Heterogeneous Value Circulation in Relational Spacetime: A Topological Formalism

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Abstract

This discussion paper develops a preliminary topological language for heterogeneous value circulation within a Generative Relational (GR) account of relational spacetime. The underlying structure is an oriented two-dimensional cell complex whose vertices are events, edges are couplings, and faces are primitive two-dimensional structures with semantics to be specified by the model. Entities and values are considered potentially emergent, historically constituted structures supported by subcomplexes; the representation does not require each event to be an exchange or each value to occupy a single face. Different value modalities may exhibit distinct topological, geometric, temporal, dynamical, and multiscale properties, while sharing portions of their relational support. Circulation is approached through causally admissible histories and transformations of value-supporting configurations, including transmission, conversion, reproduction, and changes in subsequent generative capacity. Cellular boundaries, homology, quotient constructions, discrete Morse theory, and persistent topology supply candidate analytical tools, with their interpretive conditions explicitly distinguished. The longer-term normative problem concerns how the ordering of feasible histories affects deprivation, obligations, value formation, and future possibilities. The paper treats mathematical structure as a proposed representation rather than evidence of an ultimate ontology or a validated social mechanism. This working release contains Sections 1–11 and Appendices A–E (notation, exact cellular calculations, historical subcomplex constructions, simulation reproducibility, and open mathematical extensions), including reproducible synthetic computational experiments, a preliminary justice-comparison interface, an assessment of mathematical and interpretive scope, and a conclusion; the broader ethical theory remains open.

Keywords: heterogeneous value; relational spacetime; generative relational theory; two-dimensional cell complexes; value constitution; circulation order; topological morphology; justice.

Draft status. Independent discussion paper, working draft R18, 19 October 2026. This version contains Sections 1–11, Appendices A–E, and preliminary front matter. It is neither peer reviewed nor a completed formal theory. No fieldwork or empirical calibration is claimed; the illustrative computational results are fully synthetic and reproducible. The main text and planned appendices are drafted; manuscript-wide scholarly review remains.

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Discussion Paper Note

This paper develops a preliminary mathematical representation of heterogeneous value circulation in Generative Relational (GR) spacetime. It takes a two-dimensional cell complex as an analytical starting point: vertices represent events, edges represent couplings, and faces represent primitive higher-order structures whose substantive semantics have yet to be specified. Historically extended subcomplexes may support constituted entities and values. The model permits events that have no immediate interpretation as a transfer, as well as value-supporting structures whose formation or recognition extends over time.

The paper operates with ontological suspension. Mathematical objects are proposed representations of historically consequential relational configurations; they do not establish an ultimate ontology of value, persons, institutions, or the physical world. Recognition and manifestation are distinguished from the underlying processes as far as the available evidence allows. Claims about the existence, shape, or effects of real-world relational structures would require explicit observational and interpretive procedures.

Several mathematical resources considered here have established intellectual histories: cell complexes, homology, spin-foam models, causal orders, discrete Morse theory, and persistent topology. Their use in the proposed GR value representation is a theoretical extension rather than an assertion of new mathematics. The terms *value*, *generation*, *circulation*, and *justice* may have different meanings in economics, social theory, ethics, law, and engineering. The manuscript avoids treating a mathematical translation between those fields as a substantive identification of their concepts.

The eventual normative inquiry concerns the justice of circulation order in the presence of heterogeneous needs, obligations, partially comparable values, and evolving capacities. Topological features, including holes, have their mathematical meanings before any socioeconomic interpretation is assigned. A structural descriptor does not itself establish a claim about deprivation, entitlement, moral responsibility, or the relative justice of two histories.

This working release contains Sections 1–11 and Appendices A–E. It includes mathematical foundations, the event–coupling–primitive ontology, proposed modality-dependent value constitution, historical transformation rules, circulation-order representations, and synthetic computational experiments. The computational sources and machine-readable outputs accompany the manuscript. The paper makes no claim of original field evidence, empirical calibration, or a complete normative decision procedure; Section 9 provides a preliminary framework for justice-sensitive comparisons, Section 10 examines interpretive limits, and Appendix E identifies possible mathematical extensions and their additional requirements. The main-text conclusion appears in Section 11. A manuscript-wide scholarly revision and a fully justified normative theory remain future work.

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Notices

Status. Independent theoretical discussion paper, working draft R11, 19 October 2026. This source release contains Sections 1–11, including mathematical foundations, the proposed GR ontology, value morphologies, historical transformations, and fully synthetic computational illustrations. Section 8 reports executed mathematical experiments with openly supplied Python source and numerical outputs. Section 9 introduces a provisional normative interface, and Section 10 assesses the scope and interpretive limits of the mathematics; Section 11 provides the conclusion; appendices remain planned. The draft has not been peer reviewed. Its interpretations and research agenda are provisional; no field study, empirical calibration, or normative validation is claimed.

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1 Introduction

The present paper develops a preliminary mathematical representation of heterogeneous value circulation within a Generative Relational (GR) account of relational spacetime. Its immediate objective is to establish a coherent language for events, couplings, primitive two-dimensional structures, historically constituted values, and the alternative histories through which such values may form and circulate. The inquiry proceeds by identifying the representational demands raised by heterogeneous social activity, specifying the intended interpretation of a two-dimensional cell complex, and distinguishing structural invariants from causal and normative assessments. The resulting formalism is proposed as a research architecture whose definitions, examples, and domain-specific interpretations remain open to refinement.

Value circulation is commonly encountered through apparently discrete transactions: a payment changes hands, labor is provided, information is communicated, or a resource is assigned to a recipient. These descriptions are useful at their respective levels of analysis. They can, however, compress extended histories of preparation, capacity formation, recognition, dependency, and subsequent consequence into a small number of recorded exchanges. A monetary transfer may occur at a well-defined accounting time, while the labor, knowledge, care, trust, or institutional capacity relevant to the same interaction develops through multiple activities. The timing of a recorded transfer, the duration of a value-generating process, and the moment at which a consequence becomes observable may therefore differ. A framework concerned with historical order needs to represent such distinctions explicitly.

Heterogeneity also concerns the kinds of relations through which values are generated and interpreted. Monetary amounts, hours of labor, opportunities to learn, claims to care, ecological contributions, and participation in shared institutions involve partly different units, conditions, and forms of recognition. The capacity to list these quantities in a common database does not itself provide a justified common valuation scale. A conversion between values may depend on institutional arrangements, technical capacities, power, legal entitlements, or the willingness and ability of participants to accept particular terms. Discussions of generative justice have already connected the circulation of heterogeneous labor, ecological, and expressive values with questions of alienation and the return of benefits to their generators [6]. Comparative approaches to justice likewise provide established reasons to examine substantive realizations and capabilities in addition to formally defined arrangements [14]. The present project inherits these concerns as motivating problems, without claiming to originate either tradition or to derive a normative criterion from mathematical topology.

A second difficulty arises from the status of the participants in a circulation model. Many flow diagrams begin with identifiable persons, organizations, or other entities as vertices and transfers as edges. Such representations can be effective when the boundaries and identities of the modeled participants are adequate to the specific research question. The GR interpretation considered here gives ontological priority within the *representation* to events and their couplings. A vertex denotes an event; an edge denotes a coupling between events; and a face denotes a primitive two-dimensional structure whose mathematical labels and substantive meaning require further specification. Persons, institutions, collectives, and other entities may be recognized through historically constituted sub-spacetimes rather than identified with elementary vertices. This representational choice is compatible with practical references to recognizable entities at suitable scales, while keeping their historical formation and alteration within the domain of inquiry.

A similar distinction governs the representation of value. A primitive face may carry labels, phases, weights, or other local information, but a constituted value need not correspond to one face or one instantaneous event. The working proposal associates a value modality with a potentially extended subcomplex, together with its local decorations, relevant history, and conditions of recognition. The same subcomplex can participate in several value modalities, and the supports of distinct values may overlap. Value-supporting complexes may differ in connectedness, branching, cycles, holes, weighted extent, temporal span, or multiscale organization. These are candidate structural descriptors rather than measurements of ethical worth: a large complex need not generate a large value, and an unfilled topological cycle need not denote deprivation or injustice. The hypothesis that some value-supporting structures display approxi-

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mate self-similarity likewise calls for scale-sensitive evidence rather than an assumption of universal fractality.

The use of a two-dimensional complex is informed by the established mathematics of cellular topology and, more specifically, by the combinatorial architecture of spin-foam models. Algebraic topology supplies chain groups, boundaries, and homology as descriptions of how cells assemble and how cycles relate to higher-dimensional fillings [11]. In the quantum-gravity literature, spin foams are decorated higher-dimensional complexes used to formulate histories and amplitudes of spin-network states [2]. The paper borrows the availability of event, coupling, and face-like incidence data as a structural inspiration. It introduces neither a quantum interpretation of economic value nor a claim that a physical spin label specifies an economic, ethical, or institutional quantity. The semantics of GR primitive faces and the admissible operations on their decorations remain independent choices requiring explicit definitions.

The representation of historical order introduces a further layer. The topology of a cell complex records incidences and possible cycles, whereas the realizability of a sequence of events depends on causal precedence, compatibility, and feasibility constraints. A closed 1-chain is not automatically an executable circulation process. Likewise, two causal histories with comparable final distributions may differ in how long a vulnerable party waited, which opportunities existed along the way, and which capacities their intermediate events generated or destroyed. Mathematical studies of distributed processes provide precedents for distinguishing structured state spaces from possible event executions and for using combinatorial topology to analyze concurrency [12]. Within the present project, the corresponding causal structure will be introduced as additional data on, or associated with, the relational complex; its social interpretation does not follow from the borrowed mathematics alone.

Historical order matters especially where values are constituted rather than merely transferred. Consider a stylized programme in which access to instruction, material resources, and opportunities to collaborate are distributed across different periods. Two admissible orders may produce similar end-of-period resource accounts, yet afford different intervals for learning, different risks of irreversible exclusion, and different possibilities for subsequent cooperation. A value-supporting configuration may change through those histories, and the entities whose capacities are evaluated may change with it. These possibilities motivate the distinction between endpoint comparisons and trajectory-sensitive evaluation. They also caution against constructing the admissible history space solely from vertices already labeled as transfers: observation, waiting, refusal, conflict, unsuccessful experimentation, and delayed recognition may have causal significance for both value formation and its assessment.

The formal study of these structures calls for a separation of mathematical questions that are sometimes combined under the term *circulation*. First, one may ask which subcomplexes support a specified value modality and how their primitive encodings contribute to its recognition. Second, one may study transformations that transmit, combine, reproduce, fragment, or reconfigure value-supporting structures. Third, one may characterize topological properties of the ambient complex or of particular supports using boundary operators and homology. Fourth, one may identify feasible histories and examine the effect of temporal order on their resulting configurations. A quotient construction may identify histories or cellular structures under a defined equivalence relation, but the resulting identification preserves justice-relevant distinctions only under additional conditions. In particular, homological equivalence by itself entails neither equivalence of social consequences nor equality of normative assessments.

The mathematical toolkit includes methods for simplifying and comparing complex structures. Discrete Morse theory provides a way to study cell complexes through critical cells and reduced descriptions with controlled topological properties [8]. Persistent homology and related multiscale constructions offer methods for examining the appearance and persistence of topological features as a filtration parameter changes [4]. These developments are relevant to the possible reduction of large historical complexes and to the study of morphology across scales. Their applicability to value-supporting structures depends on the definition of the filtration, the empirical or interpretive meaning of the cell weights, and the historical information preserved by the reduction. In particular, a method that preserves homology may discard the very sequence, duration, or recognition differences on which a justice judgment depends. The paper therefore treats topological reduction and historically adequate representation as separate requirements.

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The longer-term normative objective concerns the justice of heterogeneous value circulation order. A feasible history may affect the time at which needs are met, the recognition of claims, the distribution of burdens and benefits, the emergence of reciprocal obligations, and the capacities available for subsequent generation. Justice assessments may accordingly depend on several dimensions that are only partially comparable. The present proposal allows for such assessments to take values in a partially ordered space or another explicitly justified normative structure. It does not propose that Betti numbers, geometric size, spin labels, fractal dimension, or any single structural statistic measures justice. The normative evaluation of a history requires independently articulated criteria and, where real cases are considered, evidence about the relations between the formal representation and the activities it purports to describe.

The contribution sought by this paper is consequently specific. It develops a provisional event–coupling–face representation of relational spacetime; introduces historically supported subcomplexes as candidate carriers of constituted heterogeneous value; differentiates value modalities through topological, geometric, temporal, and dynamical descriptors; and places circulation within the set of causally admissible histories rather than identifying it with graph flow alone. It also establishes analytical interfaces between cellular homology, quotients, topological simplification, and the later comparison of circulation orders. These are proposed combinations and interpretations of established mathematical resources, subject to consistency checks and counterexamples. They are neither claims of novelty for the underlying mathematical constructions nor claims that the proposed GR ontology has been established empirically.

The remaining manuscript is organized to keep conceptual and mathematical responsibilities distinct. The problem setting and mathematical background precede the GR interpretation of events, couplings, and primitive faces. Subsequent sections formulate heterogeneous value supports, their transformation dynamics, and the representation of circulation histories. Illustrative constructed examples then examine alternative event orders, changing morphology, and possible multiscale descriptions. Later sections relate the resulting formal substrate to justice-oriented questions and discuss ontological restraint, evidential interpretation, identifiability, and limitations. The intended outcome is a set of definitions and analytical examples suitable for further criticism, implementation, and domain-specific inquiry.

