coordinates. They are the two relational coordinates whose joint variation the analysis proposes to treat topologically. Their substantive meaning must therefore be justified by the application rather than inherited from braid theory.

The braid chart can be many-to-one on the entire domain $\mathcal{Z}_Y$. What matters for an admissible reconstructed trajectory is whether it preserves the separations required by the selected components along that trajectory. This local requirement will be stated formally in Section 5.3.

Historical reconstructions are ordinarily discrete or irregularly sampled, whereas braid theory is formulated using continuous strands. The conversion requires a declared interpolation rule. For a finite increasing schedule $\mathbf{t} = (t_0, \dots, t_k)$ with $0 = t_0 < t_1 < \dots < t_k = 1$ and a sequence of planar positions $\mathbf{p} = (p_0, \dots, p_k) \in (\mathbb{R}^2)^{k+1}$, let the interpolation rule be the map defined in Equation 53.

$$\mathcal{I}_Y : (\mathbf{t}, \mathbf{p}) \longmapsto z_{\mathbf{t}, \mathbf{p}} \in C([0, 1], \mathbb{R}^2), \quad (53)$$

where the output is required to satisfy the interpolation conditions in Equation 54.

$$z_{\mathbf{t}, \mathbf{p}}(t_j) = p_j \quad \text{for every } j \in \{0, \dots, k\}. \quad (54)$$

The pair $(\chi_Y, \mathcal{I}_Y)$ is therefore part of the lower-level historical representation scheme. Different admissible charts or interpolation rules can generate different braid conclusions from the same record. Later robustness analysis should consequently vary these choices when they are not fixed by the domain evidence.

Definition 5.1 (Planar relational representation scheme). A *planar relational representation scheme* for record Y under fact taxonomy τ is the tuple defined in Equation 55.

$$\mathfrak{P}_Y := (\mathcal{Z}_Y, \chi_Y, \mathcal{I}_Y), \quad (55)$$

where $\mathcal{Z}_Y$ is the declared relational position domain, χ_Y is the braid chart in Equation 52, and $\mathcal{I}_Y$ is the interpolation rule in Equation 53.

Definition 5.1 separates three commitments. The fact-conditioned research-object ontology used to describe relational positions enters through $\mathcal{Z}_Y$ and is constrained by τ . The choice of which two relational coordinates will be given planar topological treatment enters through χ_Y . The treatment of unobserved or unresolved motion between reconstructed positions enters through $\mathcal{I}_Y$. None of these commitments is supplied automatically by the abstract braid group.

5.2 Typed Historical Strands

A braid requires persistent strands. Section 4.3 represented evidential identity through reconstruction witnesses, but a strand-level analysis needs a fixed registry of selected analytical components that can be realized inside each compatible history.

Fix an assumption set $A \subseteq \mathcal{A}(Y)$ and let $\mathcal{R}_A(Y)$ be the reconstruction set of Equation 47. Define the witness-augmented reconstruction set by Equation 56.

$$\tilde{\mathcal{R}}_A(Y) := \{(h, w) : h \in \mathcal{R}_A(Y), w \in W_Y(h)\}. \quad (56)$$

The pair (h, w) in Equation 56 retains the historical sequence together with one coherent temporal and identity interpretation of the record.

The fact configurations surviving the assumption set A are $\mathcal{F}_A(Y; \tau)$ from Equation 48. Define the research objects available to strand analysis by Equation 57.

$$\mathcal{O}_A(Y; \tau) := \bigcup_{F \in \mathcal{F}_A(Y; \tau)} \text{Obj}_\tau(F). \quad (57)$$

Equation 57 makes strand construction downstream from fact reconstruction rather than from names or record mentions alone.

For each $(h, w) \in \tilde{\mathcal{R}}_A(Y)$, let $\omega_{h,w} : \text{Ref}(h) \rightarrow \mathcal{O}_A(Y; \tau)$ be a declared object-interpretation map that connects witness-compatible persistent referents to research objects constructed under τ . The map is domain-specific and is not supplied automatically by braid theory.

Let $\mathcal{A}_Y^{\text{str}} = \{a_1, \dots, a_n\}$ be a finite set of strand labels selected for the analysis, with $n \geq 2$. These labels are analytical component slots. For each $(h, w) \in \tilde{\mathcal{R}}_A(Y)$, a strand-realization map is defined in Equation 58.

$$\lambda_{h,w} : \mathcal{A}_Y^{\text{str}} \longrightarrow \text{Ref}(h). \quad (58)$$

The map in Equation 58 links each analytical strand label to the persistent historical referent supplied by the witness-compatible reconstruction. Composition with $\omega_{h,w}$ links the strand to a fact-conditioned research object. Strand identity is usable for braid extraction only when $\lambda_{h,w}$ is injective and the resulting object interpretation remains well defined over the descriptive interval.

Persistent identity should be distinguished from analytical type. Let Θ_Y be a set of component types used by the study, and let the type map be defined in Equation 59.

$$\theta_Y : \mathcal{A}_Y^{\text{str}} \longrightarrow \Theta_Y. \quad (59)$$

The type in Equation 59 should represent a classification intended to remain attached to the strand over the descriptive interval. Time-varying offices, roles, or statuses should instead be represented through the role structure of Section 4.3. Treating a changing role as a fixed strand type would silently convert a historical transition into an identity assumption.

For a history h with transition length $\ell(h)$, define its state-index set by Equation 60.

$$J(h) := \{0, 1, \dots, \ell(h)\}. \quad (60)$$

A relational-position extraction for the reconstruction pair (h, w) is the map defined in Equation 61.

$$\varrho_{h,w} : \mathcal{A}_Y^{\text{str}} \times J(h) \longrightarrow \mathcal{Z}_Y. \quad (61)$$

The value $\varrho_{h,w}(a, j)$ is the relational position assigned to strand label a at state index j under the declared domain representation. The map can depend on both h and w because different witness-compatible identity assignments can attach the same record items to different persistent referents.

Let $\mathbf{t}_h = (t_0, \dots, t_{\ell(h)})$ be a strictly increasing normalized schedule. Applying the braid chart and interpolation rule componentwise gives the continuous historical strand in Equation 62.

$$z_a^{h,w} := \mathcal{I}_Y \left(\mathbf{t}_h, (\chi_Y(\varrho_{h,w}(a, j)))_{j \in J(h)} \right) \in C([0, 1], \mathbb{R}^2). \quad (62)$$

Equation 62 derives each continuous strand from objects already declared: the reconstruction pair, the selected component registry, the relational-position extraction, the braid chart, the temporal schedule, and the interpolation rule.

The unordered planar configuration traced by all selected strands is then defined in Equation 63.

$$q_{h,w}(t) := \left\{ z_a^{h,w}(t) : a \in \mathcal{A}_Y^{\text{str}} \right\}. \quad (63)$$

Equation 63 is the candidate configuration-space trajectory to which braid topology may be applied. The set notation removes coordinate ordering at the configuration level; persistent identity remains available through the strand curves themselves and through $\lambda_{h,w}$.

5.3 Braid-Admissibility in Social Reconstruction

The construction in Equation 63 produces an ordinary configuration-space trajectory only under additional conditions. It is useful to separate configuration admissibility from the stronger endpoint convention needed for the standard group-valued descriptor of the companion paper.

Let the unordered configuration space of n distinct planar points be defined in Equation 64.

$$\mathcal{U}_n := \text{UConf}_n(\mathbb{R}^2) = \left\{ \{p_1, \dots, p_n\} \subset \mathbb{R}^2 : p_i \neq p_j \text{ for } i \neq j \right\}. \quad (64)$$

Equation 64 expresses the fixed-cardinality and collision-exclusion commitments of the ordinary Artin braid construction [6].

Definition 5.2 (Configuration-admissible social reconstruction). A reconstruction pair $(h, w) \in \tilde{\mathcal{R}}_A(Y)$ is *configuration-admissible* relative to the selected strand registry and planar representation scheme when the following conditions hold:

1. the same strand-label set $\mathcal{A}_Y^{\text{str}}$ of cardinality n is represented throughout the descriptive interval;
2. the realization map $\lambda_{h,w}$ in Equation 58 is injective and each selected referent remains traceable through the interval;
3. every strand $z_a^{h,w}$ in Equation 62 is continuous;
4. the collision-exclusion condition in Equation 65 holds.

$$z_a^{h,w}(t) \neq z_b^{h,w}(t) \quad \text{for every } a \neq b \text{ and every } t \in [0, 1]. \quad (65)$$

Under Definition 5.2, Equation 63 takes values in $\mathcal{U}_n$ for every t .

Proposition 5.3 (Configuration-path criterion). *If (h, w) is configuration-admissible, then the map $q_{h,w}$ defined in Equation 63 is a continuous path $q_{h,w} : [0, 1] \rightarrow \mathcal{U}_n$.*

Proof. Configuration admissibility gives continuity of every strand. The finite product of those strand maps is therefore continuous into $(\mathbb{R}^2)^n$. Equation 65 keeps its image outside every collision diagonal. Passing to the quotient by permutations gives the continuous unordered configuration path in Equation 63. Hence its image lies in $\mathcal{U}_n$. $\square$

Proposition 5.3 establishes the path-level consequence of configuration admissibility. The standard B_n descriptor used in the companion paper additionally fixes a common base configuration. Let $c_0 \in \mathcal{U}_n$.

Definition 5.4 (Standard braid-admissible social reconstruction). A configuration-admissible pair (h, w) is *standard braid-admissible* at base configuration c_0 when Equation 66 holds.

$$q_{h,w}(0) = q_{h,w}(1) = c_0. \quad (66)$$

For a standard braid-admissible pair, the configuration path is a based loop in $\mathcal{U}_n$ and therefore determines an element of $\pi_1(\mathcal{U}_n, c_0) \cong B_n$.

Historical cases can have different initial and terminal relational configurations. The methodology should not create a loop by silently adding a closure path. When $q_{h,w}(0) \neq q_{h,w}(1)$, the path can instead be retained as an endpoint-preserving homotopy class between its two endpoint configurations. As discussed in the companion paper, such classes naturally belong to the fundamental groupoid of $\mathcal{U}_n$ rather than to one based fundamental group [6]. Any later closure or endpoint-normalization convention is an additional reconstruction assumption whose effect must be audited.

The distinction between the two admissibility levels is summarized in Table 3.

Table 3: Admissibility levels for braid-theoretic social-historical reconstruction.

Level	Required commitments	Resulting topological object
Configuration-admissible	Fixed selected component set, traceable identity, continuous interpolated strands, and pairwise collision exclusion	Continuous path in $\mathcal{U}_n$
Standard braid-admissible	Configuration admissibility plus the based-loop endpoint condition in Equation 66	Element of $\pi_1(\mathcal{U}_n, c_0) \cong B_n$
Open-endpoint configuration path	Configuration admissibility with distinct initial and terminal configurations	Morphism class in the fundamental groupoid of $\mathcal{U}_n$ unless an additional closure convention is declared

Table 3 makes the endpoint convention visible as a methodological commitment rather than treating every social trajectory as if it were naturally a loop.

5.4 Domain Semantics of Braid Crossings

A braid word can be read from a generic planar braid diagram, but the algebraic crossing symbols do not carry social meaning by themselves. A crossing records a local exchange in the chosen relational chart under a fixed diagram convention. Whether that exchange corresponds to alliance reversal, institutional substitution, role competition, recognition change, bargaining displacement, or another substantive process must be established independently.

Write the strand curve in chart coordinates as defined in Equation 67.

$$z_a^{h,w}(t) = (u_a^{h,w}(t), v_a^{h,w}(t)). \quad (67)$$

A generic diagrammatic crossing occurrence between distinct strand labels a and b is a local event $c = (t_*, a, b)$ for which the first chart coordinates coincide while the second remain distinct, as stated in

Equation 68.

$$u_a^{h,w}(t_*) = u_b^{h,w}(t_*), \quad v_a^{h,w}(t_*) \neq v_b^{h,w}(t_*), \quad (68)$$

and for which the local order in the first coordinate changes across t_* .

Let $\mathcal{C}_B(h, w)$ be the set of isolated generic crossing occurrences in the selected diagram. A fixed braid-diagram convention assigns to each crossing occurrence two algebraic data. The positional index $i(c) \in \{1, \dots, n-1\}$ records which adjacent positions in the current projected order exchange, and the crossing sign $\varepsilon(c) \in \{+1, -1\}$ records the local orientation under the declared convention. The corresponding Artin letter is defined in Equation 69.

$$\sigma(c) := \sigma_{i(c)}^{\varepsilon(c)}. \quad (69)$$

The subscript $i(c)$ in Equation 69 is a positional index in the braid diagram. It is not the identity label of a historical actor. The actor or institution pair involved in the occurrence is given separately by the strand labels $\{a, b\}$ carried by c .

To represent substantive interpretation without attaching it to the algebra by fiat, let $\mathcal{S}_Y^{\text{dom}}$ be a set of domain-level event meanings admitted by the study. Define the interpretation relation in Equation 70.

$$\text{Sem}_Y^{h,w} \subseteq \mathcal{C}_B(h, w) \times \mathcal{S}_Y^{\text{dom}}. \quad (70)$$

Membership $(c, s) \in \text{Sem}_Y^{h,w}$ means that crossing occurrence c has been assigned domain interpretation s under independently stated historical reasoning. Equation 70 is a relation rather than a required function because a crossing can support several interpretive descriptions, and some crossings may be left without a substantive label.

The separation among algebraic crossing, strand identity, and domain interpretation yields three methodological cautions. First, two crossing occurrences can have the same Artin letter while involving different historical entities and different substantive meanings. Second, one substantive interaction can occur without any crossing in the chosen relational chart. Third, changing the braid chart can change which local events appear as diagrammatic crossings even when the underlying historical reconstruction is unchanged. Accordingly, the braid word is a representation of topological organization in the chosen relational coordinate scheme, not a direct transcription of social events.

The commitments introduced in this section can be summarized by the extraction chain in Equation 71.

$$(Y, A) \mapsto (h, w) \mapsto \varrho_{h,w} \mapsto \{z_a^{h,w}\}_{a \in \mathcal{A}_Y^{\text{str}}} \mapsto q_{h,w} \mapsto \text{topological description}. \quad (71)$$

Each arrow in Equation 71 introduces a distinct commitment: evidential reconstruction, identity realization, relational-position coding, interpolation, configuration-space admissibility, and topological quotienting. The methodological value of the braid descriptor depends on keeping these commitments in-spectable rather than allowing them to disappear inside one visual braid diagram.

The next section uses this representation layer to define the set of braid or groupoid descriptions compatible with an evidential record. That set-valued construction will distinguish the existence of at least one braid-admissible reconstruction from identification of one braid class across all admissible reconstructions.

6 Set-Valued Historical Trajectory Reconstruction

This section composes the reconstruction formalism of Sections 3–4 with the relational configuration-space construction of Section 5. Its role is to define the set of topological trajectory descriptions supported by one evidential record after reconstruction assumptions, witness assignments, relational-position coding, and braid-admissibility conditions have been made explicit. The central methodological requirement is that uncertainty be propagated rather than collapsed prematurely. Section 6.1 separates uncertainty in

reconstructed histories from uncertainty in reconstruction witnesses. Section 6.2 defines the evidence-compatible braid set. Section 6.3 states braid identifiability and its fiber criterion. Section 6.4 then formalizes partial topological identifiability through derived braid invariants. The construction uses the braid descriptor developed in the companion formal paper, while the inferential claims remain relative to the evidential and representational commitments declared in the present manuscript [6].

6.1 Evidence-to-History Reconstruction with Witness Uncertainty

Fix an evidential record Y , a fact taxonomy τ , an auxiliary assumption set $A \subseteq \mathcal{A}(Y)$, a strand registry $\mathcal{A}_Y^{\text{str}}$, and a planar relational representation scheme $\mathfrak{P}_Y$ from Definition 5.1. The history-level reconstruction set $\mathcal{R}_A(Y)$ is defined by Equation 47. Section 5.2 then lifts this set to the witness-augmented reconstruction set $\tilde{\mathcal{R}}_A(Y)$ in Equation 56.

The projection from witness-augmented reconstructions to histories is defined in Equation 72.

$$\pi_H : \tilde{\mathcal{R}}_A(Y) \longrightarrow \mathcal{R}_A(Y), \quad \pi_H(h, w) := h. \quad (72)$$

For a reconstructed history $h \in \mathcal{R}_A(Y)$, define its compatible witness fiber by Equation 73.

$$\mathcal{W}_A(h; Y) := \pi_H^{-1}(\{h\}) = \{(h, w) : w \in \mathcal{W}_Y(h)\}. \quad (73)$$

Equation 73 records temporal and identity ambiguity that remains after the state-history sequence itself has been fixed.

The two uncertainty levels should remain separate. History nonidentification is present when the reconstruction set contains more than one history. Witness nonidentification conditional on a fixed history is present when that history has more than one compatible witness pair. Neither condition implies the other. A unique history can admit several identity or temporal witness assignments, while several histories can each admit a unique witness.

For later topological analysis, define the configuration-admissible and standard-braid-admissible augmented domains in Equations 74 and 75.

$$\tilde{\mathcal{R}}_A^{\text{cfg}}(Y) := \{(h, w) \in \tilde{\mathcal{R}}_A(Y) : (h, w) \text{ satisfies Definition 5.2}\}, \quad (74)$$

$$\tilde{\mathcal{R}}_A^B(Y; c_0) := \{(h, w) \in \tilde{\mathcal{R}}_A^{\text{cfg}}(Y) : (h, w) \text{ satisfies Definition 5.4 at } c_0\}. \quad (75)$$

The dependence on τ , $\mathfrak{P}_Y$, and $\mathcal{A}_Y^{\text{str}}$ is suppressed in the notation of Equations 74 and 75; both objects remain relative to those declared representation choices.

The exclusion sets in Equations 76 and 77 distinguish two different failures.

$$\mathcal{E}_A^{\text{cfg}}(Y) := \tilde{\mathcal{R}}_A(Y) \setminus \tilde{\mathcal{R}}_A^{\text{cfg}}(Y), \quad (76)$$

$$\mathcal{E}_A^B(Y; c_0) := \tilde{\mathcal{R}}_A(Y) \setminus \tilde{\mathcal{R}}_A^B(Y; c_0). \quad (77)$$

The first set contains reconstructions that fail the fixed-component, traceability, continuity, or collision-exclusion requirements. The second additionally contains configuration-admissible reconstructions that fail the based-loop endpoint condition. A later singleton braid claim should not be obtained by silently discarding either exclusion set.

6.2 History-to-Braid Reconstruction

For each $(h, w) \in \tilde{\mathcal{R}}_A^B(Y; c_0)$, Equation 63 gives a based loop $q_{h,w} : [0, 1] \rightarrow \mathcal{U}_n$. Let Q_B denote the endpoint-preserving homotopy quotient of based loops and let $\kappa_n : \pi_1(\mathcal{U}_n, c_0) \rightarrow B_n$ be the fixed braid-group isomorphism used in the companion formalism. Define the reconstruction-pair braid descriptor in Equation 78.

$$\beta_A(h, w) := \kappa_n(Q_B(q_{h,w})) \in B_n. \quad (78)$$

The subscript A in Equation 78 records that the admissible pair arose under the auxiliary reconstruction assumptions indexed by A . The braid value also remains relative to the strand registry, braid chart, interpolation rule, and base configuration fixed above.

The evidence-compatible braid set is defined in Equation 79.

$$\mathcal{B}_A(Y; c_0) := \left\{ \beta_A(h, w) : (h, w) \in \tilde{\mathcal{R}}_A^B(Y; c_0) \right\} \subseteq B_n. \quad (79)$$

Equation 79 propagates both history uncertainty and witness uncertainty into the topological description.

For a fixed reconstructed history $h \in \mathcal{R}_A(Y)$, define the history-conditioned braid set by Equation 80.

$$\mathcal{B}_A(h; Y, c_0) := \left\{ \beta_A(h, w) : (h, w) \in \mathcal{W}_A(h; Y) \cap \tilde{\mathcal{R}}_A^B(Y; c_0) \right\}. \quad (80)$$

Equation 80 isolates braid uncertainty caused by alternative witness assignments for one fixed history.

The full braid set decomposes over reconstructed histories.

