

Generative Relations in Legal Judgment: Evidence, Historical Trajectories, and Norm-Relevant Similarity

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Abstract

Legal comparison depends on the relationship between general norms and particular histories. Identical words or recorded outcomes can arise through different dependencies, while different expressions can perform comparable functions within a relationship. This discussion paper develops an account of norm-relevant similarity through the reconstruction of generative relations: the processes through which resources, expectations, interpretations, and available actions become connected over time. It distinguishes evidential support, causal reconstruction, normative characterization, and institutional authority. A fictional reimbursement rule and a controlled family of cases make the comparison criteria explicit without asserting current doctrine. A relational-field formulation then separates observation from retained history and counterfactual response, while requiring role-preserving mappings, justified time scales, and independently defended normative constraints. Completed synthetic calculations show how shared trajectories, attractors, action values, or boundary amplitudes can coexist with differences in dynamics or interior histories. These results supply mathematical examples of limited inference; they provide no empirical validation of legal decision-making. The paper grounds reason-giving, contestation, and proportionate correction in stated commitments to equal standing, protection against material error, and accountable public authority. Its contribution is an inspectable connection between historical reconstruction and issue-specific comparison, with established legal analogy and case-based reasoning as substantive baselines.

Keywords: legal judgment; evidence; relational history; legal analogy; norm-relevant similarity; interpretive correction.

Research Paper Note

This legal-philosophical discussion paper develops a framework for evidential reconstruction and norm-relevant comparison. Its rule and cases are fictional. The mathematical studies establish properties of stipulated models and have not been validated against actual legal judgments.

The project welcomes objections, corrections, and identification of related concepts in other disciplines. It claims no priority in legal analogy, context-sensitive reasoning, interpretive justice, procedural correction, or dynamical approaches to social relations. Relevant antecedents include research on forensic relevance, legal analogy, case-based legal reasoning, and attraction basins for generative justice [3, 4, 11, 14, 17]. The proposed Generative Relational vocabulary requires comparison with these traditions. Access limitations and remaining reading are stated in the source ledger.

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Good-faith criticism, correction, and independent development are welcome. Relational reconstructions should retain their supporting evidence, counterevidence, uncertainty, and normative scope. Model

outputs should remain identified as consequences of stated assumptions. Institutional use requires an argument about authority and affected interests in addition to any formal analysis.

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AI-use disclosure. The supplied exploratory human–AI conversation informed the research questions. An OpenAI coding assistant prepared the outline, initial source checks, methodological criticisms, a proposed relational-field formulation, an executable synthetic illustration, the complete first manuscript draft, and the expanded preprint revision. AI assistance involved substantive argument, analysis, and writing, beyond language editing or formatting. The illustration concerns a constructed transition system and supplies no independent empirical validation of the proposed account. A companion Paper I pilot collected 24 actual agent responses about reality attribution, and a separate spatial field experiment was run in Python. Neither evaluates legal interpretation or validates judicial performance. No such validation follows from the subsequent field-equivalence, action, and finite-group boundary calculations either. Their analytic and numerical comparisons concern specified synthetic structures. No legal-response experiment or expert study has been conducted.

Human attribution. The author is Wanhong HUANG, contact huangwanhong@serendip.ngo. AI systems are disclosed as research and drafting tools. The author attribution identifies the human contributor and does not represent independent peer review. Earlier sources and PDFs are preserved in the project version archive.

1 Introduction

This paper examines how legal judgment can compare particular histories while remaining accountable to general norms. Its purpose is to clarify the relation among evidential reconstruction, causal explanation, the selection of relevant similarities, and the authority of a conclusion. The discussion develops a fictional rule and a family of controlled cases, then uses finite transition systems and field-based mathematical examples to expose limits of compressed comparison. The resulting proposal concerns the organization and criticism of reasons; it is a discussion framework without validation against actual judicial decisions.

Two cases can end in the same recorded event and differ in the path by which that event became possible. A person may accept an arrangement after an ordinary offer, after a threat affecting access to an essential resource, or after the removal of a constraint that previously limited choice. If a legal issue concerns coercion, dependence, or the practical availability of refusal, a terminal description can omit a distinction selected by the norm. Yet the inclusion of a richer history cannot itself establish its legal

significance. An investigator must support the history with evidence and explain the rule under which the difference matters.

These obligations arise at different stages. A report is evidence about an event; a causal account proposes how events are connected; a comparison selects features shared across histories; and a judgment applies an institutionally situated standard. Errors occur when one stage silently substitutes for another. A coherent narrative can remain evidentially weak. A reliable prediction can depend on features with doubtful relevance to the issue. A close mathematical match can erase the roles or temporal conditions that make a legal distinction intelligible. The paper develops a way to keep these obligations visible within a connected account.

The phrase *generative relations* emphasizes the processes through which positions, opportunities, constraints, and responses develop. The proposed unit of comparison includes the history relevant to a stated issue, with explicit attention to the people and resources whose relations the norm selects. This emphasis has antecedents in legal analogy, case-based reasoning, causal inquiry, epistemic injustice, and relational accounts of justice. The paper claims no priority over those traditions. Its contribution is a structured discussion of their intersection with the distinct meanings of similarity and equivalence in mathematical models.

The fictional Rule R makes the inferential stages inspectable without presenting an invented rule as law in any jurisdiction. Its case family holds some descriptions fixed while varying a prior threat, a refusal path, a resource floor, and evidence of the relevant relation. A finite enumeration verifies consequences within stipulated transitions. The field examples then show how equal outputs, oriented trajectories, asymptotic states, action values, or boundary amplitudes can preserve different information. These mathematical results establish limitations of particular inference patterns. The normative importance of the lost information remains a matter for the defended rule, evidence, and institutional context.

Section 2 reviews the relevant antecedents. Sections 3 onward develop the evidential problem and the fictional cases before specifying norm-relevant comparison. The later sections examine field models, actions and boundary descriptions, public correction, and objections. Section 12 states the proposed institutional research agenda. The appendices integrate the complete finite comparison, restrictions on comparison maps, and shared mathematical studies into this paper. The companion paper, *Human and Artificial Reality: Generative Relations and the Formation of Interpretation*, addresses the philosophical problem of reality attribution and contains the live-agent response corpus. That pilot supplies no empirical legal validation for the present proposal.

2 Literature Review

This section identifies antecedents for the paper's treatment of norms, analogy, evidence, causal reconstruction, justice, and mathematical comparison. The review is selective and organized by argumentative role. It makes no claim to exhaust the jurisprudential literature or to settle differences among its traditions. Sources were verified online before citation; attributions remain within the abstract, metadata, or original passages actually consulted. The purpose is to locate the proposed framework and identify the work required for a more complete doctrinal and empirical assessment.

2.1 Norms, Principles, and the Structure of Reasons

Dworkin's *The Model of Rules* is a foundational point of reference for the structure of legal reasons. In the inspected discussion, principles possess a dimension of weight or importance, whose assessment can be controversial and does not constitute exact measurement [10]. This distinction is relevant to the temptation to turn every normative consideration into a numerical coefficient. A model that aggregates features needs an independent account of the kind of reason each feature supplies and the permissibility of compensation among them.

The present use of Rule R has a more limited methodological purpose. A stipulated rule exposes how a conclusion depends on specified conditions. It does not imply that actual legal practice consists

entirely of such rules, or that contested principles can always be reduced to a Boolean checklist. The fictional construction holds normative conditions sufficiently stable to examine evidential and temporal differences. Extension to a real controversy would require an account of legal authority, interpretation, exceptions, burdens of proof, and the relations among potentially competing reasons.

This separation preserves an important place for disagreement. Participants may agree about events and disagree about the rule, agree about the rule and disagree about the evidence, or share both while disputing whether a case falls within an exception. A useful decision record should show which dispute remains. Increasing the detail of a field model cannot resolve an unsettled issue of authority merely through greater mathematical precision.

2.2 Relational Analogy and Legal Case Comparison

Gentner's structure-mapping account of analogy distinguishes relational mapping from a comparison based only on attributes and gives higher-order systematicity a central role [15]. This is an antecedent for examining patterns of relations across cases. A mapping between a resource controller, a dependent participant, an available refusal, and an anticipated consequence can preserve information that a list of endpoint attributes omits. The normative permission to use such a mapping remains separate from its descriptive coherence.

Legal analogy has its own established analyses. Lamond distinguishes forms of analogical reasoning in common-law practice, including classificatory, close, and distant analogies [17]. Ashley's work places legal case-based reasoning within a developed computational tradition, and Ashley and Rissland examine law, learning, and representation [3, 4]. These sources rule out a novelty claim based merely on comparing cases, encoding distinctions, or attending to representation. The present proposal concerns the explicit temporal and normative conditions under which a relation preserved by a model can support a particular legal comparison.

The distinction between retrieval and justification follows from this positioning. Similarity can help locate cases for inspection. A selected case then requires explanation of the respects in which it bears on the issue. An algorithm may give a close match because many observable details agree while omitting the role relation that controls the legal question. Conversely, cases that differ in ordinary descriptive distance can share the structure selected by a norm. The discussion below treats a similarity measure as one possible instrument within a reason-giving process, with the selection and interpretation of its features available for criticism.

2.3 Evidence, Relevance, and Counterfactual Reconstruction

Research on forensic decision making already investigates the relevance of contextual information. Gardner and colleagues' study of forensic analysts addresses which information practitioners consider relevant to their decisions [14]. The present argument should therefore be read as a further specification of relevance under a selected issue. It does not portray legal or forensic inquiry as uniformly indifferent to context.

Pearl's overview of causal inference emphasizes the assumptions required for causal and counterfactual claims and distinguishes queries about interventions, counterfactuals, and related effects [21]. This antecedent clarifies the status of a reconstructed refusal path. Observed acceptance under one set of conditions does not, by itself, identify the result of a different intervention. A causal interpretation combines evidence with assumptions about the relevant response mechanism. The paper's transition model makes those assumptions explicit, allowing their consequences to be checked while leaving their real-world adequacy unresolved.

The evidential problem is especially acute when the allegedly constrained option was never attempted. A counterfactual refusal can be supported by prior communications, control over resources, observed responses to similar refusals, and evidence about feasible alternatives. Each item has a provenance and a possible competing interpretation. A field representation can organize a proposed account of these relations, but the diagram or trajectory does not add an observation merely by displaying it. The case

analysis therefore distinguishes recorded events, supported inferences, stipulated assumptions, and unresolved alternatives.

The framework also separates causal and normative thresholds. A causal contribution to an outcome can exist without satisfying the conditions of a particular legal wrong. Conversely, a norm can make a threatened consequence relevant even when a later intervention prevents that consequence from occurring. The appropriate comparison depends on what the rule selects, when it selects it, and what the evidence can establish about the affected person’s situation.

2.4 Interpretive Resources and Epistemic Injustice

Fricker’s treatment of hermeneutical injustice concerns disadvantages associated with the resources available for making experience intelligible [13]. Flores’s discussion of trauma and elite capture supplies a further examination of the politics of interpretive categories [12]. These antecedents motivate attention to whose descriptions enter a record and whose experience can be articulated through the categories an institution accepts.

The present framework treats these concerns as reasons to inspect the production of evidence as well as its final form. A missing category, an inaccessible procedure, or an asymmetric opportunity to explain a history can affect what later appears in a dataset. Enlarging a feature vector cannot automatically recover information that was never recorded or that was systematically excluded. Attention to historical relations should therefore include the history of the record itself: its collection, compression, classification, and revision.

This commitment also limits inference from fluent articulation. A coherent and institutionally familiar account can receive attention unavailable to an awkward or unfamiliar one. Neither fluency nor difficulty establishes truth by itself. The proposed decision record should retain reasons for accepting or rejecting a claim and make it possible to challenge the categories through which the claim was understood. These requirements concern evidential accountability; they leave the substantive resolution of a disputed case to its supported facts and governing norms.

2.5 Generative Justice, Public Deliberation, and Revision

Eglash and Garvey’s discussion of basins of attraction and generative justice is a direct antecedent for bringing dynamical language into a discussion of justice [11]. Both authors and the existing concept must be acknowledged. The present paper uses *generative relations* to organize a question about legal histories and comparison; it does not claim to originate generative justice or to reproduce that chapter’s full program.

Niemeyer and Dryzek’s account of meta-consensus provides an antecedent for examining common commitments amid continuing substantive disagreement [18]. Nobles and Schiff locate legal pluralism within a systems-theoretic discussion [19]. These references help frame the institutional problem of publicly defensible comparison across differences in interpretation. They do not supply a theorem guaranteeing convergence or agreement in the proposed procedure.

Revision also has a specific literature. Tutt’s revisability principle concerns a narrower problem involving identity and records [25]; it should not be treated as a general theory of judicial appeal. The extension pursued here is expressly argumentative: where a judgment relies on an evidential or interpretive compression, a later challenge should be able to identify the material assumption and explain the consequence of revising it. Questions of finality, cost, reliance, and institutional competence remain part of that assessment.

2.6 Dynamical and Variational Comparison

Behavioural systems theory and approximation metrics provide mathematical antecedents for distinguishing a representation from its accessible behaviour and for specifying an approximate comparison [16, 26]. Viability theory supplies an established setting for studying trajectories subject to constraints [5]. The

finite resource-floor example in this paper is elementary and fully enumerated; it claims no general viability result and no empirical identification of a legally appropriate floor.

Action-based comparison introduces additional structure. Gradient-flow and large-deviation accounts make clear that a functional is interpreted together with dynamics, dissipation, or probabilistic assumptions [1, 20, 22]. The worked examples below use fixed finite coefficients to distinguish shared energy, different mobilities, pathwise action values, and consistent coordinate transformations. Equality of a scalar evaluated on a path discards information about the path. Whether that discarded information matters legally depends on the issue under consideration.

Spin-foam research and coarse-graining methods provide an established context for boundary amplitudes and consistency across descriptions [6, 7, 9, 23]. The present finite-group construction uses positive weights and a directly verifiable dual expansion. It identifies a precise example in which the selected boundary data agree and interior insertions differ. The comparison serves as a mathematical illustration of information retained by a boundary description. It supplies neither a physical theory of law nor a probabilistic model fitted to legal outcomes.

2.7 Contribution and Remaining Research

The proposed contribution is a connected discipline of comparison: identify the legal issue and its authority, reconstruct the relevant history from evidence, state the correspondence permitted between cases, and disclose the distinctions a representation removes. The fictional cases and mathematical appendices make selected failures of inference inspectable. They also identify where further work belongs: doctrinal interpretation, evidence about actual decision practices, defensible measurement, and evaluation by affected participants and legal specialists.

This positioning leaves the paper open to correction by established theories and by more economical formulations. A successful relational framework should improve the clarity of reasons and counterarguments in a case. The adoption of a field, action, or state-sum formalism has value only when its defined structures contribute to that task and its assumptions can be defended.

3 General Norms and Particular Histories in Adjudication

This paper examines how a general norm becomes applicable to a particular history and how that application supports comparison across cases. Its method combines conceptual analysis, a controlled fictional case family, and limited mathematical illustrations. The central proposal is that norm-relevant similarity can depend on the processes through which resources, expectations, interpretations, and available actions became connected over time. The paper calls these productive historical connections generative relations.

Consider a supervisor's message, "You know what you need to do," followed by an employee's signed waiver of reimbursement. The message and signature may be authentic. Their authenticity leaves further questions concerning the message's understood significance, the employee's available alternatives, the supervisor's capacity to impose consequences, and the conditions under which the waiver acquired legal effect. A history of sanctions could make the words operate as a reminder of a credible threat. A history of independent planning could support a different account. The same record fragment can enter competing reconstructions.

The legal issue adds another layer. A rule concerning formal authorization can select different features from a rule concerning induced waiver. Even a well-supported causal account does not determine which rule governs, how a burden of proof operates, or which remedy follows. Legal judgment joins evidential and explanatory tasks with a claim of institutional authority. Its reasons should make the transition among those tasks available for assessment.

Existing legal reasoning already gives substantial attention to context and comparison. Lamond examines forms of analogy in common-law reasoning [17]. Ashley's work on case-based legal systems and Ashley and Rissland's later treatment of law, learning, and representation provide substantive antecedents for structured case comparison [3, 4]. The proposed framework should be assessed against

such approaches in their strongest forms. A representation that already captures temporal order, dependence, and competing reasons may accomplish much of the work described here.

The paper's proposed contribution is therefore connective and diagnostic. It brings together the reconstruction of a history, the distinction between actual and counterfactual possibilities, the selection of norm-relevant features, and the institutional conditions for correcting an interpretation. Field models make some dependencies explicit, especially when retained states and reciprocal effects matter. Their mathematical vocabulary earns a place only when it clarifies an inferential step or exposes a loss of relevant information.

The normative argument relies on stated premises. A decision-making institution that claims authority over affected persons should recognize their equal standing as participants in the justification of that authority, protect them against material errors within the institution's remit, and give reasons connecting its coercive or distributive decisions to authorized norms and supported facts. These are commitments defended as part of the proposal. They do not follow from the stability of a social system or the existence of a relational field.

The discussion remains jurisprudential and jurisdiction-neutral. Section 6 defines a fictional workplace rule so that the case comparisons have a determinate issue and burden. The resulting outcomes are consequences of those stipulations. They establish no current legal entitlement and should not be transferred to doctrines of criminal duress, coercive control, complicity, or contract validity without a separate analysis of applicable authority. This restriction gives the example a clear scope while preserving its relevance to the more general problem of norm-guided reconstruction.

4 Evidence and the Reconstruction of Relational Histories

This section separates the evidential stages through which a record supports a claim about a relationship. It considers provenance, rival histories, causal interpretation, and normative characterization in sequence. The sequence is analytical; actual investigation may move repeatedly between stages as new evidence changes the questions worth asking.

Provenance concerns the relation between an item and the event it purports to record. A message archive may accurately preserve words while omitting earlier communications. A witness may accurately recall a consequence while being mistaken about its cause. Establishing authenticity reduces one kind of uncertainty. It leaves questions of completeness, selection, and interpretation for further inquiry. An investigator should avoid treating the verified existence of a record as verification of every inference attached to it.

A relational history assembles supported events into a temporally ordered account. In the running example, relevant events might include instructions, refusals, changes in access to work, explanations of those changes, and opportunities to obtain assistance. Their significance depends partly on order. A sanction announced after a waiver can influence later conduct; it cannot, solely by occurring later, establish that the employee anticipated it earlier. Evidence of a pre-existing policy could support anticipation, but that would be an additional evidential route.

The reconstruction should retain rival explanations. A reduction in work after refusal might have resulted from retaliation, ordinary scheduling, or an independent change in demand. Temporal succession alone leaves these alternatives available. Evidence that a supervisor explicitly connected the reduction to refusal would bear differently on the alternatives from an unexplained coincidence. The aim is to identify which facts discriminate between accounts, how strongly they do so, and where the record remains inconclusive.

Generative explanation adds a mechanism: an account of how a relationship made a particular event possible or likely. The supervisor's control over assignments could make a warning credible; the employee's dependence on those assignments could make refusal costly; repeated sanctions could make an ambiguous expression intelligible as a threat. Each link is a candidate claim requiring support. A coherent narrative can organize evidence without acquiring the evidential status of the facts it organizes.

Counterfactual claims require particular care. An employee's acceptance does not directly show what

would have followed refusal. A mechanism can be supported by previous refusals, explicit policies, contemporaneous statements, or other appropriately relevant evidence. General information about a group may suggest an investigative question, yet the individual attribution needs a warranted connection to the person's circumstances. The difference between a physically possible action and a practically accessible alternative can be central, but neither should be inferred from group membership alone.

A legal characterization then asks which supported features satisfy the authorized rule. A rule may require a prior threat, a causal contribution, and the absence of a protected alternative. Another may attach consequences to a formal notice regardless of some causal details. The characterization must preserve the difference between an unresolved factual premise and a disagreement about the norm's meaning. Otherwise, an interpretive dispute can appear to be a factual finding, or an evidential gap can disappear into a broad description of the relationship.