2 Heterogeneous Value and Historical Relational Formation

This section specifies the representational problem that motivates a topological formalism of heterogeneous value circulation. Its objective is to distinguish value modalities, generative processes, historical constitution, and the temporal ordering of feasible events before introducing the mathematical machinery of cell complexes. The discussion proceeds through conceptual analysis and deliberately constructed examples. Established work on generative justice, care, comparative justice, and path dependence supplies disciplinary context; the interpretation of value as a historically supported relational configuration remains the proposal developed in the present paper. These distinctions supply constraints that subsequent formal definitions must satisfy.

2.1 Heterogeneous Values and Partial Commensurability

The term *heterogeneous value* covers differences in what is evaluated, how an evaluation is expressed, and which processes bring the evaluated phenomenon into existence. Monetary claims can be recorded in a denomination and an amount. Labor can be described by duration, skill, conditions, and outputs. Care involves relationships of attentiveness, practical response, need, and continued responsibility. Knowledge may involve situated understanding, reproducible methods, or capacities that can be shared without an equivalent subtraction from the original holder. Institutional trust, ecological contribution, and the ability to participate in collective decisions present still other structures of production, dependence, and recognition. Such descriptions may coexist within an activity even when a common unit of comparison has not been justified.

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A transaction can establish an agreed exchange ratio while leaving several value dimensions open. Consider an agreement exchanging a payment of one hundred dollars for twelve hours of labor. The agreement records contractual terms and a monetary exchange ratio. It does not, on that basis alone, determine the quality of the work, the vulnerability associated with delayed payment, the burdens of the working conditions, the knowledge acquired during performance, or the legitimacy of the bargaining situation. Further evidence and normative principles would be required to characterize those features. Equally, the contractual price need not determine the value of an emergent capability or a later relationship made possible by the activity.

Comparability is consequently an operation defined for specified purposes. Two monetary amounts denominated in the same currency and at a specified time may admit a conventional quantitative ordering. Two training opportunities may be compared with reference to an explicitly chosen educational objective. A comparison between urgent care and a commercial benefit may require a priority rule, a threshold of acceptable harm, or a rights-based restriction. The existence of one comparison does not establish that the complete underlying values share a universal scalar representation. A formalism can accordingly accommodate modality-dependent comparisons, partially defined conversion relations, and cases in which the available information supports no settled ranking.

Comparability also depends on the authority to define a conversion. Institutional rules, technical interfaces, legal entitlements, and unequal bargaining power may influence which relations become operationally recognized. A conversion mechanism is therefore itself an object of historical inquiry. Its implementation may enable cooperation, impose burdens, privilege particular forms of value, or exclude contributions that remain difficult to price. Generative justice provides a relevant prior vocabulary through its attention to the circulation and possible alienation of labor, ecological, and expressive value [6]. The present model considers the representational conditions for studying such processes; it does not treat the distinct value categories or the normative insight as new discoveries of GR.

The capability-oriented and comparative aspects of justice developed by Sen provide a further reason to retain several dimensions of substantive consequence when evaluating alternatives [14]. Likewise, Tronto's account of care situates caring practices within political relations and inequalities, illustrating why an exchange description alone can be insufficient for a social interpretation of work and responsibility [15]. These traditions have different conceptual commitments. Their inclusion here establishes points of contact concerning value, recognition, and consequence, without asserting theoretical identity.

For later formalization, the *modality* of a value will refer to a specified family of constitution, transformation, observation, and comparison conditions. The definition is deliberately relational and revisable: a modality may overlap other modalities, and a historical change in institutions or techniques may alter its recognized boundaries. Different modalities may also exhibit different supporting morphologies within a relational complex. Structural descriptors can assist comparison, but they do not independently settle a question of value magnitude or ethical priority.

2.2 Value Generation, Circulation, and Reproduction

A formal account of heterogeneous circulation requires a distinction among the operations through which value arises, changes relations, and contributes to subsequent possibilities. *Generation* concerns the constitution or alteration of value-related structures and capacities through historical activity. *Circulation* concerns the relations by which constituted values, claims, effects, or productive capacities become available in other configurations. *Conversion* concerns changes between specified modalities or modes of manifestation. *Reproduction* concerns the renewal of conditions for generating or sustaining value. These operations can overlap empirically; separating them analytically helps identify their different assumptions and consequences.

A payment between accounts supplies a relatively discrete record of a monetary transfer. The activity that the payment supports may extend across time and require a complex history of preparation, labor, tools, knowledge, and coordination. An educational interaction can simultaneously communicate existing knowledge and help constitute a new capability in the learner. The transmitted content may remain

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Table 1: Analytical processes in the historical formation and circulation of heterogeneous value.

Process	Analytical role	Possible evidential trace
Generation	Constitution or alteration of value-bearing structures and capacities	Learning record, new capability, changed practice
Circulation	Relational accessibility or transmission of an existing value-related configuration	Transfer, sharing, uptake, use
Conversion	Transformation between specified modalities or manifestations	Contract, production process, interpretation rule
Reproduction	Renewal of capacities and relations enabling later generation	Maintenance, teaching, care, institutional renewal
Capture	Concentration of benefits or control at a specified interface	Asymmetric terms, control of access, appropriation
Dissipation	Loss, degradation, or weakening of previously available capacities	Skill decay, exhausted resource, discontinued practice

accessible to the teacher; the newly formed capability may differ in scope and use from the teacher's original knowledge. A care practice may require continued attentiveness and follow-up even where a particular intervention has an identifiable completion time. These cases illustrate different possible relations among transfer, transformation, persistence, and historical dependence.

Table 1 sets out the analytical distinctions adopted at this stage. Its entries are modeling categories and possible observable traces, rather than claims of empirically established laws of value formation.

The term *circulation* consequently admits several mechanisms. A constituted monetary claim may be transferred under accounting rules. A piece of information may be copied and interpreted. A practical capacity may circulate indirectly when an experienced participant teaches others. A beneficial relationship may facilitate further relationships that retain only part of the original configuration. These cases involve different assumptions about conservation, divisibility, ownership, and replication. An additive flow equation may be appropriate to a defined resource ledger, while a nonlinear or history-dependent transformation is potentially more suitable for learning, recognition, or social trust.

The preservation of value during transfer is therefore a hypothesis specific to a mechanism. Some exchanges redistribute scarce resources; some forms of communication enable multiple participants to use related knowledge; some transfers damage productive capacity through extraction or uncompensated burdens. Likewise, apparent growth in recorded output may coexist with depletion of the relations needed for future activity. The connection between value circulation and generativity concerns the conditions under which circulating structures enable, constrain, or diminish subsequent processes of formation. The general form of this connection is an open modeling problem; an observational or ethical assessment requires more information than the occurrence of a transfer.

The distinction is also relevant to *return*. A circulating benefit may return to a contributing participant through payment, increased capability, common infrastructure, reciprocal care, or other arrangements. Different return pathways may be only partially comparable. Existing discussions of generative justice motivate inquiry into such returns [6]; the current proposal seeks to represent the histories and interfaces through which they become possible. The degree of return, its distribution, and its justice must be evaluated through separately specified criteria.

2.3 Historical Constitution of Values and Entities

The GR interpretation used in this paper assigns vertices of the underlying two-complex to *events*, edges to *couplings*, and faces to primitive two-dimensional structures. The meaning of a face and the admissible encodings associated with it will be specified at the level of a proposed model. An event need not be an exchange, a moment of value recognition, or a transaction involving already identified parties. Reading, waiting, observing, misunderstanding, repairing, excluding, and changing a rule may all be represented

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as events where the analytical domain warrants their inclusion. Events and couplings participate in the historical constitution of entities and values, rather than simply recording interactions between timeless elementary agents.

An entity can be represented at an appropriate level of description by a historical subcomplex and additional identification conditions. Those conditions may involve continuity of practice, organization, shared memory, legal recognition, physical persistence, or other domain-specific relations. A participant identifiable as one organization at a particular time may have changed its activities, membership, capacities, or obligations over an extended history. A legal designation and an internal cooperative practice need not delimit exactly the same subcomplex. The model therefore leaves room for several defensible individuations of an entity, with their observational and normative purposes made explicit.

The analogous proposal for value associates a modality with a potentially distributed historical subcomplex. A recognized skill, for example, may depend on training, unsuccessful attempts, assistance, bodily practice, and later demonstrations. A body of shared knowledge may depend on experiments, documentation, critical correction, circulation of evidence, and interpretive activity across multiple settings. Trust may be supported by repeated reliable interactions and by institutional protections that reduce the hazards of reliance. In each case, the historical support may span multiple events, couplings, and primitive faces. It may overlap the support of other values, and its boundaries may depend on the recognition question being asked.

It is useful to distinguish *constitution*, *recognition*, and *manifestation*. Constitution concerns the formation of a relevant relational structure or capacity. Recognition concerns the conditions under which an observer or institution identifies and attributes a value. Manifestation concerns the occasion and form in which a capacity or value becomes observable in action. These distinctions describe analytical roles; they do not establish that the corresponding times are always sharply separable. Recognition may itself influence constitution by changing opportunities, identities, or institutional support. Some values become recognizable only retrospectively, while the historical conditions contributing to them were distributed across preceding activities.

The inclusion of recognition raises an epistemic constraint. Different observers may delimit different supporting subcomplexes, assign different modalities, or give different accounts of causally relevant events. Differences can arise from measurement limits, situated interpretive systems, unequal access to records, and power over categories of recognition. Recording a structure at high resolution cannot eliminate these sources of uncertainty by itself. A formal model therefore needs to preserve provenance, indicate which identifications depend on interpretation, and support revisions as additional evidence becomes available. This requirement connects the topology of proposed supports with an account of how those supports are known.

A subcomplex representation should also remain optional rather than forced. Some phenomena may require weighted covers, overlapping regions, time-varying supports, or higher-order data whose representation is not captured by a single strict subcomplex. An instantaneous event may be sufficient to record a legally effective change while an extended supporting history remains useful for understanding its formation and consequences. The choice of representation must be justified locally by the research question and available evidence. The present paper therefore develops subcomplexes as a candidate formal language, with the epistemic and ontological status of each construction kept visible.

2.4 Temporal Order, Dependencies, and Generative Consequences

Historical order matters where the feasibility or consequence of an event depends on what has already occurred. Dependencies may arise from physical resources, institutional permissions, learned skills, shared information, legal obligations, or interpersonal relationships. The order in which two events are recorded is not necessarily the order required by their causal relations. For that reason, the proposed model will supplement the underlying cell complex with a separately defined precedence structure and constraints on event execution.

For a finite event set E and an acyclic precedence relation $\prec$ on E, an admissible total ordering may

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be represented by a linear extension, as expressed in Equation 1.

$$\Gamma(E, \prec) = \left\{(e_{\pi(1)}, \ldots, e_{\pi(n)}) : e_i \prec e_j \implies \pi^{-1}(i) < \pi^{-1}(j)\right\}. \tag{1}$$

Here $E = \{e_1, \ldots, e_n\}$ and π ranges over permutations of $\{1, \ldots, n\}$. The construction describes order-consistent sequences; subsequent sections will refine feasibility through event-specific resources, temporal constraints, and state-dependent transformations. A path in a cellular complex, a linear extension of an event order, and a cellular chain are distinct mathematical objects. Their relationships require explicit constructions.

Consider a setting with two requests for an indivisible service that can be provided sequentially. Suppose both requests will eventually be met and the completion of one imposes an interval of waiting on the other. If one need is time-sensitive, the scheduling order may change experienced harm even when the final ledger records two completed services. The relevant difference need not be captured by the monetary price or by a comparison of final quantities alone. A justice evaluation may attend to urgency, duration of deprivation, responsibility for delay, and the institutional conditions that make one sequence feasible. A model that compresses both histories into an identical endpoint may obscure these consequences.

The distinction can be expressed without selecting a universal loss measure. Let Π_{fin} record a particular endpoint summary, and let H record history-sensitive features such as waiting, evolving capacities, and obligations. Under an illustrative pair of histories, Equation 2 describes the possibility of endpoint agreement alongside historical difference.

$$\Pi_{\text{fin}}(\gamma_a) = \Pi_{\text{fin}}(\gamma_b), \qquad \mathsf{H}(\gamma_a) \neq \mathsf{H}(\gamma_b). \tag{2}$$

The choice of Π_{fin} is material: agreement on aggregate outcomes does not entail agreement on every eventual consequence. The expression formalizes an information distinction and carries no independent verdict about the justice of either history. A normative ranking would require justified comparisons among the consequences and claims represented by H.

The historical process may also change the future set of feasible events. An early opportunity for learning can enable later forms of work; a delayed treatment can narrow future possibilities; recognition by an institution can open access to collaboration or resources. These mechanisms suggest feedback between present ordering and future generative capacity. Historical path dependence has an established literature in economics: David's analysis of QWERTY concerns the historical interdependence of technical choices, while Arthur's increasing-returns model shows how early contingencies can affect technological adoption and possible lock-in [5, 1]. These works supply bounded precedents for treating sequence as consequential under specific mechanisms. They do not establish the GR account of value constitution, nor do they demonstrate that every difference of social event order produces path dependence.