Proposition 6.1 (Braid-set decomposition by reconstructed history). *The evidence-compatible braid set satisfies Equation 81.*

$$\mathcal{B}_A(Y; c_0) = \bigcup_{h \in \mathcal{R}_A(Y)} \mathcal{B}_A(h; Y, c_0). \quad (81)$$

Proof. Take $\beta \in \mathcal{B}_A(Y; c_0)$. By Equation 79, there exists $(h, w) \in \tilde{\mathcal{R}}_A^B(Y; c_0)$ with $\beta = \beta_A(h, w)$. Equation 56 gives $h \in \mathcal{R}_A(Y)$, and Equation 73 places (h, w) in $\mathcal{W}_A(h; Y)$. Hence $\beta \in \mathcal{B}_A(h; Y, c_0)$, so the left side is contained in the right side.

Conversely, if $\beta \in \mathcal{B}_A(h; Y, c_0)$ for some $h \in \mathcal{R}_A(Y)$, Equation 80 provides a pair $(h, w) \in \tilde{\mathcal{R}}_A^B(Y; c_0)$ with $\beta = \beta_A(h, w)$. Equation 79 then places β in $\mathcal{B}_A(Y; c_0)$. $\square$

Proposition 6.1 identifies two possible sources of topological ambiguity. Different reconstructed histories can yield different braid classes, and different witnesses attached to the same reconstructed history can also yield different braid classes because identity realization and relational-position extraction depend on the witness.

Configuration-admissible pairs can fail the based-loop condition while still carrying open-endpoint topological information. Let $\Pi_1(\mathcal{U}_n)$ denote the fundamental groupoid of $\mathcal{U}_n$. The evidence-compatible groupoid-morphism set is defined in Equation 82.

$$\mathcal{P}_A^\Pi(Y) := \left\{ [q_{h,w}]_\partial : (h, w) \in \tilde{\mathcal{R}}_A^{\text{cfg}}(Y) \right\} \subseteq \text{Mor}(\Pi_1(\mathcal{U}_n)). \quad (82)$$

Equation 82 preserves endpoint-sensitive trajectory classes without imposing an artificial closure. The standard braid set in Equation 79 is the based-loop sector obtained when every admitted pair begins and ends at the declared base configuration.

6.3 Braid Identifiability

A nonempty braid set is not sufficient for an evidential identification claim. Some compatible reconstructions can fall outside the standard braid domain, and several admissible reconstructions can yield different braid classes. The definition therefore combines full-domain admissibility with invariance across all augmented reconstructions.

Definition 6.2 (Braid identifiability from a historical record). The braid class is *identified from* (Y, A) at base configuration c_0 when the two conditions in Equation 83 hold.

$$\tilde{\mathcal{R}}_A(Y) \subseteq \tilde{\mathcal{R}}_A^B(Y; c_0), \quad |\mathcal{B}_A(Y; c_0)| = 1. \quad (83)$$

The first condition in Equation 83 prevents the identified braid from being produced by excluding evidence-compatible reconstructions for which the standard braid descriptor is undefined. The second requires one braid value across all admitted reconstruction pairs.

For $\beta \in B_n$, define the augmented braid fiber by Equation 84.

$$\mathcal{F}_B^{\text{aug}}(\beta) := \{(h, w) : (h, w) \text{ is standard braid-admissible at } c_0 \text{ and } \beta_A(h, w) = \beta\}. \quad (84)$$

The dependence on the fixed representation scheme and assumption context remains implicit.

Proposition 6.3 (Fiber criterion for historical braid identifiability). *Assume $\tilde{\mathcal{R}}_A(Y) \neq \emptyset$. The braid class is identified from (Y, A) if and only if there exists a unique $\beta \in B_n$ such that Equation 85 holds.*

$$\tilde{\mathcal{R}}_A(Y) \subseteq \mathcal{F}_B^{\text{aug}}(\beta). \quad (85)$$

Proof. Suppose the braid is identified. The first condition of Equation 83 places every augmented reconstruction in the standard braid domain. The second gives a unique braid value β . Therefore every $(h, w) \in \tilde{\mathcal{R}}_A(Y)$ lies in $\mathcal{F}_B^{\text{aug}}(\beta)$, which is Equation 85.

Conversely, suppose Equation 85 holds for a unique β . The definition of $\mathcal{F}_B^{\text{aug}}(\beta)$ places every augmented reconstruction in the standard braid domain, establishing the first condition of Equation 83. It also assigns every pair the same braid value β , so Equation 79 gives $\mathcal{B}_A(Y; c_0) = \{\beta\}$. $\square$

Proposition 6.3 makes clear that braid identifiability is weaker than identification of one complete reconstruction pair. The entire augmented reconstruction set can contain several histories, several witnesses, or both, while remaining inside one braid fiber.

Proposition 6.4 (Braid identification with reconstruction non-identification). *Assume the braid class is identified from (Y, A) . If Equation 86 holds, then one braid class is identified while the witness-augmented historical reconstruction remains nonidentified.*

$$|\tilde{\mathcal{R}}_A(Y)| > 1. \quad (86)$$

Proof. Braid identification gives $|\mathcal{B}_A(Y; c_0)| = 1$ by Definition 6.2. Equation 86 states that more than one reconstruction pair remains compatible with the evidence and assumptions. The two cardinality statements are compatible because the many reconstruction pairs can all lie in the one braid fiber specified by Proposition 6.3. $\square$

Proposition 6.4 therefore separates topological identification from reconstruction-pair identification. The converse does not follow. Identification of one history does not guarantee braid identification when several witness assignments for that history produce different admissible relational trajectories. Likewise, a unique witness-augmented reconstruction need not yield a braid class if the pair falls outside the standard braid domain. These cases motivate reporting reconstruction, admissibility, and braid-identification status separately.

The identification hierarchy is summarized in Table 4.

Table 4 prevents a topological conclusion from being mistaken for a stronger historical reconstruction claim.

6.4 Partial Identifiability through Derived Invariants

The full braid class can remain nonidentified while a coarser braid-derived quantity is stable across all admissible reconstructions. This situation is methodologically important because historical evidence can support a robust topological characteristic without supporting one complete braid element.

Let $I : B_n \rightarrow \Omega_I$ be a declared braid-derived map. The codomain Ω_I can be algebraic, combinatorial, or numerical; no universal choice is assumed. Examples developed in the companion formal paper include permutation, exponent sum, conjugacy-level descriptions, and other derived invariants [6].

Define the evidence-compatible invariant-value set by Equation 87.

$$\mathcal{I}_A(Y; c_0) := \{I(\beta) : \beta \in \mathcal{B}_A(Y; c_0)\} \subseteq \Omega_I. \quad (87)$$

Table 4: Distinct inferential statuses in set-valued historical braid reconstruction.

Status	Formal condition	Interpretation
Historical reconstruction exists	$\mathcal{R}_A(Y) \neq \emptyset$	At least one state-history sequence satisfies the declared record constraints and assumptions
Witness-augmented reconstruction exists	$\tilde{\mathcal{R}}_A(Y) \neq \emptyset$	At least one coherent temporal and identity interpretation supports a reconstructed history
Standard braid admissibility is complete	$\tilde{\mathcal{R}}_A(Y) \subseteq \tilde{\mathcal{R}}_A^B(Y; c_0)$	Every compatible reconstruction pair supports the standard based-loop braid descriptor
Braid class is identified	Equation 83	Every compatible reconstruction pair supports the same braid class
Reconstruction pair is identified	$ \tilde{\mathcal{R}}_A(Y) = 1$	One history-witness pair remains under the declared evidential and assumption regime

Definition 6.5 (Invariant identifiability from a historical record). The derived quantity I is *identified from* (Y, A) when Equation 88 holds.

$$\tilde{\mathcal{R}}_A(Y) \subseteq \tilde{\mathcal{R}}_A^B(Y; c_0), \quad |\mathcal{I}_A(Y; c_0)| = 1. \quad (88)$$

The admissibility condition in Equation 88 is retained because an invariant of B_n cannot support an unconditional evidential conclusion if some compatible reconstruction pairs do not define a braid element.

Proposition 6.6 (Braid identification implies derived-invariant identification). *If the braid class is identified from (Y, A) , then every declared map $I : B_n \rightarrow \Omega_I$ is identified from (Y, A) .*

Proof. Braid identification supplies complete standard braid admissibility and one compatible braid value, denoted by β . Equation 87 then yields one compatible invariant value, namely $I(\beta)$. Definition 6.5 follows. $\square$

The converse requires injectivity on the braid values actually compatible with the evidence.

Proposition 6.7 (Relative injectivity criterion). *Assume complete standard braid admissibility and that I is injective on $\mathcal{B}_A(Y; c_0)$. Then I is identified from (Y, A) if and only if the braid class is identified from (Y, A) .*

Proof. Braid identification implies invariant identification by Proposition 6.6. Conversely, suppose I is identified. Then $\mathcal{I}_A(Y; c_0)$ is a singleton. If $\mathcal{B}_A(Y; c_0)$ contained two distinct braid elements β_1 and β_2 , injectivity of I on this set would give $I(\beta_1) \neq I(\beta_2)$, contradicting singleton invariant identification. Hence the braid set is a singleton, and complete standard braid admissibility gives braid identifiability. $\square$

Proposition 6.7 shows that invariant-level identification can be promoted to braid identification only when the selected invariant is injective on the braid values compatible with the evidence.

When I is noninjective on the evidence-compatible braid set, a weaker form of identification can occur.

Definition 6.8 (Partial topological identifiability). The reconstruction has *partial topological identifiability through I* when the conditions in Equation 89 hold.

$$|\mathcal{B}_A(Y; c_0)| > 1, \quad |\mathcal{I}_A(Y; c_0)| = 1, \quad (89)$$

together with complete standard braid admissibility.

Definition 6.8 states that the historical evidence and reconstruction assumptions do not determine one complete braid element, yet they do determine the selected derived feature. The conclusion should therefore be reported at the invariant level rather than promoted to full braid identification.

For several derived maps $I_\lambda : B_n \rightarrow \Omega_\lambda$ indexed by a finite set Λ , define the joint invariant map by Equation 90.

$$I_\Lambda : B_n \longrightarrow \prod_{\lambda \in \Lambda} \Omega_\lambda, \quad I_\Lambda(\beta) := (I_\lambda(\beta))_{\lambda \in \Lambda}. \quad (90)$$

The evidence-compatible joint-value set is then defined in Equation 91.

$$\mathcal{I}_{A,\Lambda}(Y; c_0) := \{I_\Lambda(\beta) : \beta \in \mathcal{B}_A(Y; c_0)\}. \quad (91)$$

Adding derived maps can refine the partial identification because equality under the joint map requires equality under every component map. It cannot, however, recover distinctions that every selected invariant removes.

The set-valued framework developed in this section yields a hierarchy rather than a single binary result. A historical record can identify one braid class, several braid classes sharing one derived invariant, several incompatible invariants, or no standard braid description at all. The next section asks how these conclusions change under controlled variation of the evidence, reconstruction assumptions, relational chart, and interpolation choices.

7 Robustness under Evidential, Taxonomic, and Geometric Uncertainty

This section formalizes robustness when historical trajectories depend on evidential interpretation, fact taxonomy, reconstruction assumptions, and lower-level geometric realization. Its role is to separate three questions: whether the same conclusion survives alternative reconstruction regimes, whether it survives reasonable alternative articulations of the social facts, and whether a fixed geometric strand family remains in the same braid class under bounded perturbation. Section 7.1 defines comparable reconstruction regimes and general perturbation classes. Section 7.2 isolates fact-taxonomy uncertainty. Section 7.3 defines robust braid and invariant conclusions. Section 7.4 derives geometric stability under timing reparameterization and positional uncertainty. Section 7.5 then derives uniform certificates across set-valued reconstruction domains. The metric arguments specialize the perturbation results of the companion formal paper to the evidential and taxonomic architecture developed here [6].

7.1 Perturbation Classes for Comparable Reconstruction Regimes

Sections 4–6 derive a braid conclusion only after several commitments have been declared: the evidential record and its interpretation constraints, the auxiliary reconstruction assumptions, the strand registry, the planar relational representation scheme, and the base-configuration convention. Robustness analysis requires controlled variation of those commitments.

Let the reconstruction-constraint package associated with a record Y be the tuple defined in Equation 92.

$$\mathcal{K}_Y := \left(\{\mathcal{C}_{j,\tau}^F(Y)\}_{j \in \mathcal{J}(Y)}, \mathcal{L}_Y, \prec_Y, \equiv_Y^{\text{id}}, \#_Y^{\text{id}}, \mathcal{O}_{\text{role}}(Y), \mathcal{K}_{\text{role}}(Y), \{\mathcal{C}_a(Y)\}_{a \in \mathcal{A}(Y)} \right). \quad (92)$$

Every entry of Equation 92 was defined in Section 4. The tuple introduces no additional inference; it packages the constraints whose joint effect determines the reconstruction set.

A braid reconstruction regime is defined in Equation 93.

$$\eta := (Y, \tau, \mathcal{K}_Y, A, \mathcal{A}_Y^{\text{str}}, \mathfrak{P}_Y, c_0), \quad A \subseteq \mathcal{A}(Y). \quad (93)$$

The regime in Equation 93 supplies the fact taxonomy together with the inputs needed to construct fact configurations, witness-augmented histories, historical strands, braid admissibility, and the evidence-compatible braid set. Write these regime-dependent outputs as $\tilde{\mathcal{R}}_\eta$, $\tilde{\mathcal{R}}_\eta^B$, and $\mathcal{B}(\eta)$. They are the objects defined previously by $\tilde{\mathcal{R}}_A(Y)$, $\tilde{\mathcal{R}}_A^B(Y; c_0)$, and $\mathcal{B}_A(Y; c_0)$, with the dependence on all inputs of η made explicit.

Different regimes need not be topologically comparable. A change in the number of strands, base configuration, or braid-group identification can change the codomain in which the conclusion lives. The present robustness analysis therefore begins with a comparability requirement.

Definition 7.1 (Braid-comparable reconstruction regimes). Two reconstruction regimes η_1 and η_2 are *braid-comparable* when they use the same strand number n , the same base configuration $c_0 \in \mathcal{U}_n$, and the same fixed isomorphism $\kappa_n : \pi_1(\mathcal{U}_n, c_0) \rightarrow B_n$.

Definition 7.1 allows the fact taxonomy, evidential constraints, auxiliary assumptions, relational chart, interpolation rule, and even the record itself to vary, provided the resulting standard braid conclusions are expressed in the same group B_n under one identification convention.

Fix a reference regime η_0 . A reconstruction perturbation class is any nonempty set $\mathfrak{U}(\eta_0)$ containing η_0 and consisting of regimes braid-comparable with η_0 . The class is not assumed to be metric. Its members can represent, for example, alternative admissible fact taxonomies, date placements, reconstruction assumptions, identity resolutions, braid charts, evidence interpretations, or interpolation rules.

The aggregate braid set over a perturbation class is defined in Equation 94.

$$\mathcal{B}_{\mathfrak{U}} := \bigcup_{\eta \in \mathfrak{U}(\eta_0)} \mathcal{B}(\eta) \subseteq B_n. \quad (94)$$

The regimes in which the standard braid descriptor is incomplete on the reconstructed domain are collected in Equation 95.

$$\mathcal{E}_{\mathfrak{U}}^B := \left\{ \eta \in \mathfrak{U}(\eta_0) : \tilde{\mathcal{R}}_\eta \not\subseteq \tilde{\mathcal{R}}_\eta^B \right\}. \quad (95)$$

Equation 95 prevents robustness from being declared merely because unstable or non-braid-admissible reconstructions have been omitted from the comparison.

Regimes with no witness-augmented reconstruction are collected separately in Equation 96.

$$\mathcal{E}_{\mathfrak{U}}^\emptyset := \left\{ \eta \in \mathfrak{U}(\eta_0) : \tilde{\mathcal{R}}_\eta = \emptyset \right\}. \quad (96)$$

A regime in Equation 96 is evidentially inconsistent under its declared constraints or assumptions. It should not count as a successful robustness case merely because it contributes no competing braid value.

A useful special perturbation class comes directly from the assumption audit of Section 4.5. For a fixed record and fixed representation choices, define the leave-one-assumption-out regime class in Equation 97.

$$\mathfrak{U}_{\text{LOO}}(Y, A) := \{ \eta_A \} \cup \{ \eta_{A \setminus \{a\}} : a \in A \}, \quad (97)$$

where all regime components other than the auxiliary assumption set are held fixed. Equation 97 turns the local materiality test of Definition 4.2 into a perturbation class over which one topological conclusion can be tested jointly.

7.2 Taxonomic Robustness of Historical Trajectory Conclusions

A historical conclusion can be stable under record and geometric perturbation while remaining sensitive to the fact taxonomy used to articulate the research object. The framework therefore treats taxonomy as an explicit uncertainty dimension rather than a hidden preprocessing choice.

Fix all components of a reference reconstruction regime except the fact taxonomy. Let $\mathfrak{T}_{\text{adm}}$ be a nonempty set of fact taxonomies judged admissible for the historical question. For each $\tau \in \mathfrak{T}_{\text{adm}}$, let η_τ be the corresponding reconstruction regime and assume the regimes are braid-comparable.

Define the taxonomy-aggregated braid set in Equation 98.

$$\mathcal{B}_{\mathfrak{T}}(Y) := \bigcup_{\tau \in \mathfrak{T}_{\text{adm}}} \mathcal{B}(\eta_{\tau}). \quad (98)$$

The taxonomies with empty witness-augmented reconstructions are collected in Equation 99.

$$\mathcal{E}_{\mathfrak{T}}^{\emptyset} := \left\{ \tau \in \mathfrak{T}_{\text{adm}} : \tilde{\mathcal{R}}_{\eta_{\tau}} = \emptyset \right\}. \quad (99)$$

The taxonomies for which the standard braid descriptor is incomplete are collected in Equation 100.

$$\mathcal{E}_{\mathfrak{T}}^B := \left\{ \tau \in \mathfrak{T}_{\text{adm}} : \tilde{\mathcal{R}}_{\eta_{\tau}} \not\subseteq \tilde{\mathcal{R}}_{\eta_{\tau}}^B \right\}. \quad (100)$$

Definition 7.2 (Taxonomically robust braid identification). A braid value $\beta \in B_n$ is *taxonomically robust over* $\mathfrak{T}_{\text{adm}}$ when the three conditions in Equation 101 hold.

$$\mathcal{E}_{\mathfrak{T}}^{\emptyset} = \emptyset, \quad \mathcal{E}_{\mathfrak{T}}^B = \emptyset, \quad \mathcal{B}_{\mathfrak{T}}(Y) = \{\beta\}. \quad (101)$$

Definition 7.2 asks whether the trajectory conclusion survives reasonable alternative articulations of the relevant social facts. It does not assert that the competing taxonomies are ontologically equivalent. Two taxonomies can organize the historical material differently while still yielding the same braid-level structure.

Taxonomic instability occurs when the aggregate set contains several values or when some admissible taxonomy produces no coherent or braid-admissible reconstruction. Such instability is analytically different from ordinary measurement error: it indicates dependence of the topological conclusion on how the research object is factually articulated.

The same construction applies to a derived descriptor $I : B_n \rightarrow \Omega_I$ by replacing $\mathcal{B}_{\mathfrak{T}}(Y)$ with the union of its I -images. A coarse invariant can therefore be taxonomically robust even when the full braid is taxonomy-sensitive.

7.3 Robustness of Braid-Level Conclusions

The set-valued reconstruction of Section 6 defined identifiability inside one reconstruction regime. Robustness asks whether that identified conclusion persists over a declared perturbation class.

Definition 7.3 (Robust braid identification). The braid class is *robustly identified over* $\mathfrak{U}(\eta_0)$ when the three conditions in Equation 102 hold.

$$\mathcal{E}_{\mathfrak{U}}^{\emptyset} = \emptyset, \quad \mathcal{E}_{\mathfrak{U}}^B = \emptyset, \quad |\mathcal{B}_{\mathfrak{U}}| = 1. \quad (102)$$

The first condition in Equation 102 requires a nonempty witness-augmented reconstruction in every regime. The second requires complete standard braid admissibility, and the third requires one braid value after all evidential, reconstructive, and representational variations admitted by the perturbation class have been propagated.

Robust identification can be characterized regime by regime.

Proposition 7.4 (Regimewise criterion for robust braid identification). *The braid class is robustly identified over $\mathfrak{U}(\eta_0)$ if and only if there exists a unique $\beta \in B_n$ such that, for every $\eta \in \mathfrak{U}(\eta_0)$, Equation 103 holds.*

$$\emptyset \neq \tilde{\mathcal{R}}_{\eta} \subseteq \tilde{\mathcal{R}}_{\eta}^B, \quad \mathcal{B}(\eta) = \{\beta\}. \quad (103)$$

Proof. Assume robust braid identification. Equation 102 makes both $\mathcal{E}_{\mathfrak{U}}^{\emptyset}$ and $\mathcal{E}_{\mathfrak{U}}^B$ empty, so every regime has a nonempty witness-augmented reconstruction and complete standard braid admissibility. The aggregate braid set in Equation 94 is a singleton, say $\{\beta\}$. Every regime braid set is nonempty and is a subset of this singleton, hence equals $\{\beta\}$. This gives Equation 103.