The resulting account of evidence is relational in a precise sense. A record bears on a history through its provenance and content; a history supports a mechanism through discriminating evidence; a mechanism becomes legally significant through a norm; and the institution's authority governs how remaining uncertainty affects a decision. Each connection admits challenge at its own level. Maintaining these distinctions makes correction more targeted: an objection to timing need not reopen an agreed question of authenticity, while a change in the governing rule can alter relevance even if the events remain fixed.

5 Contextual Relevance and Interpretive Authority

This section examines the selection of contextual information and the authority involved in that selection. It distinguishes relevance from reliability, admissibility, weight, and prejudice, then considers how interpretive resources influence reconstruction. Its purpose is to connect relational inquiry with existing concerns about bias without treating access to more context as an automatic improvement.

Information is relevant to a specified task when it bears on a proposition or decision criterion that the task legitimately addresses. Reliability concerns the trustworthiness of the information or the process producing it. Admissibility concerns an institution's rules for receiving and using it. Weight concerns its contribution within the evidential record. Prejudicial effects concern possible distortions or impermissible disadvantages arising from its use. These dimensions can interact while remaining analytically distinct.

Research by Gardner, Kelley, Murrie, and Dror directly investigates forensic analysts' judgments of task relevance [14]. The present proposal therefore begins from an established recognition that relevance itself requires examination. A relational analysis adds a question about the historical structure needed for the particular inference. It does not justify exposing every analyst to every aspect of a case. Some tasks may require shielding from information that is consequential elsewhere in the institutional process.

For example, examining whether a message file has been altered can involve different information needs from assessing what an authentic message conveyed within a relationship. The first task may depend on technical provenance, while the second can require prior communications and shared expectations. An institutional design can distinguish these tasks and regulate the movement of information between them. The appropriate boundary depends on the inferential work being performed and on the risk that extraneous information will shape that work.

Relevance also changes with the legal issue. Earlier sanctions may be material to a rule addressing induced waiver and peripheral to a narrowly defined rule about whether a notice contained prescribed text. A disagreement over relevance can therefore be a disagreement over governing purpose or authority. Resolving it requires stating the norm and its interpretation. A statistical association with outcomes, standing alone, supplies an incomplete reason for admitting or weighting a feature in an authoritative judgment.

Interpretive resources influence which explanations become available. Fricker's discussion of hermeneutical injustice concerns deficits in shared interpretive resources and their unequal consequences [13]. This provides an established reason to examine whether affected persons can make their experiences intelligible in the institution's categories. The present framework also requires scrutiny of

an available category's application: a powerful description can organize a case persuasively while fitting its evidence poorly.

An institution can respond by inviting a party to explain the significance of a pattern in concrete terms. Which events established an expectation? Which resource was controlled? Which refusal was attempted, and what followed? Such questions can translate an unfamiliar account into contestable propositions without presuming its truth or requiring the party to adopt an unsupported technical vocabulary. They also give an opposing account a determinate target for response.

Interpretive authority remains situated. Investigators and adjudicators bring professional categories, precedents, and organizational incentives to the record. Acknowledging these conditions need not discredit their work. It supplies reasons to document decisions about relevance, expose assumptions that affect outcomes, and permit appropriately bounded challenge. The normative force of these practices follows from the institution's duties toward affected persons, which Section 10 develops explicitly.

6 A Fictional Rule and a Family of Comparable Cases

This section fixes a fictional institutional rule and varies a controlled case family. The purpose is to show how similar records can support different outcomes and different expressions can support the same outcome when the norm is held constant. The invented rule supplies the comparison criterion; the cases supply stipulated evidence. Neither is presented as actual law.

6.1 The Reimbursement Rule and Its Evidential Burden

The example concerns an organization with an approved employee expense and a later signed waiver of reimbursement. Under fictional Rule R, an internal tribunal restores reimbursement if the employee establishes each of three conditions. First, before the waiver, an organizational representative communicated or maintained a credible threat of a serious reduction in an essential work-related resource, conditional on refusal to waive. Second, that threatened consequence materially influenced the employee's decision to waive. Third, at the relevant time, the employee lacked a reasonably accessible way to refuse while remaining protected against that consequence.

For this exercise, each condition must be better supported by the admissible record than its denial. An unresolved balance on any required condition means that the claim under Rule R is not established. This expressly stipulated burden permits determinate treatment of incomplete cases. It does not assign numerical probabilities to narratives, derive a real jurisdiction's evidential standard, or imply that the rule is the best possible design.

The terms of the rule also require interpretation. A resource is essential here when losing it would substantially impair the employee's access to agreed work or the means needed to perform it. A threatened reduction is credible when the representative has the relevant capacity and the employee has supported grounds to anticipate its use. A protected route to refusal must be accessible in time, usable in the employee's circumstances, and capable of preventing the threatened consequence. A nominal complaints channel can fail these stipulated conditions; an effective and timely protective procedure can satisfy them.

Material influence is a separate condition. The employee can face a credible threat and nevertheless have made an independently settled decision that the threat did not affect. The inquiry concerns whether the threat made a material difference to the waiver decision, including its content or timing. A statement made after the event may provide evidence, but the fictional cases below stipulate supporting contemporaneous records where a condition is established. This avoids allowing a single retrospective assertion to determine every outcome by definition.

Rule R's rationale is protection against a specified form of organizational cost-shifting through threatened deprivation. That purpose explains its focus on dependency and refusal conditions. It also exposes the rule's limits. The rule may leave other objectionable practices outside its remedy, and an institution could defend broader or narrower protections. Those choices would change the comparison framework. For present purposes, the fixed rule makes its own inclusions and exclusions inspectable.

Table 1: Fictional case family under specified rules. An unestablished claim records the effect of the stipulated burden; it does not establish free choice or the absence of every relevant wrong. Case R6 changes the governing rule and is therefore outside the same-rule comparison.

Case	Stipulated record	Conditional outcome
R0	Ambiguous supervisor message and waiver; incomplete refusal and resource history.	Claim under R unestablished.
R1	Same message; documented prior sanctions, contemporaneous reliance on the threat, and an inaccessible protective route.	Reimbursement restored under R.
R2	Same words from a peer without sanctioning capacity; supported independent planning and protected refusal.	Claim under R unestablished.
R3	Different wording; the same supported sanction dependence, material influence, and lack of protection as R1.	Reimbursement restored under R.
R4	R1’s credible threat; independent records establish a settled waiver decision that the threat did not alter.	Material influence unestablished.
R5	Threat first communicated after the waiver; stipulated absence of an earlier threat or anticipation.	Prior-threat condition unestablished.
R6	R1’s history under Rule F, which assigns validity solely by authenticated signature and formal capacity.	Waiver valid under F.

6.2 Historical Variations and Conditional Outcomes

The controlled case family varies wording, prior history, causal influence, event order, and the governing rule. Table 1 records the cases and their conditional outcomes. The table summarizes the stipulated records; the following discussion explains the consequential differences.

R0 is the reference fragment: the supervisor’s ambiguous message is followed by a waiver, and the record lacks sufficient evidence about previous refusals and available protection. The order supports a question about influence but leaves Rule R’s conditions unresolved. The stipulated burden produces an unestablished claim. It would be inaccurate to redescribe that procedural outcome as proof that the employee chose freely. Additional evidence could change the result without changing the original words.

R1 adds records of prior resource reductions explicitly connected to refusals, evidence that the supervisor retained the relevant control, and contemporaneous communications showing that the employee anticipated the sanction and waived because of it. The protective channel could not prevent the consequence within the decision period. The case stipulates that this evidence makes each condition better supported than its denial. Reimbursement follows under Rule R. The operative difference from R0 is evidential support for the required relationship, rather than an assumption that every supervisor’s ambiguous words carry the same force.

R2 preserves the words while changing the relation. The speaker is a peer without control over the resource, and records show independent planning of the waiver together with an effective refusal option. The linguistic match with R1 is therefore insufficient for the rule. R3 makes the complementary variation: different words refer to the same established sanction practice, materially influence the decision, and operate without accessible protection. The rule supports the same result as R1 because the specified normative conditions correspond despite the surface difference.

R4 is counterevidence to an overly expansive relational inference. The supervisor has the capacity and history described in R1, but independent contemporaneous records establish that the employee had settled the waiver decision before the threat and did not alter its content or timing because of it. A general description of a dependent relationship leaves the causal condition unsatisfied in this stipulated case. The claim under Rule R remains unestablished. Other norms could evaluate the supervisor’s threat separately.

R5 changes temporal order. The threat is first made after the waiver, and the case expressly excludes evidence of an earlier communicated or anticipated threat. The later act cannot satisfy Rule R’s prior-threat condition for this decision. If subsequent evidence revealed an earlier policy known to the employee, the case would change materially. The conclusion therefore depends on the full stipulation, rather than a rule that later evidence can never illuminate earlier conditions.

R6 changes the institution's criterion. Fictional Rule F treats an authenticated signature by a formally authorized employee as sufficient for waiver validity, and the case satisfies those requirements. Applying F to R1's history therefore yields validity for the purpose governed by F. This outcome illustrates norm dependence and exposes a possible objection to F's protective adequacy. It does not establish that any real legal system adopts F, nor that choosing F is justified merely because its application is straightforward.

The family permits a discriminating conclusion. Historical relations matter when they support conditions selected by the rule; their presence does not automatically decide every issue. Similarity should preserve the normatively relevant connections, including counterevidence and temporal direction. A comparison that omits those connections can group R1 with R2 or separate R1 from R3 for reasons that fail the stipulated purpose.

6.3 Evidential Routes through the Reference Case

This subsection works through the transition from the incomplete reference record R0 to the more determinate cases. It distinguishes evidence supporting a rule condition from evidence challenging a rival account. The purpose is to expose the intermediate reasoning that a summary label such as coercive dependence could otherwise conceal.

The first question concerns the alleged resource. A record should establish what the employee expected to receive, which terms governed access, and how the threatened reduction would affect access to agreed work or its essential means. The same monetary amount can have different significance in different circumstances. Under Rule R, the institutional inquiry concerns the stipulated essential resource and serious reduction. A bare measure of financial loss cannot silently substitute for those conditions.

The second question concerns organizational control. A supervisor's title can be evidence of a role, while the actual capacity to impose the alleged consequence may depend on delegation, scheduling procedures, or another decision-maker. Records of prior reductions can help identify that capacity. Evidence that the reductions were decided independently can weaken the proposed mechanism. The reconstruction should preserve the difference between apparent authority as perceived by the employee and the supported capacity required for a credible threat under the fictional rule.

The third question concerns conditionality. A resource reduction following a refusal can support investigation without establishing that refusal caused the reduction. A contemporaneous message connecting the two, consistent treatment of comparable refusals, or a documented sanction policy can bear on that issue. Alternative explanations require examination at the same evidential level. For example, an independent scheduling constraint should be assessed through the records supporting it, rather than dismissed because it is convenient to the organization.

The fourth question concerns the employee's anticipation and decision. Evidence that a policy existed leaves open whether the employee knew of it and whether it affected the waiver. Contemporaneous communications, an attempted negotiation, or a documented change of plan could support material influence. Evidence of a settled independent decision can oppose it, as stipulated in R4. The inference should address whether the alleged threat made a material difference to the actual decision, including its timing or content.

The fifth question concerns protection. An official complaints channel may offer an effective route to refusal, or it may be unavailable within the relevant period. The distinction depends on access, timing, authority, and capacity to prevent the consequence. A route that provides compensation only after a threatened deprivation has occurred has different properties from one that prevents it. Rule R requires protection against the consequence at the decision time; a different remedial rule could recognize later restoration for another purpose.

These routes need not all demand extensive investigation in every case. Some conditions may be agreed, immaterial under the governing issue, or already supported by reliable records. Comprehensive reasoning concerns coverage of the consequential inferences. It does not require an unlimited collection of personal history. The questions identify where further evidence could change the application of the rule and where additional detail would leave it unchanged.

6.4 Counterfactual Support and Competing Histories

This subsection examines the status of a reconstructed alternative action. It distinguishes a supported counterfactual claim from a merely imaginable alternative and explains how uncertainty enters the fictional burden. The analysis also identifies reasons that a successful outcome can leave the available alternatives unresolved.

An observed waiver leaves the response to refusal unobserved. That response must be inferred from other evidence. Previous comparable refusals may be informative when the relevant authority, policy, resource, and time conditions correspond. A refusal by a person with independent protection may provide weaker evidence about a dependent employee. The case comparison used to support the counterfactual therefore requires its own account of similarity.

The inference can also be defeated by changed conditions. A prior sanction may have occurred under a policy that was rescinded before the present decision. An accessible protective route may have become effective after earlier failures. Historical explanation should preserve the possibility of such change. A theory that treats past dependence as indefinitely decisive would fail to distinguish retained expectations from the current capacity to impose a consequence.

Competing histories can agree on the observed signature while differing on these unobserved alternatives. One may describe material influence from a credible threat; another may describe independent planning within a generally dependent relationship. The tribunal's task under Rule R is to assess support for the required conditions against their denials. The existence of a conceivable rival account is insufficient by itself to defeat a claim, while coherence of the preferred account is insufficient by itself to establish it.

The threshold supplied by the fictional burden therefore matters to the conclusion. A finding can be warranted without exhaustive identification of the relationship's dynamics. Conversely, a model capable of simulating a sanction may remain too weakly connected to the record to support a finding. The distinction between mathematical possibility and evidentially supported attribution runs through both the fictional case family and the field examples.

7 Norm-Relevant Similarity and Comparison across Cases

This section generalizes the case family into an account of norm-relevant comparison. It distinguishes manifest, relational, and generative similarity, then considers the authority and limitations of mappings between cases. The account concerns reasons for comparison and does not presume a single numerical measure of legal likeness.

Manifest similarity concerns selected recorded features: words, document forms, visible conduct, or outcomes. Relational similarity concerns arrangements such as control over a resource, recognized authority, access to assistance, and dependence. Generative similarity concerns the temporal processes through which these arrangements shaped the event under judgment. A case can agree with another in one respect and differ in another. The governing norm determines which correspondences bear on the issue.

Under Rule R, R1 and R3 exhibit generative similarity because credible prior sanctions, material influence, and inaccessible protection correspond across their histories. Their different wording remains compatible with this correspondence. R1 and R4 share a broader relation of dependence while differing on material influence. Treating dependence as a complete surrogate for the rule would erase that legally selected difference. The example shows how a richer description can improve comparison only if its additional detail is organized by the issue.

A mapping between cases must preserve roles whose significance the norm recognizes. Exchanging supervisor and employee can preserve an unlabeled interaction graph while changing authority and access. Exchanging the threatened person with a bystander can preserve aggregate resource totals while changing whose interests were affected. Abstract structural similarity therefore needs a role interpretation. A map is admissible for a legal comparison when the preserved structure includes the features that the relevant

norm treats as consequential.

Temporal mapping requires comparable discipline. Rescaling time can reveal a common sequence of dependence and response. It can also conceal that one person endured deprivation for days and another for minutes. If duration bears on the rule or remedy, a map that erases it is inadequate for that question. Where timing is irrelevant within an explicitly defined range, a coarser alignment may be justified. The permissibility of the simplification follows from the norm and the evidential task.

The general-particular relation is reciprocal within institutional limits. A general rule identifies reasons for collecting evidence and selecting comparisons. A difficult case can then expose ambiguity in the rule's concepts, tension among its purposes, or an omitted circumstance. An authorized interpreter may clarify the rule through reasons that extend beyond the desired result in the particular case. The required authority and constraints depend on the institution. The present framework cannot license an adjudicator to alter a rule simply because a different formulation fits the preferred reconstruction.

Precedent adds an institutional history of comparison. A prior decision may supply an interpretation, a relevant distinction, or a limit on the reasons available in later cases. A representation of precedent should retain the issue decided and the role of the selected facts. A generic label such as "pressure" can conceal whether an earlier case turned on credibility, causal influence, available protection, or a procedural burden. Legal analogy and case-based reasoning already investigate structured comparisons [3, 4, 17]; the generative perspective proposes careful retention of temporal and counterfactual structure where those features do legal work.

Approximate similarity also needs a purpose. Small descriptive differences can cross a legally operative threshold. Large differences can concern features that the governing issue leaves irrelevant. A smooth numerical distance therefore cannot by itself determine equal treatment. An institution can use a comparison score to retrieve potentially informative cases, while the justification of a decision still addresses conditions and exceptions whose effects may be discontinuous. A mandatory protection should retain its separate force even when many other features match closely.

Norm-relevant similarity is thus a structured and revisable relation. It identifies the issue, records the correspondence, states consequential differences, and explains their effect under the rule. Revision can follow new evidence, a corrected mapping, or an authorized change in interpretation. These routes should remain distinguishable because they justify different kinds of correction and have different implications for other cases.

7.1 Precedent Selection and Issue-Specific Comparison

This subsection develops the use of a prior case as a reason in a later one. It distinguishes retrieval, explanatory comparison, and authoritative reliance, then considers how a relational reconstruction can clarify their connection. The discussion remains at the level of the fictional institution and general comparison problems.

Retrieval identifies cases worth examining. Shared wording, occupational roles, or outcomes may provide useful search features even when they do not determine the legal issue. Explanatory comparison asks whether the earlier case illuminates a mechanism or distinction in the present record. Authoritative reliance asks what force the institution permits the prior decision to have. A successful retrieval procedure should not be assumed to perform the latter tasks merely because the resulting case looks similar.

Suppose R1 becomes a prior decision and a later tribunal examines R3. The comparison should retain the interpretation of Rule R that made the prior sanctions, material influence, and inaccessible protection consequential. It should explain how the different wording in R3 bears on those same conditions. If the rule selected the communicated function rather than prescribed words, the linguistic difference could be immaterial to the issue. That conclusion follows from the rule and the supported relation, rather than a global similarity score.

Now suppose the later case resembles R4. A retrieval system could rank R1 highly because the resource arrangement and sanction history match. The distinguishing evidence concerns independent decision-making. A useful representation should bring that difference to attention, even if it occupies

a small part of the record. A numerical average over many shared features could conceal it. The distinction’s legal force depends on the causal condition of Rule R, which operates separately from overall resemblance.

The same history can support different comparisons across issues. Evidence of dependence can matter to protection against induced waiver while formal capacity matters to authorization under Rule F. A citation to an earlier decision should therefore identify the issue for which it is offered. The fact that a relationship was described one way in a prior proceeding leaves open whether the same description resolves another rule’s conditions.

A relational approach consequently needs an account of exclusions as well as inclusions. It should state which differences have been examined and found immaterial under the issue, which remain unresolved, and which defeat the proposed analogy. This structure permits disagreement about a mapping to be assessed without requiring agreement about every feature of the two histories. Its value can be tested against established case-based methods by asking whether it exposes consequential distinctions more clearly and with acceptable effort.

8 Relational Fields, Historical Dynamics, and Counterfactual Possibility

This section introduces a field formulation as a disciplined representation of distributed relations and retained history. It explains the status of the variables, develops a conditional comparison measure, and connects the formulation to completed synthetic calculations. The equations specify modelling choices; they are not empirically established laws of adjudication or social conduct.

8.1 Relational States and Local Organization

A directed relation field $R(x, y, t)$ can represent how participant or location x is connected to y at time t . Its components might encode supported estimates of resource control, communication access, or dependence. A separate local state $z(x, t)$ can record resources and available capacities, while $p(x, t)$ describes a response disposition and $m(x, t)$ retained exposure. These components require operational definitions in any application. They should not be combined into an unexplained intensity of power or justice.

The domain can be finite. With a fixed set of participants, R is a time-dependent matrix or tensor, and spatial coupling can be represented through a specified network. A continuum field requires an additional argument for the domain, measure, geometry, and meaning of proximity. Assigning arbitrary coordinates to people does not establish a physical space of social relations. The field vocabulary is useful when it makes distributed dependence and local variation explicit; a matrix description may supply the appropriate implementation.