Causal feasibility and normative priority should remain distinct. An urgent claim does not automatically make a proposed event sequence physically executable; constraints may include the availability of trained personnel, legal authorization, equipment, or the compatibility of other obligations. Conversely, feasibility alone does not determine which permitted sequence is just. The research problem is to compare accessible alternatives under stated information, constraints, and normative criteria, while allowing the realized history to change what becomes possible later.

2.5 Relational Spacetime and the Representation of Value

A topological representation is motivated by the need to retain relations extending beyond a transaction ledger. A graph can record events and pairwise couplings, and richly annotated temporal or multilayer graphs may adequately address many practical questions. A two-dimensional complex becomes useful when the model requires primitive structures associated with configurations of couplings and the boundaries of those structures. Subcomplexes can then provide a candidate support for historically constituted values, while algebraic topology can describe connectedness, cycles, and higher-dimensional closure under explicit assumptions. The additional mathematical dimension earns its place only when the face data and resulting analyses convey information relevant to the stated question.

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The GR interpretation gives the structures a fixed representational role at the primitive level: vertices stand for events, edges for couplings, and faces for primitive higher-order structures. The paper does not yet specify a universal semantic content for faces. A face may carry an abstract label or belong to a prescribed incidence relation; any economic, institutional, or ethical meaning assigned to that structure requires a domain-specific model. In particular, a two-cell should not automatically be treated as the reconciliation of two allocation orders or as the representation of a just transformation. Such choices would already encode assumptions about history and normativity into the topology to be analyzed.

A value-supporting complex may differ across modalities by connectivity, cycles, branching, temporal extent, weights, or the possibilities of transformation. Some supports may be compact relative to a chosen metric, while others may be extensive, distributed, or overlapping. A hole in a value-supporting complex has a mathematical interpretation relative to its chosen cells and boundary maps; any meaning for value circulation depends additionally on the encoding and dynamics. Equally, the size or number of cells in a support need not measure realized value. The relevant structures may be coarse-grained at different resolutions, but topological equivalence at one scale does not guarantee equivalence of timing, obligations, or recognition at another scale.

Three modeling responsibilities follow. First, a *constitution rule* must describe the conditions under which historical structures are treated as values or entities of a specified kind. Second, a *dynamical rule* must specify admissible changes in coupling, transformation, and effective accessibility, including the possibility of altered future supports. Third, an *evaluation rule* must explain which features of a history are relevant to justice and how conflicts among partially comparable claims are handled. The first two concern representation and explanation; the third requires additional ethical and institutional reasoning. Their separation allows the model to be criticized and revised without deriving justice from a topological invariant.

The resulting research strategy is modular. Cellular methods provide a language for selected structural properties; partial orders and event histories provide a language for admissible temporal succession; interpretation and evidence identify the meaning and limits of value-supporting configurations; and normative inquiry compares feasible histories. the mathematical foundations section will develop the mathematical operations needed for these connections. The present section establishes the substantive constraints: heterogeneity is multidimensional, historical formation may be extended, circulation can alter future capacity, and a useful reduction must preserve information required by the research question.

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3 Mathematical Foundations

This section develops the mathematical background required to represent heterogeneous value-supporting structures and the causal histories in which they develop. It begins with finite cell complexes, oriented incidence relations, and cellular homology; distinguishes quotient constructions from equivalence of circulation histories; then introduces partial orders, discrete Morse reductions, and multiscale topology. The discussion uses standard mathematical results and small constructed examples, with particular attention to the information that each operation preserves. The economic and ethical interpretation of these structures will be introduced in later sections: the mathematics developed here neither defines a value modality nor supplies a criterion of justice. Background on cellular and combinatorial topology is available in Hatcher [11] and Kozlov [13]; spin-foam conventions provide a related, but physically specific, use of decorated complexes [2].

3.1 Graphs, Cell Complexes, and Spin-Foam Structures

A graph records vertices and their incident edges. Its combinatorial information is sufficient for many path, flow, and reachability problems. A two-dimensional cell complex additionally specifies which edge assemblies bound primitive two-dimensional cells. The extra incidence information is central to the proposed representation because a closed sequence of edges may or may not bound a face, and the

two cases are distinguished by cellular homology. Equipping the complex with weights or representation labels adds further information without changing this elementary distinction.

For definiteness, the first mathematical constructions in this paper use a finite, regular two-dimensional CW complex K. Its sets of p-cells are denoted K_p for $p = 0, 1, 2$. Regularity means that the attaching map of each closed cell identifies its boundary with a subcomplex in the standard manner; this assumption makes the combinatorial examples transparent. Each cell is given a chosen orientation, which is a bookkeeping convention for the chain groups. A subcomplex $L \subseteq K$ is a collection of cells closed under taking incident boundary cells. The geometric realization $|K|$ is the topological space obtained from these attachments. More general cellular and stratified structures remain available for later extensions.

A finite 2-complex can carry different decorations. For example, a face f may be associated with a representation label ρ_f, a local state ψ_f, or a type-specific parameter; couplings at edges can be supplied with compatibility maps. These are additional modeling data. In physical spin-foam constructions, representation-theoretic face labels, edge intertwiners, and vertex amplitudes occur in particular state-sum formulations [2]. A GR-inspired 2-complex may borrow this combinatorial architecture while retaining the separate interpretive commitments of its own domain. In particular, the vertex-to-event, edge-to-coupling, and face-to-primitive correspondence adopted in this project is a proposed representation, not a theorem about either social activity or quantum spacetime.

Two distinctions are useful from the outset. The orientation chosen for a cell determines signs in a boundary matrix; it need not designate the causal direction of an event or the direction of a value transfer. Furthermore, the adjacency of cells specifies possible structural coupling while leaving open whether a particular history is executable. A causal relation and a set of feasible histories will therefore be specified independently in Subsection 3.5.

3.2 Oriented Cells, Chains, and Boundary Operators

For a field $\mathbb{F}$, such as $\mathbb{R}$, the pth cellular chain group $C_p(K;\mathbb{F})$ is the vector space freely generated by the oriented p-cells. A p-chain is a finite formal linear combination of those cells. It is an algebraic object determined by coefficients and cellular incidence, rather than a time-ordered execution. The chosen cellular orientations produce boundary maps $\partial_p : C_p(K;\mathbb{F}) \to C_{p-1}(K;\mathbb{F})$. In dimension two, their defining sequence is written in Equation 3.

$$C_2(K;\mathbb{F}) \xrightarrow{\partial_2} C_1(K;\mathbb{F}) \xrightarrow{\partial_1} C_0(K;\mathbb{F}), \qquad \partial_1 \circ \partial_2 = 0. \tag{3}$$

For a directed notation e_{ij} for an edge whose *chosen orientation* runs from vertex v_i to vertex v_j, $\partial_1 e_{ij} = v_j - v_i$. If a face f is attached along the ordered triangle (v_0, v_1, v_2), with e_{20} oriented from v_2 to v_0, then its oriented cellular boundary is $\partial_2 f = e_{01} + e_{12} + e_{20}$. Consequently, $\partial_1 \partial_2 f = 0$, as required by Equation 3. Reversing a cell orientation changes its sign without changing the underlying geometry.

After choosing ordered bases of oriented cells, the operators are represented by matrices B_1 and B_2. For this triangular example, Equation 4 gives the explicit incidence matrices.

$$B_1 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad B_1 B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

The columns of B_1 correspond to e_{01}, e_{12}, e_{20}. Adding the face creates the nonzero B_2 column. Omitting the face leaves the same 1-skeleton but changes C_2 to the zero space. This small difference already produces a different first homology group, as demonstrated below.

A sequence of adjacent edges $\gamma = (v_0, e_1, v_1, \ldots, e_n, v_n)$ has an associated 1-chain c_γ, obtained by adding traversed edges with signs determined by their orientations. Equation 5 records its endpoint boundary.

$$\partial_1 c_\gamma = v_n - v_0. \tag{5}$$

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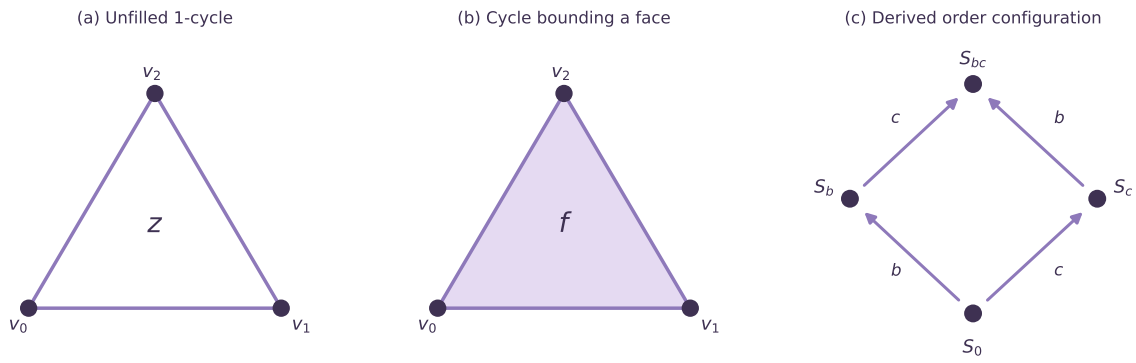

Figure 1: Cellular topology and derived history configurations. Panels (a) and (b) use event vertices v_i and coupling edges as the 1-skeleton of a candidate 2-complex; attaching face f changes the first homology. Panel (c) is a *separate, derived configuration diagram*: its nodes S_0, S_b, S_c, S_{bc} are sets of completed events, rather than vertices of the GR spacetime, and its arrows encode possible execution steps. The diagram is schematic and contains no empirical observations.

Equation 5 is a telescoping incidence identity. It can describe a path in the 1-skeleton once that path is given, but the chain itself does not preserve the list of successive events, repetitions, waiting durations, or values transformed along the route. Distinct traversals can produce the same chain, and two feasible event orders can involve cells that do not form an edge path in the original complex. A history representation should accordingly retain sequence information separately from c_γ.

3.3 Homology, Cohomology, and Topological Invariants

The pth homology group distinguishes closed chains from chains that bound cells of one higher dimension. Over the chosen field $\mathbb{F}$, the definition is given in Equation 6.

$$H_p(K;\mathbb{F}) = \frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}}, \qquad \beta_p(K) = \dim_{\mathbb{F}} H_p(K;\mathbb{F}). \tag{6}$$

Here β_p is the pth Betti number. In the finite two-dimensional case, β_0 counts connected components, β_1 counts independent first-dimensional homology classes, and β_2 counts independent two-dimensional cycles with the selected coefficients. These interpretations refer to the topology of K and are independent of a value or justice attribution. A first-dimensional hole can mean, in precise algebraic terms, a nonzero class of a 1-cycle modulo the boundaries of 2-chains.

The distinction is illustrated by Figure 1. Its first two panels have the same triangular edges, while only the second includes a two-dimensional face. With the matrices of Equation 4, $\ker B_1$ is spanned by $(1,1,1)^\top$. In the unfilled case $\operatorname{im} B_2 = \{0\}$, so $\beta_1 = 1$. With the face present, $\operatorname{im} B_2 = \ker B_1$ and $\beta_1 = 0$. The calculation is elementary; its representational importance is that an unfilled edge cycle and a bounded edge cycle have different mathematical status even where their edge structure is identical.

For a finite complex, the alternating sum of cell counts equals the alternating sum of Betti numbers. Equation 7 gives the Euler characteristic in the two-dimensional setting.

$$\chi(K) = |K_0| - |K_1| + |K_2| = \beta_0 - \beta_1 + \beta_2. \tag{7}$$

The Euler characteristic is informative but coarse: different complexes can have the same χ, and homological invariants do not specify metric size, edge labels, causal ordering, or future transformations. In a value-supporting subcomplex L_α, a nonzero $H_1(L_\alpha)$ would therefore describe a feature of that support. It would not, on its own, imply absent economic value, failed reciprocity, or an unjust outcome.

Cohomology provides a related language for observations or weights defined on cells. A p-cochain is a linear functional on $C_p(K;\mathbb{F})$. The coboundary $\delta^p : C^p(K;\mathbb{F}) \to C^{p+1}(K;\mathbb{F})$ is the dual of ∂_{p+1}.

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For a 1-cochain ω and a 2-chain F, their duality is stated in Equation 8.

$$\langle \delta^1 \omega, F \rangle = \langle \omega, \partial_2 F \rangle. \tag{8}$$

Equation 8 is the cellular version of the Stokes pairing. It can register the difference of an additive edge-level observable between two homologous candidate paths, provided those paths and the observable have been defined. If $\delta^1 \omega = 0$, the pairing with a 2-boundary vanishes. Such an observable remains insensitive to changes of a chain by 2-boundaries; it may still detect distinctions associated with nontrivial first homology. A weighted delay or deprivation assessment generally requires more than a single additive 1-cochain, since elapsed time, context, and evolving subjects can enter its domain. These modeling qualifications will matter when circulation orders are compared in the planned section on circulation histories.