Conversely, assume Equation 103 holds for every regime and one unique β . The nonempty subset relation makes both failure sets $\mathcal{E}_{\mathfrak{U}}^{\emptyset}$ and $\mathcal{E}_{\mathfrak{U}}^B$ empty. The union in Equation 94 is the union of identical singleton sets $\{\beta\}$, so it is itself $\{\beta\}$. Definition 7.3 follows. $\square$

Proposition 7.4 distinguishes robustness from identification in one favored reconstruction. A braid can be identified under the reference regime and fail robustness because another admissible regime changes the braid, makes the braid nonidentified, or introduces a reconstruction outside the standard braid domain.

The same construction applies to a derived invariant $I : B_n \rightarrow \Omega_I$. Define the perturbation-class invariant set in Equation 104.

$$\mathcal{I}_{\mathfrak{U}}^I := \{I(\beta) : \beta \in \mathcal{B}_{\mathfrak{U}}\}. \quad (104)$$

Definition 7.5 (Robust invariant identification). The derived quantity I is *robustly identified over* $\mathfrak{U}(\eta_0)$ when Equation 105 holds.

$$\mathcal{E}_{\mathfrak{U}}^{\emptyset} = \emptyset, \quad \mathcal{E}_{\mathfrak{U}}^B = \emptyset, \quad |\mathcal{I}_{\mathfrak{U}}^I| = 1. \quad (105)$$

Proposition 7.6 records that robust braid identification implies robust identification of every derived invariant.

Proposition 7.6 (Inheritance of robustness by derived braid quantities). *If the braid class is robustly identified over $\mathfrak{U}(\eta_0)$, then every declared map $I : B_n \rightarrow \Omega_I$ is robustly identified over the same perturbation class.*

Proof. Robust braid identification gives complete admissibility and a singleton aggregate braid set $\mathcal{B}_{\mathfrak{U}} = \{\beta\}$. Equation 104 then gives one invariant value $I(\beta)$. Definition 7.5 follows. $\square$

The converse need not hold because several braid classes can share one invariant value. This gives a perturbation-level version of partial topological identifiability: the full braid conclusion can vary across admissible regimes while a coarser derived quantity remains unchanged.

For the leave-one-assumption-out class in Equation 97, robust braid identification has a direct audit interpretation. The braid conclusion survives removal of every individual assumption in A while remaining completely braid-admissible. Failure identifies at least one assumption whose removal changes the topological conclusion or its domain status. This does not establish that the surviving assumptions are true; it establishes only that the reported braid is insensitive to those specified one-at-a-time removals.

7.4 Geometric Uncertainty under Timing Reparameterization

The regime-level definitions above permit qualitative variation in evidence and reconstruction choices. A separate robustness question arises after one admissible strand family has been constructed: how sensitive is its braid class to uncertainty in timing or planar position?

Fix a standard braid-admissible reconstruction pair (h, w) . For each strand label $a \in \mathcal{A}_Y^{\text{str}}$, write $z_a := z_a^{h,w}$ for the continuous strand of Equation 62, and write $q := q_{h,w}$ for the corresponding based loop.

Timing uncertainty can preserve the entire geometric trace while changing the speed or exact normalized time at which positions are traversed. Let Φ be the set of strictly increasing homeomorphisms $\phi : [0, 1] \rightarrow [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. For $\phi \in \Phi$, define the reparameterized strands by Equation 106.

$$z_a^{\phi}(t) := z_a(\phi(t)). \quad (106)$$

The corresponding configuration path is $q^{\phi}(t) = q(\phi(t))$.

Proposition 7.7 (Braid invariance under order-preserving time reparameterization). *For every $\phi \in \Phi$, the paths q and q^{ϕ} define the same braid element.*

Proof. For $s, t \in [0, 1]$, define the interpolated time map by Equation 107.

$$\phi_s(t) := (1 - s)t + s\phi(t). \quad (107)$$

Because both t and $\phi(t)$ lie in $[0, 1]$, Equation 107 maps the square continuously into $[0, 1]$. The endpoint conditions satisfy $\phi_s(0) = 0$ and $\phi_s(1) = 1$. Define $H(s, t) := q(\phi_s(t))$. Since $q(t) \in \mathcal{U}_n$ for every t ,

the image of H remains in $\mathcal{U}_n$. Moreover, $H(0, t) = q(t)$, $H(1, t) = q^\phi(t)$, and the endpoints are fixed at c_0 . Thus H is a based homotopy, so the two paths determine the same braid element. $\square$

Proposition 7.7 applies to timing uncertainty that preserves event order along the already reconstructed geometric path. It does not cover a timing revision that changes the order in which strand crossings or historically reconstructed transitions occur. Such a revision can change the braid word or the underlying braid class and belongs to the regime-level perturbation analysis rather than to simple reparameterization.

Positional uncertainty requires a metric only after the planar braid chart has been chosen. Define the minimum pairwise separation of the strand family by Equation 108.

$$\rho(h, w) := \min_{\substack{t \in [0, 1] \\ a, b \in \mathcal{A}_Y^{\text{str}}, a \neq b}} \|z_a(t) - z_b(t)\|_2. \quad (108)$$

Because the pair is configuration-admissible, the strands are continuous and pairwise distinct. The strand registry is finite and the time interval is compact, so the minimum in Equation 108 exists and is strictly positive.

Let $\tilde{z} = (\tilde{z}_a)_{a \in \mathcal{A}_Y^{\text{str}}}$ be an alternative strand family with the same labeled endpoints. Define its deviation from $z = (z_a)_a$ by Equation 109.

$$\|\tilde{z} - z\|_\infty := \max_{a \in \mathcal{A}_Y^{\text{str}}} \sup_{t \in [0, 1]} \|\tilde{z}_a(t) - z_a(t)\|_2. \quad (109)$$

Proposition 7.8 (Pair-specific positional robustness certificate). *If the endpoint-fixed alternative strand family satisfies Equation 110, then it is collision-free and represents the same braid element as the original reconstruction pair.*

$$\|\tilde{z} - z\|_\infty < \frac{\rho(h, w)}{2}, \quad (110)$$

Proof. Let $\varepsilon := \|\tilde{z} - z\|_\infty$. For $s \in [0, 1]$, define the straight-line strand interpolation by Equation 111.

$$z_a^{(s)}(t) := (1 - s)z_a(t) + s\tilde{z}_a(t). \quad (111)$$

For distinct labels a and b , the triangle inequality gives Equation 112.

$$\begin{aligned} \|z_a^{(s)}(t) - z_b^{(s)}(t)\|_2 &\geq \|z_a(t) - z_b(t)\|_2 \\ &\quad - s \|\tilde{z}_a(t) - z_a(t)\|_2 - s \|\tilde{z}_b(t) - z_b(t)\|_2 \\ &\geq \rho(h, w) - 2\varepsilon. \end{aligned} \quad (112)$$

Equation 110 makes the final quantity in Equation 112 strictly positive. Hence no collision occurs for any (s, t) . The endpoint condition keeps the base configuration fixed throughout the interpolation. Therefore the unordered configurations defined by Equation 111 form a based homotopy in $\mathcal{U}_n$, and the braid class is unchanged. $\square$

The positional certificate can be combined with timing reparameterization. For $\phi \in \Phi$, define the reparameterized reference family z^ϕ by Equation 106. Reparameterization preserves the minimum separation because ϕ is onto. The preserved margin is stated directly in Equation 113.

$$\min_{\substack{t \in [0, 1] \\ a, b \in \mathcal{A}_Y^{\text{str}}, a \neq b}} \|z_a^\phi(t) - z_b^\phi(t)\|_2 = \rho(h, w). \quad (113)$$

Equation 113 follows because the set of pairwise distances attained as t ranges over $[0, 1]$ is unchanged when t is replaced by the surjective parameter $\phi(t)$.

Corollary 7.9 (Combined timing-and-position certificate). *Let $\phi \in \Phi$ and let $\tilde{z}$ have the same labeled endpoints as z^ϕ . If Equation 114 holds, then $\tilde{z}$ represents the same braid element as z .*

$$\|\tilde{z} - z^\phi\|_\infty < \frac{\rho(h, w)}{2}, \quad (114)$$

Proof. Proposition 7.7 gives the same braid for z and z^ϕ . Equation 113 and Proposition 7.8 give the same braid for z^ϕ and $\tilde{z}$. Equality of braid elements is transitive, so z and $\tilde{z}$ represent the same braid. $\square$

Corollary 7.9 separates two uncertainties that historical data often confound. Uncertainty about the exact pace of a process can be topologically invisible when it is an order-preserving reparameterization. Uncertainty in relational position is controlled by a separation margin only after the chart and persistent strand identities have been fixed.

7.5 Uniform Certificates across Set-Valued Reconstructions

A pair-specific certificate does not automatically provide a certificate for the entire evidence-compatible reconstruction set. Each reconstruction pair can have a positive separation margin while the infimum of those margins over all compatible pairs is zero. The distinction between pointwise and uniform robustness is therefore important.

Assume one reconstruction regime η has complete standard braid admissibility. Define its uniform separation margin by Equation 115.

$$\rho_{\inf}(\eta) := \inf_{(h, w) \in \tilde{\mathcal{R}}_\eta} \rho(h, w). \quad (115)$$

Every term in the infimum of Equation 115 is positive, but the infimum itself can equal zero.

For $\varepsilon > 0$, define the endpoint-fixed geometric uncertainty family around the full reconstruction set by Equation 116.

$$\mathfrak{Z}_\varepsilon(\eta) := \left\{ \tilde{z} : \begin{array}{l} \text{there exists } (h, w) \in \tilde{\mathcal{R}}_\eta \text{ with reference family } z^{h, w}, \\ \tilde{z} \text{ has the same labeled endpoints as } z^{h, w}, \quad \|\tilde{z} - z^{h, w}\|_\infty \leq \varepsilon \end{array} \right\}. \quad (116)$$

Proposition 7.10 (Uniform geometric robustness over a reconstruction set). *Assume the braid class is identified in regime η with value β , and suppose Equation 117 holds.*

$$\rho_{\inf}(\eta) > 0. \quad (117)$$

If ε satisfies Equation 118, then every strand family in $\mathfrak{Z}_\varepsilon(\eta)$ represents β .

$$0 \leq \varepsilon < \frac{\rho_{\inf}(\eta)}{2}. \quad (118)$$

Proof. Take $\tilde{z} \in \mathfrak{Z}_\varepsilon(\eta)$. By Equation 116, there exists $(h, w) \in \tilde{\mathcal{R}}_\eta$ such that $\|\tilde{z} - z^{h, w}\|_\infty \leq \varepsilon$. Equation 115 gives $\rho_{\inf}(\eta) \leq \rho(h, w)$. Combining this inequality with Equation 118 yields $\|\tilde{z} - z^{h, w}\|_\infty < \rho(h, w)/2$. Proposition 7.8 therefore gives the same braid element for $\tilde{z}$ and $z^{h, w}$. Because the braid is identified in regime η , every reference pair (h, w) represents β . Hence $\tilde{z}$ also represents β . $\square$

When $\rho_{\inf}(\eta) = 0$, Proposition 7.10 supplies no positive common radius. This does not prove that the braid conclusion is unstable. It means only that the simple separation-margin certificate cannot certify one nonzero perturbation radius uniformly over the entire reconstruction set. Each individual reconstruction pair can still possess its own positive local certificate.

The certificate is conditional on the representational domain. Table 5 states the main commitments held fixed by the geometric argument.

Table 5 separates geometric robustness from robustness to interpretive choices. If an alternative braid chart can be shown to move every strand by at most a known planar bound while preserving labels and

Table 5: Domain assumptions underlying the metric robustness certificate.

Commitment	Held fixed by the certificate	Uncertainty requiring a different analysis
Strand number and registry	The same finite set $\mathcal{A}_Y^{\text{str}}$ is tracked throughout	Birth, death, fusion, branching, or alternative component selection
Persistent identity	Each perturbed strand retains the same label and historical referent	Alternative identity resolutions or witness assignments
Relational chart	Distances are measured in the same planar braid chart χ_Y	Competing charts or different substantive coordinate constructions
Endpoint convention	Labeled endpoints and base configuration are fixed	Endpoint uncertainty, alternative closure rules, or open-endpoint comparison
Interpolation family	The perturbed strands are compared as continuous planar curves	Alternative interpolation rules whose outputs are not bounded in the same sup norm
Evidential domain	All compatible reconstruction pairs are included in the set over which the margin is computed	Missing admissible reconstructions or changes in the evidential compatibility relation

endpoints, it can be included in the metric certificate. Without such a bound, chart uncertainty belongs to the regime-level perturbation class of Section 7.1. The same distinction applies to identity changes and altered endpoint conventions.

The present section therefore yields two complementary robustness notions. Regime robustness asks whether a braid or invariant conclusion survives a declared class of alternative evidential and representational commitments. Metric robustness asks whether one or many fixed-domain geometric reconstructions remain in the same topological class under bounded positional error and order-preserving timing variation. The next section studies the complementary case in which admissible perturbations approach a topology-changing boundary and the braid classification becomes fragile.

8 Topology-Changing Boundaries and Changing Social Ontologies

This section studies the complementary case to the robustness results of Section 7: compatible historical reconstructions that approach a topological boundary or undergo a change in the fact structure through which the research object is constituted. Its role is to distinguish geometric braid-class change from changes in fact configuration or taxonomy. Section 8.1 defines the collision boundary in fixed-endpoint strand space. Section 8.2 derives its exact sup-norm distance. Section 8.3 separates fact-configuration change from fact-taxonomy change. Section 8.4 then states the historical interpretation limits.

8.1 Collision Boundaries in Reconstructed Strand Space

Fix one standard braid-admissible reconstruction pair (h, w) and write $z = (z_a)_{a \in \mathcal{A}_Y^{\text{str}}}$ for its labeled strand family. The historical braid construction of Section 5 uses the unordered configuration path associated with these strands, but boundary analysis is clearer in the labeled ambient strand space because a collision is the equality of two strand positions.

Let the labeled endpoint data of z be denoted by $e(z) = ((z_a(0), z_a(1)))_{a \in \mathcal{A}_Y^{\text{str}}}$. Define the fixed-endpoint strand space in Equation 119.

$$\mathcal{Z}_{e(z)} := \left\{ \tilde{z} = (\tilde{z}_a)_a : \begin{array}{l} \tilde{z}_a \in C([0, 1], \mathbb{R}^2) \text{ for every } a, \\ \tilde{z}_a(0) = z_a(0), \quad \tilde{z}_a(1) = z_a(1) \end{array} \right\}. \quad (119)$$

The space in Equation 119 is equipped with the strand-family sup norm introduced in Equation 109.

The collision set inside this fixed-endpoint space is defined in Equation 120.

$$\Sigma_{e(z)}^{\text{coll}} := \{\tilde{z} \in \mathcal{Z}_{e(z)} : \exists t \in [0, 1], \exists a \neq b \text{ such that } \tilde{z}_a(t) = \tilde{z}_b(t)\}. \quad (120)$$

A member of Equation 120 does not define a path in the configuration space $\mathcal{U}_n$ at the collision time. The standard braid descriptor is therefore undefined there.

The collision-free sector is the complement defined in Equation 121.

$$\mathcal{Z}_{e(z)}^{\text{cf}} := \mathcal{Z}_{e(z)} \setminus \Sigma_{e(z)}^{\text{coll}}. \quad (121)$$

For every $\tilde{z} \in \mathcal{Z}_{e(z)}^{\text{cf}}$ whose unordered configuration path has the required based-loop endpoints, the companion formalism assigns a braid element. The resulting map is locally constant under any continuous deformation that remains inside the collision-free sector and keeps the endpoint assignment fixed.

Proposition 8.1 (Collision necessity for braid-class change). *Let $s \mapsto z^{(s)}$ be a continuous path in $\mathcal{Z}_{e(z)}$ with respect to the sup norm, and suppose $z^{(0)}$ and $z^{(1)}$ are standard braid-admissible. If their braid elements are different, then Equation 122 holds for at least one $s_* \in [0, 1]$.*

$$z^{(s_*)} \in \Sigma_{e(z)}^{\text{coll}}. \quad (122)$$

Proof. Assume, for contradiction, that $z^{(s)} \in \mathcal{Z}_{e(z)}^{\text{cf}}$ for every $s \in [0, 1]$. Define $H : [0, 1]^2 \rightarrow \mathcal{U}_n$ by taking at (s, t) the unordered configuration formed by the strand positions $\{z_a^{(s)}(t)\}_a$. Continuity of $s \mapsto z^{(s)}$ in the sup norm and continuity of every strand imply continuity of H . Collision freedom keeps the image inside $\mathcal{U}_n$, and the fixed labeled endpoint data make the endpoint configurations constant in s . Thus H is an endpoint-preserving homotopy between the two configuration paths. They therefore define the same braid element, contradicting the hypothesis. Hence Equation 122 must hold. $\square$

Proposition 8.1 is conditional on the fixed representational sector. A change in strand registry, identity assignment, braid chart, endpoint convention, or component number can change the descriptor without producing a collision in the previous chart. Such changes belong to the regime-level analysis of Section 7.1 rather than to the fixed-domain collision boundary of Equation 120.

For a reconstruction regime η , let $\tilde{\mathcal{R}}_\eta^{\text{dom}}$ denote the subset of witness-augmented reconstructions for which the strand registry, labeled endpoints, braid chart, and interpolation rule are all defined, whether or not the resulting strands are collision-free. Define its collision-boundary subset by Equation 123.

$$\mathcal{C}_{\text{coll}}(\eta) := \{(h, w) \in \tilde{\mathcal{R}}_\eta^{\text{dom}} : \rho(h, w) = 0\}. \quad (123)$$

Equation 123 uses the minimum pairwise separation of Equation 108. Configuration-admissible pairs satisfy $\rho(h, w) > 0$; Equation 123 records the adjacent representational cases at which collision exclusion fails.

8.2 Collision-Boundary Distance in Strand Space

Section 7.4 established the sufficient stability radius $\rho(h, w)/2$. The same quantity has a sharper geometric interpretation: it is the exact sup-norm distance from the fixed-endpoint strand family to the collision set.

For a standard braid-admissible strand family z , define its distance to the collision set by Equation 124.

$$d_{\text{coll}}(z) := \inf_{\tilde{z} \in \Sigma_{e(z)}^{\text{coll}}} \|\tilde{z} - z\|_\infty. \quad (124)$$

Proposition 8.2 (Exact distance to the collision boundary). *For every standard braid-admissible reconstruction pair (h, w) with strand family $z = z^{h,w}$, Equation 125 holds.*

$$d_{\text{coll}}(z) = \frac{\rho(h, w)}{2}. \quad (125)$$

Proof. Write $\rho := \rho(h, w)$. First, let $\tilde{z} \in \Sigma_{e(z)}^{\text{coll}}$. There exist $t_* \in [0, 1]$ and distinct labels a, b such that $\tilde{z}_a(t_*) = \tilde{z}_b(t_*)$. The triangle inequality gives Equation 126.

$$\begin{aligned} \rho &\leq \|z_a(t_*) - z_b(t_*)\|_2 \\ &\leq \|z_a(t_*) - \tilde{z}_a(t_*)\|_2 + \|z_b(t_*) - \tilde{z}_b(t_*)\|_2 \\ &\leq 2\|\tilde{z} - z\|_\infty. \end{aligned} \tag{126}$$

Equation 126 implies $d_{\text{coll}}(z) \geq \rho/2$.

For the reverse inequality, fix any $\varepsilon > \rho/2$. By the definition of the minimum separation and continuity of the pairwise distances, there exist an interior time $t_0 \in (0, 1)$ and distinct labels a, b such that Equation 127 holds.

$$\|z_a(t_0) - z_b(t_0)\|_2 < 2\varepsilon. \tag{127}$$

The interior choice is immediate when a minimizing time lies in $(0, 1)$. If the minimum occurs only at an endpoint, continuity supplies interior times with pairwise distance arbitrarily close to ρ , which is sufficient because $2\varepsilon > \rho$.