A schematic retained-state model is given in Equation 1. The functions F_R and F_p stand for proposed evolution rules, K for the input to retention, u for specified external influences, and $\tau_m > 0$ for a retention time scale. Couplings across participants are included in these functions and must be specified in an executable model.

$$\partial_t R = F_R(R, z, p, u), \quad \partial_t p = F_p(R, z, p, m, u), \quad \tau_m \partial_t m = K(R, z, p, u) - m. \quad (1)$$

Equation 1 separates present accessible conditions from retained influence while permitting feedback. It does not imply that every social relation follows a smooth deterministic law. Discrete events, strategic changes, unknown inputs, and stochastic effects can require a different formulation. Its role is to expose which variables and mechanisms a reconstruction presupposes. The equation supplies no numerical estimate until those choices and their evidence have been provided.

The interpreter can be represented as part of the process. An institution’s classification may change access, sanctions, or future reporting, thereby altering the relations that later records describe. The interpretation itself can then respond to those records. This feedback makes the model boundary consequential. Treating the institution as an external observer can omit effects of its own intervention. Legal systems

theory already offers resources for examining legal plurality and structural relationships [19]; the present formulation claims no priority for the general recognition of institutional coupling.

8.2 Observations, Counterfactuals, and Identification

The figures in the retained-history appendix separate the constructed result from its computational support. Figure 2 shows the matched observation interval and later separation; Figure 3 reports the mechanism controls; and Figure 1 displays temporal and spatial refinement. Their inclusion permits inspection of the mathematical example used in this section. None of these figures measures a legal relationship or the experience of a participant.

Observation selects a projection of a represented history. A signed waiver may reveal the action while leaving the employee's anticipated sanction and available protection hidden. The completed three-field study provides a controlled mathematical example: opposite retained exposures coexist with exactly matching accessible fields during a gated interval, then produce different outputs after the same gate opens. The agreement is engineered through a reset and a stipulated retention mechanism. Its legal relevance is the logical possibility of omitted historical state, whose presence in a real case still requires evidence.

An independently executed finite transition model gives a simpler counterfactual illustration. Two mechanisms generate the same acceptance observations, retaining three resource units throughout. Under refusal, one mechanism preserves those units and the other temporarily reduces them to zero. Both restore the resource to three at closure. With a stipulated floor of one unit, refusal satisfies the path constraint in the first mechanism and violates it in the second. Refusal remains physically available in both. The comparison distinguishes possible action, acceptable trajectory under a chosen constraint, and terminal restoration.

The enumeration examined all sixteen combinations of integer floors and refusal losses from zero through three, with ten named checks. Its result follows from the authored transition rules. It does not establish the prevalence of threatened deprivation, infer a real person's alternatives, or measure justice. Viability theory provides an established mathematical setting for state constraints over time [5]; the finite example evaluates selected paths and does not claim to compute a general viability kernel.

Table 2 gives the full floor-loss comparison. Its entries make visible the dependence of a path's admissibility on both the stipulated transition and the selected constraint, including combinations for which the two refusal paths receive the same classification.

Counterfactual controls must be interpreted with similar care. In a legal reconstruction, a possible refusal can be described analytically even when experimentally imposing its consequences would be inappropriate or impossible. The investigator may need observational evidence, institutional records, or naturally occurring contrasts. A simulated intervention estimates the response of the stipulated model. It becomes evidence about the world only through support for the model and its application conditions.

Identification remains a central obstacle. The scalar-field calculations show that two heat-decay laws can agree on an entire first-mode initial subspace while differing on a second-mode probe. Within their known two-parameter family, two mode rates identify the parameters. Actual relational inquiry rarely starts with so sharply delimited a family. Several different mechanisms may fit the same record and respond differently outside it. A responsible use of the framework therefore retains warranted alternatives and identifies which further evidence could discriminate among them.

8.3 Normative Projections and Conditional Dissimilarity

Table 3 distinguishes the preserved structures considered in the mathematical appendices. Figure 4 shows why a coordinate map must carry the readout with it; Figure 5 shows the consequences of changing the clock. Figure 6 illustrates how a further preparation distinguishes laws that agree on a restricted domain. Figure 7 makes the dependence of approximate agreement on the metric, tolerance, and horizon explicit. For legal comparison, these are examples of choices that must preserve the roles and temporal conditions selected by the issue.

A numerical comparison requires an explicitly defined projection of each candidate history onto issue-relevant features. Let H_1 and H_2 be histories already placed on the same role and time domains by fixed admissible mappings. Let $\Phi_N(H_i)(t)$ be a vector of features selected for norm N , expressed in justified units and square-integrable on $[0, T]$, and let W_N be a fixed positive semidefinite weight matrix. Equation 2 defines a conditional dissimilarity over a common finite horizon T .

$$d_{N,T}(H_1, H_2) = \left[\int_0^T (\Phi_N(H_1)(t) - \Phi_N(H_2)(t))^T W_N (\Phi_N(H_1)(t) - \Phi_N(H_2)(t)) dt \right]^{1/2}. \quad (2)$$

Equation 2 makes feature selection, weighting, and duration inspectable. With fixed projections, weights, and alignment, it induces a pseudometric on histories: distinct histories can have zero distance because the projection omits their differences. A separate optimization over mappings for each pair requires additional conditions before metric properties can be asserted. The expression also leaves instantaneous events outside its ordinary integral unless the representation adds an appropriate discrete component or event measure.

The weights cannot derive their authority from predictive convenience alone. A weight determining the significance of threatened deprivation requires reasons connected to the legal issue and the affected interests. Features with different units require justified scaling. Under Rule R, moreover, the required conditions remain separately assessed. A close average match cannot compensate for a missing prior threat or an established lack of material influence. The distance can help inspect resemblance; it does not replace the rule or its burden.

Uncertain reconstructions can be compared by evaluating the score and rule conditions across warranted candidate histories. The set of candidates should reflect support and counterevidence, rather than arbitrary parameter proliferation. Stability across that set can identify a conclusion whose justification depends little on a disputed premise. Variation can identify the premise that matters. The proportion of sampled models supporting an outcome is not an evidential probability unless the sampling and probabilistic interpretation have independently been justified.

8.4 Transient Burdens and Long-Term Agreement

Shared long-term behaviour provides a further test of the comparison's scope. In the executed field example, two decaying fields approach the same zero attractor while differing in their rate of decay. Their initial condition is $1 + 0.4 \cos x$, their diffusion coefficient is 0.05, and their decay rates are 0.2 and 1. The time integrals of the spatial mean over the infinite horizon are 5 and 1. The means remain above 0.4 for approximately 4.58145 and 0.91629 time units, respectively.

Figure 8 displays these differences. The field values have no independently established interpretation as human welfare or legal injury. If an application had supported such an interpretation, however, the example shows why a common eventual state would leave an accumulated-burden question unresolved. Terminal restoration in the finite model makes the same inferential point through a different construction.

Eglash and Garvey's work on attraction basins for generative justice is an existing connection between dynamical concepts and normative inquiry [11]. The present paper's conditional use differs in scope and requires comparison with that antecedent. Its immediate point is that stability, convergence, and restoration are descriptive properties. Their value depends on what persists, who bears the transient conditions, and the norms under which those conditions are assessed.

9 Actions, Boundary Descriptions, and the Limits of Equivalence

This section evaluates action-based and spin-foam-type comparisons as possible extensions of the field framework. It distinguishes a functional over histories from an individual functional value, then examines exact boundary agreement with different interior observables. The purpose is to clarify mathematical resources and their inferential limits before any social interpretation is proposed.

9.1 Energy, Mobility, and Path Functionals

The action appendix supplies distinct comparisons for the claims developed here. Figure 9 displays motions generated by a shared energy and different mobilities. Table 4 evaluates each motion under several residual functionals. Figure 10 presents distinct fixed-endpoint histories with the same scalar action. The sequence identifies the specific information preserved by each equality, allowing its relevance to a historical judgment to be assessed separately.

An energy functional assigns a value to a state, while a dynamical rule specifies how the state changes. In gradient-flow modelling, the energy and the dissipation or mobility structure jointly contribute to that rule [22]. The executed action study uses a finite invariant two-mode field representation with coefficient vector a and energy $E(a) = a^\top K a / 2$, where $K = \text{diag}(0.55, 1)$. Different positive mobility matrices give the law $\dot{a} = -M K a$ and therefore different motions despite the shared energy.

For $M = I$ and $M = 2I$, the motions follow the same oriented orbit at different speeds under the identity state map. For $M = \text{diag}(0.4, 2)$, the direction in the original coefficient coordinates changes as well. A nonlinear homeomorphism nevertheless relates the first and third flows by same-clock topological conjugacy on this finite coefficient space. These statements coexist because they preserve different structures. An applied model must decide which state interpretation and time correspondence its question requires.

A residual path action measures departure from a selected evolution law. For a constant positive mobility, the study uses $I_M[a] = \frac{1}{4} \int_0^T (\dot{a} + M K a)^\top M^{-1} (\dot{a} + M K a) dt$. A solution of its own deterministic law has zero residual. Different laws can therefore give zero values to different paths under different functionals. The common number does not identify the motion or the model.

The study also compares distinct paths under the same functional and fixed zero endpoints. Two explicitly constructed sinusoidal histories have equal action, approximately 0.459675138, while differing at intermediate times. Equality of one scalar path summary therefore leaves history unresolved even before different models are compared. A coordinate change preserves the functional value only when the relevant energy, mobility, and metric are transformed consistently. Holding the wrong metric fixed changes the computed quantity.

The stochastic interpretation remains limited. For the associated finite-dimensional small-noise process, the residual action has a large-deviation role concerning path neighbourhood probabilities [1]. Its value is not a normalized finite-noise probability density. The calculation does not show that people choose paths minimizing a social action, that legal norms are energy minima, or that a fitted cost function acquires normative authority. Those would be additional modelling and justificatory claims.

Actions can nevertheless organize useful questions. They make explicit which histories are permitted, which departures are penalized, what boundary conditions are imposed, and which quantities a comparison preserves. In a legal application, such choices would need evidential support and reasons tied to the issue. A theory that selects an action solely because it produces a desired judgment would leave the judgment's justification circular.

9.2 Boundary State Sums and Interior Observables

The finite-group study examines a positive $\mathbb{Z}_2$ lattice-gauge model on a strip of faces. Edge variables take values $+1$ or -1 , and each face variable U_f is the product of its boundary edge variables. A face weight is $w_f(U_f) = 1 + t_f U_f$, with $|t_f| < 1$. Summation over internal edges uses normalized Haar weights. Character expansion produces face labels and internal edge constraints, giving a genuine finite-group spin-foam-type representation [7, 9].

For this strip family, the source-free boundary amplitude is exactly $A(B) = 1 + B \prod_f t_f$, where B is the boundary holonomy. The two-face choices $(t_1, t_2) = (0.2, 0.8)$ and $(0.4, 0.4)$ have the same product and therefore the same complete boundary kernel. At positive boundary holonomy, both amplitudes equal 1.16; at negative holonomy, both equal 0.84. These equalities concern a specified normalized boundary description.

The interior comparison differs. Conditional on $B = +1$, the first-face expectation is approximately 0.862069 for the first pair and 0.689655 for the second. Interior source probes therefore distinguish the models. Figure 11 juxtaposes boundary agreement and interior separation. The absolute boundary amplitudes depend on the normalization convention; conditional expectations cancel an overall normalization, while equality across differently normalized boundary models would require separate examination.

Figure 12 shows the boundary-preserving family and its varying interior expectations. The family demonstrates that agreement of the selected boundary kernel can persist across a range of internal parameters. An application that assigns significance to an interior observable must include that observable in the comparison or justify its exclusion.

Spin-foam models in quantum-gravity research involve further representation-theoretic, geometric, and quantum structures [23]. The finite example uses a controlled portion of the state-sum machinery. Its relevance here is that summing over unobserved interiors can preserve a boundary description while discarding distinctions accessible to additional observables. Treating a case file as a boundary record is, at present, an analogy. A literal legal state-sum model would require a warranted history space, weights, composition rule, and account of what its values mean.

The exact checks cover primal–dual agreement, gauge invariance, boundary configurations, and composition under gluing. They establish the internal correctness of the finite construction. They do not validate the selection of legal evidence or assign probabilities to rival case histories. Source-free boundary equality, source-dependent equality, and equality of full interior laws remain different claims. The broader lesson for comparison is to name the observable algebra or probe family whose agreement has actually been established.

10 Public Commonality and Interpretive Correction

This section develops the institutional implications of uncertain relational reconstruction. It begins with explicit normative grounds, identifies commitments that can be shared amid disagreement, and specifies a proportionate account of correction. The argument addresses institutions claiming authority over affected persons; its conclusions depend on the stated grounds of that authority.

10.1 Equal Standing and Accountable Reasons

The first premise is equal standing in the justification of an institution’s exercise of authority. A decision can impose a burden on a person even when that person rejects its factual or normative account. Equal standing requires the institution to address the person’s relevant interests and objections through reasons appropriate to its mandate. It permits differences in outcome when authorized norms and supported circumstances justify them. It opposes assigning lesser justificatory importance to a person’s position merely because that position is inconvenient or unfamiliar.

The second premise is protection against material error. Institutions often decide under uncertainty, so eliminating every possibility of error is unavailable. The relevant demand is to use procedures reasonably suited to identifying consequential mistakes, in light of the affected interests and the institution’s resources. Where a single disputed historical premise controls a serious burden, the reason to examine that premise is stronger than where the same disagreement could make no difference under the governing rule.

The third premise is accountable authority. A decision-making institution should connect its outcome to a norm it is authorized to apply and to facts supported through its accepted evidential process. The connection allows a reviewing body or affected person to distinguish an evidential disagreement from an unauthorized change of rule. A mathematically elaborate model can assist this connection only if its assumptions remain available for inspection. Technical complexity does not extend the institution’s mandate.

Together, these premises support reason-giving and targeted contestation. If a decision turns on a reconstructed dependency, the reasons should identify the relevant resource, the evidence of control, the

claimed temporal mechanism, and the condition of the rule that the mechanism supports. If counterevidence would defeat the condition, the decision should address it. This requirement follows from the significance of the inference to the authoritative outcome, without requiring a complete public history of every private interaction.

The premises also distinguish an epistemic observation from a normative conclusion. The fact that interpretation is situated supports attention to possible limitations in access and categories. A duty to offer correction arises only when that observation is combined with commitments concerning standing, material error, and authority. Other institutional purposes or competing rights can constrain how the duty is implemented. Those constraints should receive reasons of their own.

10.2 Common Commitments amid Substantive Disagreement

Public commonality concerns commitments sufficient for people with different interpretations to participate in a reasoned process. Agreement can concern the question to be decided, the provenance of a record, the admissibility of an objection, or the relevance of a possible counterexample. Such common ground leaves room for disagreement about the best reconstruction or the preferred rule. Niemeyer and Dryzek's treatment of meta-consensus provides an established deliberative antecedent for examining shared commitments beyond agreement on outcomes [18].

In the fictional case family, parties might agree that the message archive is authentic while disagreeing about whether prior scheduling changes were sanctions. They might agree that a threat existed while disagreeing about material influence. Stating these locations of disagreement narrows the work required. A demand for consensus on the whole narrative could obscure the specific evidence capable of resolving the decisive point.

The framework therefore recommends an inspectable connection between each material finding and its grounds. For Rule R, an institutional record could state the evidence supporting each required condition, the strongest counterevidence, the treatment of uncertainty under the stipulated burden, and the resulting remedy. The record need not assign a scalar score to the employee's entire relationship. Its purpose is to expose the consequential inference at a level both parties can understand and challenge.

Model sensitivity can serve the same purpose when modelling is justified. If all well-supported reconstructions agree that the threat followed the waiver and none supplies an earlier anticipated threat, the timing conclusion may be robust to other uncertainty. If reasonable reconstructions differ on the effectiveness of the protective route, that issue may require additional evidence. Robustness across selected models is evidence about dependence on assumptions. It does not amount to a vote by models and cannot alter the burden of proof through the number of parameter settings an analyst happens to generate.

Participation also requires usable interpretive access. A technically available model can remain practically inaccessible if a party cannot understand its variables, obtain the inputs, or challenge a mapping. Appropriate assistance, intelligible explanations, and access to material records can therefore matter to the force of a supposed opportunity to contest. Their form should fit the proceeding's stakes and resources. A complex simulation that creates an unnecessary barrier can undermine the justificatory purpose for which it was introduced.

10.3 Material Correction, Finality, and Proportionality

Correction should address identified routes by which an outcome could be materially mistaken. New evidence of a prior threat can change R0's support. Evidence of independent decision-making can change the application of Rule R to a case initially understood as R1. A correction to the governing rule can affect several cases whose records remain unchanged. These changes call for different procedures because they concern different grounds of the decision.

Finality also has value. A prolonged dispute consumes resources, imposes uncertainty, and can prevent reliance on an institution's decisions. A fallible interpretation therefore does not by itself justify indefinite reopening. The case for review depends on the significance of the alleged error, the quality and availability of the challenge, the consequences of delay, and the institution's authority to revisit the

matter. A proportionate process can require a threshold showing of materiality before imposing further investigative costs.

Urgency can require provisional decisions. An institution might preserve a resource while investigating whether its withdrawal would defeat meaningful review. The justification would depend on the applicable norm, the expected consequences of action and inaction, and the reversibility of available measures. The field examples contribute only the descriptive reminder that temporary conditions can matter even when later states converge. A duty to provide interim protection requires the independent normative and institutional argument.

Privacy and confidentiality impose further limits. A relational account can tempt an investigator to seek an expansive history of personal associations. The proper scope follows the material issue and the evidential value of the proposed inquiry. Access can be limited, sensitive details can be protected where compatible with meaningful response, and weakly relevant demands can be refused. Equal standing includes interests in protection from disproportionate inquiry as well as interests in exposing an omitted history.

This account of correction is compatible with decisive judgment. Once the institution has applied its authorized norm, addressed material evidence, and provided the appropriate opportunity for challenge, it can reach an outcome despite residual uncertainty. The reasons should state what has been established and how the burden resolves what remains. Such closure preserves the difference between justified institutional decision and a claim to exhaustive knowledge of the relationship.

10.4 A Reviewable Decision Record

This subsection translates the account of reasons into a possible decision record for the fictional tribunal. It distinguishes the status of factual premises, the interpretation of the rule, and the treatment of unresolved questions. The example specifies a reviewable form of justification without prescribing a universal administrative template.

For a finding corresponding to R1, the record would first identify the issue as restoration under Rule R. It would state the three conditions and the stipulated burden. It would then identify the supported resource and threatened reduction, the representative's capacity, and the evidence that refusal was linked to the consequence. The finding on material influence would address the employee's decision separately. The finding on protection would explain the accessibility and timing of the available route.

The record would also state the strongest supported counteraccount. If an alternative scheduling explanation had been advanced, the tribunal would identify the records bearing on it and explain their effect on conditionality or capacity. If independent planning were supported, it would address the causal condition. The response need not disprove every imaginable history. It should explain why the materially supported alternatives do or do not change the result under the burden.

For R0, the same form could produce a shorter decision. Authenticity of the message and waiver may be established while the relevant history remains unresolved. The tribunal would identify which required conditions lack sufficient support and what follows under the burden. It would avoid redescribing the procedural result as a positive finding of free choice. If a further inquiry were authorized and proportionate, the record could identify the evidence needed to resolve the material gap.

For R6, the rule difference would appear explicitly. A conclusion under Rule F would identify authenticated signature and formal capacity as its conditions. An objection that F provides inadequate protection would remain available as an objection to the rule or its institutional authorization. It would not be answered merely by repeating that the signature exists. Distinguishing application from justification helps locate the proper level and route of challenge.

10.5 Model Governance and Evidential Revision

This subsection considers how a model could enter the decision process without obscuring responsibility. It identifies disclosure, reproducibility, sensitivity, and revision requirements proportionate to the role

the model plays. These are proposed institutional conditions, distinct from the completed mathematical checks in this project.

A model used only to suggest investigative questions has a different role from one offered as evidence of a hidden dependency. The latter use requires a stronger connection between variables, assumptions, and records. The decision should identify which outputs depend on stipulated parameters and which parameters have independent support. A visually persuasive trajectory cannot supply the missing connection by itself.