3.4 Quotient Spaces and Equivalence Relations

A quotient identifies objects according to a specified equivalence relation. Several quotients can arise in a topological study, and their informational effects differ. A topological quotient $|K|/\sim$ identifies points of a geometric realization. A chain-group quotient identifies algebraic chains modulo a selected subspace. Homology, in particular, identifies cycles modulo boundaries, as specified in Equation 6. A quotient of circulation histories, by comparison, would require a relation defined directly on those histories. None of these identifications supplies its own social or normative semantics.

A useful criterion emerges for path comparison. Suppose γ and γ' are two edge paths with the same endpoints. Their difference $d = c_\gamma - c_{\gamma'}$ is a 1-cycle, by Equation 5. The difference has trivial first-homology class precisely when $d \in \operatorname{im} \partial_2$. Equivalently, there exists a 2-chain F with $d = \partial_2 F$. The statement concerns cellular incidence; identifying the two trajectories in a historical or ethical analysis requires additional grounds.

Proposition 3.1 (Criterion for descending a history assessment)**.** *Let Γ be a set of feasible histories, $\sim$ an equivalence relation on Γ, and $J : \Gamma \to Q$ a function to any set Q. A function $\overline{J} : \Gamma/\sim \to Q$ satisfying $J = \overline{J} \circ q$, where $q : \Gamma \to \Gamma/\sim$ is the quotient map, exists if and only if J is constant on each equivalence class. When it exists, $\overline{J}$ is unique.*

Proof. If $J = \overline{J} \circ q$, equivalent histories have the same image under q and hence under J. Conversely, when J is constant on equivalence classes, define $\overline{J}([\gamma]) = J(\gamma)$. The definition is independent of the representative and is unique because q is surjective. □

Proposition 3.1 is elementary, yet it supplies an exact requirement for justice-sensitive reduction: a proposed equivalence may be used to quotient histories only when the assessment relevant to the research question is invariant under it. For example, identifying two event orders by matching their final allocations may erase different waiting times or irreversible losses. Conversely, if an evaluation is shown to be invariant under a specific rearrangement, that rearrangement may be treated as immaterial for that evaluation. The criterion does not settle which assessment is ethically justified.

The physical face of a GR-inspired two-complex also needs to be kept distinct from an equivalence relation between histories. Faces remain primitive structures with semantics to be established. A separately constructed *history configuration complex* may contain cells representing admissible reorderings, but these would be derived cells in a new object. Attaching such a cell to K itself would alter its original topology and would require an explicit model assumption. This separation is central to the interpretation of holes in later sections.

3.5 Causal Partial Orders and Admissible Histories

A partial order $\preceq$ on a finite event set E encodes reflexive, antisymmetric, and transitive precedence. Incomparable events need not be simultaneous; they may represent steps for which the given model imposes no precedence requirement. A *linear extension* is a total ordering consistent with the partial order.

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Its purpose is to identify a candidate sequential execution while retaining the distinction between necessary and contingent ordering. Mathematical approaches to concurrent and directed processes provide background for this separation between order and topology [12, 10].

For the minimal example in Figure 1(c), two events $b, c \in E$ are incomparable. A configuration is a downward-closed subset of completed events. The admissible configurations are $S_0 = \varnothing$, $S_b = \{b\}$, $S_c = \{c\}$, and $S_{bc} = \{b, c\}$. They permit the two sequences bc and cb. The diagram has two distinct directed routes between the same initial and final configurations. Their common endpoint records completed events, while their histories retain the different order in which completion occurred. If execution takes time or changes later feasibility, additional state variables and transition rules are required to assess their consequences.

To make the distinction formal, write $\mathrm{Lin}(E, \preceq)$ for the set of linear extensions. If each completed prefix is required to satisfy model-specific feasibility constraints $\mathcal{C}$, we may define a set of admissible histories by Equation 9.

$$\Gamma_{\text{adm}}(E, \preceq, \mathcal{C}) = \big\{\gamma \in \mathrm{Lin}(E, \preceq) : \text{each prefix of } \gamma \text{ satisfies } \mathcal{C}\big\}. \tag{9}$$

Equation 9 is a minimal finite model. It assumes that all events in E eventually occur and that an execution can be represented as a sequential list. In social activity, events may be optional, mutually incompatible, overlapping in time, or conditionally enabled by earlier changes. Event structures with conflict constraints, configuration posets, or richer transition systems can represent these cases. The choice should follow the question and available evidence, rather than being inferred from a planar diagram.

The directed configuration diagram is also distinct from the ambient GR 2-complex. Its vertices are *configurations of occurred events*, whereas GR spacetime vertices themselves denote *events*. A triangle in the undirected 1-skeleton of an event complex may define a nontrivial first cycle even when its event precedence is acyclic. This distinction prevents an ordinary homology class from being interpreted as an executable directed circulation. Directed algebraic topology provides additional methods when forward-oriented paths and irreversibility are integral to the model [10].

3.6 Discrete Morse Theory and Topological Reduction

A large cell complex can contain extensive incidence information that is unnecessary for a particular topological computation. Discrete Morse theory provides a systematic way to analyze a finite regular CW complex using matched cell pairs and critical cells. Forman's construction is an established discrete analogue of smooth Morse theory; subsequent combinatorial accounts describe its algorithmic and homotopical use [8, 13].

A discrete Morse function m assigns a real number to each cell. For a p-cell σ, at most one incident $(p+1)$-cell τ may satisfy $m(\tau) \leq m(\sigma)$, and at most one incident $(p-1)$-cell η may satisfy $m(\eta) \geq m(\sigma)$. A cell without either such exception is critical. Equivalently, admissible matched pairs can be organized into an acyclic discrete gradient matching on the face poset, with the remaining cells critical. The resulting Morse chain complex is chain-homotopy equivalent to the original cellular chain complex, preserving ordinary homology under the stipulated assumptions.

A reduction may therefore replace a large cellular calculation with one involving fewer critical generators. Yet the reduction is selected for algebraic-topological purposes. It does not guarantee the preservation of event labels, coupling types, face decorations, times, feasible histories, or a normative history assessment. In particular, a Morse function on cells should not be called a social cost, utility, urgency, or justice measure unless it has been explicitly defined and justified in those terms.

For the present research, an appropriate reduction problem will eventually require two maps: a topological simplification $R : K \to K_M$, and a rule relating the historical and value-related data on K to data on the reduced object. The latter may be impossible without an enriched target or a retained provenance map. Homology preservation gives one formal guarantee; preservation of the variables used in circulation-order comparisons is a separate requirement. The two-level design is useful even when no particular reduction algorithm has yet been chosen.

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3.7 Multiscale Geometry, Persistent Homology, and Fractal Structures

A value-supporting structure may be observed at several resolutions or reconstructed from data subject to thresholds. The mathematical starting point for a scale study is a *filtration*: a nested family of subcomplexes indexed by an ordered parameter. Equation 10 describes the finite case.

$$K^{(a_0)} \subseteq K^{(a_1)} \subseteq \cdots \subseteq K^{(a_m)} = K, \qquad a_0 \leq a_1 \leq \cdots \leq a_m. \tag{10}$$

In a cellular filtration, admitting a cell at a given threshold requires admitting its boundary faces as well, so every stage remains a subcomplex. The inclusion $K^{(a_i)} \hookrightarrow K^{(a_j)}$ for $a_i \leq a_j$ induces a map on homology. Persistent homology studies the resulting family of maps, identifying classes that persist across thresholds and representing their birth and death in barcodes or persistence diagrams [4, 9].

Several candidate parameters should be kept distinct. A calendar-time filtration describes an accumulating historical record, provided each stage is a subcomplex and the record is monotone. A coupling-strength filtration includes cells when measured or modeled interaction weights pass an established criterion. A metric filtration instead changes geometric resolution. These parameters answer different questions and can generate different persistence summaries for the same underlying activity. Likewise, new records of past events may change an epistemic reconstruction without constituting new past events. The parameter and its observational meaning must therefore be documented before persistence has a social interpretation.

The terms *scale* and *fractal* also require separation. Nested institutions, repeated production cycles, and multiple levels of aggregation are forms of multiscale organization. Fractal dimension concerns particular geometric scaling properties of a metric realization or a limiting family of structures. Given a metric set X and a suitable box-counting function $N_X(\varepsilon)$, a box-counting dimension, when the limit exists, is defined by Equation 11.

$$\dim_{\mathrm{B}} X = \lim_{\varepsilon \downarrow 0} \frac{\log N_X(\varepsilon)}{\log(1/\varepsilon)}. \tag{11}$$

The limit in Equation 11 presupposes a metric and a specified covering convention. A conventional finite geometric realization containing ordinary two-dimensional faces will generally have box-counting dimension two under the corresponding local Euclidean metric. Evidence of nontrivial fractal scaling would require an appropriate finer family, limiting construction, or empirically resolved scaling range. A branching picture or visually repeated motif supplies a hypothesis for investigation rather than a calculated fractal dimension.

Coarse-graining addresses a complementary problem: constructing an effective description from finer cells or trajectories. It may preserve selected aggregate features while suppressing microhistorical detail. The validity of a coarse-grained value representation consequently depends on its target observables and the questions being asked. A reduction designed to retain Betti numbers need not retain temporal deprivation, just as one designed to retain total transfers need not retain the historical formation of value-supporting subcomplexes.

Table 2 summarizes the analytical responsibilities of the mathematical concepts introduced in this section. It identifies their principal objects and principal preservation limits; the latter are particularly relevant where topology will later be used to study circulation-order justice.

The mathematical background yields a modular set of requirements for the subsequent GR construction. The 2-complex specifies event, coupling, and primitive-face incidence; supplementary causal structure selects feasible histories; selected subcomplexes may support emerging value modalities; and topological observables describe only those properties determined by their definitions. Quotient and reduction operations can be used only after the distinctions that matter for the research question have been identified. The following section will assign the proposed GR-level semantics to these formal objects, and the later investigation of circulation histories will specify how their transformations and ordering are represented.

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Table 2: Mathematical operations, their immediate objects, and historical information requiring separate preservation.

Operation or structure	**Mathematical object**	**Information outside its automatic guarantee**
Cellular boundary	Oriented cell incidences and endpoints	Event duration, value type, execution feasibility
Homology	Cycles modulo higher-dimensional boundaries	Normative meaning, temporal sequence, magnitude
Quotient	Equivalence classes under a stated relation	Differences discarded by the equivalence
Partial order	Necessary precedence among events	Feasibility, durations, complete social causality
Discrete Morse reduction	Critical cells and homology-preserving complexes	Primitive decorations, timing, provenance
Persistent homology	Classes tracked through a selected filtration	Interpretation of the filtration parameter
Fractal dimension	Metric scaling of a specified set or limit	Value magnitude and social meaning

4 Relational Spacetime and Primitive Structures

This section assigns a preliminary Generative Relational (GR) interpretation to the mathematical objects developed in Section 3. Its objective is to specify the analytical roles of event vertices, coupling edges, and primitive two-dimensional faces; distinguish those objects from historically constituted entities; and state how causal accessibility can be represented without identifying topological incidence with temporal precedence. The construction is a structural formalization under ontological suspension. It defines the domain of a proposed representation, while leaving substantive face semantics, rules of value constitution, and empirically parameterized dynamics to subsequent sections. A finite regular CW complex is retained as the baseline; extensions to more general cellular and evolving structures remain possible.

4.1 Event-Based Relational Spacetime

Let K be a finite, oriented, regular two-dimensional CW complex. Its zero-cells K_0 are interpreted as *events*. An event is a model-indexed occurrence or episode selected because of its relevance to the relational domain under investigation. Events may involve an observation, an attempt, a decision, the transmission of information, an exclusion, a repair, a material change, a legal act, or an activity whose value significance becomes apparent only later. Consequently, eventhood does not depend on the immediate transfer of an identified value. At a chosen observational resolution, a prolonged activity may be encoded by one event vertex; at a finer resolution, it may be reconstructed through many vertices and intervening cells. The zero-dimensional character of a vertex is a property of the mathematical representation, rather than evidence that the represented occurrence has zero physical duration.

The first formal object is an *event-decorated relational complex*. Equation 12 records the minimal data used in this section.

$$\mathfrak{K} = (K, \Lambda, \prec, \mathscr{A}), \qquad \Lambda = (\Lambda_0, \Lambda_1, \Lambda_2). \tag{12}$$

In Equation 12, K supplies the cells and their attaching relations; Λ_p denotes a specified assignment of annotations to p-cells; $\prec$ is a strict partial order on a selected event set, introduced independently of cell orientations; and $\mathscr{A}$ collects model-specific admissibility conditions. The annotations may be types, records, constraints, representation labels, or interpretive metadata, and need not take numerical values. The object is deliberately general: a single universal group, metric, value unit, or stochastic interpretation is not assumed. Each instantiation must state which annotations are observed, inferred, posited, or normatively assigned.

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The complex may be read as a relational spacetime in a limited, methodological sense. The term *spacetime* refers here to the structural representation of historical relations among events, including possible temporal and causal annotations. It does not assert that the social domain possesses a background Lorentzian manifold, quantum-gravitational dynamics, or a physically fundamental cellular ontology. A metric can be introduced when the domain warrants one, although topology and incidence alone determine neither distance nor duration. Similarly, an oriented edge is an algebraic choice for cellular calculations; a causal direction must be established by additional assumptions or evidence.