By continuity of the pairwise distance, choose an open interval $J \subset (0, 1)$ containing t_0 such that $\|z_a(t) - z_b(t)\|_2 < 2\varepsilon$ for every $t \in J$. Choose a continuous function $\lambda : [0, 1] \rightarrow [0, 1]$ with support contained in J and $\lambda(t_0) = 1$. Define the time-dependent midpoint $m(t) := (z_a(t) + z_b(t))/2$ and modify only strands a and b according to Equation 128.

$$\begin{aligned} \tilde{z}_a(t) &:= (1 - \lambda(t))z_a(t) + \lambda(t)m(t), \\ \tilde{z}_b(t) &:= (1 - \lambda(t))z_b(t) + \lambda(t)m(t). \end{aligned} \tag{128}$$

All other strands are unchanged. Because the support of λ is contained in the interior interval J , the endpoint data are preserved. At t_0 , Equation 128 gives $\tilde{z}_a(t_0) = \tilde{z}_b(t_0) = m(t_0)$, so $\tilde{z} \in \Sigma_{e(z)}^{\text{coll}}$. For $t \notin J$ there is no displacement. For $t \in J$, each modified strand moves by $\lambda(t)\|z_a(t) - z_b(t)\|_2/2 < \varepsilon$. Hence Equation 129 holds.

$$\|\tilde{z} - z\|_\infty < \varepsilon. \tag{129}$$

Because Equation 129 holds for every $\varepsilon > \rho/2$, $d_{\text{coll}}(z) \leq \rho/2$. Combining the two inequalities proves Equation 125. $\square$

Proposition 8.2 shows that the radius used in Proposition 7.8 is not merely an arbitrary sufficient constant. In the fixed-endpoint sup-norm geometry, it is the exact distance to the nearest collision boundary. The open ball of radius $\rho(h, w)/2$ is collision-free, while collision-producing perturbations can be found at arbitrarily small excess over that radius.

For a set-valued historical reconstruction, define the uniform distance to collision by Equation 130.

$$d_{\text{coll}}^{\text{inf}}(\eta) := \inf_{(h, w) \in \tilde{\mathcal{R}}_\eta} d_{\text{coll}}(z^{h, w}). \tag{130}$$

Corollary 8.3 (Uniform boundary distance). *If regime η has complete standard braid admissibility, then Equation 131 holds.*

$$d_{\text{coll}}^{\text{inf}}(\eta) = \frac{\rho_{\text{inf}}(\eta)}{2}. \tag{131}$$

Proof. Apply Proposition 8.2 to every $(h, w) \in \tilde{\mathcal{R}}_\eta$ and then take the infimum. Pulling out the positive constant $1/2$ gives Equation 131. $\square$

Equation 131 provides a precise sense in which a set-valued reconstruction can be uniformly close to a topology-changing boundary even when every individual reconstruction pair is collision-free. If $\rho_{\text{inf}}(\eta) = 0$, the reconstruction set contains braid-admissible representatives whose distance to collision can be made arbitrarily small.

For a declared geometric uncertainty scale $\varepsilon > 0$, define boundary exposure by Equation 132.

$$\rho_{\inf}(\eta) < 2\varepsilon. \quad (132)$$

When Equation 132 holds, Corollary 8.3 implies that at least one compatible reconstruction pair has collision boundary at distance less than ε . When $2\varepsilon < \rho_{\inf}(\eta)$, Proposition 7.10 certifies collision avoidance and braid preservation throughout the corresponding uncertainty family. The equality case $2\varepsilon = \rho_{\inf}(\eta)$ requires attention to whether the infimum is attained and whether the uncertainty set is open or closed.

Boundary exposure is weaker than demonstrated braid instability. Reaching the collision set is necessary for a continuous fixed-domain path between distinct braid classes by Proposition 8.1, but the existence of a nearby collision does not by itself prove that the declared uncertainty family contains collision-free representatives on two different sides of that boundary. The formal conclusion at Equation 132 is therefore proximity to a topology-changing boundary, not automatic multiplicity of braid classes.

8.3 Fact-Configuration Change and Fact-Taxonomy Change

A braid-domain change can arise without any collision in the previously selected strand chart. Two cases should be separated.

A *fact-configuration change* occurs when the taxonomy remains fixed while the instantiated facts change. Let τ be fixed and let $F_t \in \mathfrak{C}_\tau$ be the fact configuration at historical position t . Configuration change is represented by Equation 133.

$$F_{t_1} \neq F_{t_2}, \quad \tau \text{ fixed.} \quad (133)$$

An institutional split, merger, succession, or change of status can belong to this class when the existing taxonomy remains adequate.

A *fact-taxonomy change* occurs when the category system itself changes or becomes inadequate. Equation 10 already permits $\mathcal{T}_F(t_1) \neq \mathcal{T}_F(t_2)$. Such a change can alter which research objects $\text{Obj}_\tau(F)$ are available, how many strand-like entities exist, or what continuity means across the regime boundary.

The two changes are analytically distinct. A fixed- n braid can fail because one object splits into two under a stable taxonomy, or because the earlier taxonomy no longer supplies a faithful object category. In either case, the correct response is not to interpret the failure automatically as a collision inside the old chart.

Proposition 8.4 (Fixed-strand continuation under fact-regime change). *Suppose two adjacent historical regimes use strand registries $\mathcal{A}_1^{\text{str}}$ and $\mathcal{A}_2^{\text{str}}$. Ordinary fixed-strand continuation across the boundary requires a declared bijection between the registries that is compatible with the research-object constructions on both sides.*

Proof. A standard braid on fixed n requires the same number of continuously traceable strands throughout the interval. If no bijection exists between the registries, strand number or identity cannot be continued one-to-one across the boundary. If a bijection exists but is incompatible with the research-object constructions, the same strand label would refer to different or noncorresponding historical objects. Either condition prevents ordinary fixed-strand continuation. $\square$

Proposition 8.4 connects the topology-change analysis to the fact-regime architecture. Local braid descriptions can remain valid on both sides of the boundary while the transition itself requires an intersector or regime-changing representation.

8.4 Historical Interpretation of Boundary Proximity

The boundary objects defined above belong to the chosen historical representation. They require a strand registry, a planar relational chart, an interpolation rule, a fixed endpoint sector, and a metric on that chart.

Table 6: Sources of proximity to a topology-changing boundary and their interpretive status.

Source	Formal manifestation	Interpretation required before substantive use
Positional uncertainty	Small $\rho(h, w)$ or small $\rho_{\text{inf}}(\eta)$ in a fixed braid chart	Whether near-coincidence in the chart corresponds to a meaningful relation in the historical domain
Temporal ambiguity	Alternative admissible event placements change crossing order or bring two transitions close to exchange	Whether the unresolved chronology is evidential uncertainty, a coding choice, or substantive simultaneity
Identity ambiguity	Alternative witness assignments change which persistent referents define the strands	Whether the records justify continuity of the entities whose trajectories are being compared
Chart uncertainty	Alternative plausible maps χ_Y move reconstructions toward or away from collision	Whether the planar coordinates preserve the relational distinctions relevant to the research question
Interpolation uncertainty	Alternative admissible interpolation rules approach different collision boundaries	Whether the evidence supports the assumed within-interval path geometry
Domain change	Strand birth, death, branching, fusion, or endpoint-sector change	A new representational sector rather than a collision perturbation inside the previous fixed- n braid space

A small value of $\rho(h, w)$ or $\rho_{\text{inf}}(\eta)$ therefore supports a statement about the fragility of the topological classification under those commitments. It does not, without additional domain argument, establish that the underlying social system was near a physical collision, causal bifurcation, political crisis, phase transition, or other substantive form of historical criticality.

Within the present framework, boundary proximity can arise through several analytically different routes. The distinctions are summarized in Table 6.

Table 6 prevents several distinct uncertainties from being compressed into one scalar fragility claim. The minimum separation is meaningful only after the domain-specific representation has been justified.

A topology-changing boundary can nevertheless be historically informative when its semantics are supplied independently. Suppose the relational chart has been constructed so that equality of two planar positions corresponds to a domain-relevant relational indifference, tie, or coincidence. A small separation margin can then indicate that the evidence-compatible reconstruction lies near a change in the topological ordering of those relations. Even in this case, the topological statement remains descriptive. An explanation of why the historical process approached that boundary requires causal, institutional, interpretive, or other domain-specific evidence.

The same caution applies to the language of “critical configurations.” In this manuscript, a topologically critical configuration means only a reconstructed strand family on the collision boundary of Equation 120, relative to a fixed representation scheme. The term carries no automatic implication of critical phenomena in statistical physics, dynamical-systems bifurcation, or crisis theory.

For reporting purposes, a boundary-proximity claim should therefore include at least four items: the representation scheme under which the boundary is defined, the evidential and auxiliary assumptions supporting the relevant reconstruction set, the quantitative or qualitative uncertainty class against which proximity is assessed, and the strongest topological conclusion actually demonstrated. In particular, a

study should distinguish among the three statuses listed in Equation 134.

$$\begin{aligned}
\text{certified separation} & : 2\varepsilon < \rho_{\text{inf}}(\eta), \\
\text{boundary exposure} & : \rho_{\text{inf}}(\eta) < 2\varepsilon, \\
\text{demonstrated class multiplicity} & : |\mathcal{B}_{\mathcal{U}}| > 1 \text{ for a declared comparable regime class.}
\end{aligned} \tag{134}$$

Equation 134 orders the claims by what has actually been established. Boundary exposure does not imply demonstrated class multiplicity, and class multiplicity does not by itself identify the historical mechanism responsible for the divergence.

The boundary analysis therefore complements the robustness analysis rather than replacing it. Section 7 identifies conditions under which one topological conclusion survives declared uncertainty. The present section identifies the boundary that such a conclusion must approach before a continuous fixed-domain braid change becomes possible and states how close the evidence-compatible reconstructions lie to that boundary. The next section uses these distinctions comparatively across historical cases, where similarity can be assessed at the level of braid classes, derived invariants, and joint descriptors without presuming identical generators.

9 Comparative Historical Trajectory Analysis

This section develops a comparative methodology for using the set-valued historical trajectory descriptions constructed in Sections 6–8. Its role is to clarify what can and cannot be compared across historical cases when the cases can differ in evidential record, reconstruction assumptions, strand registry, braid index, relational chart, endpoint convention, and degree of topological identification. The objective is not to force all cases into one common braid group. Instead, the section distinguishes direct braid comparison from comparison through common derived descriptors, defines joint comparative resolution, and formalizes recurrent structural motifs as repeated descriptive values rather than as causal mechanisms. Section 9.1 states the conditions for direct braid-class comparison. Section 9.2 defines common invariant families across heterogeneous cases. Section 9.3 derives the resolution of joint comparative descriptors. Section 9.4 defines recurrence and robust recurrence across cases.

9.1 Braid-Class Comparison

Let $\mathcal{C}$ be a finite set of historical cases. For each case $c \in \mathcal{C}$, let Y_c be the evidential record, A_c the auxiliary reconstruction assumptions, $\mathcal{A}_c^{\text{str}}$ the strand registry, $\mathfrak{P}_c$ the planar relational representation scheme, and $c_{0,c}$ the selected base configuration when the standard braid descriptor is used.

The case-specific reconstruction regime is defined in Equation 135.

$$\eta_c := (Y_c, \mathcal{K}_{Y_c}, A_c, \mathcal{A}_c^{\text{str}}, \mathfrak{P}_c, c_{0,c}). \tag{135}$$

Let $\mathcal{B}(c)$ denote the evidence-compatible braid set $\mathcal{B}_{A_c}(Y_c; c_{0,c})$ from Equation 79.

A raw comparison $\beta_c = \beta_d$ between two cases is meaningful only when the two braid values inhabit a common algebraic and representational setting. The first requirement is equal strand number. If case c uses n_c strands and case d uses n_d strands with $n_c \neq n_d$, then $\beta_c \in B_{n_c}$ and $\beta_d \in B_{n_d}$ are elements of different groups. Equality is then not a well-typed comparative statement.

Even when $n_c = n_d$, the same Artin word can encode different substantive entities when the strand registries differ. Direct comparison therefore requires a declared strand correspondence. Let $n_c = n_d = n$ and let $\varphi_{dc} : \mathcal{A}_c^{\text{str}} \rightarrow \mathcal{A}_d^{\text{str}}$ be a bijection between the strand registries. The correspondence is *type-compatible* when Equation 136 holds.

$$\theta_{Y_d}(\varphi_{dc}(a)) = \theta_{Y_c}(a) \quad \text{for every } a \in \mathcal{A}_c^{\text{str}}. \tag{136}$$

Equation 136 does not require the two cases to contain the same historical individuals. It requires the comparative coding to justify which strands occupy corresponding analytical types.

A second requirement concerns the braid chart and crossing convention. Let χ_c and χ_d be the case-specific braid charts. A *direct-comparison protocol* between cases c and d is a declared tuple defined in Equation 137.

$$\mathfrak{C}_{dc}^{\text{dir}} := (n, \varphi_{dc}, \mathfrak{o}_{dc}, \mathfrak{e}_{dc}, T_{dc}), \quad (137)$$

where $\mathfrak{o}_{dc}$ records the common crossing-orientation convention, $\mathfrak{e}_{dc}$ records the endpoint comparison convention, and T_{dc} is an explicitly declared braid-group isomorphism from the codomain presentation used for case c to the common copy of B_n used for comparison with case d . The map T_{dc} must be justified by the same presentation, orientation, and endpoint conventions; a historical strand correspondence φ_{dc} does not by itself determine an algebraic transport on braid values.

The transported case braid set is defined in Equation 138.

$$\widehat{\mathcal{B}}_{dc}(c) := T_{dc}(\mathcal{B}(c)) \subseteq B_n. \quad (138)$$

For the reference case d , take T_{dd} to be the identity.

Definition 9.1 (Direct braid comparability). Cases c and d are *directly braid-comparable* when a justified protocol $\mathfrak{C}_{dc}^{\text{dir}}$ exists and both cases have complete standard braid admissibility under their declared reconstruction regimes.

Definition 9.1 separates algebraic equality from substantive comparability. Equal elements of two formally identified braid groups support a direct topological comparison only after the strand types, crossing convention, and endpoint convention have also been aligned.

For directly comparable cases, define the comparative braid-intersection set in Equation 139.

$$\mathcal{B}_{\cap}(c, d) := \widehat{\mathcal{B}}_{dc}(c) \cap \mathcal{B}(d). \quad (139)$$

If both cases have identified braids, say $\widehat{\mathcal{B}}_{dc}(c) = \{\beta_c\}$ and $\mathcal{B}(d) = \{\beta_d\}$, then $\mathcal{B}_{\cap}(c, d) \neq \emptyset$ is equivalent to $\beta_c = \beta_d$.

When the case braid sets are non-singleton, Equation 139 supports a weaker statement. A nonempty intersection means that at least one braid class is compatible with both cases under the comparison protocol. It does not establish that the two cases have the same identified braid.

9.2 Invariant Families across Cases

Direct braid comparison can be unavailable or unnecessarily fine-grained. A comparative question may instead concern a derived structural property that can be defined consistently across cases.

Let Ω_{λ} be a common comparative codomain and let $I_{\lambda,c} : B_{n_c} \rightarrow \Omega_{\lambda}$ be a case-specific map implementing one declared comparative descriptor for case c . The maps can differ by braid index while sharing the same codomain and interpretation.

For each case, define the evidence-compatible comparative value set in Equation 140.

$$\mathcal{V}_{\lambda}(c) := \{I_{\lambda,c}(\beta) : \beta \in \mathcal{B}(c)\} \subseteq \Omega_{\lambda}. \quad (140)$$

Equation 140 propagates the case-specific braid uncertainty into the shared comparative codomain.

Definition 9.2 (Comparative identifiability through a descriptor). The comparative descriptor λ is *identified for case c* when the case has complete standard braid admissibility and Equation 141 holds.

$$|\mathcal{V}_{\lambda}(c)| = 1. \quad (141)$$

Definition 9.2 allows two cases to be compared through λ even when their full braid classes are not identified, provided the derived comparative value is identified in both cases.

For cases $c, d \in \mathcal{C}$, define the pairwise comparative-value intersection in Equation 142.

$$\mathcal{V}_{\lambda}^{\cap}(c, d) := \mathcal{V}_{\lambda}(c) \cap \mathcal{V}_{\lambda}(d). \quad (142)$$

Table 7: Levels of comparative historical trajectory description.

Comparative level	Required common structure	Inferential content
Direct braid comparison	Same braid index together with justified strand, orientation, and endpoint correspondence	Comparison of complete braid classes under the declared transport protocol
Derived comparative descriptor	Case-specific maps into one common codomain Ω_λ	Comparison of one selected structural feature even when the full braids differ or remain nonidentified
Joint comparative descriptor	A finite family of compatible derived descriptors	Simultaneous comparison at the intersection of their induced resolutions
Groupoid-level comparison	Compatible endpoint-sensitive descriptor scheme	Comparison of open-endpoint trajectory classes without artificial braid closure

When both sets are singletons, $\mathcal{V}_\lambda^\cap(c, d) \neq \emptyset$ means that the identified comparative values agree. When either set contains several values, the intersection expresses overlap among compatible structural descriptions rather than equality of identified case-level structures.

The distinction between direct and derived comparison is summarized in Table 7.

Table 7 emphasizes that comparative resolution is chosen rather than given. A coarser descriptor can be methodologically preferable when it matches the historical question and is more robustly identifiable across the available cases.

9.3 Joint Comparative Resolution

A comparative study commonly uses several structural descriptors simultaneously. Let Λ be a finite index set of comparative descriptors. For each case c , define the joint comparative map by Equation 143.

$$J_c : B_{n_c} \longrightarrow \prod_{\lambda \in \Lambda} \Omega_\lambda, \quad J_c(\beta) := (I_{\lambda,c}(\beta))_{\lambda \in \Lambda}. \quad (143)$$

The case-specific joint value set is defined in Equation 144.

$$\mathcal{V}_\Lambda(c) := \{J_c(\beta) : \beta \in \mathcal{B}(c)\}. \quad (144)$$

For a fixed case c , each comparative map induces an equivalence relation on the compatible braid set. Define the relation induced by descriptor λ in Equation 145.

$$\beta_1 \equiv_{\lambda,c} \beta_2 \iff I_{\lambda,c}(\beta_1) = I_{\lambda,c}(\beta_2). \quad (145)$$

Likewise, let $\equiv_{\Lambda,c}$ be the equivalence induced by J_c .

Proposition 9.3 (Resolution of a joint comparative descriptor). *For each case c , the joint comparative equivalence satisfies Equation 146.*

$$\equiv_{\Lambda,c} = \bigcap_{\lambda \in \Lambda} \equiv_{\lambda,c}. \quad (146)$$

Proof. Two braid elements have equal values under J_c exactly when all corresponding coordinates in the product codomain are equal. By Equation 143, coordinatewise equality is equivalent to equality under every $I_{\lambda,c}$. This is precisely membership in every relation $\equiv_{\lambda,c}$, which gives the intersection in Equation 146. $\square$

Proposition 9.3 gives a formal basis for combining complementary structural comparisons. Adding a descriptor can refine the joint comparative resolution because it can separate braid elements that the previous descriptor family identified. It cannot recover a distinction erased by every component map.

Joint comparative identifiability is defined in Equation 147.

$$|\mathcal{V}_\Lambda(c)| = 1. \quad (147)$$

When Equation 147 holds together with complete standard braid admissibility, the selected vector of comparative features is identified for case c even if the complete braid is not.

Joint identification is stronger than separate singleton claims only when the descriptor values are evaluated on the same compatible braid set. The construction in Equation 144 enforces that requirement and prevents coordinate values drawn from mutually incompatible braid reconstructions from being assembled into one fictitious joint profile.

For a pair of cases, define the joint comparative intersection by Equation 148.

$$\mathcal{V}_\Lambda^\cap(c, d) := \mathcal{V}_\Lambda(c) \cap \mathcal{V}_\Lambda(d). \quad (148)$$

As before, a nonempty intersection indicates a shared compatible profile; equality of identified profiles requires both case sets to be singletons.

9.4 Recurrent Structural Motifs

Comparative historical analysis often seeks recurrent relational patterns. The present framework defines such recurrence descriptively rather than causally.

Fix a comparative descriptor family Λ and let v belong to the common joint codomain $\prod_{\lambda \in \Lambda} \Omega_\lambda$. Define the support set of v across cases by Equation 149.

$$\text{Supp}_\Lambda(v) := \{c \in \mathcal{C} : v \in \mathcal{V}_\Lambda(c)\}. \quad (149)$$

Equation 149 records cases for which the profile v is compatible with the current reconstruction regime.