Reproducibility concerns the calculation conditional on its inputs. An opposing analyst should be able to determine whether the reported result follows from the stated equations and data, subject to justified confidentiality arrangements. Empirical adequacy concerns whether the representation is supported for the case. A reproducible error in representation remains an error; an apparently plausible representation with an unverifiable calculation leaves a different weakness. Review should keep the two questions distinct.

Sensitivity analysis should focus on assumptions that are both uncertain and consequential. Varying a parameter over a convenient numerical interval has limited relevance if that interval lacks evidential justification. Conversely, a discrete alternative such as the absence of a prior communicated threat may matter more than fine adjustments to a fitted coefficient. A reviewable analysis should explain why its candidate set includes the alternatives it does and how excluded possibilities were treated.

Revision needs a material trigger. New evidence can change the support for a parameter, a competing mechanism, or the governing interpretation. A software correction can change a reported calculation while leaving the underlying record intact. A revision record should state which route occurred and identify affected conclusions. This discipline supports proportionate correction and protects against quietly altering a model to preserve a preferred outcome.

The institution remains responsible for its judgment. A modelling team can explain technical consequences and identify uncertainty, while the authorized decision-maker must connect those consequences to the norm and burden. Public authority cannot be transferred to an optimization routine through an unexplained choice of objective. The field framework is defensible in this setting only when it makes the relevant reasons easier to inspect and challenge.

11 Objections and Institutional Limits

This section examines objections to the framework's necessity, reliability, proportionality, and normative adequacy. Each objection identifies a condition under which the proposed approach could fail or add little. The responses define limits that a future institutional study should test directly.

The strongest redundancy objection is that careful legal analysis already reconstructs histories, evaluates counterfactuals, and compares reasons under norms. Case-based representations can encode temporal and causal factors, while ordinary evidential reasoning can preserve uncertainty without a field equation. This objection sets the appropriate baseline. The proposed framework has added value only if it makes a consequential dependency clearer, reveals an omitted alternative, improves a comparison, or reduces a demonstrable error. A simpler method that accomplishes these tasks should be preferred where the additional machinery supplies no corresponding benefit.

A narrative objection concerns persuasive overfitting. An investigator can assign enough latent states and feedback mechanisms to explain almost any observed sequence. A detailed relational account may then seem rigorous because it contains many connections, even though the evidence poorly constrains them. The response requires explicit rival histories, scrutiny of unsupported links, and attention to evidence that could contradict the proposed mechanism. In a formal model, sensitivity and identification analysis should precede strong claims about an estimated hidden state.

A stereotype objection concerns the transfer from group-level patterns to individual cases. Information about structural vulnerability can motivate an inquiry into access or resource dependence. It does not establish that a particular employee lacked protection, anticipated a threat, or acted because of it. The framework should preserve the individual evidence bearing on those conditions and recognize coun-

terevidence such as R4. Otherwise, relational terminology can reproduce the categorical shortcuts it was intended to expose.

A measurement objection challenges numerical weights and field values. Some features lack a defensible interval scale, and others concern rights whose force resists compensatory aggregation. The weighted comparison in Equation 2 has an appropriate use only when its inputs and interpretation are justified. Qualitative structured comparison can remain preferable. Mandatory conditions, protected statuses, and procedural burdens should retain their separate operation even where a quantitative model helps organize surrounding evidence.

An institutional objection concerns cost and asymmetry. A well-resourced party may produce sophisticated reconstructions that are difficult to challenge, while a poorly resourced party cannot obtain the records needed to demonstrate a simple dependency. More modelling can consequently increase imbalance. Evaluation should examine who can inspect inputs, contest assumptions, and obtain assistance. The proposed commitments to standing and accountable reasons support reducing unnecessary complexity and addressing consequential access barriers within the institution's remit.

A conservatism objection targets commonality. Shared procedures and categories can preserve entrenched exclusions, particularly when those disadvantaged by the process cannot challenge its terms. Agreement therefore provides a limited indicator of legitimacy. The proposed normative premises also require scrutiny of whose interests and objections receive standing and whether the rule itself can be criticized through an authorized route. A stable consensus can coexist with a defective rule, as the comparison between R and F illustrates at a fictional level.

A relativism objection raises the converse concern: if every interpretation is situated and every comparison is norm-relative, arbitrary judgments might appear equally defensible. Norm relativity does not supply that conclusion. A judgment can misstate the applicable rule, rely on an inauthentic record, reverse event order, ignore counterevidence, or use a mapping inconsistent with its own criterion. These are available grounds of criticism within a defined institutional problem. Disagreement about the justice of the rule remains a further normative question.

Finally, a mathematization objection concerns the attraction of field theory, actions, and spin foams. Mathematical sophistication can create an impression of explanation without an empirically supported correspondence to the case. The completed studies establish exact examples and checked simulations within stipulated structures. Their legal application remains conditional. A future empirical study should be able to show precisely where the formal representation changes a reason, identifies evidence, or improves a task relative to a strong qualitative baseline. A figure or a low action value alone would be insufficient evidence of such an improvement.

12 Conclusions and Institutional Research

This section states the paper's principal conclusions and the conditions for further evaluation. The argument connects historical reconstruction with norm-relevant comparison, while the completed computations clarify limited forms of equivalence. Institutional recommendations rest on the explicit normative commitments developed above.

Legal similarity depends on the issue and on the evidentially supported processes that bear on it. Identical words can arise within different dependencies; different words can activate comparable expectations; and a shared result can conceal different available alternatives. A generative-relational account reconstructs these connections while preserving provenance, rival explanations, temporal direction, and uncertainty. Its adequacy depends on the strength of those connections and their relevance under an authorized norm.

The fictional case family demonstrates this discipline in a controlled setting. Prior sanctions and inaccessible protection matter under Rule R only when the required causal influence is supported. Independent planning and changed temporal order can defeat that condition or another required element. A change to Rule F alters the comparison itself. These results are conditional applications of invented rules, giving readers an inspectable example of the framework's operation and limits.

The field, action, and finite-group calculations show several ways in which a selected equivalence can leave relevant differences unresolved. Projected observation can omit retained state, a common attractor can omit transient burden, a shared energy can omit mobility, and a boundary kernel can omit interior observables. A legal application must establish which, if any, of these structures corresponds to its evidence and norms. Their mathematical validity supplies no independent validation of adjudication.

Further research should compare the framework with strong legal-factor and causal-reconstruction baselines on a specified issue and jurisdiction. Expert evaluation could assess whether the representation improves identification of material evidence, treatment of counterevidence, explanation of distinctions, and calibration of uncertainty. The design should include cases where historical reconstruction changes the answer and cases where the governing rule makes the additional history immaterial. Burden, disclosure, privacy, and review arrangements should be specified before institutional use.

No such legal-response experiment or expert study has been completed in this project. The existing 24-agent pilot concerns reality attribution and supplies no evidence of legal performance. The present paper offers a discussion framework and a set of explicit mathematical comparisons. Its continued use should depend on whether it improves accountable reasoning under realistic evidential and institutional constraints.

12.1 Materials and Reproducibility

This subsection identifies the preprint’s complete technical support and its underlying files. Appendix A works through all finite refusal-path comparisons, and Appendix B develops the restrictions on normative comparison maps. The subsequent appendices reproduce the complete field-equivalence, action, and finite-group analyses, including their derivations, numerical results, and figures.

The finite transition illustration is preserved in `experiments/finite_relations.py` with its protocol, enumerated states, and results. The complete three-field study appears in Appendix D. The dynamical, action, and finite-group appendices include their own definitions and calculations, with scripts, arrays, exact checks, and exported figures indexed by `experiments/field_comparison/README.md`. These materials support inspection of the synthetic claims without treating the fictional case family as observed legal data.

Appendix C incorporates the fuller relational-field formulation and its observation and comparison assumptions. Source ledgers in `research/` record internet verification before citation and the access level supporting each attribution. Further close comparison with neighbouring scholarship, independent human review, and validation in a specified legal setting remain outstanding.

A Finite Histories and Counterfactual Refusal

This appendix specifies the finite illustration used to distinguish observed action, counterfactual response, path constraints, and terminal restoration. It works through the complete small parameter family. This stipulated mathematical construction does not instantiate an actual legal finding or estimate a person’s available resources.

The model has a decision stage, a response stage, and closure. The recorded action is acceptance or refusal. A resource starts at three units. Under acceptance, all compared mechanisms retain three units throughout. Under refusal, a mechanism with loss parameter $\ell \in \{0, 1, 2, 3\}$ reduces the resource to $3 - \ell$ at the response stage and restores it to three at closure. The loss is a stipulated response law. The acceptance observation therefore contains no information that distinguishes values of ℓ within this family.

The physically available choices are the same across the compared mechanisms. A separate path condition requires the resource to remain at least f at every stage, where $f \in \{0, 1, 2, 3\}$ is a stipulated floor. Refusal satisfies that condition exactly when $3 - \ell \geq f$. Table 2 gives the complete family. The result follows by inspecting the response stage, since the other resource values are three.

Table 2: Refusal paths satisfying the stipulated resource floor. Every acceptance path satisfies each listed floor. Every refusal path returns to three units at closure, so a terminal-only check omits the distinctions shown [here](#).

Floor f	Losses satisfying the path condition	Losses violating the path condition
0	0, 1, 2, 3	None
1	0, 1, 2	3
2	0, 1	2, 3
3	0	1, 2, 3

At floor one, the zero-loss and three-unit-loss mechanisms have exactly matching acceptance observations and different classifications of refusal paths. Their unlabeled transition graphs also have matching adjacency. The difference becomes available only when the resource labels and floor are retained. The example identifies information lost through two reductions: observing acceptance alone and comparing a graph after dropping its consequential labels.

Closure does not recover the omitted history. The same final value is compatible with a path that stayed above the floor and a path that temporarily crossed it. The numerical restoration is a reset of one variable. It supplies no evidence that a prior consequence, lost opportunity, or experiential harm has been reversed. A richer complete state could retain such history; the example concerns its omission by the selected terminal observation.

The comparison is related to Rule R only through an explicitly conditional interpretation. If a resource variable and floor were evidentially and normatively justified, the difference in refusal paths could bear on available protection. The finite model itself establishes neither credible communication nor material influence on an actual waiver. Those remain separate requirements of the fictional rule. This limitation explains why a useful counterfactual model can contribute to one condition without deciding the complete claim.

The saved enumeration includes all sixteen floor–loss combinations and ten checks. Exact evaluation establishes the properties of the stipulated transition family. A real application would require evidence for the response rule, the resource interpretation, the relevant floor, and the person’s knowledge or anticipation where the norm selects it. The program and raw enumerated states remain available in the project archive.

B Restrictions on Norm-Relevant Comparison Maps

This appendix develops the mathematical limitations of a norm-relative dissimilarity. It separates projection, tolerance, role correspondence, time mapping, and noncompensatory conditions. The constructions clarify requirements for a comparison; they do not derive a legal norm from a metric.

For fixed feature maps, alignment, horizon, and positive semidefinite weights, Equation 2 is the pull-back of a weighted L^2 seminorm. Symmetry and the triangle inequality follow from that seminorm. Distinct histories can have zero distance when their selected features agree almost everywhere, or when their difference lies in the null directions of the weights. Thus, zero distance identifies an observational class under the declared comparison, while leaving excluded history unresolved.

An instantaneous event can have zero contribution to an ordinary time integral. If the norm makes the occurrence or timing of that event consequential, an event indicator, atomic measure, or separate condition is required. Similarly, if the identity of an affected person matters, aggregation over participants can lose a decisive distinction. A mathematically valid metric can therefore remain unsuitable for the particular legal issue.

Tolerance introduces another limitation. For constant scalar feature histories with values 0, $0.75\varepsilon/\sqrt{T}$, and $1.5\varepsilon/\sqrt{T}$ and unit weight, adjacent distances under Equation 2 are 0.75ε , while the first-to-third distance is 1.5ε . The relation of lying within tolerance ε is consequently nontransitive. A chain of approximate matches cannot automatically justify treating all members as interchangeable.

Mappings can produce a further loss. A permutation that exchanges a resource controller with the person dependent on that resource may preserve an unlabeled graph and alter every normatively meaningful role. If a comparison permits permutations, it must state the role conditions they preserve. An optimization over unrestricted correspondences can obtain a small numerical discrepancy by removing the distinction the legal issue requires.

Time transformations have an analogous effect. Replacing a trajectory $z(t)$ with $z(ct)$ for $c > 0$ preserves the order of its states while changing residence times. Where a finite exposure integral is defined on corresponding horizons, the change of variables supplies a factor $1/c$. Equal oriented orbits can therefore carry different accumulated quantities. Whether that difference matters depends on the interpretation of the quantity and the norm; the map itself supplies no answer.

Noncompensatory conditions should retain separate operation. Under Rule R, a missing prior threat cannot be offset by close agreement on many contextual features. A weighted sum may support retrieval or descriptive inspection while the rule's required conditions still determine its application. A more elaborate model should make these conditions clearer, rather than absorb them into a score whose tradeoffs lack institutional authority.

C Relational Fields and Norm-Relevant Historical Comparison

This section develops the field representation proposed for legal comparison. It distinguishes relational variables, their evidential reconstruction, and the normative conditions governing comparison. The construction specifies an analytical framework whose empirical and doctrinal adequacy require separate evaluation. The numerical experiment accompanying Paper I supplies a limited example of hidden historical state; the legal application below has not been estimated or tested against adjudicated cases.

C.1 Relational Domains and Field Variables

This subsection defines the objects distributed over the comparison domain and identifies the assumptions introduced by their representation. The purpose is to preserve relationships between participants alongside attributes of individual participants.

Let Ω denote an explicitly described domain of participants or institutional positions. A relation field $R(x, y, t)$ assigns a vector of directed relational properties to the ordered pair (x, y) at time t . Possible components concern control of access to income, housing, information, or permitted participation. Each component requires its own operational definition and evidential provenance. Dependence in one direction can coexist with a different dependence in the reverse direction; symmetry is an empirical or substantive assumption.

A finite participant domain produces a time-dependent collection of matrices. A continuous domain introduces a field approximation whose aggregation, measure, and resolution require justification. Physical distance, institutional proximity, and semantic resemblance describe different relations. A diffusion operator is appropriate only after specifying an exchange process and its relevant geometry. The periodic spatial ring used in the companion synthetic experiment carries no presumption of suitability for legal relations.

Individual resources and available actions can be recorded in a local field $z(x, t)$. An interpretive field $p(x, t)$ may represent specified expectations or classifications, provided that an observation procedure or explicit latent-variable interpretation is supplied. Such variables describe a proposed reconstruction. Their names alone supply neither measurements nor causal explanations. In particular, an investigator's designation of a variable as coercive cannot substitute for a legal argument establishing coercion.

C.2 Evidence, Memory, and Competing Reconstructions

This subsection relates observed records to candidate field histories. It identifies the information omitted by an observation procedure and the additional assumptions needed for causal reconstruction.

An evidential record consists of observations generated through a stated procedure, with missingness, selection, provenance, and measurement error described separately. Candidate histories of R, z, p can agree about a recorded sentence while differing in prior dependence, anticipated sanctions, and available alternatives. The admissible candidates must remain answerable to evidence and disconfirmation. An unconstrained collection of possible stories would supply little assistance to judgment.

Historical dependence may be represented through an explicit memory variable or through a time-dependent integral of earlier states. These representations can be mathematically equivalent under stated assumptions. A complete state that retains the relevant history therefore remains compatible with the proposed approach. The criticism concerns inadequate observation or compression of that state. A signature at a terminal time is a particularly limited projection of a history when the disputed issue concerns the circumstances of agreement.

The same observed history may also be compatible with distinct mechanisms. Evidence that an action occurred does not identify the consequences of refusal. A counterfactual reconstruction must specify the altered action, the mechanism held fixed, the time horizon, and the evidence supporting the resulting response. The completed finite transition illustration in the project isolates this distinction through authored alternatives. It supplies a mathematical counterexample to unique identification from the realised path, with no factual finding about an actual person's options.

C.3 Comparison Maps and Normative Relevance

This subsection specifies how two reconstructed histories can be compared and locates the legal choices within that operation. The construction serves as a diagnostic representation of selected differences.

Before comparing cases, identify a legal issue ν , a justified mapping between relevant participants or roles, and a common temporal reference. The mapping must retain attributes material to that issue, including asymmetries of authority or vulnerability when legally relevant. A graph isomorphism that exchanges protected and controlling roles may erase precisely the distinction under examination. A non-linear alignment of time can similarly remove the duration or order of a material event. Every such transformation requires an explicit rationale.

For two histories already represented on a common domain and interval, an illustrative diagnostic is given in Equation 3.

$$d_\nu^2(R_1, R_2) = \int_0^T \int_{\Omega \times \Omega} \sum_{k=1}^K w_{\nu,k}(x, y, t) [R_{1,k}(x, y, t) - R_{2,k}(x, y, t)]^2 d\mu(x) d\mu(y) dt. \quad (3)$$

In Equation 3, k indexes defined relational components, μ is a declared measure on the domain, and the fixed nonnegative weights $w_{\nu,k}$ incorporate any required scaling of units. For a finite participant set, the integrals over participants become weighted sums. For fixed weights and mappings, d_ν is the distance induced by a weighted seminorm on histories with finite weighted integral; zero distance can leave differences in unweighted components unresolved. The numerical scale depends on those choices.

The weights express a proposed selection of relevance. They cannot acquire legal authority from numerical fit or mathematical convenience. Even a well-supported diagnostic distance may omit decisive thresholds, noncompensatory rights, exceptions, or procedural rules. A sufficiently close value consequently establishes similarity only for the selected representation. A legal conclusion requires an additional argument connecting the preserved relations to the governing norm. Where such aggregation conceals the norm's structure, componentwise comparison or an explicit rule-based analysis is preferable.

Reconstruction uncertainty also remains material. Comparison should expose whether its result changes across the histories supported by the record and across defensible relevance mappings. Counting candidate models does not establish the probability of a legal fact. The choice of models, their evidential support, and any probabilistic assumptions require separate justification.

C.4 Interpretive Feedback and Institutional Assessment

This subsection identifies the potential contribution of field dynamics to the general–particular relation and its limits as a theory of justice. It distinguishes changes within an institution from the justification of the institution’s response.

A legal classification can alter later access to resources, expected sanctions, and available interactions. A field model can represent those effects through specified couplings between interpretive decisions and relational conditions. Conversely, evidence about a particular history may reveal that a general category omits a normatively material relation. This reciprocal movement is a proposed object of analysis; a differential equation supplies no authority to revise doctrine.

Dynamic persistence and stability concern the evolution of a stipulated system. They leave the acceptability of the persistent arrangement open. A stable relation can contain domination, and a justified intervention can disrupt an established pattern. Normative assessment therefore requires publicly defensible premises concerning the issue, affected interests, institutional authority, and opportunities for contestation. The field representation contributes by making selected dependencies and omissions inspectable. Its added value over a competent temporal, causal, or factor-based representation remains a comparative research question.

This proposal welcomes criticism, correction, and identification of disciplinary precedents. It claims no conceptual priority. AI assistance, publication licensing, and rights reservations are recorded in the paper’s accompanying `i_note.tex`.

D Retained History in a Spatial Field Model

This appendix contains the complete retained-history construction, its derivation, numerical verification, and mechanism controls. The same project calculation is included in both papers for independent inspection; this reuse constitutes no additional experiment.

D.1 Theoretical Scope and Modelling Commitments

This section locates the model within the account of historically formed interpretation. It distinguishes the spatial field construction, its neighbouring mathematical traditions, and the explanatory limits of the selected variables. The purpose is to make a dependence on relations, history, and observation conditions explicit enough to inspect.

A field assigns a value to every position in a specified domain. Here, local evolution and neighbourhood exchange generate trajectories of environmental and interpretive states. The resulting object is a spatial field model with infinitely many degrees of freedom before discretisation. The numerical grid approximates that domain. A vector field on a finite state space becomes available after discretisation, but spatial locality, boundary conditions, and grid convergence retain independent roles in the present construction.