Event selection constitutes an epistemic operation. A documented meeting may contain many interactions, while a dataset may retain only one recorded event. An institution may record a completed service and omit the preparatory labor, waiting, misunderstanding, or negotiation that shaped its occurrence. Such omissions can alter the geometry and homology of the constructed complex. The event inventory should therefore be accompanied by provenance, resolution, and inclusion criteria. These criteria can differ between an analysis of knowledge formation and an analysis of temporal deprivation, even when both use the same general mathematical language.

4.2 Couplings and Higher-Dimensional Primitive Faces

An edge $e \in K_1$ is interpreted as a *coupling* involving its incident event vertices. Coupling is a broad structural relation that may express dependence, coordination, mediation, co-occurrence, constraint, transmission, or another relationship selected by the modeling protocol. The occurrence of an edge does not establish that something valuable was transmitted, that an obligation was satisfied, or that the two endpoint events occurred in the edge's chosen algebraic orientation. Several couplings may connect the same event pair if the chosen cell-complex representation supports them as distinct cells, and interactions involving more than two events can be represented through interconnected edges and higher-dimensional attachments.

A face $f \in K_2$ is a *primitive two-dimensional structure* attached along a closed sequence of oriented edges. In the baseline regular-complex setting, a face has a well-defined cellular boundary. Equation 13 expresses its incidence structure, with $[f : e]$ denoting the signed incidence coefficient.

$$\partial_2 f = \sum_{e \in K_1} [f : e]\, e, \qquad [f : e] \in \mathbb{Z}. \tag{13}$$

Equation 13 is a mathematical statement about attachment. The term *primitive* indicates that the face belongs to the chosen representational building blocks, not that its sociological significance is immediately knowable. A face might eventually be assigned compatibility constraints, a field-like state, a representation label, or a domain-specific interaction among couplings. The present manuscript does not choose between those possibilities. In particular, a face is not defined as an institution, a value, a justice-preserving interchange of histories, or a guarantee that its boundary couplings can be traversed in causal order.

The distinction has consequences for topological interpretation. Consider an oriented closed edge chain z with $\partial_1 z = 0$. If a combination of primitive faces has boundary z, then $[z]$ is trivial in first homology. If no such two-chain exists, the class may be nontrivial. This difference is determined by the specified face attachments. Neither case alone provides a conclusion about the completeness of mediation, the feasibility of value exchange, or the justice of a circulation order. The model must first explain why its face data are relevant to a particular value-supporting historical structure. The incidence identity $\partial_1 \partial_2 = 0$ establishes a topological consistency property, as reviewed in Section 3.2, while substantive interpretation remains an additional task.

Figure 2 illustrates three distinct operations on a small decorated complex: identifying primitive cells, selecting a recorded historical subcomplex, and associating selected regions with a candidate historical entity. The figure intentionally labels vertices as events. The shaded regions in its final panel are candidate supports for interpretation; they are not drawn as independently existing agents occupying individual vertices.

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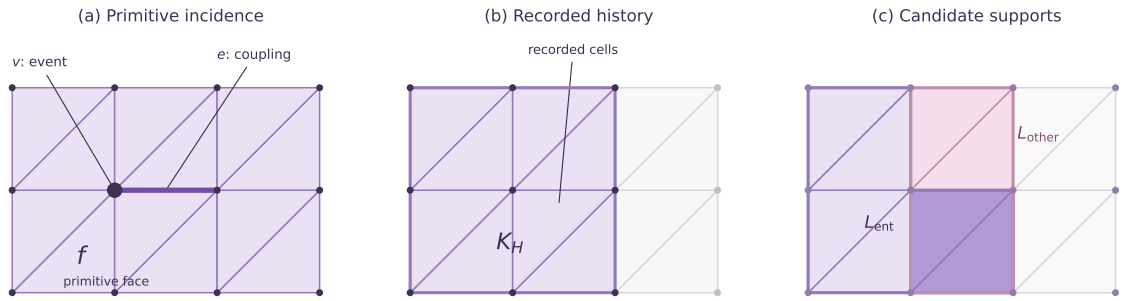

Figure 2: Event, coupling, and primitive-face incidence (left); one recorded historical subcomplex (middle); and overlapping candidate supports for the historical constitution of an entity and another relational configuration (right). The latter supports are derived selections within the event complex, not primitive agents or fixed value quantities. The panels are schematic and do not establish causal directions or face semantics.

Table 3: Structural roles and interpretive commitments in the proposed GR complex.

Object	**Specified role**	**Additional conditions required**
Vertex $v \in K_0$	Event, at the chosen observational resolution	Event identification, timing, evidential provenance
Edge $e \in K_1$	Coupling of incident events	Coupling type, admissibility, and any causal relevance
Face $f \in K_2$	Primitive two-dimensional incidence structure	Face-label semantics and interpretive relevance
Historical $K_H \subseteq K$	Recorded event–coupling–face configuration	Selection rule, documentary completeness, resolution
Entity support $L \subseteq K_H$	Candidate history associated with entity constitution	Identity, continuity, and recognition conditions
Value support $L_\alpha \subseteq K_H$	Candidate history associated with value modality α	Modality-specific constitution and evidence

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Table 3 differentiates the structural types required to prevent an emergent entity or value from being mistaken for a primitive vertex or face. This distinction is carried forward into the constitution maps developed below.

4.3 Local Encodings and Spin-Foam-Inspired Representations

Spin-foam theory supplies a well-developed example in which two-dimensional complexes support local decorations. In the quantum-gravity formulation discussed by Baez [2], faces may carry group-representation labels, edges intertwiners, and local amplitudes contribute to a state-sum or transition-amplitude construction. The present framework borrows the organizational possibility of decorating face, edge, and vertex data. Physical representations, quantum amplitudes, and probabilities are retained as optional formal analogies, rather than inherited economic interpretations.

A general local encoding can assign to each face f a state $\lambda_f \in \mathcal{A}_f$, where $\mathcal{A}_f$ is a specified state space. Incident faces may be subject to coupling relations at an edge. The proposed interface is described in Equation 14.

$$\mathscr{I}_e \subseteq \left(\prod_{f \succ e} \mathcal{A}_f \right) \times \mathcal{C}_e, \tag{14}$$

Here $f \succ e$ means that e lies in the cellular boundary of f, and $\mathcal{C}_e$ holds any relevant coupling state or constraint record. A pair belongs to $\mathscr{I}_e$ when the incident labels and the coupling state are

compatible under the model's selected rules. Equation 14 uses a relation because nonlinear, context-dependent, or partially defined couplings need not admit a single-valued linear operator. If the domain and representation justify a linear map, an intertwiner-like construction may be introduced as a special case. Where an event also changes the incident decorated configuration, an event-level operation may be defined on its local neighborhood. Its dynamics and historical consequences will be considered in Section 6, after value-supporting structures have been specified.

The label spaces $\mathcal{A}_f$ can differ from face to face. A possible representation-theoretic specialization is $\lambda_f = (\rho_f, \psi_f)$ with ρ_f a selected representation and ψ_f an associated state; an optional phase variable could be introduced within an explicitly defined amplitude model. Such a label does not identify a value modality by itself. For example, a numerical phase is periodic and has no inherent interpretation as a quantity of care, knowledge, or labor. A spin label similarly encodes a mathematical representation of a group; interpreting it as value requires a separately motivated translation. The descriptive distinctions developed in Section 2 allow a value-supporting subcomplex to assemble many primitive face encodings together with historical interactions and contextual interpretation.

The present model consequently admits several distinct levels of structure: bare cellular incidence, annotated local states, compatibility of adjacent cells, historical realization, and interpretive identification of emergent structures. A state-sum amplitude over decorations would constitute a further model choice requiring a declared configuration space, local weights, and a normalization or probabilistic interpretation if one is intended. The existence of a decorated complex alone supplies none of those additional results.

4.4 Historical Subcomplexes and Entity Constitution

GR treats an entity as potentially constituted through a historically extended relational configuration. The relevant support may include observations, material continuities, communicative practices, institutional constraints, and other events and couplings, together with selected primitive faces. A candidate entity-supporting subcomplex thus represents an analytical reconstruction of constitution over a history. Its boundary is determined by evidence and identification criteria, and it need not coincide with the boundaries of an administrative label or a single legal status.

A subcomplex must include the boundaries of all cells it contains. A mere subset of observed events does not therefore determine a unique two-dimensional historical support. We introduce a *history-realization assignment* to make the difference explicit. Let $E \subseteq K_0$ be the modeled event set, and let $\mathscr{I}(E, \prec)$ denote the set of finite, causally down-closed event subsets: $H \in \mathscr{I}(E, \prec)$ implies that $w \prec v$ and $v \in H$ entail $w \in H$. A possible recorded-history assignment is stated in Equation 15.

$$\mathscr{K} : \mathscr{I}(E, \prec) \longrightarrow \mathrm{Sub}(K), \qquad H \longmapsto K_H, \tag{15}$$

The intended baseline conditions for Equation 15 are $(K_H)_0 = H$, closure under cellular faces, and monotonicity: $H \subseteq H'$ implies $K_H \subseteq K_{H'}$. These conditions define an accumulating historical representation. A sufficient finite construction using cell prerequisites is supplied in Appendix C. They do not follow from the ambient topology automatically. In particular, the assignment must decide when an edge coupling or a primitive face is included, given that its incident events have occurred. An observation process may document some such cells only retrospectively; a source revision can change the reconstructed assignment without altering the past event history. Historical occurrence, active coupling, and available documentary evidence are therefore different layers of the model.

An active relational configuration may be represented by a selected collection A_H of currently operative cells or annotations within the recorded K_H. Activity can cease or change while the historical record accumulates. When A_H is represented by a cell subset that is not closed under boundaries, it should be treated as an annotation or active-cell set rather than automatically called a CW subcomplex. An institution can lose a collaboration, for example, while the corresponding former coupling remains significant for explaining its present configuration. This distinction is relevant to subsequent discussions of dissolution, fragmentation, and reproduction.

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Entity identification is introduced as a *set-valued interpretive construction*. For a declared identification protocol κ and an available historical record K_H, Equation 16 denotes the admissible identified supports.

$$\mathscr{E}_\kappa(H) \subseteq \big\{(L,\xi) : L \in \mathrm{Sub}(K_H),\ \xi \in \mathrm{IdData}_\kappa(L,H)\big\}. \tag{16}$$

Here L is a candidate supporting subcomplex and ξ is a record of the identity conditions used by κ: continuity markers, evidence, institutional recognition, or other criteria relevant to the particular domain. Equation 16 is not an ontological existence theorem. Its set-valued form admits ambiguous boundaries, plural defensible individuations, or an absence of sufficiently supported identifications. The protocol may itself be affected by the historical processes it seeks to identify. These issues become important where some participants possess disproportionate power to record, delimit, or recognize other participants.

Continuity of entity identity across historical stages also requires a relation beyond inclusion of subcomplexes. Given $H \subseteq H'$, the two sets $\mathscr{E}_\kappa(H)$ and $\mathscr{E}_\kappa(H')$ can be linked by a protocol-specific correspondence that records persistence, transformation, branching, merger, or dissolution. Shared support by itself need not entail identity; conversely, an entity's recognized continuity may survive the replacement of many constituent events, practices, or organizational roles. The model consequently treats an entity as a relationally and historically identified object, potentially maintained through changing supports.

Values can also have historical supports, but their identification follows different conditions. A researcher may become part of an identifiable laboratory or collaboration through a long series of activities, while a research skill generated through some of the same activities is a separate emergent value-related structure. Their candidate supporting complexes may overlap without being identical. The modality-dependent value-constitution operation will be specified in Section 5. This separation prevents the mathematical language of subcomplexes from collapsing the concepts of entity, value, and event into one another.

4.5 Causal Structure and Historical Accessibility

Causal order is an additional relation on modeled events. A strict partial order $\prec$ is irreflexive and transitive; it represents the precedence dependencies asserted by a given analysis. A pair of incomparable events need not be physically simultaneous: incomparability indicates that the current model does not require one to precede the other. The direction of a coupling edge is an algebraic orientation and may differ from the direction or existence of a precedence relation. Cyclic sequences in the 1-skeleton can therefore coexist with an acyclic event precedence structure. Models that include feedback through time should represent distinct occurrences as distinct event vertices and assign the relevant precedence to those occurrences.

For an event $v \in E$, define its causal predecessor set as in Equation 17.

$$\mathrm{Past}_\prec(v) = \{w \in E : w \prec v\}, \qquad \mathrm{Acc}(v) = \mathscr{K}\big(\mathrm{Past}_\prec(v) \cup \{v\}\big). \tag{17}$$

Equation 17 defines an *accessible recorded support* of an event under the selected history-realization rule. Because $\mathrm{Past}_\prec(v) \cup \{v\}$ is down-closed in a strict partial order, the expression lies in the domain of $\mathscr{K}$. The construction is intentionally modest. It gathers the historical cells admitted by the model at that event; it does not prove that those cells are all causally efficacious, observed by the event's participants, or sufficient to determine its effects. A claim of causal sufficiency would require mechanisms and assumptions absent from this topological construction.

A chronology may additionally attach an interval $\tau(v) = [a_v, b_v]$ to an event, with timestamps or bounds supported by the records. If the intended notion of $\prec$ entails chronological separation, the model must impose a corresponding compatibility condition such as $b_w \le a_v$ whenever $w \prec v$. Some causal models allow overlapping processes and therefore require a different convention. Time labels, causal precedence, and cell orientation should be stored as distinct features so that the selected interpretation remains auditable.