Compatibility is weaker than case-level identification. Define the identification support set in Equation 150.

$$\text{ISupp}_\Lambda(v) := \{c \in \mathcal{C} : \mathcal{V}_\Lambda(c) = \{v\} \text{ and case } c \text{ has complete standard braid admissibility}\}. \quad (150)$$

Definition 9.4 (Recurrent comparative motif). A joint descriptor value v is a *recurrent comparative motif* at identification level when Equation 151 holds.

$$|\text{ISupp}_\Lambda(v)| \geq 2. \quad (151)$$

The threshold in Equation 151 expresses repetition across at least two cases. It is descriptive and carries no claim of statistical prevalence.

A motif can also be required to survive case-specific perturbation classes. Let $\mathfrak{U}_c$ be a declared reconstruction perturbation class for case c , as in Section 7.1. For each regime $\eta \in \mathfrak{U}_c$, let $\mathcal{V}_\Lambda(c; \eta)$ be the corresponding joint comparative value set. Define the robust identification support in Equation 152.

$$\text{RISupp}_\Lambda(v) := \left\{ c \in \mathcal{C} : \begin{array}{l} \text{every } \eta \in \mathfrak{U}_c \text{ has a nonempty reconstruction,} \\ \text{every compatible reconstruction is standard braid-admissible,} \\ \mathcal{V}_\Lambda(c; \eta) = \{v\} \text{ for every } \eta \in \mathfrak{U}_c \end{array} \right\}. \quad (152)$$

Definition 9.5 (Robust recurrent comparative motif). A value v is a *robust recurrent comparative motif* relative to the declared perturbation classes when Equation 153 holds.

$$|\text{RISupp}_\Lambda(v)| \geq 2. \quad (153)$$

Definitions 9.4 and 9.5 separate three claims that should not be merged. A value can be compatible with several cases, identified in several cases, or robustly identified in several cases. Each level is stronger than the preceding one.

The same motif value also does not establish a shared generator. Let $\mathcal{G}_c^*(v)$ denote the candidate generators for case c whose realized histories are compatible with the record and whose admitted trajectory description has joint comparative value v . Even if two cases share the same identified motif, their candidate-generator sets can remain disjoint. The descriptive recurrence in Equation 151 therefore supports a claim about common trajectory structure, not a claim about common causal production.

This distinction prepares the inverse use developed in the next section. Comparative trajectory structure can constrain candidate explanations by excluding generators that cannot realize the observed structural profile. It does not, by itself, identify the mechanism that produced the history.

10 Inverse Constraints on Candidate Generative Explanations

This section formalizes the inverse use of reconstructed trajectory structure. Its role is deliberately limited: an identified braid, invariant, or comparative motif can constrain a declared class of candidate generative explanations by excluding candidates whose forward realizations cannot reproduce the observed structural target. The objective is not to invert the braid descriptor into a unique historical cause. Section 10.1 distinguishes individual generative specifications from explanatory hypotheses that admit sets of realizations. Section 10.2 defines the forward structural image of a hypothesis and derives conservative compatibility and refutation criteria. Section 10.3 proves why structural compatibility ordinarily remains weaker than causal identification and states the redundancy of reusing a descriptor after exact history-level compatibility has already been imposed. Section 10.4 extends the construction to common explanatory templates across several historical cases.

10.1 Candidate Generator Hypothesis Classes

Section 3.2 represents a candidate generative class by a subset $\mathcal{U}^* \subseteq \mathcal{U}(\mathfrak{D})$. An element $u \in \mathcal{U}^*$ is a fully specified generative specification: it fixes an initial state together with a finite mechanism word and therefore has one realized history under Gen. Historical explanations are often coarser than one fully specified element. A mechanism hypothesis can leave parameters, initial conditions, timing, or local mechanism choices unresolved while still making structural predictions.

Let $\mathfrak{H}^*$ be a declared class of explanatory hypotheses. Each $H \in \mathfrak{H}^*$ is represented by a nonempty subset $\mathcal{U}_H \subseteq \mathcal{U}^*$ of generative specifications admitted by that hypothesis. Define the history realization set of H in Equation 154.

$$\Gamma(H) := \{\text{Gen}(u) : u \in \mathcal{U}_H\} \subseteq \Gamma(\mathfrak{D}). \quad (154)$$

Equation 154 turns an explanatory hypothesis into a set of forward-realizable histories. A singleton $\mathcal{U}_H$ recovers the fully specified case.

The mapping $H \mapsto \mathcal{U}_H$ is part of the explanatory model class and must be stated by the substantive inquiry. The present formalism does not infer which historical mechanisms belong in a hypothesis. It only records the consequences of a declared hypothesis once its generative realization set has been specified.

The evidence-compatible part of a hypothesis is defined in Equation 155.

$$\Gamma_A(H; Y) := \Gamma(H) \cap \mathcal{R}_A(Y). \quad (155)$$

A hypothesis is history-compatible with the current reconstruction regime when $\Gamma_A(H; Y) \neq \emptyset$. Define the history-compatible hypothesis class in Equation 156.

$$\mathfrak{H}_A^{\text{hist}}(Y) := \{H \in \mathfrak{H}^* : \Gamma_A(H; Y) \neq \emptyset\}. \quad (156)$$

Equation 156 is a strong evidential filter when $\mathcal{R}_A(Y)$ contains detailed path-level information. Later subsections show that a trajectory descriptor derived from the same reconstruction cannot add information after this full history-compatibility condition has already been imposed.

10.2 Forward Structural Reachability

Inverse structural screening is most transparent when formulated through a general trajectory descriptor. Let $D : \Gamma_D \rightarrow \Omega_D$ be a descriptor defined on the history domain Γ_D . The descriptor can be the braid map, a braid invariant, a joint comparative descriptor, or another trajectory-level representation justified by the study.

For a hypothesis $H \in \mathfrak{H}^*$, define its descriptor-admissible realization set in Equation 157.

$$\Gamma_D(H) := \Gamma(H) \cap \Gamma_D. \quad (157)$$

The forward structural image of the hypothesis is defined in Equation 158.

$$\mathcal{S}_D(H) := \{D(h) : h \in \Gamma_D(H)\} \subseteq \Omega_D. \quad (158)$$

Equation 158 contains every descriptor value reachable by at least one realization admitted by the hypothesis.

Let $\mathcal{O}_D(Y, A) \subseteq \Omega_D$ be a nonempty structural target extracted from the historical evidence under the declared reconstruction and descriptor regime. For example, $\mathcal{O}_D(Y, A)$ can be a singleton identified braid, a set of evidence-compatible braid classes, or a singleton derived invariant.

Possible structural compatibility is defined by nonempty overlap.

Definition 10.1 (Structural possibility). A hypothesis H is *possibly compatible* with $\mathcal{O}_D(Y, A)$ when Equation 159 holds.

$$\mathcal{S}_D(H) \cap \mathcal{O}_D(Y, A) \neq \emptyset. \quad (159)$$

Definition 10.1 is existential. It says that the hypothesis can realize at least one trajectory whose selected structure is compatible with the historical target. It does not say that every realization of the hypothesis has that structure.

A stronger condition requires the entire forward realization set to remain inside the descriptor domain and all descriptor values to lie inside the target set.

Definition 10.2 (Uniform compatibility). A hypothesis H is *uniformly compatible* with $\mathcal{O}_D(Y, A)$ when both conditions in Equation 160 hold.

$$\Gamma(H) \subseteq \Gamma_D, \quad \mathcal{S}_D(H) \subseteq \mathcal{O}_D(Y, A). \quad (160)$$

Definition 10.2 is substantially stronger than the possibility condition in Equation 159. It states that every history produced by the declared hypothesis is descriptor-admissible and has a target-compatible description.

Structural refutation requires equal care. If some realization of a hypothesis lies outside Γ_D , the descriptor alone cannot evaluate that realization. A missing descriptor value should not be converted into evidence against the hypothesis. The conservative refutation criterion is therefore defined only for hypotheses whose entire realization set is inside the descriptor domain.

Definition 10.3 (Structural refutation). A hypothesis H is *structurally refuted* by $\mathcal{O}_D(Y, A)$ when Equation 161 holds.

$$\Gamma(H) \subseteq \Gamma_D, \quad \mathcal{S}_D(H) \cap \mathcal{O}_D(Y, A) = \emptyset. \quad (161)$$

Definition 10.3 prevents domain exclusion from masquerading as contradiction. A hypothesis for which some forward realizations fall outside the descriptor domain can remain unevaluated by that descriptor even when its currently describable realizations miss the target.

Define the structurally refuted class in Equation 162.

$$\mathfrak{H}_D^{\text{ref}}(Y, A) := \{H \in \mathfrak{H}^* : H \text{ satisfies Definition 10.3}\}. \quad (162)$$

Table 8: Descriptor-level statuses for candidate generative explanations.

Status	Formal condition	Interpretation
Possible compatibility	$\mathcal{S}_D(H) \cap \mathcal{O}_D(Y, A) \neq \emptyset$	At least one realization of the hypothesis reaches the observed structural target
Uniform compatibility	$\Gamma(H) \subseteq \Gamma_D$ and $\mathcal{S}_D(H) \subseteq \mathcal{O}_D(Y, A)$	Every declared realization is descriptor-admissible and target-compatible
Structural refutation	$\Gamma(H) \subseteq \Gamma_D$ and $\mathcal{S}_D(H) \cap \mathcal{O}_D(Y, A) = \emptyset$	The hypothesis cannot realize the target anywhere inside its declared realization set
Descriptor-indeterminate	$\Gamma(H) \not\subseteq \Gamma_D$ and no compatible described realization is available	The descriptor lacks sufficient domain coverage to refute the hypothesis

The corresponding nonrefuted class is defined in Equation 163.

$$\mathfrak{H}_D^{\text{surv}}(Y, A) := \mathfrak{H}^* \setminus \mathfrak{H}_D^{\text{ref}}(Y, A). \quad (163)$$

Membership in $\mathfrak{H}_D^{\text{surv}}(Y, A)$ means only that the structural target has not excluded the hypothesis under the declared descriptor and model class.

The three inferential statuses are summarized in Table 8.

Table 8 makes survival weaker than support. A hypothesis can survive because it reaches the target, or because the descriptor does not cover enough of the hypothesis’s realization set to justify exclusion.

10.3 Structural Constraint under Causal Underdetermination

The forward structural image can eliminate candidate explanations, but overlap with an observed target ordinarily does not identify one generator. Define the possibly compatible hypothesis class in Equation 164.

$$\mathfrak{H}_D^{\text{pos}}(Y, A) := \{H \in \mathfrak{H}^* : \mathcal{S}_D(H) \cap \mathcal{O}_D(Y, A) \neq \emptyset\}. \quad (164)$$

If the set in Equation 164 contains more than one hypothesis, the structural target is compatible with several candidate explanations. A singleton set remains only a relative result because the declared explanatory class $\mathfrak{H}^*$ can be incomplete and because the descriptor can collapse distinctions among realized histories.

The descriptor-induced observational relation between hypotheses is defined in Equation 165.

$$H_1 \equiv_D^{\text{hyp}} H_2 \iff \mathcal{S}_D(H_1) = \mathcal{S}_D(H_2). \quad (165)$$

Hypotheses in the same class of $\equiv_D^{\text{hyp}}$ are indistinguishable by any conclusion that depends only on their complete forward image under D .

Proposition 10.4 (Structural observational equivalence). *If $H_1 \equiv_D^{\text{hyp}} H_2$, then for every target $\mathcal{V} \subseteq \Omega_D$, Equation 166 holds.*

$$\mathcal{S}_D(H_1) \cap \mathcal{V} \neq \emptyset \iff \mathcal{S}_D(H_2) \cap \mathcal{V} \neq \emptyset. \quad (166)$$

Proof. Equation 165 gives $\mathcal{S}_D(H_1) = \mathcal{S}_D(H_2)$. Their intersections with any common target $\mathcal{V}$ are therefore equal. Nonemptiness of one intersection is equivalent to nonemptiness of the other. $\square$

Proposition 10.4 formalizes one reason inverse trajectory analysis is constraining rather than uniquely reconstructive. When distinct causal hypotheses have the same descriptor-level forward image, no observation stated only at that descriptor resolution can distinguish them.

A second limitation is subtler and methodologically important. A descriptor derived from a fully specified historical reconstruction cannot be counted as independent evidence against hypotheses that have already been required to reproduce that same reconstruction.

Assume $\mathcal{R}_A(Y) \subseteq \Gamma_D$ and define the descriptor image of the full reconstruction set by Equation 167.

$$\mathcal{O}_D^{\text{full}}(Y, A) := \{D(h) : h \in \mathcal{R}_A(Y)\}. \quad (167)$$

Proposition 10.5 (Redundancy after full history-level compatibility). *Assume $\mathcal{R}_A(Y) \subseteq \Gamma_D$. If $H \in \mathfrak{H}_A^{\text{hist}}(Y)$, then Equation 168 holds.*

$$\mathcal{S}_D(H) \cap \mathcal{O}_D^{\text{full}}(Y, A) \neq \emptyset. \quad (168)$$

Proof. Because $H \in \mathfrak{H}_A^{\text{hist}}(Y)$, Equation 156 gives a history $h \in \Gamma(H) \cap \mathcal{R}_A(Y)$. The assumption $\mathcal{R}_A(Y) \subseteq \Gamma_D$ places h in the descriptor domain. Hence $D(h) \in \mathcal{S}_D(H)$ by Equation 158 and $D(h) \in \mathcal{O}_D^{\text{full}}(Y, A)$ by Equation 167. Therefore the intersection in Equation 168 is nonempty. $\square$

Proposition 10.5 prevents double counting. If a candidate explanation has already been tested against the complete reconstructed history set, applying a trajectory descriptor to that same reconstruction yields a summary of information already contained in the history-level compatibility test. The descriptor can still be valuable because it provides a lower-dimensional structural criterion, supports comparison across cases, or matches the predictive resolution of a model. It should not be presented as additional independent evidence merely because the summary is mathematically different from the original history.

The inverse method is therefore most informative in at least three situations. First, candidate models can be tested at the trajectory-structural resolution without requiring them to reproduce every fine-grained historical detail. Second, comparative work can test whether one explanatory template can realize structural targets across several cases. Third, a descriptor can supply a common comparison level when full histories inhabit heterogeneous or noncommensurable representations.

Descriptor refinement can also change the strength of elimination. Suppose two descriptors D_f and D_c share a common domain and satisfy $D_c = \phi \circ D_f$ for some map ϕ . Then D_f is the finer descriptor and D_c is its coarsening. Let $\mathcal{O}_f$ be a fine target and $\mathcal{O}_c := \phi(\mathcal{O}_f)$ the corresponding coarse target.

Proposition 10.6 (Refutation under descriptor refinement). *Assume $\Gamma(H) \subseteq \Gamma_{D_f} = \Gamma_{D_c}$. If H is structurally refuted by $\mathcal{O}_c$ under D_c , then it is structurally refuted by $\mathcal{O}_f$ under D_f .*

Proof. Suppose, toward a contradiction, that H is not refuted at the fine level because some $h \in \Gamma(H)$ satisfies $D_f(h) \in \mathcal{O}_f$. Applying ϕ gives $D_c(h) = \phi(D_f(h)) \in \phi(\mathcal{O}_f) = \mathcal{O}_c$. Thus $\mathcal{S}_{D_c}(H)$ intersects $\mathcal{O}_c$, contradicting coarse structural refutation. $\square$

Proposition 10.6 shows that a contradiction visible at a coarser descriptor cannot disappear merely by retaining additional structural detail, provided the targets are related by the same coarsening map. The converse need not hold: a fine descriptor can exclude a hypothesis whose coarse image still overlaps the coarse target.

10.4 Comparative Elimination of Candidate Explanations

Comparative analysis can strengthen structural screening by requiring one explanatory template to remain compatible with several case-specific targets. The construction must distinguish a common explanatory template from a claim that the cases literally share one fully specified generator.

Let Θ^* be a declared class of cross-case explanatory templates. For each $\theta \in \Theta^*$ and each case $c \in \mathcal{C}$, let $\mathcal{U}_{c,\theta} \subseteq \mathcal{U}(\mathcal{D}_c)$ be the set of case-specific generative specifications admitted by template θ . Define the case-specific realization set in Equation 169.

$$\Gamma_c(\theta) := \{\text{Gen}_c(u) : u \in \mathcal{U}_{c,\theta}\}. \quad (169)$$

For each case, let $D_c : \Gamma_{D_c} \rightarrow \Omega_c$ be the descriptor used for inverse screening and let $\mathcal{O}_c \subseteq \Omega_c$ be the case-specific structural target. For the comparative elimination result below, assume that the declared template class is descriptor-complete in every screened case, as stated in Equation 170.

$$\Gamma_c(\theta) \subseteq \Gamma_{D_c} \quad \text{for every } c \in \mathcal{C} \text{ and } \theta \in \Theta^*. \quad (170)$$

When Equation 170 does not hold, templates with unevaluable realizations must retain the descriptor-indeterminate status of Table 8 rather than being eliminated from the absence of an observed target value.

The forward structural image of template θ in case c is defined in Equation 171.

$$\mathcal{S}_c(\theta) := \{D_c(h) : h \in \Gamma_c(\theta) \cap \Gamma_{D_c}\}. \quad (171)$$

The template is possibly compatible with case c when $\mathcal{S}_c(\theta) \cap \mathcal{O}_c \neq \emptyset$. Define the case-compatible template set in Equation 172.

$$\Theta_c^{\text{pos}} := \{\theta \in \Theta^* : \mathcal{S}_c(\theta) \cap \mathcal{O}_c \neq \emptyset\}. \quad (172)$$

For a case set $C \subseteq \mathcal{C}$, define the jointly compatible template set in Equation 173.

$$\Theta_C^{\text{pos}} := \bigcap_{c \in C} \Theta_c^{\text{pos}}. \quad (173)$$

Under Equation 170, Equation 173 retains exactly the templates not structurally refuted by any screened case and able to realize every case-specific target, possibly through different case-specific generative specifications.

Adding cases cannot enlarge the jointly compatible template class.

Proposition 10.7 (Comparative monotonicity of structural screening). *If $C_1 \subseteq C_2 \subseteq \mathcal{C}$, then Equation 174 holds.*

$$\Theta_{C_2}^{\text{pos}} \subseteq \Theta_{C_1}^{\text{pos}}. \quad (174)$$

Proof. Equation 173 defines $\Theta_{C_2}^{\text{pos}}$ as an intersection over every case in C_2 . Since $C_1 \subseteq C_2$, all constraints used in $\Theta_{C_1}^{\text{pos}}$ also occur in the larger intersection. Adding intersections can only preserve or reduce the resulting set. $\square$

Proposition 10.7 gives the formal sense in which comparative cases can eliminate explanatory templates. It does not establish that a surviving template is causally correct. Survival means only that the template has at least one realization compatible with each declared structural target.

A recurrent comparative motif from Section 9.4 can be used as a shared target, but the logical direction remains one-way. If several cases robustly identify the same motif value v , a template that cannot realize v in one of those cases is incompatible with the motif recurrence. A template that can realize v in all cases remains only structurally possible. Common trajectory structure therefore narrows a candidate explanatory class without collapsing descriptive recurrence into a claim of common causal production.

The next section positions this inverse use relative to process tracing, sequence-sensitive analysis, path dependence, and comparative-historical interpretation. Those methods can supply causal and contextual evidence that a trajectory descriptor alone does not provide, while the present formalism supplies a disciplined way to state which structural consequences a candidate explanation must be able to realize.

11 Methodological Interfaces with Historical Social Science

This section locates the proposed trajectory formalism relative to established modes of historical social inquiry. Its role is methodological rather than competitive: braid-theoretic trajectory description supplies a formal account of selected relational structure in reconstructed histories, while process tracing,

sequence-sensitive analysis, path-dependence analysis, and comparative-historical interpretation address causal, temporal, and contextual questions that the braid descriptor does not settle by itself. The objective is to state the interfaces precisely enough that the same evidence is not counted twice, sequence sensitivity is not confused with full event-sequence preservation, and recurrent topological motifs are not promoted into causal explanations. Section 11.1 treats process tracing. Section 11.2 distinguishes event sequences from braid words and braid classes. Section 11.3 states the relation to path dependence. Section 11.4 places the formalism within comparative-historical interpretation. Section 11.5 consolidates the distinct inferential roles.

11.1 Process Tracing

Process tracing is centrally concerned with careful within-case description, diagnostic evidence, and the temporal ordering of observations used to evaluate causal claims [2]. The present framework intersects with that practice at the level of reconstructed process structure, but it does not supply a substitute for causal-process evidence.