Spatially extended nonlinear neural fields provide an established example of mathematical models whose patterns depend on coupling and stimulus [2]. Coupled nonlinear population-response models form another neighbouring tradition [27]. These references acknowledge mathematical precedents. Their neuronal interpretations do not validate the social interpretation proposed here. The present equations are phenomenological assumptions, with a direct derivation of the specific comparison result given below. A microscopic theory, empirical calibration, and variational action principle remain outside this construction.

D.2 Domain, State Variables, and Relational Coupling

This section defines the spatial domain, interprets the signed state variables, and gives the equations and parameters used in the experiment. The definitions expose the assumptions that an empirical application would have to justify.

The domain is a periodic ring of unit circumference, represented by $x \in [0, 1)$ with the endpoints identified. The coordinate represents position among neighbouring sites in an artificial interaction environment. Distances and neighbourhood exchange refer to this chosen ring metric. The construction does not identify an observed social or semantic metric, and no relativistic interpretation is attached to the coordinate or to simulation time.

The real-valued fields $r(x, t)$, $p(x, t)$, and $m(x, t)$ represent relational provision in the local environment, an interpretive disposition, and retained exposure, respectively. Each value is a dimensionless signed deviation from a stipulated reference state. The signs distinguish directions of deviation; they have no assigned moral valence. The local output $j(x, t) = \tanh p(x, t)$ is an observable response. Its scale is arbitrary, and it is neither a probability nor a measure of reality.

The evolution equations are specified in Equation 4.

$$\begin{aligned}\tau_r \partial_t r &= D_r \partial_{xx} r - r + a(x) \tanh p + u(x, t), \\ \tau_p \partial_t p &= D_p \partial_{xx} p - p + b(x) \tanh r + \gamma q(t) m, \\ \tau_m \partial_t m &= r - m.\end{aligned}\tag{4}$$

In Equation 4, u is a controlled external exposure and $q \in [0, 1]$ is a context gate that controls the current relevance of retained exposure. Periodic boundary conditions apply to r, p, m ; the spatial derivatives of r and p also agree at the identified endpoints. The memory field evolves locally and has no diffusion. Positive D_r, D_p describe exchange that smooths local differences. The effective diffusion rates are D_r/τ_r and D_p/τ_p .

The primary parameter values are $\tau_r = 1$, $\tau_p = 2$, $\tau_m = 8$, $D_r = 0.02$, $D_p = 0.01$, and $\gamma = 0.35$. Spatial heterogeneity is fixed by $a(x) = 0.45[1 + 0.35 \cos(2\pi x)]$ and $b(x) = 0.65[1 + 0.25 \sin(2\pi x)]$. The coefficients remain unchanged within a condition. The path from r to p represents environmental formation of the disposition, while the path from p to r represents interpretive feedback mediated by the assumed response $\tanh p$. The field r is a summary of relational conditions; it is not a complete representation of pairwise social relationships. The model therefore formalises changing relational conditions and disposition states while leaving the architecture and interpretation of their coupling supplied by the investigator.

D.3 Historical Dependence and Stability

This section establishes the model's memory mechanism and a conservative stability bound. The calculations separate dependence on prior exposure from claims about permanent historical identity or normative desirability.

Solving the third line of Equation 4 with an integrating factor gives the exact representation in Equation 5.

$$m(x, t) = e^{-(t-t_0)/\tau_m} m(x, t_0) + \frac{1}{\tau_m} \int_{t_0}^t e^{-(t-s)/\tau_m} r(x, s) ds.\tag{5}$$

Equation 5 shows that present exposure and an exponentially weighted history jointly determine m . The enlarged state (r, p, m) admits local-in-time evolution. A description retaining only r, p generally requires an additional historical term when the gate is active. Many histories can produce the same retained-memory field, so m neither records an entire biography nor uniquely identifies its generating path.

For two solutions under common controls and parameters, let $\delta r, \delta p, \delta m$ denote their differences. Define the weighted supremum distance by Equation 6.

$$W(t) = \max \left\{ \frac{\|\delta r(\cdot, t)\|_\infty}{0.8}, \|\delta p(\cdot, t)\|_\infty, \frac{\|\delta m(\cdot, t)\|_\infty}{0.9} \right\}.\tag{6}$$

The comparison in Equation 6 combines the three fields using explicitly chosen mathematical weights. Since $\tanh$ is 1-Lipschitz, $a \leq 0.6075$, $b \leq 0.8125$, and periodic diffusion is nonexpansive in the supre-

mum norm, the upper right derivative of W obeys Equation 7 between preparation interventions.

$$D^+W(t) \leq -cW(t),$$

$$c = \min \left\{ 1 - \frac{0.6075}{0.8}, \frac{1 - 0.8125(0.8) - 0.35(0.9)}{2}, \frac{1 - 0.8/0.9}{8} \right\} = \frac{1}{72}. \quad (7)$$

To obtain Equation 7, take the component attaining the maximum in Equation 6, bound each incoming difference by its weight times W , and use the nonpositive diffusion contribution at a spatial extremum. The three component bounds are 0.240625, 0.0175, and approximately 0.0138889. Integrating the inequality gives $W(t) \leq e^{-c(t-s)}W(s)$ for common controls between resets. The argument also applies to the centred spatial discretisation before numerical time integration because its neighbour weights are nonnegative. It is a comparison bound for the equations, separate from the empirical accuracy checks on the time integrator.

The reaction terms are globally Lipschitz in the fields for bounded q . Combining their integral equations with the periodic heat evolution gives successive approximations on a sufficiently short time interval; the Lipschitz bound makes those approximations contractive. The same bound extends the solution over finite intervals for bounded controls. This construction justifies unique evolution within each smooth phase; the declared reset supplies the next initial condition.

Contraction of the complete state permits an initially hidden difference to become visible in a projection. A difference stored mostly in m can decrease in the full distance while becoming expressed through p and j . The bound therefore coexists with the observable divergence examined below. It also implies that the specified common forcing eventually erases these state differences. Persistent identity, multistable interpretation, and the desirability of an attracting state receive no support from this calculation.

D.4 Preparation and Observation Regimes

This section specifies the experiment's two histories, preparation reset, observation window, and shared probe. An analytic comparison then establishes the restricted sense in which their observable trajectories agree.

Both histories start from zero fields at $t = -8$. During $-8 \leq t < 0$, the gate is closed and the exposure is $u_\sigma(x, t) = 0.8\sigma[1 + \cos(2\pi(x - 0.25))]/2$, with $\sigma \in \{-1, +1\}$. At $t = 0$, the investigator imposes the preparation $r(x, 0^+) = p(x, 0^+) = 0$ and retains $m(x, 0^+) = m(x, 0^-)$. The reset is an explicit intervention on accessible states. It is a strong idealisation whose feasibility in a human or artificial system has not been established.

For $0 \leq t \leq 4$, set $u = 0$ and $q = 0$. The observation procedure reveals the complete fields r, p , and hence j , throughout this interval, while withholding m and the earlier exposure record. At $t = 4$, the shared probe sets $q = 1$ and retains $u = 0$ through $t = 16$. A control branch keeps $q = 0$. All other equations and primary parameters remain common across the two histories.

Proposition D.1. *The two prepared histories have identical accessible fields throughout $[0, 4]$, and different observable responses immediately after the shared probe wherever their retained-memory fields differ at $t = 4$.*

Proof. With $q = u = 0$ and $r = p = 0$ initially, the first two equations have the unique solution $r = p = 0$. The third equation reduces to local exponential decay. The equations and preparation are invariant under simultaneous reversal of the state and exposure signs, so the two retained-memory fields have opposite signs. A nonzero nonnegative exposure creates nonzero nonnegative memory in the positive branch. At the instant the gate opens, the difference in the right derivatives of j is given by Equation 8.

$$\partial_t(j_+ - j_-)(x, 4^+) = \frac{\gamma}{\tau_p}(m_+ - m_-)(x, 4). \quad (8)$$

Equation 8 follows because $p = r = 0$ at the boundary, $\partial_{xx}p = 0$, and $\tanh'(0) = 1$. A nonzero right derivative yields different outputs for sufficiently short subsequent times at that position. $\square$

The proposition is an existence result under deliberately constructed controls. The matched window is designed through the reset and gate. Direct observation of m , disclosure of the exposure history, or use of the probe expands the observation regime and removes the stated agreement. The construction establishes neither universal indistinguishability nor spontaneous formation of complete interpreters.

D.5 Numerical Method and Verification

This section describes the discretisation, recorded outcomes, and checks against exact cases and refined computations. These checks assess the implementation of the stipulated equations; empirical adequacy requires additional evidence.

Use $N = 128$ equally spaced sites and centred second differences with periodic boundaries. In Fourier coordinates, the discrete Laplacian has symbol $\lambda_k = -4N^2 \sin^2(\pi k/N)$. The diffusion substep applies its exact exponential, using the effective diffusion rates defined above. Each full step composes a diffusion half-step, a classical fourth-order Runge–Kutta reaction step, and a second diffusion half-step. This Strang-type splitting has second-order accuracy in time within the smooth phases. The reaction approximation is not itself time symmetric. The primary step size is $\Delta t = 0.02$. Each gate change and reset coincides with a phase boundary. Recorded profiles are spaced by 0.1 time unit, with the phase endpoints retained.

The spatial RMS observable distance is defined in Equation 9.

$$d_j(t) = \left\{ \frac{1}{N} \sum_{i=0}^{N-1} [\tanh p_+(x_i, t) - \tanh p_-(x_i, t)]^2 \right\}^{1/2}. \quad (9)$$

Equation 9 is a quadrature approximation to an L^2 distance on the unit ring. It is descriptive and supplies no sampling standard error. Reported peaks are maxima on the recorded temporal grid.

A single Fourier heat-decay mode agrees with the exact discrete solution to maximum error 2.28×10^{-10} or less. Its continuum RMS error falls from 2.99603×10^{-4} at $N = 32$ to 1.87256×10^{-5} at $N = 128$, with successive ratios approximately 4.000. The pure local memory-decay case agrees with its exact exponential to maximum error 8.41×10^{-14} . Zero-state invariance and sign reversal are exact within the recorded floating-point outputs.

For the full positive-history run, time-step refinements 0.04, 0.02, 0.01, 0.005 at $N = 128$ give successive final-state RMS difference ratios 3.99987 and 3.99997. Grid refinements 32, 64, 128, 256 at $\Delta t = 0.005$, compared at coincident sites, give ratios 4.00347 and 4.00087. The primary time step differs from the finest time step by a final-state RMS of 3.14688×10^{-7} . The $N = 128$ run at the fine time step differs from $N = 256$ by 1.06532×10^{-6} . The ratios support the expected second-order convergence for this smooth example. They are not rigorous error bounds over all possible inputs. Figure 1 displays the numerical comparisons.

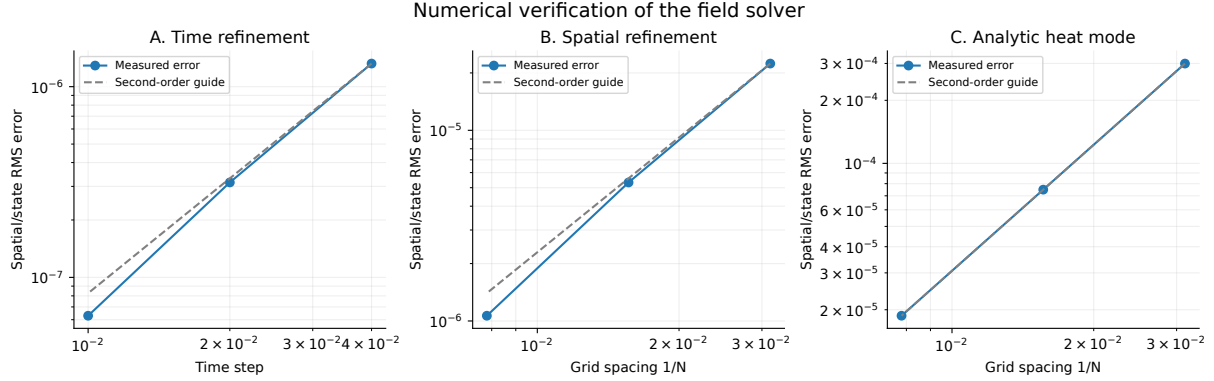
D.6 Field Responses and Mechanism Ablations

This section reports the primary comparison and every prespecified mechanism and retention-time variant. The outcomes describe the chosen parameter regime and preserve the distinction between an analytic existence argument and a numerical magnitude.

The maximum difference in both accessible fields is exactly zero throughout the matched window, as is the difference in j . At the probe boundary, the spatial RMS memory magnitude in each branch is 0.182721; the inter-branch memory distance is therefore 0.365442. The sampled peak observable distance after the probe is 0.0970694 at $t = 9.1$, and the distance at $t = 16$ is 0.0695686. Both no-probe controls retain identical zero outputs through $t = 16$. Figure 2 displays the prepared fields and their subsequent spatial and temporal responses.

Each ablation reruns its own exposure and preparation phases. Removing interpretive feedback by setting $a = 0$ reduces the positive branch's sampled peak RMS output from 0.0485347 to 0.0325108, while removing memory transmission by setting $\gamma = 0$ leaves all post-reset output at zero. Replacing

Figure 1: Numerical verification. Temporal and spatial refinement compare the complete final state with the finest corresponding computation. The heat mode compares the numerical relational field with its analytic continuum solution. The dashed lines give second-order reference slopes.



a, b with their spatial means gives a sampled peak RMS output of 0.0470643. The feedback comparison includes its effect on the prepared memory as well as its effect during the probe. It consequently estimates the total difference between these stipulated mechanisms; it does not isolate a direct effect holding prepared memory fixed.

Changing τ_m to 4 gives a sampled peak RMS output of 0.0319091 at $t = 7.6$, while changing it to 16 gives 0.0463872 at $t = 10.9$. The primary value 8 gives 0.0485347 at $t = 9.1$. A longer memory time constant also slows exposure accumulation in Equation 5; the observed peak therefore depends jointly on exposure duration, dormant interval, and retention rate. This sensitivity supports no general monotonic claim about longer memory. Figure 3 reports all variants.

D.7 Interpretive Limits and Empirical Development

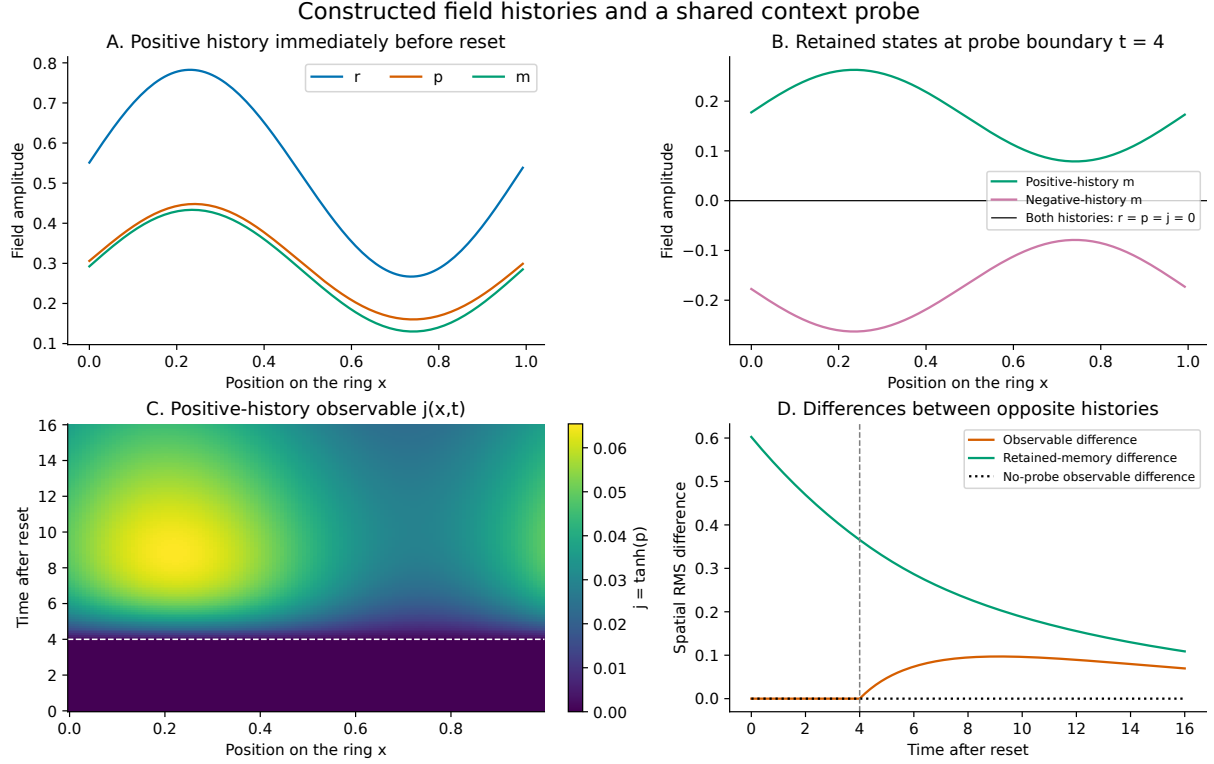
This section identifies the model’s contribution to the philosophical paper and the evidence required for further application. It examines the observation regime, the supplied interpretive architecture, and the relation to the independent live-agent study.

The model supplies a precise case in which evolving relational conditions form a disposition state, retained exposure affects later responses, and interpretive output participates in subsequent relational conditions. Observable agreement is indexed to the disclosed variables and available controls. This conditional lesson can inform Paper I’s discussion of manifest agreement and generative history. The simulation cannot determine which aspects of an actual human or artificial system have these dynamics.

The reset and context gate are decisive idealisations. A practical analogue would require evidence that an intervention can equate the accessible state while preserving a relevant latent state. Measurement leakage, incomplete reset, and direct access to memory would change the comparison. A scalar disposition with a predetermined response function captures only state formation within an interpreter design supplied by the modeller. It does not explain the emergence of categories, semantic content, language, consciousness, or a complete system of evidential standards. The interaction metric, coefficient heterogeneity, and fixed coupling architecture are also assumed. Empirical field modelling would require justified aggregation, independently observable inputs and responses, and competing models tested under interventions.

The live-agent experiment elicits language-model responses to specified prompts. Its responses may bear on sensitivity to disclosed histories and comparison criteria. No mapping from those responses to r, p, m has been estimated or validated. The two exercises therefore make different kinds of contribution: a restricted response study and a constructed mathematical illustration. Their juxtaposition does not establish a common mechanism. Neither permits an inference from dynamic convergence to justice, moral standing, or a hierarchy of reality.

Figure 2: Synthetic histories and the shared context probe. Panel A shows the positive-history fields immediately before the explicit reset of r, p . Panel B shows the different retained memories at the end of the matched window, while all accessible fields remain zero. Panel C displays the positive-history output after the gate opens at $t = 4$. Panel D compares observable and memory distances across histories; the no-probe observable distance remains zero. Agreement is imposed through the declared preparation and context gate.



D.8 Reproducibility and Source Access

This section identifies the executable record and the scope of source verification. The simulation is deterministic and requires Python, NumPy, and Matplotlib. From the repository root, run `python experiments/field_theory/simulate.py`. The local protocol was saved before execution; it was not registered externally. The output manifest records UTC timestamps, configuration, software versions, source and protocol hashes, and hashes of generated outputs. Compressed arrays retain every recorded field profile, and JSON and CSV files retain all reported conditions and verification results. PDF and PNG figures are standalone exports. All prespecified conditions were executed; the numerical checks passed without a change to model parameters or protocol.