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Historical accessibility alone does not determine executable circulation order. A feasible history also depends on enabling conditions, resource constraints, institutional permissions, and the evolving decorated state. In a finite fixed-event approximation, linear extensions of $\prec$ filtered by a declared feasibility condition $\mathscr{A}$ provide a history set of the kind defined in Equation 9. A richer representation can admit optional and incompatible events, concurrent activities, newly enabled actions, or changes to the later event set. Such flexibility is relevant where value generation changes which future actions and couplings become possible, but those dynamics are introduced only after the value modalities have been defined.

The history-based representation also makes a methodological distinction between causal occurrence and epistemic reconstruction. A documented event may be retrospectively reclassified, a previously unknown coupling may become visible, or an apparent face may be reinterpreted as an artifact of an observational convention. Such revisions may alter the modeled subcomplex and its homology. They do not by themselves establish that the historical world underwent the same topological change at the time of the later reinterpretation. For empirical use, the formalism should retain versioned records and identification rationales so that competing reconstructions can be compared without erasing their provenance.

The resulting ontology provides a platform for the remainder of the paper. Events, couplings, and primitive faces specify the baseline relational complex; local decorations supply optional structure; historical realization records the accessible past; and interpretive protocols identify emergent entities from subcomplexes. Value-supporting subcomplexes can now be developed without assigning exchange semantics to every event, reducing entities to vertices, or presupposing that a topological hole has a moral interpretation. Section 5 takes up that value-specific task by examining modality-dependent constitution and the topology, geometry, duration, and scale of heterogeneous value structures.

5 Topological Morphology of Heterogeneous Values

This section develops the value-specific structures anticipated by the event–coupling–primitive ontology of Section 4. Its objective is to describe heterogeneous values through historically supported configurations while distinguishing their topological morphology, geometric extent, temporal development, and modality-specific recognition. The construction begins with the selection of value-supporting subcomplexes and a partial constitution rule; proceeds to structural descriptors, overlap, and multiscale descriptions; and concludes with the separation of constitution, observation, and manifestation. These definitions are proposed analytical devices. They supply a domain for the transformation laws developed in Section 6 and the circulation-history analysis of Section 7, without identifying homological features with economic magnitude or normative standing.

5.1 Value-Supporting Subcomplexes

Let K denote the oriented two-dimensional relational complex of Section 4, equipped with a fixed modeling protocol and a causal event precedence relation $\prec$. Vertices are events, edges are couplings, and faces remain primitive cells. For a down-closed event history $H \in \mathscr{I}(E, \prec)$, the history-realization assignment $\mathscr{K}(H) = K_H$ of Equation 15 supplies a recorded subcomplex. A candidate value-supporting structure is selected *within* K_H. The selection may depend on events of preparation, failed attempts, training, mediation, and recognition, even where none of those events is individually an identifiable transfer of value.

Fix a set $\mathcal{A}$ of proposed value modalities and a modality $\alpha \in \mathcal{A}$. Examples may include monetary claims, the provision of care, research capabilities, practical know-how, ecological regeneration, and forms of institutional trust. The examples are domain-dependent categories of analysis; the mathematics does not assume that they are exhaustive or stable across historical settings. A *support-selection protocol* κ_α specifies which kinds of evidence and relational features may justify associating a historical configuration with modality α. Rather than assume a unique support, define the set of candidates in

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Equation 18.

$$\mathscr{S}_{\alpha,\kappa_\alpha}(H) \subseteq \big\{(L,\xi) : L \in \mathrm{Sub}(K_H),\ \xi \in \mathrm{SuppData}_{\alpha,\kappa_\alpha}(L,H)\big\}. \tag{18}$$

The record ξ contains the supporting reasons, identification uncertainties, and evidence provenance admitted by the declared protocol. Equation 18 may have several outputs or an empty output set. A value-supporting subcomplex is therefore an *identified analytical support*, not an autonomous value-object inserted into the primitive ontology. Its cells include the necessary incident boundaries. A disconnected support is permitted, for example when a capability develops through geographically separated events whose connection is represented in the broader historical context. A support whose relevant evidence extends beyond the observed complex must be marked partial rather than silently completed.

The selected support L also has an inclusion $\iota_L : L \hookrightarrow K_H$. That inclusion matters because the same intrinsic cell pattern can occupy different positions in the surrounding coupling structure. A local pattern of training events, for example, may contribute to a different future capability when it participates in an institution offering subsequent access, support, or recognition. Intrinsic topology alone omits such ambient conditions. The present section retains both L and its ambient embedding as components of the representation.

The support of a value and the support of an entity are identified by distinct protocols. An individual or institution may be historically constituted through a subcomplex represented by $\mathscr{E}^{\circ}_{\kappa}(H)$ in Equation 16; a research capability may have another support covering some of the same events. Shared cells do not establish identity between the entity and the value. Moreover, neither object requires a single distinguished event vertex at which its entire constitution occurred. The distinction is important for subsequent accounts of circulation between historically emergent entities.

5.2 Modality-Dependent Value Constitution

A support-selection rule identifies candidate evidence for a modality, while a *constitution rule* specifies what value-related state is attributed to the supported configuration. The two operations should be distinguished. Let $\Lambda|_L$ denote the restrictions to L of the ambient primitive encodings and coupling annotations introduced in Section 4.3. Let η_H be declared contextual information, including any relevant institutional or interpretive conditions and the available temporal record. For each modality α, choose a structured codomain $\mathfrak{V}_\alpha$ and an admissible domain $\mathscr{D}_\alpha$. Equation 19 describes a potentially set-valued, partial constitution relation.

$$\mathcal{G}_\alpha : \mathscr{D}_\alpha \rightrightarrows \mathfrak{V}_\alpha, \qquad (L, \Lambda|_L, H, \eta_H) \longmapsto \mathcal{G}_\alpha(L, \Lambda|_L, H, \eta_H). \tag{19}$$

Here $\mathscr{D}_\alpha$ contains only configurations meeting the conditions specified by κ_α and by the intended domain of $\mathcal{G}_\alpha$. The codomain $\mathfrak{V}_\alpha$ need not be a vector space, an ordered group, or a set of real-valued quantities. Its elements may describe capacities, claims, relations of trust, structured resources, or other modality-specific states. The set-valued form accommodates several defensible interpretations of one history. When a particular empirical protocol supports a unique output, a single-valued map may be selected as a special case. Equation 19 currently supplies a specification of what must be defined, rather than a computable model or an existence claim for every social value.

For example, the relation between a monetary payment and twelve hours of labor can be represented by distinct modality spaces, $\mathfrak{V}_{\text{money}}$ and $\mathfrak{V}_{\text{labor}}$. A locally accepted exchange rate or wage relation may establish a *conversion interface* between specified portions of these spaces. Such an interface does not determine how working conditions, exhaustion, care responsibilities, or the later production of expertise enter other value modalities. Commensuration can itself be a consequential social process, rather than a neutral preexisting common metric [7]. The formalism accordingly preserves modality-specific data and makes proposed conversion operations explicit.

A constitution rule also requires a time and evidence convention. One study may describe a *capacity available at a historical stage*; another may describe an *observed manifestation* or an *institutionally recognized claim*. Those outputs can differ even where their supporting subcomplexes overlap extensively.

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Table 4: Dimensions of heterogeneous value morphology and candidate mathematical descriptions. Descriptors require explicit observation, weighting, and identification conventions.

Dimension	**Mathematical candidate**	**Descriptive role**
Topology	Homology, connectivity, cycles	Global incidence features of selected support
Geometry	Distances, measures, weighted cell structures	Extent and relationships within an assigned geometry
Temporal extent	Duration and historical dependencies	Periods of formation, continuation, and relevant precedence
Scale	Multiscale decompositions and coarse-graining	Structural descriptions at different resolutions
Dynamics	Transformation, reproduction, fragmentation	Changes in support and generative conditions
Modality	Type-dependent constitution and interpretation	Criteria for recognizing value-related states

The formalism leaves room for such differences by retaining η_H and ξ rather than suppressing the identifying procedures. In particular, assigning a state to $\mathfrak{V}_\alpha$ does not settle who benefits from it, who controls its use, or how its generation ought to be evaluated. Those questions require the circulation dynamics and normative arguments developed later.

5.3 Topological and Geometric Heterogeneity

Values of different modalities may be supported by complexes with different intrinsic topology, embedding, weights, durations, and patterns of coupling. Table 4 records the principal dimensions retained in the proposed representation. The table names mathematical candidates and their descriptive roles; it supplies neither a unique descriptor for each modality nor a ranking of values. In particular, a complex that supports care may be spatially compact but temporally extensive, while a knowledge-producing collaboration may consist of geographically distributed structures whose relevant interactions are coordinated through comparatively few couplings.

For a finite support L, standard topological descriptors include its Betti numbers $\beta_p(L) = \dim_{\mathbb{F}} H_p(L;\mathbb{F})$ for a stated coefficient field $\mathbb{F}$, and its Euler characteristic. Optional metric and weight descriptors require a chosen metric or cellular weighting. Equation 20 collects one possible *descriptive signature*, with μ_α a declared structural measure and D_α an optional graph- or geometry-based extent descriptor.

$$\mathfrak{M}_\alpha(L) = \big(\beta_0(L), \beta_1(L), \beta_2(L), \chi(L), \mu_\alpha(L), D_\alpha(L)\big), \qquad \chi(L) = \beta_0(L) - \beta_1(L) + \beta_2(L). \quad (20)$$

The equality in Equation 20 holds for finite CW complexes with homology taken over a field. The two optional descriptors are not inferred from incidence alone: μ_α requires a weighting or measure and D_α requires a metric convention; values of D_α may be undefined for disconnected supports unless a componentwise convention is adopted. Likewise, cell-level orientation does not determine direction of physical movement or causal influence. A structural signature is useful for comparing representations constructed under the *same* protocol, but it omits many characteristics relevant to formation and justice.

Figure 3 depicts three finite square-cell complexes: a compact filled patch, a branching filled patch, and a filled ring surrounding an unattached central region. They are simplified patterns of primitive incidence. Each has one connected component; the first two have vanishing first Betti number, while the third has one independent first-homology class. The branching pattern illustrates why visible shape can vary without producing a distinct homology group. A topological hole in such a value-supporting complex indicates a missing two-dimensional filling *relative to the chosen cell structure*. Its economic or normative meaning depends on the independent support and face semantics and is not supplied by β_1 alone.

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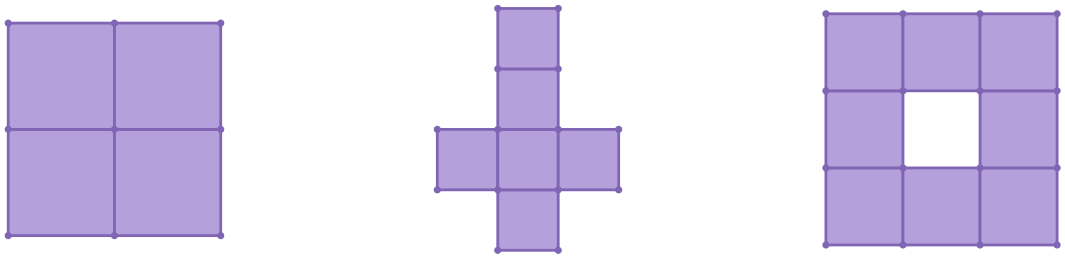

Figure 3: Three illustrative finite value-supporting two-dimensional subcomplexes. Shaded square cells denote selected primitive faces and all incident edges and vertices are included. The compact and branching examples each have $\beta_0 = 1$ and $\beta_1 = 0$; the ring has $\beta_0 = 1$ and $\beta_1 = 1$. Their suggested morphologies are not assignments of specific economic values, and no valuation or causal directions are encoded by the shading.

The embedding $L \hookrightarrow K_H$ can also matter. Two supports with isomorphic intrinsic cell structures can have different boundaries of coupling to surrounding events, access to resources, or positions in a wider institutional system. To distinguish an intrinsic descriptor from such ambient properties, one may record the set of couplings in K_H incident to cells of L that are not themselves included in L. That interface is a separate relational object and is not determined by $H_*(L)$ alone. An apparent change in morphology may therefore result from adding new support cells, revising the observational boundary, or altering couplings outside the selected support. These possibilities require different explanations.

5.4 Structural Extent, Value Magnitude, and Temporal Duration

The word *size* admits several mathematically distinct readings. For a finite subcomplex L, a combinatorial size may consist of cell counts; a geometric size may be the measure of its realization under a chosen metric; and a historical size may describe how long relevant processes continued. None of these quantities is a modality-independent measure of how valuable the resulting structure is. A comparatively brief intervention may prevent severe harm, while a large and prolonged system of activity may accomplish much less under a particular evaluative criterion. A single numeric valuation would require its own domain, scale, and justification.