To make the interface explicit, distinguish two evidential roles. Let Y_R denote the record material used to construct a trajectory reconstruction and let Y_C denote evidence used to evaluate a candidate causal hypothesis. The two collections can overlap, but they need not be identical. Under the notation of Sections 4–6, the reconstruction side produces $\mathcal{R}_A(Y_R)$ and, when the relevant admissibility conditions hold, a topological target such as $\mathcal{B}_A(Y_R; c_0)$ or a derived invariant-value set.

Let $\text{PT}(H; Y_C)$ denote the process-tracing assessment of a candidate hypothesis H relative to the causal evidence Y_C . No universal codomain for PT is imposed here because process-tracing designs can use different evidential tests and claim types. The important structural point is that the two evidential routes in Equation 175 have different inferential roles.

$$Y_R \mapsto \mathcal{R}_A(Y_R) \mapsto \mathcal{O}_D(Y_R, A), \quad (H, Y_C) \mapsto \text{PT}(H; Y_C). \quad (175)$$

The left side of Equation 175 constructs a structural target from reconstructed history. The right side evaluates a causal hypothesis through diagnostic evidence and within-case reasoning.

Section 10.3 already established a non-double-counting result. When a hypothesis has been required to reproduce a complete history in $\mathcal{R}_A(Y_R)$, a descriptor derived from that same history adds no independent evidential content merely by being algebraically different from the history representation. The same caution applies at the process-tracing interface. If Y_C contains no evidential information beyond what was already used to construct $\mathcal{R}_A(Y_R)$, the braid or invariant should be described as a structural summary or model-checking target rather than as a second independent causal test.

The interface becomes methodologically stronger when the two roles are genuinely complementary. Process-tracing evidence can support or weaken claims about a mechanism that is only one member of the structurally compatible hypothesis class from Section 10. Conversely, a robustly identified trajectory structure can exclude a candidate mechanism whose forward image cannot realize the observed structural target before more fine-grained causal evidence is considered. The two methods then constrain different parts of the same inquiry.

11.2 Sequence-Sensitive Analysis

Sequence-sensitive social analysis treats temporal order and event organization as analytically consequential rather than as incidental ordering of otherwise exchangeable observations [1]. Braid-theoretic trajectory description belongs to this broad family of order-sensitive representations, but its retained information is more selective than a raw event sequence.

Let $s = (e_1, \dots, e_m)$ be a temporally ordered sequence of coded events admitted by a reconstruction. A domain-specific crossing-coding rule can assign a braid word $W(s) \in \Sigma_n^*$, where Σ_n^* is the free monoid on the Artin crossing alphabet defined in the companion formal paper [6]. The braid element associated with the coded sequence is stated in Equation 176.

$$\Phi_B(s) := \text{ev}_B(W(s)) \in B_n. \quad (176)$$

Equation 176 exhibits three distinct objects: the historical event sequence s , the braid word $W(s)$, and the braid element $\Phi_B(s)$. They should not be identified.

The passage from word to braid element deliberately quotients some word-level differences through the Artin relations. For example, crossings involving sufficiently separated adjacent positions commute. Under the ordinary braid presentation, Equation 177 holds when $|i - j| \geq 2$.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2. \quad (177)$$

Thus two coded event sequences whose only difference is the order of independent distant crossings can yield the same braid element. Braid description is therefore order-sensitive without preserving every local ordering distinction.

The adjacent braid relation introduces another controlled equivalence. Equation 178 states the standard relation.

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \quad (178)$$

Equation 178 shows that even certain different three-step crossing words represent the same topological braid. A braid class therefore records an equivalence class of ordered interactions at a selected topological resolution rather than a verbatim event chronology.

This distinction creates a useful methodological choice. A study concerned with the exact ordering of every coded event should retain the event sequence or braid word alongside the braid element. A study concerned with topological organization under admissible deformation can use the braid class as a more robust quotient. The two representations answer different questions.

The sequence-to-braid map in Equation 176 can also be many-to-one because the crossing-coding rule W itself can discard noncrossing events, semantic content, durations, or interactions that do not alter the selected relational ordering. Sequence analysis and braid analysis should therefore be treated as complementary resolutions rather than as interchangeable techniques.

11.3 Path Dependence

Historical-sociological accounts of path dependence treat some sequences as causally consequential because earlier events condition later possibilities and because mechanisms of reproduction, reaction, or institutional persistence can make temporal order matter substantively [7]. A nontrivial braid records path structure, but it does not by itself establish path dependence in this causal sense.

The distinction can be expressed using the hypothesis-level structural equivalence from Equation 165. Let $PD : \mathfrak{H}^* \rightarrow \{0, 1\}$ be a declared predicate indicating whether a candidate explanatory hypothesis satisfies the substantive path-dependence criterion used by the study. The predicate is external to the braid descriptor and must be justified by the historical theory of the domain.

Suppose two hypotheses $H_1, H_2 \in \mathfrak{H}^*$ satisfy the conditions in Equation 179.

$$H_1 \equiv_D^{\text{hyp}} H_2, \quad PD(H_1) \neq PD(H_2). \quad (179)$$

Proposition 10.4 then implies that every target stated solely through D treats the two hypotheses identically with respect to possible structural compatibility. The descriptor alone cannot decide which path-dependence status in Equation 179 is correct.

This yields a general methodological conclusion. Topological path-history and causal path dependence occupy different analytical levels. A braid can record that relational positions have intertwined in a historically ordered way. A path-dependence claim additionally concerns how earlier events altered later reachable possibilities or sustained a process through a specified causal mechanism.

A trajectory descriptor can nevertheless contribute to path-dependence analysis. It can provide a formal index of the realized path whose causal significance is then investigated; it can identify structurally distinct histories that end in similar observed conditions; and it can serve as a forward prediction that a proposed path-dependent mechanism should be capable of reproducing. These uses remain supplementary to the causal argument.

11.4 Comparative-Historical Interpretation

Comparative historical analysis is an established mode of social-scientific inquiry with a long tradition of combining historical evidence and cross-case comparison [8]. The trajectory framework enters that tradition at an intermediate descriptive level. It provides explicit objects for comparing reconstructed relational structure while leaving substantive case interpretation and causal explanation to the broader comparative design.

Section 9 formalized several levels of comparison. Direct braid equality requires a justified common braid index, strand correspondence, crossing convention, and endpoint convention. Derived comparative descriptors allow cases with different full braid descriptions to be compared through a shared codomain. Recurrent motifs record repeated structural values across cases.

These formal comparisons require contextual interpretation. The same braid element can involve different historical entities, institutions, or domain meanings. Conversely, two cases that are substantively interpreted as variants of one broader process can receive different braid descriptions because their relational charts, strand registries, or realized trajectories differ. Comparative interpretation therefore supplies the domain-specific warrant for deciding which formal correspondences are meaningful.

A useful comparative claim can be stated at several strengths:

1. the same structural value is *compatible* with more than one case;
2. the same value is *identified* in more than one case;
3. the same value is *robustly identified* across declared perturbation classes;
4. one explanatory template can realize the case-specific structural targets in every compared case.

The fourth claim is stronger than the first three because it introduces a cross-case causal model class. Even then, Section 10.4 shows that template survival is a compatibility result rather than unique causal identification.

The formalism is therefore most useful when comparative interpretation treats a braid or invariant as a controlled intermediate representation: sufficiently abstract to permit cross-case structural comparison, yet explicitly downstream from the evidence, reconstruction assumptions, and domain semantics that give the comparison historical meaning.

11.5 Distinct Inferential Roles

The methodological interfaces developed above can be summarized by separating five inferential roles. Table 9 states the primary question addressed by each role and the contribution made by trajectory description.

Table 9 supports a non-substitution principle. Identification at one inferential role does not generally imply identification at another. A braid can be identified while the causal mechanism remains underdetermined. A process-tracing argument can favor a mechanism while several braid reconstructions remain possible. A path-dependent process can share a braid class with a process generated by a different causal structure. A recurrent comparative motif can occur without a common generator.

The relations among the roles are therefore better represented as constrained interfaces than as a hierarchy in which one method replaces the others. Historical reconstruction supplies admissible trajectories; trajectory description supplies selected structural equivalence classes; causal methods evaluate mechanisms; comparative interpretation determines which formal correspondences are substantively meaningful. The inverse framework of Section 10 connects the descriptive and causal levels by eliminating generators unable to realize an observed structural target, while preserving the distinction between nonrefutation and causal confirmation.

The next section turns from methodological complementarity to interpretive discipline. It examines how domain semantics enter the reconstruction, why topological objects should not be reified as social entities, how missing records and selection processes can bias the reconstruction set, and how changing social ontologies can challenge fixed strand identities and relational charts.

Table 9: Distinct inferential roles at the interface between braid-theoretic trajectory description and historical social science.

Inferential role	Primary question	Relation to trajectory description
Historical reconstruction	Which histories remain compatible with the interpreted record and declared assumptions?	Supplies the admissible history and witness sets on which any trajectory descriptor must operate
Sequence-sensitive description	Which aspects of temporal and relational ordering are retained?	Braid classes preserve selected intertwining structure while quotienting some word-level and geometric differences
Process tracing	Which diagnostic evidence supports or weakens a causal-process claim?	Can use trajectory structure as a constraint or summary but supplies causal evidence not contained in braid topology itself
Path-dependence analysis	How do earlier events alter later possibilities or sustain a historical process?	Can use the realized braid path as a structural index, while causal dependence requires additional generator-level or historical argument
Comparative-historical interpretation	Which similarities and differences across cases are historically meaningful?	Uses transported braids, invariants, or motifs as intermediate comparison objects under explicit contextual protocols

12 Interpretive Discipline in Historical Reconstruction

This section states the interpretive constraints required when a topological trajectory descriptor is used on historical social material. Its role is to prevent a formally correct braid construction from acquiring substantive meanings that were not supplied by the historical domain, to separate missing evidence from demonstrated historical absence, and to make visible cases in which the social ontology used by the reconstruction changes during the period under study. The objective is therefore disciplinary rather than expansionary: the section restricts what can be inferred from the formalism and identifies where additional historical interpretation is required. Section 12.1 formalizes the dependence of substantive meaning on domain interpretation. Section 12.2 gives a factorization criterion for claims that are genuinely determined by a braid description. Section 12.3 treats evidential omission and record-selection sensitivity. Section 12.4 develops a regime-based treatment of changing actor categories, strand identities, and relational coordinate systems.

12.1 Domain-Derived Semantics

Section 5.4 introduced the interpretation relation $\text{Sem}_{Y'}^{h,w}$ to keep algebraic crossing data separate from substantive historical meaning. The present subsection extends that separation from individual crossings to complete topological descriptions.

Let $\tilde{\mathcal{R}}_A^B(Y; c_0)$ be the standard braid-admissible reconstruction-pair set of Equation 75. Let $\mathcal{S}_Y^{\text{hist}}$ be a set of historically justified interpretive statements admitted by the study. Examples can concern institutional succession, alliance reconfiguration, role displacement, recognition, bargaining position, or other domain-specific meanings. No universal semantics are imposed on $\mathcal{S}_Y^{\text{hist}}$.

Define the reconstruction-level interpretation relation in Equation 180.

$$\text{Int}_{Y,A} \subseteq \tilde{\mathcal{R}}_A^B(Y; c_0) \times \mathcal{S}_Y^{\text{hist}}. \quad (180)$$

Membership $((h, w), s) \in \text{Int}_{Y,A}$ means that the historical interpretation s has been justified for reconstruction pair (h, w) using domain evidence and argument. Equation 180 does not derive s from the braid element.

For a reconstruction pair (h, w) , define its admitted interpretation set by Equation 181.

$$\mathcal{S}_{Y,A}(h, w) := \left\{ s \in \mathcal{S}_Y^{\text{hist}} : ((h, w), s) \in \text{Int}_{Y,A} \right\}. \quad (181)$$

The evidence-compatible interpretation set is then defined in Equation 182.

$$\mathcal{S}_{Y,A}^{\text{comp}} := \bigcup_{(h,w) \in \tilde{\mathcal{R}}_A^B(Y; c_0)} \mathcal{S}_{Y,A}(h, w). \quad (182)$$

Equation 182 preserves interpretive plurality across admissible reconstructions. A substantive statement can be supported for some reconstruction pairs and unsupported for others even when all pairs share one braid class. Accordingly, braid identifiability does not by itself imply semantic identifiability.

Definition 12.1 (Reconstruction-stable interpretation). A historical interpretation s is *reconstruction-stable* under the current regime when the standard braid-admissible reconstruction set is nonempty and Equation 183 holds.

$$\tilde{\mathcal{R}}_A^B(Y; c_0) \neq \emptyset, \quad s \in \bigcap_{(h,w) \in \tilde{\mathcal{R}}_A^B(Y; c_0)} \mathcal{S}_{Y,A}(h, w). \quad (183)$$

Definition 12.1 is deliberately stronger than membership in the union of Equation 182: the interpretation must remain justified for every compatible braid-admissible reconstruction pair, and semantic stability is not declared vacuously when no such pair exists.

The same requirement can be applied to individual crossing meanings. A stable crossing interpretation requires not only an identified braid class but also a justified correspondence among the relevant crossing occurrences across reconstruction pairs. Equality of Artin letters alone does not provide that correspondence because the positional index $i(c)$ does not identify the historical entities involved in the crossing.

12.2 Historical Attribution from Topological Representation

A braid element is a descriptive object produced by the extraction chain of Equation 71. It should therefore support only those historical claims that are invariant across every reconstruction pair mapped to the same braid value.

Let X_B be any declared set of standard braid-admissible reconstruction pairs and let $\beta : X_B \rightarrow B_n$ be the braid descriptor on that set. Let $P : X_B \rightarrow \Omega_P$ be a proposed historical property or interpretive classification. The codomain Ω_P can contain qualitative labels, institutional states, causal classifications, or any other declared property values.

Definition 12.2 (Braid-determined historical property). The property P is *braid-determined* on X_B when there exists a map $\bar{P} : \beta(X_B) \rightarrow \Omega_P$ satisfying Equation 184.

$$P = \bar{P} \circ \beta \quad \text{on } X_B. \quad (184)$$

Definition 12.2 states that the property can be read from the braid value alone on the declared reconstruction domain.

Proposition 12.3 (Interpretive factorization criterion). *The property P is braid-determined on X_B if and only if Equation 185 holds.*

$$\beta(x_1) = \beta(x_2) \implies P(x_1) = P(x_2) \quad \text{for every } x_1, x_2 \in X_B. \quad (185)$$

Proof. Assume first that Equation 184 holds. If $\beta(x_1) = \beta(x_2)$, then applying $\bar{P}$ to the common braid value gives $P(x_1) = P(x_2)$. Thus Equation 185 holds.

Conversely, assume Equation 185. For $b \in \beta(X_B)$, choose any $x \in X_B$ with $\beta(x) = b$ and define $\bar{P}(b) := P(x)$. The fiber-constancy condition makes this definition independent of the chosen representative. Equation 184 then follows. $\square$

Proposition 12.3 gives a formal non-reification test. A proposed historical meaning can be attributed to the braid value itself only when the meaning is constant on the relevant braid fibers. If two reconstruction pairs share one braid but differ in the proposed property, the property belongs to information outside the braid descriptor.

The criterion applies directly to causal and normative language. If one braid fiber contains reconstructions with different mechanisms, institutional meanings, power relations, normative evaluations, or actor interpretations, none of those properties can be inferred from the braid class alone. Additional domain evidence can still justify the property for a particular reconstruction or case. The inferential source then lies in that additional evidence and interpretation rather than in the topology by itself.

The fact-regime architecture of Section 2 adds a second non-reification requirement. A historical-event fact and a social-recognition fact should not be identified merely because the latter concerns the former. A person can be historically prior in the formulation of a concept while later attribution, citation, indexing, institutional recognition, or educational transmission fails to stabilize the corresponding social-recognition fact. Governance can therefore alter fact-formation conditions without altering the underlying historical event. The present paper treats this as a distinction among fact configurations and manifestation mechanisms rather than as evidence that the historical fact itself has changed.

The same caution applies to derived invariants. If $I : B_n \rightarrow \Omega_I$ is a coarsening of braid description, any property determined by $I \circ \beta$ must be constant on the larger fibers induced by the invariant. Coarser descriptors therefore support a narrower class of descriptor-determined substantive claims.

12.3 Selection Sensitivity under Incomplete Records

The reconstruction formalism treats the available record as an input. Historical archives and datasets can nevertheless omit events, actors, interactions, or failed alternatives. The absence of an item from the observed record should therefore be distinguished from evidence that the corresponding historical event did not occur.

A first audit can be performed entirely within the observed record. Let $\mathcal{J}(Y)$ be the record-item index set from Section 4.1. For a subset $J \subseteq \mathcal{J}(Y)$, let $Y|_J$ denote the reduced record obtained by retaining only items indexed by J , while holding the interpretation rules for the retained items fixed. Let $\mathcal{R}_{A,J}(Y)$ be the reconstruction set generated from that reduced record together with the same auxiliary assumption set A .

Removing record constraints can only preserve or enlarge the admissible reconstruction set.

Proposition 12.4 (Monotonicity under record deletion). *If $J_1 \subseteq J_2 \subseteq \mathcal{J}(Y)$, then Equation 186 holds.*

$$\mathcal{R}_{A,J_2}(Y) \subseteq \mathcal{R}_{A,J_1}(Y). \quad (186)$$

Proof. Every record-content constraint imposed by J_1 also appears among the constraints imposed by J_2 . The temporal, identity, role, and auxiliary assumptions are held fixed. Hence every history compatible with the larger record subset J_2 also satisfies the constraints associated with J_1 . $\square$

Proposition 12.4 supports a leave-one-record-out audit. The procedure parallels the assumption audit in Section 4.5. A topological conclusion that disappears after one evidential item is removed depends materially on that item, even when the item itself is historically credible.

Record deletion does not model genuinely unobserved evidence because the omitted item is already known to the analyst. To represent uncertainty about archival incompleteness, let $\mathfrak{C}_{\text{evid}}(Y)$ be a declared family of admissible evidence completions. Each $Y^+ \in \mathfrak{C}_{\text{evid}}(Y)$ contains the observed record together

with additional hypothetical or externally recovered items considered possible under the study's archival assumptions.

For each completion Y^+ , let $\mathcal{D}_A(Y^+)$ denote its compatible descriptor set. Equation 187 collects these sets across the completion family.

$$\mathcal{D}_A^{\text{comp}}(Y; \mathfrak{C}_{\text{evid}}) := \bigcup_{Y^+ \in \mathfrak{C}_{\text{evid}}(Y)} \mathcal{D}_A(Y^+). \quad (187)$$

Definition 12.5 (Completion-robust descriptive conclusion). A descriptor value ω is *completion-robust* relative to $\mathfrak{C}_{\text{evid}}(Y)$ when every admissible completion has a nonempty descriptor-admissible reconstruction and Equation 188 holds.

$$\mathcal{D}_A^{\text{comp}}(Y; \mathfrak{C}_{\text{evid}}) = \{\omega\}. \quad (188)$$

Definition 12.5 is conditional on the declared completion family. It does not prove that unknown archival material could not overturn the conclusion; it states that the conclusion survives the specified missing-record scenarios.

Missingness should also be distinguished from selection bias. A record-selection process is potentially biasing for the present analysis when the mechanism determining which events or actors become observable depends on historical features that also affect the trajectory descriptor or the reconstruction constraints. The current formalism does not infer such dependence from missingness alone. Instead, possible selection processes should be introduced as additional reconstruction regimes or completion families and tested for their effect on the descriptor set.

This distinction preserves the epistemic status of absence. An event absent from Y can be coded as historically absent only when the evidence and source criticism warrant that stronger statement. Otherwise, the formalism should represent the event as unresolved or include alternative completions in $\mathfrak{C}_{\text{evid}}(Y)$.

12.4 Reflexive Treatment of Changing Social Ontologies

The standard braid construction assumes a fixed strand registry, persistent identities, and one relational representation scheme over the descriptive interval. Historical social systems can violate these assumptions because actor categories, institutional boundaries, offices, social roles, and the coordinates used to describe relational position can themselves change.

To represent such changes explicitly, partition the historical interval into r descriptive regimes with boundaries $0 = t_0 < t_1 < \dots < t_r = 1$. For regime $j \in \{1, \dots, r\}$, define the local representational specification in Equation 189.

$$\mathfrak{D}_j := (\tau_j, \mathcal{A}_j^{\text{str}}, \mathcal{Q}_j, \mathcal{Z}_j, \chi_j), \quad (189)$$

where au_j is the fact taxonomy used to articulate the research objects in regime j , $\mathcal{A}_j^{\text{str}}$ is the strand registry, $\mathcal{Q}_j$ the local role vocabulary, $\mathcal{Z}_j$ the relational-position domain, and χ_j the braid chart used during that regime.