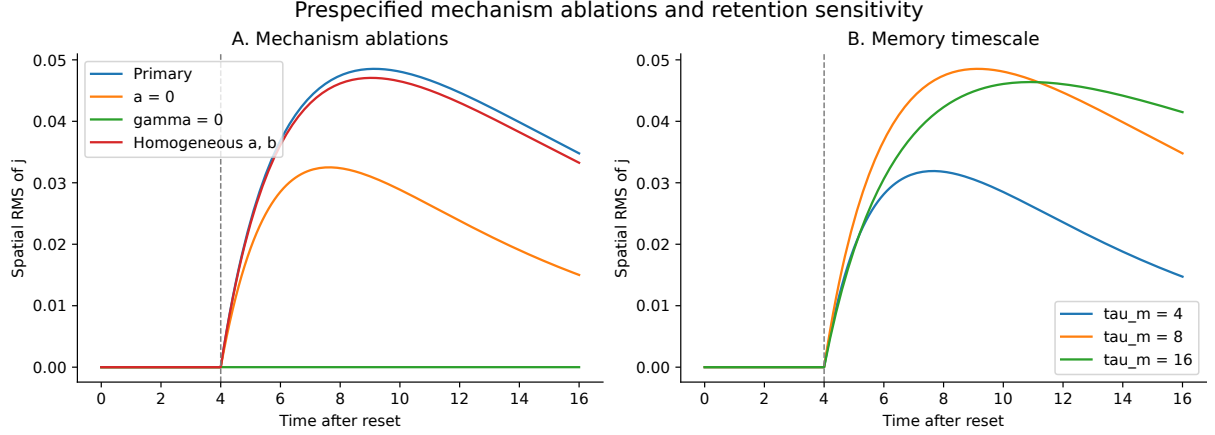
Bibliographic details and the permitted scope of each attribution were verified online before insertion. The Amari article was checked through its publisher metadata and abstract; no full-text derivation from that article is claimed. The original Wilson–Cowan article was inspected in the university-hosted PDF. The source ledger records both access limits.

E Comparison Objects and Preserved Structure

This section identifies the objects compared and defines the principal relations used in the experiments. The method combines explicit mathematical definitions with counterexamples to overly broad inference from a shared property.

A field model includes a domain, state space, evolution law, admissible initial and boundary data, controls, and observation procedures. Its numerical approximation additionally specifies spatial and temporal resolution. A comparison can concern two states within a model, two complete trajectories,

Figure 3: Mechanism ablations and retention-time sensitivity for the positive history. Each curve includes its own exposure and preparation phases. The vertical line marks the shared gate change at $t = 4$. The zero-memory- transmission curve lies at zero throughout. RMS values summarise the spatial observable field and are not statistical uncertainty measures.



two evolution laws, or two descriptions of an underlying system. Equality at one level can leave another unresolved. The distinction between behaviour, representation, and latent variables has an established place in systems theory [26].

For a deterministic autonomous model, write Φ_i^t for its evolution on an explicitly chosen space X_i , with $t \geq 0$. Dissipative field equations often supply forward semiflows; an invertible backward evolution on the same function space is an additional assumption. For a homeomorphism $h : X_1 \rightarrow X_2$, time-preserving conjugacy is the relation in Equation 10.

$$h \circ \Phi_1^t = \Phi_2^t \circ h \quad (t \geq 0). \quad (10)$$

Equation 10 preserves evolution under the specified correspondence and common clock. A coordinate translation must carry the initial data, coefficients, controls, and observation locations together. Physical interpretation remains an additional constraint on admissible maps: exchanging two social roles can change the matter being investigated even when the equations admit that map.

Oriented-orbit equivalence also permits a strictly increasing change of time along trajectories. Its simplest form replaces the second clock by ct for $c > 0$. This preserves the order of states while potentially changing deadlines, dwell times, exposure integrals, and the response to controls scheduled in an external clock. A claim of failure of orbit matching under a fixed physical identification leaves open the existence of a different, possibly interpretively inadmissible, state transformation. Shared attractors preserve still less information about approach paths or elapsed time.

E.1 Observations, Interventions, and Approximation

This subsection defines a finite-horizon output comparison and explains its quantifiers. It distinguishes certified relations among models from agreement in a finite set of simulated tests.

Suppose the models have common output space Y , specified readouts O_i , an initial-state map h , and a control map Ψ . For an initial set K and allowed input class $\mathcal{U}$, define the discrepancy in Equation 11.

$$d_{T,K,\mathcal{U}}^{h,\Psi} = \sup_{z \in K, u \in \mathcal{U}, 0 \leq t \leq T} \|O_1\Phi_1(t, z; u) - O_2\Phi_2(t, hz; \Psi u)\|_Y. \quad (11)$$

Equation 11 requires well-defined solutions over the stated horizon and compatible output units. Zero discrepancy gives equality for the declared initial states, controls, times, and observation procedure under these maps. A small value is an explicitly indexed approximation. A computed maximum over sampled states, times, or inputs supplies a diagnostic; a uniform bound over an unsampled class requires further

Table 3: Comparison relations and their scope. The rows specify different preserved structures; they do not form a universal linear hierarchy.

Comparison	Preserved structure	Further information required
Coordinate conjugacy	Complete forward evolution under an invertible map and a common clock	Meaning-preserving readouts, control maps, and admissible state identification
Oriented orbits	Trajectory images and direction under a declared time change	Physical clock, durations, scheduled inputs, and exposure
Observable behaviour	Output paths or laws under specified preparations and controls	Hidden state, excluded interventions, and wider horizons
Approximate similarity	A declared error bound under fixed metric and quantifiers	Tolerance choice, coverage, and consequences of residual error
Attractor agreement	A specified asymptotic set or pattern	Transient paths, rates, basins, and finite-time consequences
Energy or action comparison	A specified functional, its stationary structure, or a scalar evaluated on a path	Mobility, variations, metric or noise, boundary terms, and measure
Boundary state sums	An identified boundary functional or operator	Interior insertions, normalization, and untested boundaries or refinements
Norm-relevant similarity	Legally or ethically material properties under a defended rule	Authority, evidence, exceptions, and contestability of the rule

analysis. The equality of output distributions in a stochastic model would require a comparison of laws, with the initial and noise distributions also specified.

Simulation and bisimulation in formal systems theory additionally require matching transitions while preserving observations; a bisimulation supplies matching in both directions. Approximate forms relax observational equality through a defined metric [16]. The numerical studies below do not implement a general bisimulation algorithm. Their exact claims follow from direct identities for the constructed models; their sampled metrics retain their limited domains.

An error threshold generally defines a similarity relation with different logical properties from equivalence. For example, scalar values 0 , 0.75ε , and 1.5ε have successive distances below ε while their endpoint distance exceeds it. Approximate similarity at fixed tolerance is therefore generally nontransitive. Optimising over transformations can further change the meaning of the comparison. The admissible transformation class and the quantities it preserves should be specified before ranking models by their apparent similarity.

E.2 Variational, Boundary, and Normative Comparisons

This subsection locates energy, action, and boundary comparisons within the same account of preserved structure. Table 3 records the distinctions used throughout the comparative appendices.

An energy functional, an action functional, and an accumulated scalar value describe different objects. Equality of energies leaves mobility and dissipation unspecified. Equality of an action on one path leaves its value on other paths and its variational derivatives unresolved. Equality of full action functionals under a specified path map is a stronger statement whose consequences also depend on boundary conditions, admissible variations, and, for stochastic or quantum state sums, the measure and normalization.

Boundary equivalence compares complete boundary functionals or operators after internal variables have been integrated. It can hold while insertion observables in the interior distinguish the models. A renormalization or refinement-consistency claim must identify the coarse maps and the observables they carry. Such a claim for one finite family does not establish invariance across every complex or a continuum limit. These structural requirements are explicit in spin-foam coarse-graining work [6].

F Comparison Structures for Field Evolution

This section distinguishes mathematical relations between field evolutions through explicit constructions on a periodic ring. Each comparison specifies the field state space, observation map, temporal convention, and admissible transformation. Exact conclusions follow from direct operator identities. The numerical examples illustrate their consequences and verify the implementation; empirical applications require an additional interpretation of the field variables and observations.

F.1 State Space and Observation Metrics

This subsection fixes the domain and distance conventions used throughout the constructions. Let the spatial domain be the ring $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and the state space be real $L^2(\mathbb{T})$. The spatial norm is normalised by the circumference. Two horizon-dependent comparisons are specified in Equation 12. The supremum convention controls the largest discrepancy over the interval, while the time-averaged convention measures accumulated squared discrepancy per unit time.

$$\begin{aligned} \|f\|_{\text{rms}}^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^2 dx, \\ d_{\infty,T}(u, v) &= \sup_{0 \leq t \leq T} \|u(t) - v(t)\|_{\text{rms}}, \\ d_{2,T}(u, v) &= \left[\frac{1}{T} \int_0^T \|u(t) - v(t)\|_{\text{rms}}^2 dt \right]^{1/2}. \end{aligned} \tag{12}$$

An observation operator can replace the full field in either comparison. For instance, the spatial mean discards every nonconstant Fourier mode. Equality of this observation defines classes of trajectories that may remain distinct in the field state space. Approximate similarity at tolerance ε requires both the metric and the horizon; it generally lacks transitivity. Constant fields with values $0, 0.75\varepsilon$, and 1.5ε demonstrate this failure even before any observation reduction.

F.2 Fixed Spatial Coordinate Conjugacy

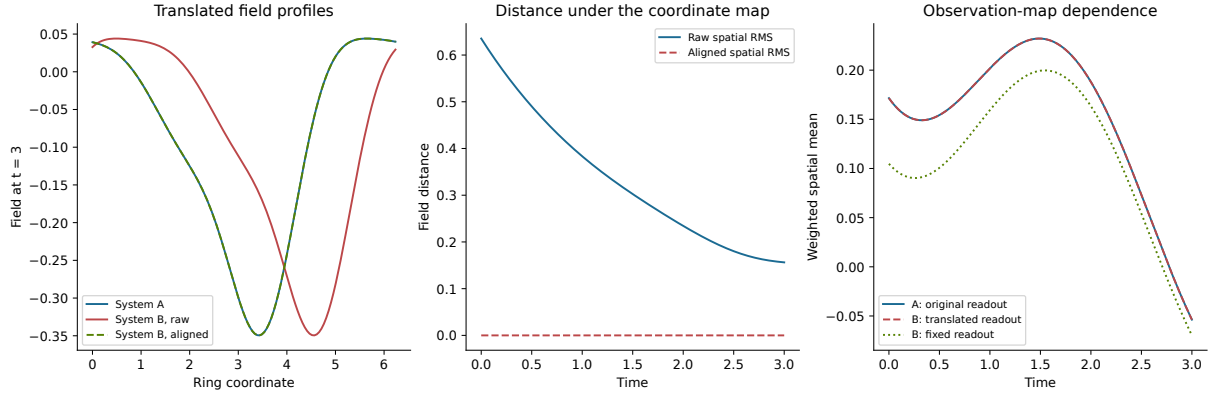
This subsection constructs a field equivalence under a fixed translation and examines the corresponding observation map. Define $(T_\delta f)(x) = f(x - \delta)$, an invertible isometry on the periodic state space. The autonomous part of the first system is $L_A = D\partial_x^2 - V_A(x)$ with domain $H^2(\mathbb{T})$, $D > 0$, and bounded positive potential. The second system uses $V_B = T_\delta V_A$. Spatial translation preserves the operator domain and commutes with the Laplacian. The resulting operator and semigroup identities are given in Equation 13.

$$L_B T_\delta = T_\delta L_A, \quad S_B(t) T_\delta = T_\delta S_A(t), \quad t \geq 0. \tag{13}$$

For controlled equations $\partial_t u_j = L_j u_j + b_j(x)g(t)$, the relation $u_B(t) = T_\delta u_A(t)$ holds for every common scalar control history g when both $u_B(0) = T_\delta u_A(0)$ and $b_B = T_\delta b_A$. A weighted spatial-mean observation $O_A(u) = \langle w_A u \rangle$ is preserved by the corresponding choice $w_B = T_\delta w_A$, because the ring measure is translation invariant. This preservation concerns transformed controls and observations; fixing a spatially selective observation weight defines a different comparison.

The numerical example takes $D = 0.04$, $V_A(x) = 0.4 + 0.2 \cos x$, $b_A(x) = 0.8 + 0.3 \sin 2x$, $g(t) = 0.3 \sin(1.7t)$, and $w_A(x) = 1 + 0.5 \cos(x - 0.2)$. The initial field is $0.7 \cos x + 0.25 \sin 3x$. A translation by 23 of 128 grid cells gives $\delta = 1.129010$. Figure 4 shows a maximum sampled raw field distance of 0.635315, exactly zero aligned field distance, and a transformed-readout discrepancy below 5.56×10^{-17} . The fixed-readout discrepancy reaches 0.066752. These observations illustrate the exact identities in Equation 13; the numerical equality alone would be insufficient to establish an identity for arbitrary initial fields and controls.

Figure 4: Fixed spatial coordinate conjugacy with translated coefficients, forcing profiles, initial fields, and observation weights. Raw field distance remains positive while the specified alignment preserves the complete field evolution and transformed observation.



F.3 Autonomous Time Reparameterisation

This subsection compares the temporal ordering of states with their physical timing. For $L = D\partial_x^2 - \lambda$ with $D, \lambda > 0$, define $L_B = cL_A$ for constant $c > 0$. The heat-decay semigroups satisfy Equation 14 for every initial field. Consequently the complete oriented forward orbits coincide under a positive rescaling of time.

$$S_B(t)u_0 = S_A(ct)u_0, \quad t \geq 0. \quad (14)$$

The construction concerns forward semigroups, with no assumption of well-posed backward heat evolution. It establishes equality after the declared clock change and leaves same-time observations free to differ. Externally timed interventions, deadlines, and accumulated durations would require their own transformation rules. The identity makes no assertion about the existence or absence of additional state-space conjugacies.

Figure 5 uses $D = 0.12$, $\lambda = 0.3$, $c = 2.4$, and initial field $\cos x + 0.45 \sin 2x + 0.2 \cos 3x$. Its maximum sampled same-time RMS discrepancy is 0.236746, while the discrepancy after exact time alignment is zero. At physical time $t = 1$, the discrepancy is 0.230174. The plotted two-mode projection is a visualisation of the analytically established orbit identity, with the full third mode retained in all saved field arrays.

F.4 Observed Trajectories and Generator Identification

This subsection constructs complete trajectory agreement under one initial condition and separation under a declared probe. A Fourier mode $\cos kx$ under the heat-decay generator has rate $\mu_k = Dk^2 + \lambda$. The parameter pairs $(D_A, \lambda_A) = (0.1, 0.6)$ and $(D_B, \lambda_B) = (0.5, 0.2)$ give identical mode-one rate 0.7. Their restriction to the mode-one invariant subspace therefore agrees exactly, including every linear combination of $\sin x$ and $\cos x$. Complete noiseless observation of a mode-one trajectory does not determine the two parameters.

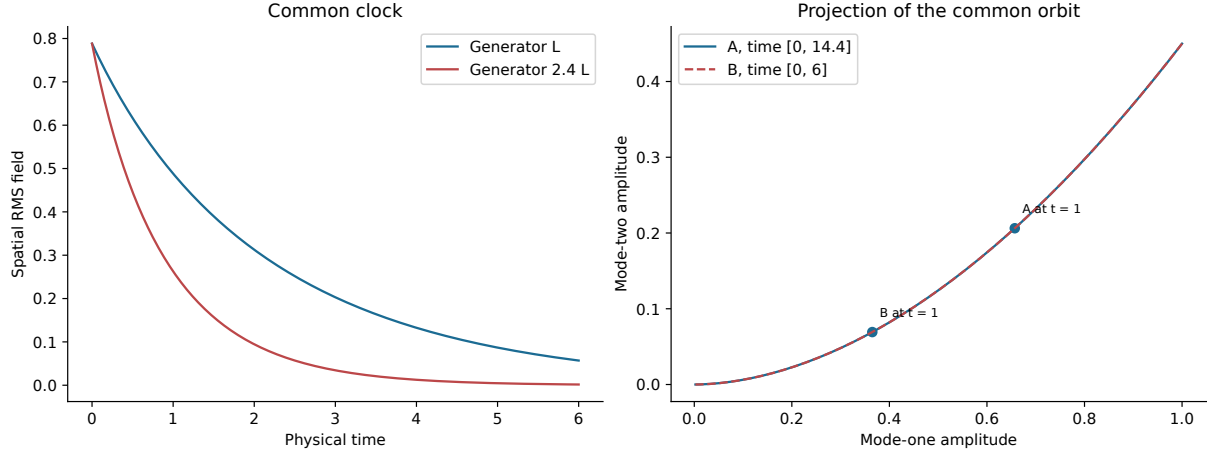
A reset to the common initial field $\cos 2x$ excites rates $\mu_{2,A} = 1.0$ and $\mu_{2,B} = 2.2$. The continuous-time probe discrepancy and its maximising time are given in Equation 15. Both quantities follow directly from the modal solution and the normalised spatial norm.

$$d_{\text{probe}}(t) = \frac{e^{-t} - e^{-2.2t}}{\sqrt{2}}, \quad (15)$$

$$t_* = \frac{\log(2.2)}{1.2} = 0.657048 \dots, \quad d_{\text{probe}}(t_*) = 0.199936 \dots$$

Figure 6 distinguishes the baseline agreement from the probe separation. The full field observation is complete with respect to space and time for the baseline initial condition; it remains restricted with

Figure 5: Autonomous field evolutions with generators L and $2.4L$. The oriented forward orbit is shared, while observations at a common physical time differ. The phase plot displays a two-mode projection of the complete field trajectory.



respect to the set of initial conditions. This distinction supplies an explicit counterexample to identifying the generator from one agreeing trajectory.

Under the additional assumption that the generator belongs to this known two-parameter family, rates for two distinct modes identify the parameters. For modes one and two, $D = (\mu_2 - \mu_1)/3$ and $\lambda = (4\mu_1 - \mu_2)/3$. The audit recovers the chosen parameters from projections of saved field arrays to within 8×10^{-15} . This conditional identification result does not establish unrestricted mechanism identification from finite empirical observations.

F.5 Horizon, Readout, and Admissible Translation Classes

This subsection quantifies approximate similarity under distinct comparison conventions. Consider the common initial field $0.2 + \cos x$, decay $\lambda = 0.008$, diffusion $D = 0.002$, and a relative drift $\omega = 0.12$. The first field solves the heat-decay equation; the second has the additional transport term $-\omega \partial_x u$. Their solutions and raw spatial discrepancy are specified in Equation 16, with $a = D + \lambda = 0.01$.

$$\begin{aligned} u_A(x, t) &= 0.2e^{-\lambda t} + e^{-at} \cos x, \\ u_B(x, t) &= 0.2e^{-\lambda t} + e^{-at} \cos(x - \omega t), \\ \|u_A(t) - u_B(t)\|_{\text{rms}} &= \sqrt{2}e^{-at} |\sin(\omega t/2)|. \end{aligned} \tag{16}$$

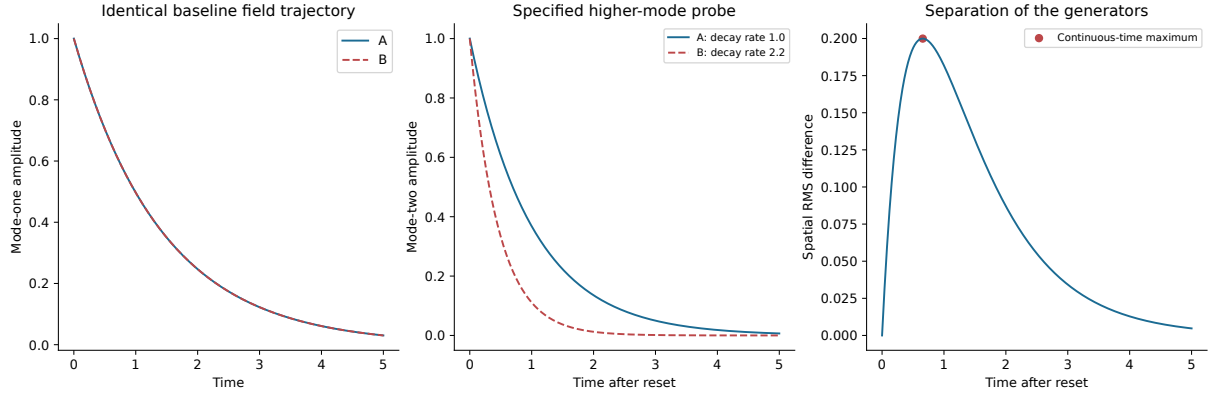
The spatial means agree at every time, while the full spatial observations differ. For the declared raw supremum metric, the distance increases from 0.042208 at horizon 0.5 to 0.165946 at horizon 2. Tolerance 0.1 therefore accepts the first comparison interval and rejects the second. The continuous supremum over every longer horizon is computed from the analytic expression, including its first and largest peak at $t = 23.427461$.