A minimal structural extent descriptor is stated in Equation 21. Its temporal component is defined only when the event records used to constitute L carry a meaningful time label τ; the extrema should be replaced by interval bounds when events are interval-valued.

$$\mathrm{Ext}(L;\tau) = \big(|L_0|, |L_1|, |L_2|, \Delta_\tau(L)\big), \qquad \Delta_\tau(L) = \max_{v \in L_0} \tau(v) - \min_{v \in L_0} \tau(v). \tag{21}$$

Equation 21 is intended for nonempty L_0 with point-valued event times. Its numerical values change under subdivision, observation resolution, and selection of temporal bounds. A refinement of the same social activity may contain many more modeled cells without producing more value. Likewise, a value-supporting history may contain long periods of inactivity or preparation; the elapsed span need not represent active labor time or time of receipt by a beneficiary. Claims about duration therefore require an explicit measurement convention.

Value magnitude, where meaningful, is an additional modality-dependent evaluation m_α defined on some subset of $\mathfrak{V}_\alpha$. In certain economic domains, monetary amounts have a well-defined unit and ordinary addition; in others, a partial ordering, a multiattribute description, or qualitative distinctions may be more faithful to the underlying practice. The mathematical representation should not postulate an additive scalar for trust, care, dignity, knowledge, or institutional legitimacy solely to support optimization.

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A value morphology may be described with geometric quantities even when there is no agreed scalar magnitude of value. The debate over commensuration underscores the social importance of the choice of metrics [7].

Duration also raises a question of constitution. A complex may accumulate relevant events over an extended interval before an analyst can justify identifying a capability or claim. The date at which a protocol first returns an identifiable state may depend on when evidence becomes sufficient, rather than on an ontologically sharp instant of creation. A subsequent period of circulation or recognition may follow. The present formalism consequently allows a value to be constituted gradually, retrospectively identified, and later transformed or dissipated, while keeping those processes analytically distinct.

5.5 Overlapping and Distributed Value Structures

A single historical complex can support several modalities simultaneously. Consider a set of events involving a patient, a volunteer, and hospital staff, understood as historically constituted participants in the ambient relational system. Practical care, trust, medical knowledge, organizational coordination, and costs may depend on some of the same events and couplings. A modality-specific analysis may select different subcomplexes for each of these descriptions, with partial overlap and different boundaries. There is no general reason to require that the supports partition the historical record.

Let $L_\alpha, L_\beta \subseteq K_H$ be selected supports for two modalities. Because intersections and unions of cellular subcomplexes are themselves subcomplexes, Equation 22 defines an elementary overlap descriptor when the union contains at least one cell. The symbol $|L|_c$ denotes the number of cells in L, in all dimensions.

$$\mathsf{O}_c(L_\alpha, L_\beta) = \frac{|L_\alpha \cap L_\beta|_c}{|L_\alpha \cup L_\beta|_c}, \qquad 0 \le \mathsf{O}_c \le 1. \tag{22}$$

The index in Equation 22 describes combinatorial overlap under a fixed resolution. It neither measures value similarity nor demonstrates that two modalities can be substituted for each other. A weighted alternative may be appropriate where cell types and observational resolution differ, but those weights require justification. Figure 4 illustrates two selected supports over one ambient grid. Some primitive cells are shared, while other cells participate in only one candidate modality. The shared region represents common historical support, not a claim that the recognized values have identical content or beneficiaries.

The topological features of an intersection or union do not follow by simply adding the Betti numbers of its component supports. Standard homological tools, including Mayer–Vietoris constructions, could analyze some such relationships if sufficient information about the inclusion maps is retained [11]. For the present purpose, the narrower observation is that a shared historical support can contribute to distinct modality-dependent constitution processes. It may also mediate dependencies: the reproduction of one value could condition another, or attempts to appropriate one modality might affect the viability of the shared structure. The mechanisms of these interactions require the explicit dynamics deferred to Section 6.

5.6 Multiscale Organization and Fractal Morphologies

Value-supporting complexes can be described at several observational or computational resolutions. A research activity, for example, may be modeled through many small episodes, then aggregated into a collaborative project, and then described as part of a broader institutional programme. These aggregate levels require explicit reconstruction or coarse-graining operations; they are not automatically nested subcomplexes of a single discretization. A refinement map, a quotient, or a relation between differently discretized supports must specify which event distinctions and primitive attachments are retained.

For scales $s \preceq s'$, one may propose a map $q_{s,s'} : K^{(s)} \to K^{(s')}$ together with a transformation of decorations and value-constitution data. Here s' denotes a coarser level by convention. If $q_{s,s'}$ is a cellular map, it induces homology maps, but it need not be a homotopy equivalence; Betti numbers and holes may change across scales. A quotient can erase historical distinctions important for causal ordering

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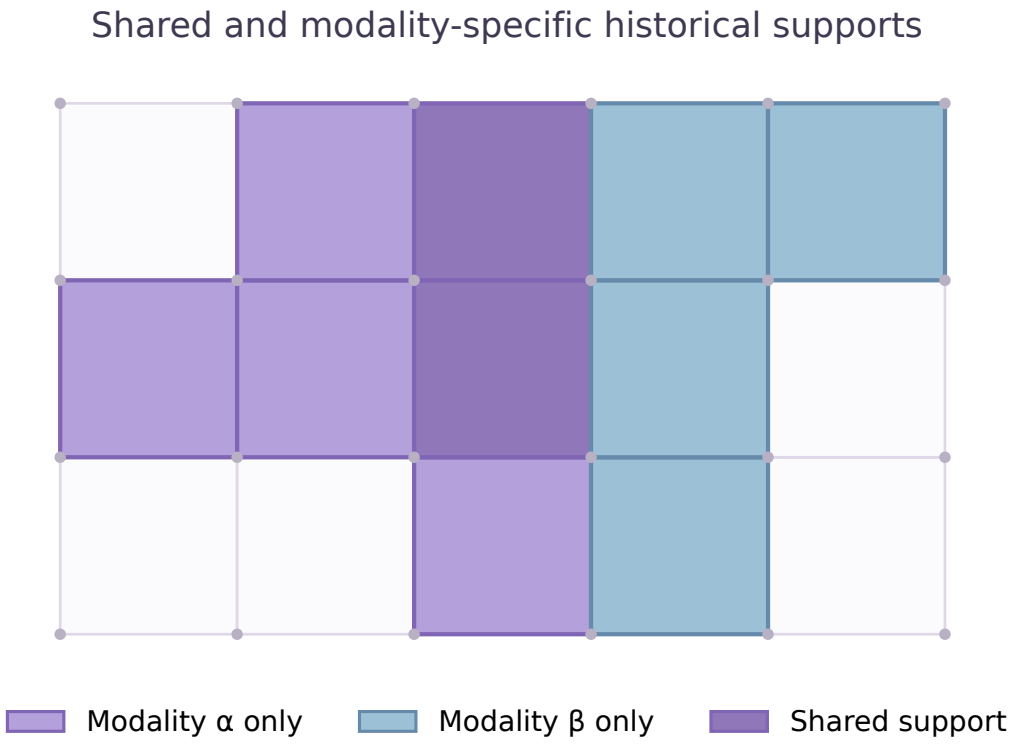

Figure 4: Two modality-specific supports drawn within a common schematic event–coupling–face complex. Mauve and blue faces belong to distinct selected supports; darker shared faces belong to their intersection. The underlying cells remain events, couplings, and primitives. Overlap is a relation among analytical supports and does not entail interchangeability or additive equivalence of the constituted values.

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or justice. A valid effective model therefore needs a declared preservation target, such as an observable, a family of admissible histories, or a specified comparison of value states.

A second form of scale dependence is supplied by a filtration $L_{\alpha,a} \subseteq L_{\alpha,b}$ for ordered thresholds $a \leq b$. The threshold may refer to observational confidence, selected coupling strength, or historical inclusion, provided the subcomplexes genuinely satisfy nesting. Persistent homology can then describe the birth and persistence of selected topological classes across that filtration [4, 9]. Filtrations derived from increasing recorded histories need to be distinguished from the evolving *active* relational state: an active coupling may disappear while remaining present in the accumulated historical record. A nonnested series of active supports does not itself define the required persistence filtration without additional construction.

Fractal geometry is a more specific possibility. Repeated branching, hierarchical organization, or nested generative processes may motivate testing for approximate scaling relations. Deterministic network models demonstrate that iterative structural mechanisms can yield hierarchical patterns [3]; those examples provide mathematical precedents for examining scale, not evidence that the value structures of social activity are fractals. A finite 2-complex has finitely many modeled cells, and a geometric fractal dimension requires a chosen realization, metric, covering scheme, and scaling regime. Apparent self-similarity over a few observational scales supports at most a bounded empirical description.

It is equally important to distinguish structural repetition from generative reproduction. A programme can reproduce capacities for future research while its geometric support changes substantially. Conversely, two support patterns may be geometrically similar while their enabling conditions, beneficiaries, or effects differ. A useful multiscale account should specify what persists or recurs: incidence motifs, causal dependencies, transformation rules, value-generating capacities, or merely a feature of the observed diagram. The paper leaves these alternatives open for the dynamical treatment of Section 6.

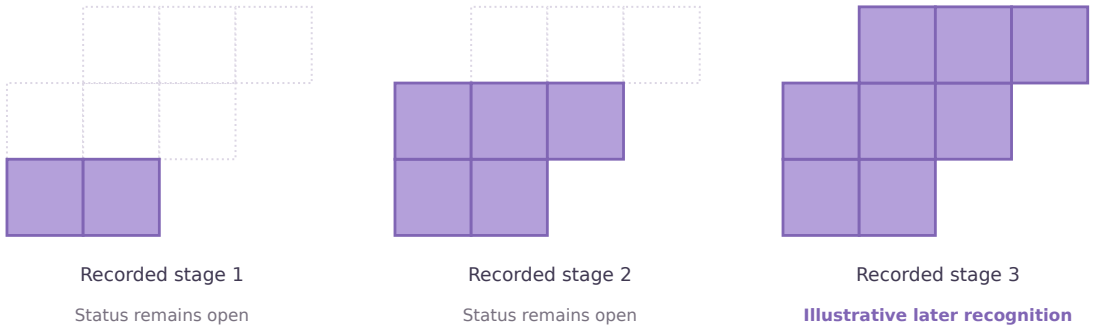

Figure 5: Schematic accumulation of historical value-supporting cells across three recorded stages. The later recognition marker represents a possible change in evidential interpretation rather than an instantaneous origin of the entire value. Shading denotes selected primitive faces within a historical record; actual active couplings and manifestation events may evolve nonmonotonically.

5.7 Value Recognition, Interpretation, and Manifestation

Constitution, recognition, and manifestation should be distinguished in any historical account of heterogeneous value. The constitution rule $\mathcal{G}_\alpha$ in Equation 19 assigns a value-related state under a declared analytical protocol. An observer or institution, however, may have access only to an incomplete evidential record and may use a different interpretive protocol. A recognizable capability can also remain unexercised for a period, while a documented output may appear before its broader future significance can be assessed. These differences are epistemic and temporal features of the intended representation.

For observer or institution r, let $O_{r,H}$ be the available evidence at historical stage H and $\kappa^{\mathrm{rec}}_{\alpha,r}$ a recognition protocol. The schematic recognition relation in Equation 23 maps evidence and selected context to a set of possible judgments $\mathfrak{R}_{\alpha,r}$.

$$\mathcal{R}_{\alpha,r} : (O_{r,H}, \eta_{r,H}) \rightrightarrows \mathfrak{R}_{\alpha,r}, \qquad \eta_{r,H} \in \mathscr{C}_{\alpha,r}. \tag{23}$$

Equation 23 expresses a second analytic layer and does not posit direct observer access to an observer-independent value essence. The codomain may contain provisional recognition, withheld judgment, contested classification, or other domain-specific categories. Its evidential inputs and rules must be stated for an empirical implementation. Recognition may be delayed or revised, and different observers may apply non-equivalent criteria. Such variation can bear on who controls a conversion interface, whose work becomes visible, and which historically constituted capacities receive support. These questions are relevant to value circulation and power, but no claim of injustice follows solely from disagreement between recognition protocols.

A *manifestation* is a contextually observed expression of a value-related state: a demonstrated skill, a completed repair, an instance of care, an articulated scientific result, or an exercised institutional capacity. Manifestations can furnish evidence for recognizing a value; they may also change the supporting complex through new events and couplings. Figure 5 represents accumulating recorded cells across three stages and an illustrative later recognition marker. It makes no assertion that value begins precisely at the recognition stage. Nor does it assume that growth of the recorded support implies monotone growth in value magnitude: subsequent activity could undermine trust, exhaust capacity, or transform a modality into another.

The distinction also allows temporally extended processes to be compared under different resolutions. One analyst may describe a developing skill through weekly observations; another may reconstruct years of training, setbacks, and collaborations. Neither record by itself determines a universal birth event for the value. A defensible study would preserve the historical provenance of the reconstructed supports and the recognition criteria used at each stage. It would also distinguish retrospective reinterpretation of an existing record from a contemporaneous transformation of social activity.

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The proposed morphology formalism therefore consists of a historical support-selection procedure, modality-dependent constitution, optional structural and geometric descriptors, and observer-relative recognition protocols. The same events may support several modalities, and different modalities may be recognized on different timescales. The construction identifies a domain for studying how supports change when events occur and couplings are reconfigured. Section 6 introduces those dynamical transformations; Sections 7 and 9 consider what their ordering may imply for the circulation of values and for justice-related comparisons of feasible histories.