A transition between adjacent regimes requires an explicit lineage relation $\mathcal{L}_{j+1,j}$ defined in Equation 190.

$$\mathcal{L}_{j+1,j} \subseteq \mathcal{A}_j^{\text{str}} \times \mathcal{A}_{j+1}^{\text{str}}. \quad (190)$$

Membership $(a, b) \in \mathcal{L}_{j+1,j}$ means that strand label b in the later regime is treated as a justified continuation of strand label a in the earlier regime.

Ordinary fixed- n braid continuation across the boundary requires more than the existence of some lineage relation.

Proposition 12.6 (Fixed-strand continuation criterion across ontological regimes). A standard fixed-strand braid representation can continue across the boundary between regimes j and $j+1$ only if $\mathcal{L}_{j+1,j}$ is the graph of a bijection $\lambda_{j+1,j} : \mathcal{A}_j^{\text{str}} \rightarrow \mathcal{A}_{j+1}^{\text{str}}$. In particular, Equation 191 is necessary.

$$|\mathcal{A}_j^{\text{str}}| = |\mathcal{A}_{j+1}^{\text{str}}|. \quad (191)$$

Table 10: Interpretive and evidential disciplines for braid-theoretic historical reconstruction.

Discipline	Formal test or object	Claim protected against
Domain-derived semantics	$\text{Int}_{Y,A}$ and reconstruction-stable interpretation in Equation 183	Reading social meaning directly from Artin symbols or braid classes
Non-reification	Factorization criterion in Proposition 12.3	Attributing to topology a property that varies inside one braid fiber
Record sensitivity	Record-deletion monotonicity in Proposition 12.4	Treating one selected evidential corpus as if it were complete by construction
Missing-record sensitivity	Completion family $\mathfrak{C}_{\text{evid}}(Y)$ and Equation 188	Treating unobserved material as incapable of altering the descriptor conclusion
Ontological regime change	$\mathfrak{D}_j$, $L_{j+1,j}$, and Proposition 12.6	Forcing changing actors, roles, or coordinate systems into one fixed strand ontology

Proof. A standard braid strand has one persistent identity throughout the interval. Continuation across the regime boundary therefore requires every earlier strand to have exactly one later continuation and every later strand to descend from exactly one earlier strand. This is precisely the condition that $L_{j+1,j}$ be the graph of a bijection. A bijection can exist only when the two registries have equal cardinality, giving Equation 191. $\square$

Proposition 12.6 does not state that social continuity is impossible when the condition fails. It states that an ordinary fixed-strand braid is no longer an adequate representation of that continuity. Birth, merger, fission, institutional succession with ambiguous lineage, and newly constituted actor categories require a richer inter-sector or categorical description.

Even a bijective lineage relation is insufficient when the relational-position domains or charts change incompatibly. Suppose there is a declared transport map $\psi_{j+1,j} : \mathcal{Z}_j \rightarrow \mathcal{Z}_{j+1}$. Chart compatibility across the boundary can be stated through the existence of a planar map $\Psi_{j+1,j} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying Equation 192.

$$\chi_{j+1} \circ \psi_{j+1,j} = \Psi_{j+1,j} \circ \chi_j \quad \text{on the declared transition domain.} \quad (192)$$

Equation 192 makes comparability of the two relational charts an explicit assumption. Without a justified transport, one continuous planar strand should not be created merely by plotting values from two historically different coordinate systems on the same axes.

Reflexivity enters because the analytical categories used by the researcher can themselves be historically situated. A label such as an institutional role, actor type, fact category, or relational coordinate can change meaning over the period under study or can be imposed retrospectively by the analyst. The regime family $\{\mathfrak{D}_j\}$ therefore records both changes in instantiated fact configurations and changes in the taxonomy or representation used to articulate them. Section `efsubsec:fact-configuration-taxonomy-change` distinguishes these two kinds of change explicitly.

The appropriate response to such change is not automatic reclassification into one fixed braid. When the transition conditions are justified, local braid segments can be transported and composed. When they are not justified, the analysis should retain separate regime-specific descriptions and represent the boundary itself as part of the historical object. This preserves ontological restraint and makes the cost of cross-regime continuity visible.

Table 10 summarizes the principal interpretive disciplines introduced in this section.

Table 10 locates the main discipline of the method: every topological claim remains conditional on the evidential reconstruction, representational choices, and semantic interpretation that make the descriptor meaningful. The next section develops worked examples in which these distinctions can be observed

directly, including cases of braid identification with historical ambiguity, partial invariant identification, sensitivity to omitted evidence, and failure of fixed-strand representation under changing social ontology.

13 Formal Worked Examples

This section gives synthetic worked examples of the principal inferential distinctions developed in Sections 6–12. Its role is diagnostic rather than empirical. The examples do not represent claims about any particular historical case; they show how the formal objects behave when reconstruction ambiguity, partial topological identification, evidential deletion, ontological regime change, and comparative explanatory screening are made explicit. Each example therefore specifies only the structure required for the methodological point at issue. Section 13.1 exhibits braid identification with nonunique historical reconstruction. Section 13.2 gives partial topological identifiability through exponent sum. Section 13.3 shows how an omitted record can destroy braid identification and how evidence completions expose that sensitivity. Section 13.4 gives a fixed-strand failure under a changing social ontology. Section 13.5 illustrates comparative elimination of candidate explanatory templates.

13.1 Braid Identification with Historical Ambiguity

Fix one evidential record Y , one auxiliary assumption set A , a two-strand registry, and a standard braid base configuration c_0 . Suppose the history-level reconstruction contains two distinct histories, as stated in Equation 193.

$$\mathcal{R}_A(Y) = \{h_1, h_2\}, \quad h_1 \neq h_2. \quad (193)$$

Assume each history has one compatible witness, denoted w_1 and w_2 , and that both pairs satisfy standard braid admissibility. The witness-augmented reconstruction is then given by Equation 194.

$$\tilde{\mathcal{R}}_A(Y) = \{(h_1, w_1), (h_2, w_2)\} \subseteq \tilde{\mathcal{R}}_A^B(Y; c_0). \quad (194)$$

Suppose the relational trajectories differ in geometric or non-topological historical detail but produce the same braid element in B_2 . Let their braid values be specified by Equation 195.

$$\beta_A(h_1, w_1) = \beta_A(h_2, w_2) = \sigma_1^2. \quad (195)$$

Equation 79 therefore gives the singleton braid set in Equation 196.

$$\mathcal{B}_A(Y; c_0) = \{\sigma_1^2\}, \quad \left| \tilde{\mathcal{R}}_A(Y) \right| = 2. \quad (196)$$

The first relation in Equation 196, together with complete standard braid admissibility in Equation 194, satisfies Definition 6.2. The second relation shows that the witness-augmented reconstruction remains nonidentified. This is a concrete instance of Proposition 6.4.

Generator uncertainty can remain larger still. Suppose there are two candidate generative specifications u_1 and u_2 satisfying Equation 197.

$$\text{Gen}(u_1) = h_1, \quad \text{Gen}(u_2) = h_2, \quad u_1 \neq u_2. \quad (197)$$

The evidence can therefore identify one braid while leaving both the reconstructed history and the candidate generator nonunique. The example instantiates the methodological claim that a selected trajectory structure can be identified at a coarser resolution than either the complete history or its generative explanation.

13.2 Partial Topological Identifiability through Exponent Sum

The second example shows that a derived invariant can be identified while the full braid remains non-identified. Consider a three-strand reconstruction regime with two compatible witness-augmented reconstructions x_1 and x_2 . Let their braid values be the elements of B_3 defined in Equation 198.

$$\beta_1 := \sigma_1 \sigma_2, \quad \beta_2 := \sigma_2 \sigma_1. \quad (198)$$

Assume the pair-level descriptor of Equation 78 assigns β_1 to reconstruction pair x_1 and β_2 to reconstruction pair x_2 .

Using the standard permutation homomorphism $\pi_{\text{perm}} : B_3 \rightarrow S_3$ recalled in the companion formal paper, the two braids have different permutation images, as stated in Equation 199 [6].

$$\pi_{\text{perm}}(\beta_1) = s_1 s_2, \quad \pi_{\text{perm}}(\beta_2) = s_2 s_1, \quad s_1 s_2 \neq s_2 s_1. \quad (199)$$

Equation 199 implies $\beta_1 \neq \beta_2$. The evidence-compatible braid set is therefore the two-element set in Equation 200.

$$\mathcal{B}_A(Y; c_0) = \{\beta_1, \beta_2\}. \quad (200)$$

Now let $e : B_3 \rightarrow \mathbb{Z}$ be the exponent-sum homomorphism. Equation 201 follows directly from the two words in Equation 198.

$$e(\beta_1) = e(\beta_2) = 2. \quad (201)$$

The evidence-compatible exponent-sum set is consequently given by Equation 202.

$$\mathcal{I}_A^e(Y; c_0) = \{2\}. \quad (202)$$

Equations 200 and 202 satisfy Definition 6.8, assuming complete standard braid admissibility. The correct conclusion is that exponent sum is identified while the complete braid is not. Promoting Equation 202 to a claim that one braid has been reconstructed would exceed the evidential resolution.

The example also shows why the choice of invariant matters. The permutation map distinguishes the two compatible braids while the exponent-sum map does not. Different downstream descriptors therefore induce different identification resolutions on the same historical reconstruction set.

13.3 Missing-Evidence Sensitivity under Record Deletion

The third example makes record sensitivity explicit. Let the interpreted record contain two items, r_1 and r_2 , and let the full and reduced record-index sets be defined in Equation 203.

$$J_{\text{full}} := \{r_1, r_2\}, \quad J_{\text{red}} := \{r_1\}. \quad (203)$$

Suppose the corresponding reconstruction sets satisfy Equation 204.

$$\mathcal{R}_{A, J_{\text{full}}}(Y) = \{h_+\}, \quad \mathcal{R}_{A, J_{\text{red}}}(Y) = \{h_+, h_-\}. \quad (204)$$

Equation 204 is consistent with Proposition 12.4: deleting the constraint associated with r_2 enlarges the admissible reconstruction set.

Assume that both reconstructed histories admit standard two-strand braid descriptions and that the corresponding braid values are specified by Equation 205.

$$\beta(h_+) = \sigma_1, \quad \beta(h_-) = \sigma_1^{-1}. \quad (205)$$

The full and reduced braid sets are then stated in Equation 206.

$$\mathcal{B}_{\text{full}} = \{\sigma_1\}, \quad \mathcal{B}_{\text{red}} = \{\sigma_1, \sigma_1^{-1}\}. \quad (206)$$

The full record identifies one braid, while deletion of r_2 destroys that identification.

The same construction can be stated as a missing-evidence problem. Treat the reduced record as the observed record Y and let the admissible completion family contain two completions, Y^+ and Y^- , that resolve the missing item in opposite ways. Suppose their descriptor sets are defined in Equation 207.

$$\mathcal{D}_A(Y^+) = \{\sigma_1\}, \quad \mathcal{D}_A(Y^-) = \{\sigma_1^{-1}\}. \quad (207)$$

The completion aggregate from Equation 187 is therefore given by Equation 208.

$$\mathcal{D}_A^{\text{comp}}(Y; \{Y^+, Y^-\}) = \{\sigma_1, \sigma_1^{-1}\}. \quad (208)$$

Equation 208 fails the singleton requirement in Definition 12.5. The observed absence of r_2 therefore cannot be treated as evidence that either orientation is historically correct. The example separates archival nonobservation from historical absence and shows how an apparently identified conclusion can depend on one record item.

13.4 Ontological Regime Change across Strand Sectors

The fourth example illustrates a representational failure caused by changing social ontology rather than by ordinary positional uncertainty. Consider two consecutive historical regimes with strand registries defined in Equation 209.

$$\mathcal{A}_1^{\text{str}} := \{a, b\}, \quad \mathcal{A}_2^{\text{str}} := \{a, b_1, b_2\}. \quad (209)$$

Interpret the transition as persistence of strand a together with a historically justified fission of the earlier entity represented by b into two later entities represented by b_1 and b_2 . The lineage relation is defined in Equation 210.

$$\mathcal{L}_{2,1} := \{(a, a), (b, b_1), (b, b_2)\}. \quad (210)$$

The lineage relation in Equation 210 is not the graph of a function from $\mathcal{A}_1^{\text{str}}$ to $\mathcal{A}_2^{\text{str}}$ because b has two successors. The registry cardinalities also satisfy Equation 211.

$$|\mathcal{A}_1^{\text{str}}| = 2, \quad |\mathcal{A}_2^{\text{str}}| = 3. \quad (211)$$

By Proposition 12.6, no ordinary fixed- n braid can continue across this boundary while faithfully representing the declared lineage.

The failure does not imply that either local regime is topologically undescribable. Suppose the first regime admits a local path $q_1 : [0, 1] \rightarrow \mathcal{U}_2$ and the second admits a local path $q_2 : [0, 1] \rightarrow \mathcal{U}_3$. Their codomains are stated in Equation 212.

$$q_1([0, 1]) \subseteq \mathcal{U}_2, \quad q_2([0, 1]) \subseteq \mathcal{U}_3. \quad (212)$$

Equation 212 supports separate regime-specific trajectory descriptions. What fails is the attempt to concatenate the two regimes into one ordinary braid with one persistent strand registry. A richer inter-sector or categorical description is required if the fission event itself is part of the historical object.

This example also marks a limit of robustness analysis. No arbitrarily small Euclidean perturbation of a two-strand path can create a third persistent strand while remaining inside the same fixed configuration space. Ontological regime change is therefore not a large positional perturbation of an ordinary braid; it changes the representational domain.

13.5 Comparative Screening of Explanatory Templates

The final example illustrates how a recurrent structural target can eliminate explanatory templates without uniquely identifying a cause. Consider two historical cases, c and d , and let the comparative descriptor be the exponent-sum value. Suppose both cases robustly identify the same target value, as stated in Equation 213.

$$\mathcal{O}_c = \mathcal{O}_d = \{2\}. \quad (213)$$

Let the declared explanatory-template class be $\Theta^* = \{\theta_1, \theta_2, \theta_3\}$, and assume descriptor completeness for these templates in both cases as required by Equation 170.

Suppose the case-specific structural images are defined by Equation 214.

$$\begin{aligned}\mathcal{S}_c(\theta_1) &= \{2\}, & \mathcal{S}_d(\theta_1) &= \{2, 4\}, \\ \mathcal{S}_c(\theta_2) &= \{2\}, & \mathcal{S}_d(\theta_2) &= \{0\}, \\ \mathcal{S}_c(\theta_3) &= \{0\}, & \mathcal{S}_d(\theta_3) &= \{2\}.\end{aligned}\tag{214}$$

Equation 172 then gives the two case-level compatible template sets in Equation 215.

$$\Theta_c^{\text{pos}} = \{\theta_1, \theta_2\}, \quad \Theta_d^{\text{pos}} = \{\theta_1, \theta_3\}.\tag{215}$$

The jointly compatible set is therefore stated in Equation 216.

$$\Theta_{\{c,d\}}^{\text{pos}} = \Theta_c^{\text{pos}} \cap \Theta_d^{\text{pos}} = \{\theta_1\}.\tag{216}$$

The second case eliminates θ_2 , while the first eliminates θ_3 . This is an explicit instance of Proposition 10.7.

Equation 216 does not establish that θ_1 is the historically true causal explanation. The result is conditional on the declared template class, descriptor, forward realization sets, and structural targets. The surviving template is the only one in the declared class that can realize both case-level targets. Additional causal evidence or an expanded hypothesis class can change that conclusion.

13.6 Taxonomic Robustness across Alternative Fact Articulations

The sixth example shows how two admissible fact taxonomies can disagree about fine-grained fact articulation while supporting the same trajectory-level conclusion.

Let τ_1 and τ_2 be two admissible taxonomies for the same evidential record Y . Suppose their compatible fact-configuration sets differ as stated in Equation 217.

$$\mathcal{F}_{\tau_1,A}(Y) \neq \mathcal{F}_{\tau_2,A}(Y).\tag{217}$$

Assume, however, that both taxonomies construct two persistent research objects a and b and that every compatible reconstruction under either taxonomy yields the same one-exchange braid σ_1 . The case-specific braid sets are therefore given by Equation 218.

$$\mathcal{B}_{\tau_1,A}(Y) = \mathcal{B}_{\tau_2,A}(Y) = \{\sigma_1\}.\tag{218}$$

For the admissible taxonomy family $\mathfrak{T}_{\text{adm}} = \{\tau_1, \tau_2\}$, Equation 98 then gives Equation 219.

$$\mathcal{B}_{\mathfrak{T}}(Y) = \{\sigma_1\}.\tag{219}$$

Assuming nonempty reconstruction and complete braid admissibility for both taxonomies, Equation 219 satisfies Definition 7.2. The braid conclusion is therefore taxonomically robust even though the fine-grained fact reconstructions differ.

A contrasting case occurs when the second taxonomy constitutes the research objects differently. Suppose τ_2 treats one institutional object as two successor objects, producing a three-strand regime with no direct braid comparability to the two-strand regime under τ_1 . Then the taxonomy family fails the comparability assumption required by Definition 7.2. The correct conclusion is not that the braid has changed from one element to another; rather, the trajectory description is taxonomy-sensitive at the level of research-object constitution.

This example shows why taxonomic robustness is stronger than robustness to a small perturbation of coordinates. The uncertainty concerns what the social-scientific objects are and how they are factually articulated, not merely where fixed objects are placed in one chart.

Table 11: Formal worked examples and their methodological conclusions.

Example	Formal result	Methodological conclusion
Nonunique reconstruction, one braid	$ \tilde{\mathcal{R}}_A(Y) = 2$ and $ \mathcal{B}_A(Y; c_0) = 1$	Braid structure can be identified while the historical reconstruction and generator remain nonunique
Two braids, one exponent sum	$ \mathcal{B}_A(Y; c_0) = 2$ and $ \mathcal{I}_A^e(Y; c_0) = 1$	A derived invariant can be identified without identification of the full braid
Record deletion and evidence completion	Full record gives one braid; reduced record and alternative completions give two	Descriptor identification can depend materially on one evidential item and can fail completion robustness
Two-to-three strand ontology change	The lineage relation is non-bijective and registry cardinality changes	Local braid descriptions can remain valid while one fixed-strand representation across the boundary fails
Two-case explanatory screening	$\Theta_{\{c,d\}}^{\text{pos}} = \{\theta_1\}$ within the declared template class	Comparative structural constraints can eliminate candidate explanations without establishing unique causal truth

13.7 Inferential Summary of the Worked Examples

Table 11 consolidates the five synthetic examples and the inferential distinction each one isolates.

Table 11 shows that the formalism produces several distinct kinds of conclusion rather than one generic notion of “reconstruction.” Identification can occur at the braid level, invariant level, or comparative template level while remaining absent at finer historical or causal levels. Conversely, evidential deletion or ontological regime change can remove the conditions under which an ordinary braid conclusion is warranted. The next section collects the principal scope boundaries and failure modes of the methodology before the paper closes with its conclusion.

14 Scope Boundaries of Historical Trajectory Analysis

This section consolidates the principal conditions under which braid-theoretic historical trajectory analysis can produce a warranted conclusion and the conditions under which the analysis should retreat to a weaker descriptive level. Its role is diagnostic rather than defensive: a failure of reconstruction, braid admissibility, identifiability, robustness, or semantic stability is itself a methodological result. The objective is to prevent those distinct failures from being compressed into the binary judgment that the method either “works” or “fails.” Section 14.1 treats empty or inconsistent reconstruction sets. Section 14.2 treats descriptor-domain failure and changing representational ontology. Section 14.3 distinguishes non-identification from lower-resolution identification. Section 14.4 treats sensitivity to evidential and representational perturbation. Section 14.5 states semantic and causal limits. Section 14.6 combines these results into an explicit reporting profile.

14.1 Reconstruction Existence under Evidential Constraints

The first boundary occurs before topology enters. The reconstruction set $\mathcal{R}_A(Y)$ from Equation 47 can be empty because the interpreted records, temporal constraints, identity assignments, role constraints, auxiliary assumptions, or historical model cannot be jointly satisfied.

Define the reconstruction-failure set in Equation 220.

$$\mathfrak{F}_{\text{rec}} := \{(Y, A) : \mathcal{R}_A(Y) = \emptyset\}. \quad (220)$$

Membership in $\mathfrak{F}_{\text{rec}}$ does not identify which reconstruction component is responsible. Section 4.5 therefore treats emptiness as a diagnostic requiring examination of the record coding and assumption set.