The effect of an admissible fixed translation is evaluated with the time-averaged metric. Write $I_0 = \int_0^T e^{-2at} dt$ and $I_\omega = \int_0^T e^{-2at} e^{i\omega t} dt$. For the convention $T_\delta f(x) = f(x - \delta)$, the optimal shift is $\delta_* = -\arg I_\omega$. Equation 17 gives the resulting minimum and the corresponding unaligned cost.

$$\begin{aligned} \inf_{\delta \in \mathbb{T}} d_{2,T}(u_A, T_\delta u_B)^2 &= \frac{I_0 - |I_\omega|}{T}, \\ d_{2,T}(u_A, u_B)^2 &= \frac{I_0 - \text{Re } I_\omega}{T}. \end{aligned} \tag{17}$$

At horizon 20, the raw space-time RMS is 0.734627 and the optimal single-translation RMS is 0.427173. A time-dependent translation $\delta(t) = -\omega t$ produces zero aligned distance throughout the interval. This

Figure 6: Trajectory agreement on the excited mode-one invariant subspace and separation by a mode-two initial-condition probe. The continuous-time maximum is derived analytically and is distinguished from the maximum on the saved time grid.



time-dependent map introduces a moving frame and belongs to a larger comparison class than fixed coordinate conjugacy. Treating drift as removable can therefore discard transport information of substantive interest. Figure 7 separates the horizon effect, observation reduction, and the two translation classes; its left and right panels use the distinct metrics defined in Equation 12.

F.6 Attractors and Accumulated Observations

This subsection compares asymptotic agreement with path-sensitive quantities. Two heat-decay equations with $D = 0.05$ and decay rates $\lambda_A = 0.2$, $\lambda_B = 1$ have the same singleton zero global attractor on periodic L^2 . The energy estimate in Equation 18 establishes exponential attraction of every bounded initial set. The derivative identity applies to regular solutions and extends by the usual semigroup approximation to the L^2 contraction bound.

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{\text{rms}}^2 = -D \|\partial_x u(t)\|_{\text{rms}}^2 - \lambda \|u(t)\|_{\text{rms}}^2, \quad \|u(t)\|_{\text{rms}} \leq e^{-\lambda t} \|u(0)\|_{\text{rms}}. \quad (18)$$

For the common initial field $1 + 0.4 \cos x$, the spatial mean is $e^{-\lambda t}$. Its infinite-horizon integral is $1/\lambda$, giving 5 and 1, while the duration above a stipulated threshold $q = 0.4$ is $-\log(q)/\lambda$, giving 4.581454 and 0.916291. Figure 8 displays the transient and accumulated observations. Equal attracting sets therefore do not ensure equal histories, integrated exposures, or threshold durations.

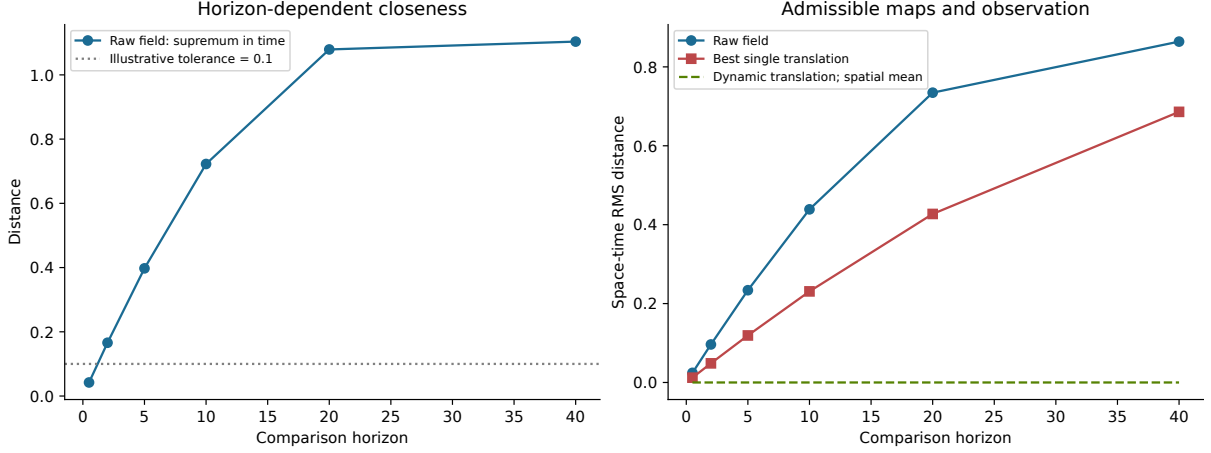
The exposure interpretation is illustrative. A connection to harm, opportunity, or obligation requires separately justified measurements and a normative rule. Attractor equality alone provides no such rule. The result instead identifies a mathematical distinction that a history-sensitive empirical or legal comparison may need to retain.

F.7 Numerical Evidence and Analytical Scope

This subsection records the evidential status and reproducibility of the computations. The heterogeneous translated system uses 128 periodic grid points, a centred second-order spatial Laplacian, and fourth-order Runge–Kutta time integration at step 0.0025. The remaining comparisons evaluate exact Fourier solutions. Their central identities and separating examples follow analytically; finite-grid observations provide illustrations and implementation checks.

Successive time-step refinement ratios are 16.5045 and 16.2502, while spatial consistency ratios are 3.9655, 3.9913, and 3.9978. The independent saved-output audit reproduces the fixed-translation costs by quadrature to 9.09×10^{-10} and verifies the modal rates, observation transformations, and contraction bounds. All eleven suite checks and fourteen audit checks pass. Source and configuration hashes, execution metadata, compressed field arrays, numerical tables, and both PNG and PDF figures are retained in `experiments/field_equivalence/`.

Figure 7: Approximate comparison of drifting field profiles under different horizons, observation maps, and admissible translations. The left panel uses a supremum-in-time field norm; the right panel uses time-averaged field RMS. Spatial-mean observation and time-dependent translation each produce zero distance for different reasons.



The scalar fields isolate comparison structures alongside the project’s earlier three-field phenomenological model. They make no assertion that human and artificial systems instantiate the same equations. They establish specified identities and counterexamples, while leaving empirical field construction, admissible transformations, and norm-relevant observations to further substantive argument.

G Action, Mobility, and Field Comparison

This appendix develops action-based comparisons for a specified class of dissipative fields. Its method combines a variational specification with analytically controlled numerical examples, separating stationary states, trajectories, physical clocks, stochastic temporal laws, and scalar action values. The construction supplies mathematical support for the discussion of generative relations. Application to social or interpretive fields requires independent justification of the state variables, dynamics, observables, and admissible interventions.

G.1 Variational Structures and Comparison Data

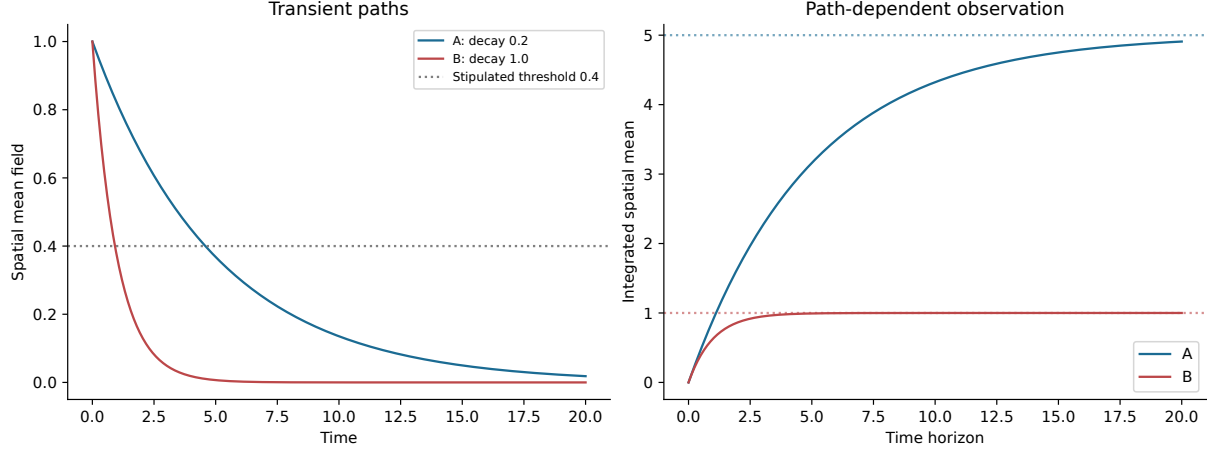
This subsection specifies the variational objects used in the comparison and their domains of interpretation. A classical stationary-action formulation selects histories by the vanishing first variation of an action under prescribed boundary variations. A gradient-flow formulation specifies an energy together with a dissipation mechanism; the latter converts energy derivatives into velocities [22, 24]. Consequently, an energy alone leaves temporal evolution underdetermined. The numerical construction below uses positive symmetric mobility and makes no claim that the programme’s separate nonsymmetric, coupled three-field system admits this structure.

Let $\Omega = [0, 2\pi)$ be a dimensionless periodic ring. The real field is restricted to the invariant subspace $u(x, t) = a_1(t)e_1(x) + a_2(t)e_2(x)$, where $e_k(x) = \cos(kx)/\sqrt{\pi}$. The basis is orthonormal in $L^2(\Omega)$. The energy and its gradient flow are given in Equation 19.

$$E[u] = \frac{1}{2} \int_{\Omega} (0.15|\partial_x u|^2 + 0.4u^2) dx = \frac{1}{2} a^\top K a, \quad \dot{a} = -M K a, \quad K = \text{diag}(0.55, 1). \quad (19)$$

The three mobilities are $M_1 = I$, $M_2 = 2I$, and $M_3 = \text{diag}(0.4, 2)$. Each is positive. On the full ring, the third operator is defined as a Fourier multiplier with these eigenvalues on the selected modes and unit eigenvalues on remaining modes. It is therefore a specified nonlocal mobility. The chosen initial

Figure 8: Heat-decay fields with a common zero global attractor and distinct transient observations. The threshold and integrated spatial mean are explicitly stipulated path observables, without empirical or normative calibration.



condition has support only in the displayed modes, so their evolution is an exact invariant field restriction. This example does not depend on a spatial discretisation approximation to the evolution equation.

For a continuously differentiable trial history on $[0, T]$, define the residual functional in Equation 20.

$$I_M[a] = \frac{1}{4} \int_0^T (\dot{a} + MKa)^\top M^{-1} (\dot{a} + MKa) dt. \quad (20)$$

Equation 20 is a nonnegative least-squares residual once a drift and metric have been selected. Its zero set consists of the solutions of Equation 19 in the specified admissible class. Prescribing the initial field then fixes the deterministic solution. Generic stationary points of this path functional, with arbitrary endpoint constraints, need not have zero residual; those endpoint constraints may exclude every deterministic solution.

The same expression is the small-noise rate action for the finite-dimensional diffusion $da = -MKa dt + \sqrt{2\varepsilon M} dW$, conditional on its initial state. In this use, the rate controls leading logarithmic asymptotics of probabilities of path neighbourhoods as ε tends to zero. It does not supply a normalized finite-noise path density. Stochastic fluctuation-path formulations have established antecedents [20]; their relations to gradient flow require attention to the stochastic model and dissipation structure [1]. The stochastic claim here concerns a finite spectral system with constant nondegenerate diffusion. A continuum stochastic field theory would require a separately specified noise covariance and function-space analysis.

Expanding the square connects residual action to dissipation and endpoint energy, as shown in Equation 21.

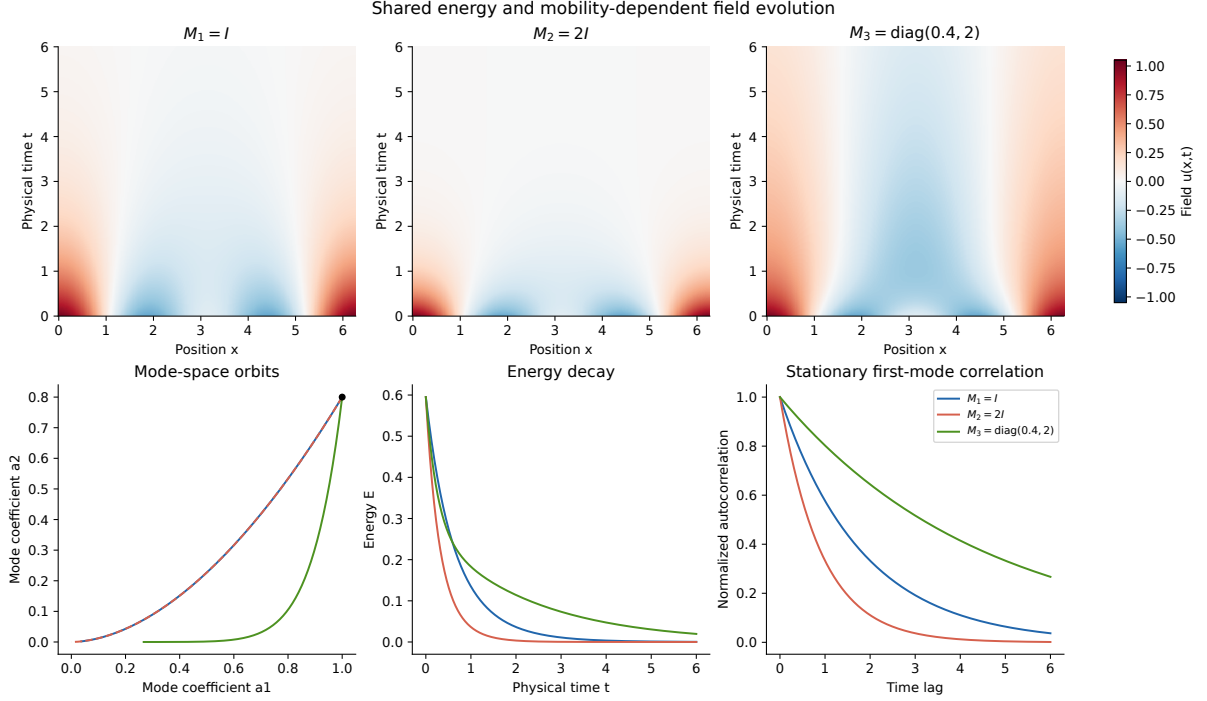
$$I_M[a] = \frac{1}{4} \int_0^T (\dot{a}^\top M^{-1} \dot{a} + (Ka)^\top MKa) dt + \frac{1}{2} (E[a(T)] - E[a(0)]). \quad (21)$$

Equation 21 follows directly from $\dot{a}^\top Ka = dE[a]/dt$. It distinguishes a time-integrated nonnegative dissipation expression from the endpoint contribution. For a deterministic relaxation, these terms cancel to give zero residual action. An isolated scalar integral therefore requires its functional definition and boundary data before comparison.

G.2 Shared Energy and Temporal Structure

This subsection compares the exact field evolutions generated by the three mobilities. All runs use $a(0) = (1, 0.8)$ and the physical interval $[0, 6]$. Exact mode amplitudes $a_k(t) = a_k(0) \exp(-m_k K_{kk} t)$ provide reference trajectories. Figure 9 presents their spatial histories, mode-space orbits, energies, and exact stochastic correlation functions.

Figure 9: Shared energy and mobility-dependent field evolution. The upper panels show exact spatial fields over a common physical interval. The lower panels compare geometric orbits under identity state identification, energy decay, and normalized stationary first-mode autocorrelation for the finite-dimensional diffusion extension. The reference and doubled-mobility orbits overlap geometrically; their physical clocks differ.



All three mobilities share the energy functional and its unique stationary field, $u = 0$. Along each solution, $dE/dt = -(Ka)^T M K a \leq 0$. The equality $a^{(2)}(t) = a^{(1)}(2t)$ holds exactly, establishing orbit equivalence under identity state identification when a uniform rescaling of time is permitted. At the same physical time, their maximum field L^2 separation on the sampled interval is 0.307034. Thus physical-time trajectory equivalence imposes a stronger requirement than this particular orbit comparison.

The unequal-mode mobility changes the initial tangent direction: the sine of its angle with the reference tangent is 0.448998. Positive scalar time reparameterisation preserves tangent directions, so it cannot identify these two orbits through the common initial field under the identity state map. Their maximum sampled physical-time field distance is 0.329727. This statement fixes the physically interpreted field coordinates. Equivalence under an unrestricted change of state variables is a separate mathematical question and requires its own allowed-map class.

An explicit alternative identification clarifies this qualification. On the selected coefficient space, the homeomorphism $h(a_1, a_2) = (\text{sgn}(a_1)|a_1|^{0.4}, \text{sgn}(a_2)|a_2|^2)$ gives a time-preserving topological conjugacy between the M_1 and M_3 flows: $h(\Phi_1^t(a)) = \Phi_3^t(h(a))$. This identity follows by applying the indicated powers to the exponential mode solutions. The map changes the field interpretation and the prescribed initial-state correspondence; in particular, $h(1, 0.8) = (1, 0.64)$. It is a homeomorphism and need not be differentiable at zero. Thus identity-map orbit comparison and topological conjugacy supply different answers for the same pair of equations. Transporting physical observables would require a further explicit interpretation of h .

The diffusion extension illustrates an additional distinction. For each mobility its stationary law is Gaussian with covariance $\Sigma = \varepsilon K^{-1}$. This follows by direct substitution in the Lyapunov equation $(MK)\Sigma + \Sigma(MK)^T = 2\varepsilon M$. With $\varepsilon = 0.1$, the diagonal variances are 0.181818 and 0.1. The normalized first-mode correlations at lag one are respectively 0.576950, 0.332871, and 0.802519. Equal stationary marginal distributions therefore coexist with unequal temporal laws at fixed physical time. These values are exact stochastic moments; stochastic sample paths were not simulated.

Table 4: Residual action for deterministic field histories under comparison generators. Rows identify the mobility generating the history; columns identify the mobility used in the residual and its metric. All histories begin at $(1, 0.8)$ and use $[0, 6]$.

Generating mobility	Action under M_1	Action under M_2	Action under M_3
M_1	0	0.07432806	0.10179066
M_2	0.07437521	0	0.21999995
M_3	0.09745974	0.20430053	0

G.3 Scalar Action and Historical Distinction

This subsection evaluates action as a comparison statistic and constructs histories occupying the same action level set. Table 4 first evaluates each deterministic history under each candidate generator, using Equation 20 with a common field identification, initial condition, and physical interval.

Every history has zero residual action under its own generator. Equality of these diagonal entries expresses successful evolution under three separately specified equations. It does not establish equality of those equations. The off-diagonal comparisons supply additional information about disagreement under a common state interpretation. Their asymmetry also shows that this cross-evaluation is a directed discrepancy, rather than a metric on models. A similarity analysis may use it only with an explicit reference generator, path ensemble, time interval, and normalization.

To test scalar action under one common functional, consider two trial paths on $[0, 4]$ with zero initial and final fields. Set $a^{[1]}(t) = (\sin(\omega t), 0)$ and $a^{[2]}(t) = (0, c \sin(\omega t))$, with $\omega = \pi/4$ and $c = \sqrt{(\omega^2 + 0.55^2)/(\omega^2 + 1)} = 0.754059$. Direct integration gives $I_{M_1}[a^{[1]}] = I_{M_1}[a^{[2]}] = 0.459675138$. Figure 10 displays their different spatial patterns and intermediate action accumulation. Their maximum field L^2 separation is 1.252440. These paths are admissible candidate fluctuations rather than zero-noise deterministic solutions. The construction demonstrates that a scalar action value, even with common endpoints and a common functional, leaves the intermediate history undetermined.

G.4 Coordinate Maps and Boundary Contributions

This subsection identifies structural information needed when action comparisons permit changes of representation. For an invertible constant linear coordinate map $v = Sa$, the transformed energy and mobility are given in Equation 22.