An empty witness-augmented reconstruction can also occur even when a state-history sequence is present at a coarser level. Define the witness-failure set in Equation 221.

$$\mathfrak{F}_{\text{wit}} := \{(Y, A) : \mathcal{R}_A(Y) \neq \emptyset \text{ and } \tilde{\mathcal{R}}_A(Y) = \emptyset\}. \quad (221)$$

Equation 221 marks cases in which histories survive the record-level constraints but no coherent temporal and identity witness supports the strand construction required by the relational configuration-space model.

Neither failure in Equations 220–221 should be repaired by assigning an arbitrary trajectory merely to keep the braid analysis running. The appropriate output is that the declared reconstruction regime has not produced a coherent admissible historical object at the required level.

14.2 Descriptor-Domain Compatibility

A nonempty reconstruction set does not imply that the reconstructed histories belong to the braid domain. Section 5 separated configuration admissibility from standard based-loop braid admissibility. The corresponding exclusion sets are $\mathcal{E}_A^{\text{cfg}}(Y)$ and $\mathcal{E}_A^B(Y; c_0)$ from Equations 76 and 77.

Define complete configuration coverage by the two conditions in Equation 222.

$$\tilde{\mathcal{R}}_A(Y) \neq \emptyset, \quad \tilde{\mathcal{R}}_A(Y) \subseteq \tilde{\mathcal{R}}_A^{\text{cfg}}(Y). \quad (222)$$

Define complete standard braid coverage by the two conditions in Equation 223.

$$\tilde{\mathcal{R}}_A(Y) \neq \emptyset, \quad \tilde{\mathcal{R}}_A(Y) \subseteq \tilde{\mathcal{R}}_A^B(Y; c_0). \quad (223)$$

The nonemptiness clauses in Equations 222 and 223 prevent domain coverage from being declared vacuously when no coherent witness-augmented reconstruction exists. Standard braid coverage is stronger than configuration coverage because the standard braid descriptor additionally requires the declared endpoint/basepoint condition.

Several qualitatively different failures can violate these conditions:

1. the number of represented components changes;
2. historical identity cannot be traced continuously enough to define persistent strands;
3. the chosen interpolation creates or requires collisions;
4. the relational-position chart is undefined for some reconstructed states;
5. the initial and terminal configurations do not satisfy the based-loop convention;
6. an ontological regime change makes one fixed strand registry or coordinate chart historically inappropriate.

These are representational-domain problems, not merely noisy observations.

When Equation 222 holds but Equation 223 fails only because endpoints differ, Equation 82 supplies an endpoint-sensitive groupoid description. When configuration coverage itself fails, an ordinary configuration-space trajectory language is incomplete for at least one compatible reconstruction and no unconditional braid-level conclusion follows.

Proposition 12.6 gives a sharp domain boundary for ontology change. If lineage across adjacent regimes is not the graph of a bijection, ordinary fixed-strand continuation is unavailable. Section 13.4 gives a worked example in which local braid descriptions remain valid on each side while the transition itself requires a richer inter-sector representation.

14.3 Resolution Management under Incomplete Identification

Complete descriptor coverage still does not guarantee identification. Under complete standard braid admissibility, the compatible braid set $\mathcal{B}_A(Y; c_0)$ can contain several elements. This is a failure of full braid identification, not necessarily a failure of all topological description.

Define braid nonidentification in Equation 224.

$$|\mathcal{B}_A(Y; c_0)| > 1. \quad (224)$$

Section 6.4 showed that a derived map $I : B_n \rightarrow \Omega_I$ can nevertheless be identified when its evidence-compatible value set is a singleton.

This motivates an explicit resolution-retreat principle.

Corollary 14.1 (Resolution retreat under full braid nonidentification). *Assume complete standard braid coverage in Equation 223 and braid nonidentification in Equation 224. Let $I : B_n \rightarrow \Omega_I$ be a declared derived descriptor. If Equation 225 holds, then the warranted topological conclusion is the identified invariant value rather than a unique braid class.*

$$|\mathcal{I}_A(Y; c_0)| = 1. \quad (225)$$

Proof. Equation 224 rules out Definition 6.2. Equation 225, together with complete standard braid coverage, satisfies Definition 6.5. The available identification therefore occurs at the invariant resolution and not at the complete braid resolution. $\square$

Corollary 14.1 formalizes a general reporting discipline: state the strongest conclusion actually identified by the reconstruction, not the finest descriptor available in principle.

The same principle applies beyond one invariant. A joint descriptor can be identified while the braid is not, and an endpoint-sensitive groupoid class can remain available when the standard based-loop braid is undefined. These alternatives are not interchangeable, because they preserve different structures. The appropriate retreat depends on the question and on which descriptor remains both defined and identifiable.

A further boundary concerns excessive coarsening. An invariant that is always constant on the candidate braid set can be perfectly identified while carrying little discriminatory information. Identification alone therefore does not establish explanatory usefulness. Section 9.3 addresses this by making comparative resolution explicit rather than treating every singleton invariant as equally informative.

14.4 Robustness Sensitivity across Reconstruction Regimes

Identification under one reconstruction regime can fail under plausible alternative regimes. Section 7 therefore defined perturbation classes $\mathcal{U}(\eta_0)$ and robust identification through regimewise agreement.

A braid identified under the reference regime but not across the perturbation class should be reported as regime-sensitive. Define the regime-sensitive braid status in Equation 226.

$$|\mathcal{B}(\eta_0)| = 1, \quad |\mathcal{B}_{\mathcal{U}}| > 1. \quad (226)$$

Equation 226 means that one braid is identified under the reference commitments but alternative declared commitments generate more than one braid value.

A different sensitivity occurs when some perturbation regimes become inconsistent or leave the braid domain. The failure sets $\mathcal{E}_{\mathcal{U}}^{\varnothing}$ and $\mathcal{E}_{\mathcal{U}}^B$ from Equations 96 and 95 must therefore be reported separately from braid-value disagreement.

Metric robustness has an equally bounded scope. The separation-margin certificate protects one fixed strand interpretation against bounded positional perturbation. It does not test uncertainty in strand identity, event coding, chart selection, role interpretation, or missing records. A large geometric margin can coexist with serious evidential or semantic uncertainty.

The boundary-proximity result from Section 8 should also be read cautiously. A small distance to the collision boundary means that the selected topological classification is close to a topology-changing configuration under the chosen geometric representation. It does not imply political crisis, causal criticality, or imminent historical transition without additional substantive theory.

Robustness claims are therefore always indexed by an uncertainty class. Let $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$ be two declared perturbation classes. Agreement on $\mathfrak{U}_2$ is stronger than agreement on $\mathfrak{U}_1$. A conclusion described as robust should identify which evidential, temporal, positional, semantic, or representational variations were included and which remained fixed.

14.5 Semantic Scope of Causal Claims

The trajectory formalism is descriptive unless additional semantic and causal structure is supplied. Section 12 formalized this boundary through the interpretation relation $\text{Int}_{Y,A}$ and the factorization criterion of Proposition 12.3.

A property P that varies inside one braid fiber cannot be attributed to the braid class alone. This includes, unless separately shown to factor through the descriptor, claims about actor intention, normative legitimacy, coercion, institutional meaning, responsibility, or causal mechanism. A braid can organize relations among reconstructed trajectories while remaining silent about those additional properties.

The same restriction applies to inverse explanation. Section 10 defines structural compatibility as overlap between a hypothesis's forward structural image and an observed structural target. Structural survival means only that the hypothesis has not been eliminated at the chosen descriptor resolution.

Let $\mathfrak{H}_D^{\text{pos}}(Y, A)$ be the possibly compatible hypothesis class from Equation 164. Even if Equation 227 holds, the resulting identification remains relative to the declared candidate class.

$$|\mathfrak{H}_D^{\text{pos}}(Y, A)| = 1. \quad (227)$$

Equation 227 does not establish that the surviving hypothesis is the unique historically possible cause outside $\mathfrak{H}^*$. An omitted alternative model can alter the result.

The redundancy result in Proposition 10.5 adds another scope condition. When the same reconstructed history has already been used to screen a causal hypothesis at full history level, a braid or invariant computed from that history is not independent evidence. It can still improve compression, comparison, visualization, or model specification, but causal evidential weight should not be counted twice.

Comparative recurrence has the same boundary. A robust recurrent motif demonstrates repeated trajectory structure at the declared comparative resolution. It does not establish a common generator, shared institutional mechanism, or universal historical law. Those stronger claims require additional causal and contextual argument.

14.6 Applicability Profile for Historical Trajectory Claims

The preceding boundaries can be consolidated into an analysis profile without collapsing distinct failure modes into one score. Two distinctions are especially important. First, the existence of a state-history reconstruction and the existence of a coherent witness-augmented reconstruction should remain separate. Second, robustness and substantive interpretation can be unassessed rather than simply false.

Define the three-valued assessment set in Equation 228.

$$\mathbb{T} := \{0, 1, \text{NA}\}, \quad (228)$$

where NA means that the corresponding robustness or semantic assessment is outside the claim currently being reported.

For a declared reconstruction regime η , define the applicability profile in Equation 229.

$$\mathbf{a}(\eta) := (a_F, a_H, a_W, a_{\text{cfg}}, a_B, a_{\text{id}}, a_{\text{tax}}, a_{\text{rob}}, a_{\text{sem}}) \in \{0, 1\}^6 \times \mathbb{T}^3. \quad (229)$$

The coordinates have the following meanings:

1. $a_F = 1$ when $\mathcal{F}_A(Y; \tau) \neq \emptyset$;
2. $a_H = 1$ when $\mathcal{R}_A(Y) \neq \emptyset$;
3. $a_W = 1$ when $\tilde{\mathcal{R}}_A(Y) \neq \emptyset$;
4. $a_{\text{cfg}} = 1$ when the nonvacuous complete configuration coverage condition in Equation 222 holds;
5. $a_B = 1$ when the nonvacuous complete standard braid coverage condition in Equation 223 holds;
6. $a_{\text{id}} = 1$ when the standard braid is identified;
7. $a_{\text{tax}} = 1$ when a declared family of admissible fact taxonomies has been assessed and the reported braid value is taxonomically robust in the sense of Definition 7.2, $a_{\text{tax}} = 0$ when that assessment fails, and $a_{\text{tax}} = \text{NA}$ when no taxonomic-robustness claim is being made;
8. $a_{\text{rob}} = 1$ when a declared perturbation class has been assessed and the braid is robustly identified, $a_{\text{rob}} = 0$ when that assessment fails, and $a_{\text{rob}} = \text{NA}$ when no broader robustness claim is being made;
9. $a_{\text{sem}} = 1$ when a substantive interpretation being reported is reconstruction-stable in the sense of Definition 12.1, $a_{\text{sem}} = 0$ when a nonempty braid-admissible reconstruction set exists but the interpretation fails that stability test, and $a_{\text{sem}} = \text{NA}$ when the analysis reports no substantive semantic claim beyond the formal descriptor or when semantic stability is not evaluable at the claimed descriptive level.

Equation 229 is a reporting device rather than a scalar quality score. Its coordinates are not interchangeable and should not be summed.

The profile makes several logically distinct outcomes visible. A study can have $a_F = 1$ and $a_H = 0$, meaning that record-compatible fact configurations exist but none can be organized into an admissible history under the current articulation relation. It can have $a_H = 1$ and $a_W = 0$, meaning that state-history candidates exist while no coherent temporal and identity witness supports the strand construction. It can have $a_B = 0$ and $a_{\text{cfg}} = 1$, indicating that an endpoint-sensitive groupoid description remains available even though the standard braid does not. It can have $a_{\text{id}} = 1$ and $a_{\text{rob}} = 0$, indicating braid identification under the reference regime without robustness across the declared perturbation class. It can also have $a_{\text{id}} = 1$ and $a_{\text{sem}} = 0$, meaning that the topological class is identified while the substantive historical interpretation remains reconstruction-dependent.

The reporting rule is stated in Definition 14.2.

Definition 14.2 (Maximally warranted reporting). A historical trajectory analysis follows *maximally warranted reporting* when it reports the most informative descriptive conclusion that is simultaneously defined, identified at the claimed resolution, and qualified by the robustness and semantic conditions actually established.

Definition 14.2 favors neither finer nor coarser descriptors automatically. A finer descriptor is preferable when the evidence and reconstruction support it. A coarser invariant is preferable when it is the strongest stable conclusion. A groupoid description is preferable when endpoint variation is historically substantive. A topological conclusion should be withheld when the reconstruction or descriptor domain is incoherent.

Table 12 summarizes the principal failure modes and the corresponding methodological response.

Table 12 gives the scope boundary of the paper in operational form. The method is most informative when the historical evidence supports a nonempty and auditable reconstruction set, the relational representation is independently justified, the relevant trajectory descriptor covers the compatible reconstructions, and the resulting structural conclusion remains stable at the resolution actually reported. Outside those conditions, the formalism remains useful mainly as a diagnostic of where information, admissibility, or interpretive warrant is missing.

Table 12: Principal scope boundaries and reporting responses in braid-theoretic historical trajectory analysis.

Boundary or failure mode	Diagnostic condition	Warranted response
Reconstruction inconsistency	$\mathcal{R}_A(Y) = \emptyset$ or no coherent witness	Revisit evidential coding, assumptions, or historical model; do not force a trajectory
Configuration-domain failure	Some compatible reconstruction lies outside $\tilde{\mathcal{R}}_A^{\text{cfg}}(Y)$	Report incomplete trajectory-domain coverage or adopt a richer representation
Based-loop failure	Configuration coverage holds but standard braid coverage fails	Retain endpoint-sensitive groupoid information where justified
Full braid nonidentification	$ \mathcal{B}_A(Y; c_0) > 1$	Report the compatible braid set or retreat to an identified invariant/joint descriptor
Taxonomic sensitivity	Admissible fact taxonomies yield different braid values, incompatible strand sectors, or failed reconstructions	Report dependence on fact articulation and avoid presenting one taxonomy as uniquely compelled by the record
Regime sensitivity	Reference regime gives one value but the perturbation class does not	Qualify the result as assumption- or reconstruction-sensitive
Semantic instability	Historical interpretation varies across compatible reconstructions	Report the topological result separately from the unstable substantive interpretation
Generator underdetermination	Several candidate explanations remain structurally compatible	Use topology as an eliminative constraint, not as unique causal identification
Comparative recurrence	Several cases share an identified or robust motif	Report recurrent structure; require separate causal argument for a common mechanism

The final section consolidates the methodological contribution and states the role of braid-theoretic trajectory description within the wider programme of historical social dynamics.

15 Conclusion

This paper has developed a preliminary methodological framework for braid-theoretic trajectory description in historical social inquiry. Its central architecture separates the generative production of a social-historical process from the epistemic reconstruction through which inquiry approaches that process. The revised framework adds a fact-regime layer between evidence and history: records constrain compatible fact configurations under a declared taxonomy; those fact configurations support candidate histories and research objects; those histories support relational trajectories; and only then can braid-theoretic description be applied.

The three-level GR reality architecture gives this reconstruction a controlled ontological scope. Ontological reality remains suspended. Relational reality contains historically constituted and efficacious beings, structures, institutions, roles, statuses, and relations. Manifestation reality contains context-dependent forms through which relational objects become expressed, recognized, recorded, measured, or otherwise accessible. Fact taxonomies are not additional reality levels. They are inquiry-relative articulation schemes operating across relational and manifestation reality.

This distinction permits the paper to treat social facts, institutional facts, legal facts, political facts, economic facts, historical facts, and other fact families as overlapping rather than mutually exclusive. It also separates fact family, fact category, fact token, and fact regime. A sixfold classification such as entity, property, state, event, process, and relation can therefore be used as one fact-form taxonomy without being elevated into a privileged GR ontology. The four GR meta-primitives - constraint, change, structure, and difference - remain the weaker representational layer beneath such taxonomies.

The resulting reconstruction architecture is set-valued at more than one level. A record can support several compatible fact configurations; a fact configuration can support several histories; a history can support several witness assignments; and a compatible history can remain consistent with several candidate generators. The paper's central epistemic thesis follows from this layered structure: selected trajectory information can be identified even when fact configuration, complete history, or generator remain underdetermined.

Relational configuration spaces provide the bridge from reconstructed social-scientific objects to braid theory. A strand need not represent a biological individual. It can represent an institution, office, organization, legal person, coalition, role-bearing entity, or another research object constituted through the selected fact configuration. A planar braid chart is therefore a declared representation of relational position rather than an assumed geography. Standard braid admissibility requires persistent traceability, fixed component number, collision exclusion, and an endpoint convention; open-endpoint cases can be retained at the fundamental-groupoid level.

The robustness analysis now distinguishes evidential, taxonomic, and geometric uncertainty. Evidential robustness asks whether a conclusion survives changes in records, assumptions, temporal assignments, identities, charts, or interpolation rules. Taxonomic robustness asks whether the conclusion survives reasonable alternative articulations of the relevant social facts. Geometric robustness asks whether a fixed strand realization preserves its braid class under bounded perturbation. These are different questions and can yield different answers. A braid can be geometrically stable while remaining taxonomically fragile, or taxonomically robust while fine-grained fact reconstructions differ.

Topology-changing analysis must likewise distinguish collision boundaries from changes in the constitution of the research object. A braid class can change through collision inside a fixed representational sector, but fixed-strand continuation can also fail because an institution splits, entities merge, lineage becomes non-bijective, or the fact taxonomy itself changes. Local braid descriptions can remain valid on adjacent historical regimes even when no single fixed-strand braid spans the transition.

The comparative framework uses explicit transport rather than raw symbol matching. Direct braid comparison requires compatible strand number, research-object correspondence, orientation convention, endpoint convention, and braid-codomain isomorphism. Derived descriptors permit comparison at coarser shared resolutions. Recurrent motifs record repeated structure across cases, while robust recurrent motifs require stability across declared perturbation classes. Neither kind of recurrence establishes

a shared generator without separate causal argument.

The inverse framework constrains candidate explanations through their forward structural images. A candidate hypothesis can be structurally compatible, uniformly compatible, refuted at the selected descriptor level, or left descriptor-indeterminate. Distinct hypotheses with the same forward image remain observationally equivalent at that resolution. The paper also establishes a non-double-counting discipline: a braid or invariant derived from a history already used in full to evaluate a causal hypothesis is a structural summary, not an independent second item of evidence.

These formal objects complement established historical methods. Process tracing contributes diagnostic causal evidence; sequence-sensitive analysis can retain distinctions quotiented by braid equivalence; path-dependence analysis concerns the causal conditioning of later possibilities; comparative-historical interpretation supplies contextual warrant for cross-case correspondence. Braid-theoretic trajectory description contributes an intermediate relational representation with an explicit domain, resolution, and inferential standing.

Interpretive discipline remains essential. A property can be attributed to a braid value only when it is constant on the relevant braid fibers. A historical event, an evidential record, an attribution fact, a social-recognition fact, and a legal fact concerning the same object should not be collapsed into one fact. This distinction is especially relevant where institutions, archives, citation systems, registries, search systems, or legal mechanisms affect whether socially consequential facts become visible, stabilized, contested, or corrected without changing the underlying historical event.

The scope analysis therefore extends maximally warranted reporting to the fact layer. A responsible report states the strongest conclusion that is actually supported at the relevant sequence of levels: fact reconstruction, history reconstruction, witness coherence, configuration coverage, braid coverage, identification, taxonomic robustness, broader perturbation robustness, and semantic stability. Where a fine-grained conclusion fails, the analysis can retreat to a coarser invariant, an endpoint-sensitive groupoid description, a compatible set of structures, or a diagnostic statement that the evidence does not warrant a topological conclusion.

The framework remains intentionally conditional. It does not choose the uniquely correct ontology of social facts, derive legal or political categories from topology, or recover causal mechanisms from braid form. Its contribution is a disciplined interface between social-scientific fact articulation, historical reconstruction, and trajectory structure. Where several fine-grained fact configurations, histories, or generators remain open while one relational trajectory class is stable, braid-theoretic description can preserve historically meaningful path information that endpoint descriptions omit while demanding less than complete recovery of the process that produced it.

Future work can extend this architecture toward governance, jurisprudence, attribution, commons provenance, and fact-formation power. Such work can ask how institutions regulate the formation, authentication, stabilization, contestation, correction, and propagation of socially consequential facts, while keeping separate the historical event, its manifestations, its social recognition, and its legal consequences. The present paper supplies the descriptive and epistemic architecture needed before those normative and governance questions are formalized.

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