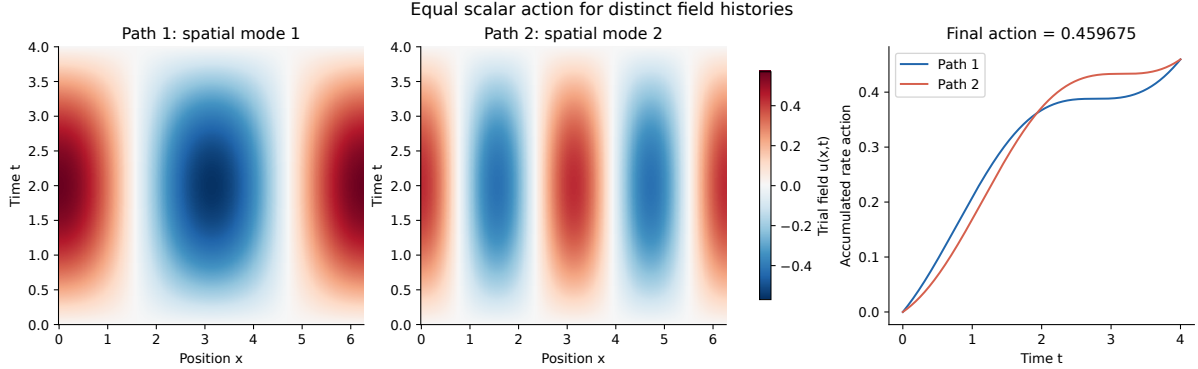
$$K_v = S^{-\top} K S^{-1}, \quad M_v = S M S^\top, \quad \dot{v} = -M_v K_v v, \quad I_{M_v, K_v}[v] = I_{M, K}[a]. \quad (22)$$

Equation 22 follows from transforming both the drift and the quadratic residual metric. The diffusion covariance transforms to $2\varepsilon M_v$. Applying $S = \text{diag}(2, 0.5)$ to a smooth two-mode test path gives original and transformed actions of 1.083807336. Keeping an identity residual metric after changing the coordinates gives 1.994733600. A coordinate transformation consequently preserves the comparison only when the associated metric and covariance are transformed. The calculation covers constant linear maps. Nonlinear Itô coordinate changes require the corresponding second-derivative correction to the drift.

Classical stationary-action equivalence involves a different comparison. Adding a total derivative to a Lagrangian shifts its action by $F(a(T), T) - F(a(0), 0)$ and preserves the bulk Euler–Lagrange equations under fixed endpoint variations [24]. If endpoint variations are allowed, boundary conditions require renewed analysis. Numeric equality of action values is therefore unnecessary for this particular equivalence of bulk variational equations. Conversely, the equal-action histories in Figure 10 show that numeric equality is insufficient for history identity. Comparison of action formulations should accordingly specify the complete functional, admissible histories, boundary data, transformations, and intended observables. A stochastic or quantum extension additionally requires its measure and interpretation of weights.

A direct calculation also qualifies the information supplied by a complete rate functional restricted to a boundary class. For $k > 0$, define $I_+[a] = \frac{1}{4} \int_0^T (\dot{a} + ka)^2 dt$ for drift $-ka$ and $I_-[a] = \frac{1}{4} \int_0^T (\dot{a} - ka)^2 dt$

Figure 10: Distinct field histories with equal final scalar action. Both trial fields begin and end at zero on $[0, 4]$ and are evaluated under the reference mobility. Their spatial modes differ. An analytically selected amplitude matches their total rate action while allowing distinct intermediate action accumulation and field histories.



for drift $+ka$, both with noise amplitude $\sqrt{2\varepsilon}$. Expansion gives $I_+[a] - I_-[a] = \frac{k}{2}(a(T)^2 - a(0)^2)$. Consequently, the functionals agree on every continuously differentiable zero-endpoint loop, although the deterministic drifts have opposite stability. Equality on this restricted path domain therefore leaves the forward generator undetermined. Initial-value evolution from other states or path classes with varied endpoints supplies additional discriminating information. This example is an analytic boundary-term calculation; it introduces no additional numerical run.

G.5 Verification and Scope of Interpretation

This subsection documents numerical verification and the scope of the illustrative conclusions. The protocol preceded the numerical execution. Main trajectories use exact exponentials; analytic time derivatives and integrals independently check residual-action quadrature. The maximum cross-action quadrature error is 3.55×10^{-7} . Periodic-grid basis orthonormality has maximum error 8.34×10^{-17} . Energy balance, the uniform clock map, equal-action construction, coordinate invariance, and the covariance Lyapunov identity agree at floating-point precision. An independent RK4 solver, with step sizes 0.2, 0.1, and 0.05, gives successive final-state error ratios from 16.54 to 18.04, approaching the fourth-order factor 16.

A separately timestamped post-run audit records the subsequently observed software environment and hashes of the preserved output files. It is a retrospective artifact audit, distinct from the execution-time record embedded in the numerical results. The analytic conjugacy and boundary-class qualifications above were added during integration without rerunning or replacing those outputs.

The simulation supports carefully scoped mathematical distinctions: a shared equilibrium or energy leaves mobility open; a shared orbit leaves its physical clock open; a shared stationary marginal leaves temporal correlations open; and a shared scalar action leaves the intermediate history open. A field similarity claim should state which of these structures it preserves, which transformations it admits, which histories or interventions it tests, and which norm and tolerance it uses. The examples introduce no new empirical claim about consciousness or normative standing. Their role in the discussion papers is to discipline the meaning of relational and historical comparison. Established variational and stochastic theories supply antecedents; conceptual priority is not asserted, and objections and corrections remain welcome. The paper's AI-use disclosure, licensing notice, and Responsible Use and Rights Reservation apply to this computational appendix.

H Finite-Group State Sums and Boundary Equivalence

This section constructs an exactly enumerable field model that separates boundary equivalence from interior statistics. It first identifies the representation-theoretic structure, then derives the boundary amplitude

and gluing rule, and finally compares two distinct microscopic models. The construction uses a classical positive-weight finite-group system. Its relation to spin foams is mathematical and explicitly limited.

H.1 Group Variables, Action, and Representation Labels

This subsection defines the degrees of freedom, measure, and action. It specifies the structural content that makes a dual spin-foam-type description available.

Take a strip of N square faces forming a disk. Each edge carries $g_e \in \mathbb{Z}_2 = \{-1, +1\}$, and each face has holonomy $U_f = \prod_{e \in \partial f} g_e$. Inversion leaves a group element unchanged. A vertex gauge transformation acts by $g_e \mapsto h_{s(e)} g_e h_{t(e)}^{-1}$ and preserves every face holonomy and every closed boundary holonomy. Open arcs can transform at their endpoints. Integration over an internal edge uses the normalized Haar average, with weight $1/2$ for each sign.

For $|t_f| < 1$, use the strictly positive face weights and dimensionless action in Equation 23.

$$\begin{aligned} w_{t_f}(U_f) &= 1 + t_f U_f = \frac{\exp(\kappa_f U_f)}{\cosh \kappa_f}, & \kappa_f &= \operatorname{arctanh} t_f, \\ S_t[g] &= - \sum_f \log(1 + t_f U_f) = - \sum_f \kappa_f U_f + \sum_f \log \cosh \kappa_f. \end{aligned} \quad (23)$$

Equation 23 defines the statistical weight e^{-S} exactly. Each face weight has Haar average one. Dropping the local normalization factors changes the absolute boundary amplitude by a model-dependent multiplier. Although that multiplier cancels from conditional interior expectations within a fixed model, it cannot be silently removed when comparing absolute amplitudes between models.

The irreducible characters are $\chi_0(g) = 1$ and $\chi_1(g) = g$. Thus each face has character coefficients $1, t_f$. Integrating an internal edge annihilates terms whose two incident character labels sum to one modulo two. Surviving labels agree across that edge; this is the finite-group invariant-tensor constraint. The description therefore has actual representation labels and edge constraints, as in established finite-group spin-foam constructions [7, 9]. These labels are not geometric angular momenta of an $SU(2)$ gravitational model.

H.2 Boundary Functionals and Composition

This subsection derives the complete boundary functional and identifies the effect of eliminating internal variables. The chosen normalization is retained throughout the calculation.

Let B be the product of the external boundary edges. The strip has $N - 1$ internal edges. Its amplitude is given in Equation 24.

$$A_t(B) = 2^{-(N-1)} \sum_{\{g_{\text{internal}}\}} \prod_{f=1}^N (1 + t_f U_f) = 1 + B \prod_{f=1}^N t_f. \quad (24)$$

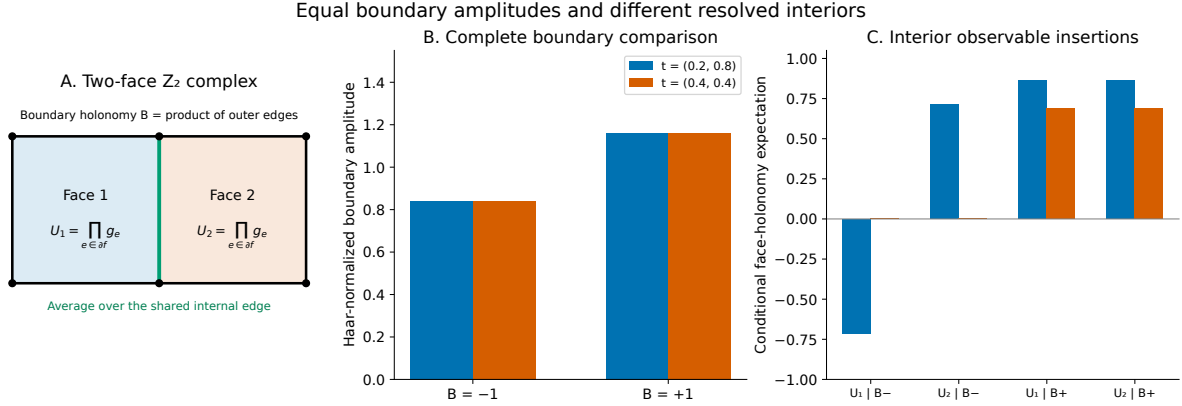
In Equation 24, the character constraints force all face labels on the connected strip to coincide. The all-zero label contributes one; the all-one label contributes the product of the couplings and the boundary holonomy. This establishes the formula for every strip length and coupling within the stated family. The subsequent finite enumerations check its implementation.

The kernel $K_t(a, b) = 1 + tab$ acts with Haar measure on boundary labels $a, b \in \mathbb{Z}_2$. When these labels describe open boundary arcs, their product $ab = B$ is gauge invariant. The individual arc labels retain their endpoint convention. Kernel composition obeys Equation 25.

$$\frac{1}{2} \sum_{b=\pm 1} K_s(a, b) K_t(b, c) = K_{st}(a, c), \quad \operatorname{spec}(K_t) = \{1, t\}. \quad (25)$$

The spectrum in Equation 25 refers to the operator with normalized Haar integration, whose ordinary matrix is $K_t/2$. The characters diagonalize this operator. The multiplication rule also makes composition

Figure 11: A two-face finite-group complex with identical complete boundary amplitudes and different conditional interior observables. Haar averaging uses the declared normalized face weights. The boundary comparison retains both possible closed holonomies; the interior comparison retains the individual face identities.



associative. A coarse single-face parameter $t_{\text{eff}} = \prod_f t_f$ reproduces the fine strip's boundary amplitude. This is a restricted compatibility under elimination of internal edges. General cylindrical consistency would require specified maps and compatible measures across a wider family of complexes [6].

H.3 Identical Boundaries and Distinct Interior Laws

This subsection compares two microscopic coupling assignments and the observables retained or lost by their common boundary description. It reports both the exact analytical comparison and the executed enumeration.

The assignments $(t_1, t_2) = (0.2, 0.8)$ and $(0.4, 0.4)$ share $t_1 t_2 = 0.16$. Each therefore gives $A(+1) = 1.16$ and $A(-1) = 0.84$, and the same complete identified boundary operator. This is stronger than agreement in a single chosen boundary test. In this special normalized one-parameter boundary family, one sector value happens to determine the product; such identifiability does not extend automatically to a larger boundary space.

An interior insertion has conditional expectation $\mathbb{E}[U_1 | B] = (t_1 + B t_2) / (1 + B t_1 t_2)$. For $B = +1$, the two models give 0.8620689655 and 0.689651724, a difference of 0.1724137931. For $B = -1$, the corresponding values are -0.7142857143 and 0. Their conditional interior laws therefore differ under the same face identification. Figure 11 displays the common boundary result and the distinct insertions.

The corresponding stronger comparison can be expressed by introducing sources for interior observables. Define a source-dependent amplitude and its first insertion through Equation 26.

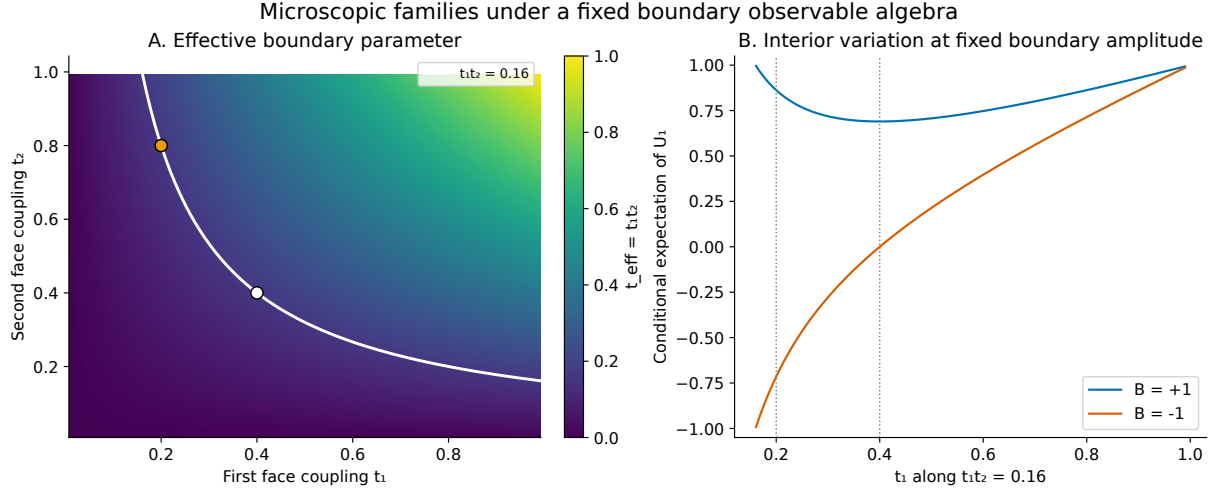
$$A_t(B; J) = 2^{-(N-1)} \sum_{\{g_{\text{internal}}\}} \prod_f (1 + t_f U_f) \exp \left(\sum_f J_f U_f \right), \quad (26)$$

$$\left. \frac{\partial}{\partial J_1} \log A_t(B; J) \right|_{J=0} = \mathbb{E}[U_1 | B].$$

Equation 26 makes the logical distinction explicit: equality at $J = 0$ leaves the derivatives with respect to independently specified interior sources undetermined. The selected models agree on their boundary functional while differing on this enlarged observable algebra. An effective description can preserve the retained observables without reconstructing the microscopic coupling factorization.

Figure 12 displays the continuous family $t_1 t_2 = 0.16$ within positive couplings. The boundary parameter is constant along that curve, while the conditional first-face expectation varies. Coarse boundary equivalence consequently relates a family of microscopic descriptions rather than a unique interior model.

Figure 12: Microscopic couplings with a common effective boundary parameter. The white curve in the left panel fixes $t_1 t_2 = 0.16$; the right panel evaluates an interior insertion along that curve. The family illustrates loss of resolved interior information under the specified boundary comparison.



H.4 Verification and the Scope of Spin-Foam Methods

This subsection records the finite checks and the interpretation permitted by the constructed model. It distinguishes the direct mathematical transfer from claims about gravitational dynamics.

Direct edge enumeration and dual character evaluation were compared for 48 strip conditions with one through eight faces, positive, signed, and zero couplings, and both boundary signs. Their maximum absolute discrepancy was 5.56×10^{-16} . All 64 two-face boundary-edge assignments agreed with the boundary-holonomy formula. Gauge invariance was exact across 8,192 edge-configuration and vertex-transformation combinations. Conditional expectations agreed with the analytic formula within 6.67×10^{-16} . Haar gluing and associativity were checked on 81 coupling pairs and 729 triples. The complete primary enumerations, action values, conditional probabilities, auxiliary checks, figure arrays, and source/output hashes are retained in `experiments/spinfoam_toy/`.

This finite construction uses a two-complex, group characters, invariant constraints, a boundary kernel, and a controlled sum over internal labels. These structures support precise comparisons between representations and resolutions. In gravitational spin-foam models, boundary states and amplitude composition require additional geometric and representation-theoretic data [23]. Even semiclassical connections to an action depend on the model and boundary regime: the EPRL four-simplex asymptotics studied by Barrett and colleagues contain multiple Regge-related contributions and prefactors [8]. A universal replacement of every amplitude by a single exponential of a chosen action would therefore omit material structure.

For the discussion papers, the transferable proposal is to make boundary data, hidden histories, admissible maps, and insertion observables explicit. A social interpretation of the group variables would require its own evidence and mapping. The present positive weights define a classical statistical model; they supply no oscillatory quantum amplitudes, gravitational geometry, or explanation of consciousness. The demonstrated boundary and interior distinction remains mathematically informative within that deliberately limited scope.

I Interpretation and Computational Records

This appendix records the scope and reproducibility of the shared mathematical studies.

I.1 Relational Interpretation and Issue-Relevant Equivalence

This section connects the mathematical comparisons to the companion discussion papers. It identifies the proposed analytical use and the substantive premises that the formal results leave open.

For Paper I, a reality comparison should identify which observations, relations, or interventions justify treating two entities alike. An equal recorded history need not identify its generator, whereas a coordinate difference can arise between descriptions that preserve the complete dynamics. Both distinctions matter: different appearance can accompany an equivalent description, and matching appearance can accompany different mechanisms. The appropriate conclusion depends on the declared comparison, rather than the word equivalence alone.

The action and finite-group calculations extend this argument to history-sensitive descriptions. An action can organise possible paths, and a boundary state sum can aggregate unresolved interiors. Their availability does not make every retained difference empirically accessible or relevant to a reality judgment. Conversely, equality of the selected aggregate cannot settle all questions about those interiors. The original three-field memory experiment remains a distinct example with its own state reset and context gate; the new studies examine equivalence between model descriptions more broadly.

For Paper II, an admissible comparison map must preserve the roles, time scales, options, and evidential distinctions material to the governing issue. A clock rescaling that removes a period of deprivation can destroy legal relevance while preserving an orbit. A boundary record that retains only a final agreement can erase the history through which agreement became available. The field representation should therefore make its observation map and omitted information inspectable. The legitimacy of a legal classification remains dependent on a norm and its institutional authority.

The numerical examples are constructed counterexamples and exact model identities. They are independent of the earlier 24-agent pilot and have no fitted relationship to its language outputs. They do not identify social dynamics as gravitational fields, establish a quantum mechanism of interpretation, or demonstrate consciousness. Their contribution is a disciplined set of comparison choices that can be tested and criticised before empirical or normative application.

I.2 Computational Records and Verification

This section identifies the executable record and the distinction between analytic claims and numerical checks. The field-equivalence, action, and finite-group experiments each preserve a local pre-run protocol, source code, arrays or exact enumerations, software metadata, results, and exported PNG/PDF figures. They use deterministic calculations and no external agent responses or fitted parameters.

The directories are `experiments/field_equivalence/`, `experiments/field_action/`, and `experiments/spinfoam_toy/`. The programme index in `experiments/field_comparison/README.md` links their reports and figures. Numerical refinements check discretized evolution; exact Fourier and finite-group calculations check the corresponding closed expressions. Finite tests supplement the analytic arguments and leave their stated function spaces, controls, and normalization intact. Source ledgers record the internet verification completed before citations were inserted.

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