6 Generative Dynamics of Value-Supporting Structures

This section specifies a preliminary dynamical layer for the historical value-supporting structures introduced in Section 5. Its objective is to describe changes in couplings, local face encodings, admissible supports, and modality-dependent value states as a history develops. A recorded event can alter a value-supporting structure, its enabling context, or the possibilities of later activity; some events have no presently identifiable value consequence. The formal construction therefore begins with histories and relational configurations, develops local and extended transformations, and then distinguishes circulation, conversion, fragmentation, reproduction, and feasibility. The rules below define classes of possible models, not empirical laws of social activity. The special assumptions needed for deterministic, conservative, stochastic, or optimization-based versions are indicated where relevant.

6.1 Local Coupling and Event-Level Transformations

The relational substrate is the oriented two-dimensional complex K from Section 4, whose vertices are events, edges are couplings, and faces are primitive higher-order cells. The historical-realization map $\mathscr{K}$ of Equation 15 associates a recorded subcomplex K_H with a causally down-closed set of events H. A subcomplex records that certain couplings and primitive faces participated in the modeled past; a further activity annotation is required to distinguish operative relations from recorded ones. The existence of an event vertex in this ontology does not assert that the event transfers a commodity, creates a value, or even bears a value-related interpretation.

For each H, let $\mathscr{X}_H$ be a declared set of admissible decorated states over K_H. An element $x_H \in \mathscr{X}_H$ contains the historical cell record, a possibly non-subcomplex active-cell annotation A_H, primitive encodings λ_H, evidence and contextual records η_H, and any provisional support identifications s_H. These fields can be incomplete; their admissibility is determined by a modeling protocol. A state is consequently more informative than its underlying topological space, while remaining a representation of a historical configuration rather than a complete ontological inventory.

Suppose an event $v \notin H$ is permitted by the causal dependencies, and $H' = H \cup \{v\}$ is down-closed. Equation 24 defines an event transition as a relation between admissible states over the two recorded histories.

$$\mathscr{R}_v^H \subseteq \mathscr{X}_H \times \mathscr{X}_{H'}, \qquad (x_H, x_{H'}) \in \mathscr{R}_v^H. \tag{24}$$

The superscript H in Equation 24 permits an event's admissible effects to depend on its historical position. A relation, rather than an everywhere-defined function, allows several outcomes, insufficient evidence, and constraints that exclude some event-state combinations. When a deterministic approximation is appropriate, one can select a partial map $T_v^H : \mathscr{X}_H \rightharpoonup \mathscr{X}_{H'}$ whose graph lies in $\mathscr{R}_v^H$. Probabilistic models require an explicitly specified transition kernel and associated measurable spaces; no probability distribution follows from the cell complex alone.

A local implementation can delimit an event neighborhood $N(v)$ consisting of v, selected incident couplings, their incident primitive faces, and declared contextual dependencies. A proposed locality condition is that the state annotations outside this domain remain unchanged, apart from the history-record update. This condition is an approximation whose adequacy depends on the phenomenon: a public announcement, regulatory change, or institutional rupture may have relevant consequences outside an immediate cellular neighborhood. Moreover, in a CW complex, a face incident to an edge may have an

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attaching boundary extending far beyond the edge's nearest vertices. The support required for a local rule therefore needs an explicit incidence and context convention.

Two descriptions of a coupling should also be separated. Its cellular orientation fixes signs in the boundary operators of Section 3; the assigned active state and local encoding describe whether and how the coupling currently participates in a modeled transformation. Neither cellular orientation nor incidence determines a causal direction of value transmission. If spin-foam-inspired representation spaces are used for faces and intertwiners for edges, an additional interpretation and compatibility rule is needed to relate their mathematical contraction to observed social activity [2]. In the current GR proposal, event transformations remain the general dynamical mechanism, with representation-theoretic coupling as one optional realization.

6.2 Historical Formation and Structural Evolution

The sequence of recorded historical complexes is monotone under the baseline history-realization assumption: $H \subseteq H'$ entails $K_H \subseteq K_{H'}$. The contemporaneously active relational configuration and the support selected for a particular value may change nonmonotonically. An operative relationship can cease, a research capability can be lost, and an institution can discontinue an activity, while the prior relevant events and couplings remain in the historical record. A change in the recorded topology should therefore be distinguished from a change in operative relations and from a change in how a particular modality is identified.

Fix a modality α and the support-selection protocol κ_α of Equation 18. At histories H and H', the selected candidate pairs (L, ξ) and (L', ξ') need not satisfy $L \subseteq L'$. For a specified event transition $\mathscr{R}_v^H$, we introduce a possible *support-update correspondence* in Equation 25.

$$\mathscr{U}_{\alpha,v}^{H} \subseteq \mathscr{S}_{\alpha,\kappa_\alpha}(H) \times \mathscr{S}_{\alpha,\kappa_\alpha}(H'), \qquad H' = H \cup \{v\}. \tag{25}$$

Equation 25 is required to be consistent with at least one admissible state transition and the declared recognition and selection protocols. This consistency must be specified in a concrete model. For example, if an event changes the active coupling configuration, then a corresponding support update may enlarge one value support, split another, or leave a third unchanged. The set-valued construction retains multiple interpretations where the evidence does not select a unique support. When a value becomes unidentifiable under the chosen protocol, disappearance from the output should be distinguished from an established historical claim that the value ceased to exist.

A useful change record therefore contains more than two homology groups. It can include newly recorded cells $K_{H'} \setminus K_H$, activated and deactivated annotations, revised modality-specific supports, contextual changes, and a provenance relation linking portions of the earlier and later support. Such a record can express the continuation, branching, merger, fragmentation, or dissolution of a value-related configuration, without requiring a common numerical magnitude. A structural update that preserves Betti numbers may still be causally decisive. Conversely, an increase in the first Betti number may result from a newly included coupling loop without producing any demonstrable change in the relevant value modality.

Figure 6 illustrates these distinctions on a small cell structure. An accumulated historical support is rendered separately from highlighted operative and modality-selected faces. The figure is synthetic; its spatial positions encode combinatorial visualization rather than empirically measured social distance.

In one modeling regime, supports are nested and a persistent-homology filtration is available. Under the general update correspondence of Equation 25, however, support nesting is not guaranteed; ordinary inclusion-induced persistence maps cannot then be assumed. Comparing the topology of changing supports would require specified correspondences, a common ambient filtration, or a method designed for non-nested changes. This issue will recur in the topological treatment of circulation histories in Section 7.

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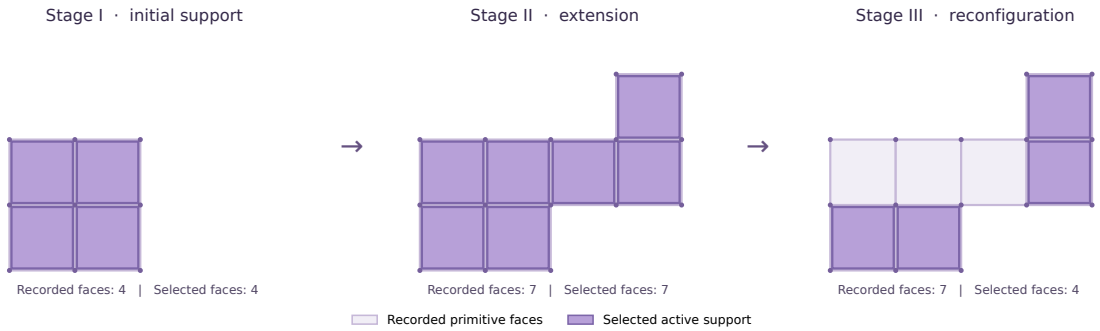

Figure 6: Illustrative relational dynamics across three recorded stages. Pale faces represent cells retained within an accumulating recorded complex; darker faces mark one selected operative value support. An initial compact support develops a branch and later exhibits separated active regions while earlier cells remain recorded. Diagram arrows order the three historical observations and do not denote directed cellular couplings or quantitative value flow. The construction distinguishes expansion of the record from changes in active support and makes no normative assignment to the resulting morphologies.

6.3 Value Transmission and Circulation

Within the proposed ontology, circulation is a derived relation among historical changes in value-supporting configurations. A cell or coupling does not itself announce the transfer of a value to an entity; values and entities are constituted by separate historically situated protocols. Consequently, a circulation claim requires evidence that some change in a value-related configuration is linked to another part of the relational history, together with declared criteria for continuity, access, or participation. The relevant events may include preparation, movement, interpretation, practice, delay, and recognition, in addition to explicit transfers.

For a finite admissible event sequence $\gamma = (v_1, \ldots, v_n)$, let $\mathbf{x}_\gamma = (x_0, \ldots, x_n)$ be a compatible trajectory of states under the event relations of Equation 24. A modality-dependent *circulation trace* is a selected relational record p_α linking value-support candidates at possibly several stages. Equation 26 defines the intended output as a set of admissible traces, given a declared provenance and circulation protocol π_α.

$$\mathrm{Circ}_{\alpha,\pi_\alpha}(\mathbf{x}_\gamma) \subseteq \mathrm{Trace}_{\alpha,\pi_\alpha}(\mathbf{x}_\gamma). \tag{26}$$

The trace space in Equation 26 is a model-specific set of relational correspondences among event occurrences, couplings, support identifications, and modality-specific states. A member may substantiate access to an existing capacity, transfer of a claim, reproduction of an informational pattern, or the creation of a new capability through interaction. The chosen protocol specifies which phenomena qualify as circulation. Without that protocol, Equation 26 remains a type specification. In particular, topological connectivity or a chain with a given boundary does not independently demonstrate that a value was transmitted or received.

The case of knowledge clarifies this distinction. An individual may share a procedure with a colleague while retaining their own ability to perform it. A subsequent collaboration may improve both participants' capacities. The event history can consequently contain a copying mechanism, practical learning, and joint generative transformation. A rule for an exclusive monetary claim might instead specify that one recognized holder loses a claim when another gains it. Such cases require different transition semantics. The condition that a quantity is conserved across events is appropriate only where a material or accounting invariant has been defined and empirically justified.

Return and recirculation are similarly historical notions. A benefit can return to a participant through a later chain of institutions and activities even if there is no directed cycle in the ambient 1-skeleton. Conversely, an algebraically closed cellular 1-chain may consist of couplings never jointly traversed in one feasible causal history. The analysis of return thus needs correspondences among emergent entity

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supports and value traces over time, with potential reciprocity and asymmetry recorded separately. Generative justice is one relevant motivation for attending to productive and reproductive circulation [6]; the present dynamics do not presuppose a unique operational definition of a just return.

6.4 Heterogeneous Value Conversion and Combination

Circulation may involve configurations from distinct modality spaces $\mathfrak{V}_\alpha$, as introduced in Equation 19. A monetary payment, a period of care, an institutional permission, and an increment of practical knowledge can have different units, histories, and identification criteria. Their joint occurrence does not require them to be components of a universal scalar measure. Conversion operations must therefore specify their input modalities, admissible contexts, output modalities, and mechanisms of recognition.

For finite index sets I and J of input and output modalities, a proposed contextual conversion or combination relation is stated in Equation 27.

$$\mathcal{C}^{\eta}_{I,J} \subseteq \left(\prod_{\alpha\in I}\mathfrak{V}_\alpha\right) \times \left(\prod_{\beta\in J}\mathfrak{V}_\beta\right), \qquad \eta \in \mathscr{C}_{I,J}. \tag{27}$$

The contextual parameter η in Equation 27 may include capacities, institutional rules, technological means, legal permissions, environmental limits, time, and the current configuration of couplings. The relation permits one-to-many outputs, several required inputs, incomplete conversion, and cases where no conversion is admissible. Its applicability to a concrete history requires support-level correspondences and actual event transitions compatible with the specified input and output states. Multiplication or addition of heterogeneous values is not defined merely by writing a Cartesian product of their state spaces.

Consider the stylized exchange of one hundred dollars for twelve hours of labor. A particular agreement may establish a recognized money-to-labor interface with a wage, schedule, and enforceable obligations. The resulting labor may subsequently contribute to care, learning, or an artifact through activities whose outcomes depend on context and prior histories. The initial price supplies information about that agreement, while the later constitution of knowledge or trust requires separate criteria. In institutional settings, control over an interface can affect which efforts are recognized, which losses are compensated, and whose capacities are maintained. The sociology of commensuration provides reasons to treat the choice of conversion rules as a social process in its own right [7].

For analytical clarity, Table 5 distinguishes several mechanisms and their additional requirements. These names denote modeling roles rather than universally separated empirical categories; one recorded activity can instantiate more than one mechanism.

6.5 Fragmentation, Dissipation, and Regeneration

The changing morphology of a support is one possible indicator of dynamics, but its semantic interpretation requires a model of relevant capacities and activity. A support may divide into several connected components after some active couplings cease. Yet a disconnected collection can still sustain a distributed capability through relations available in the ambient complex; a connected collection can support a capacity that has become inoperative. Fragmentation is therefore specified relative to the active relational and modality-support rules, with topology supplying a descriptor of the change rather than its complete explanation.

Dissipation can denote loss of material, attenuation of a capability, degradation of trust, decreased access, or failure to reproduce the conditions of a practice. Each candidate definition requires its own state comparison or admissibility relation. If a modality offers a preorder $\sqsubseteq_\alpha$ representing a declared capacity or persistence criterion, one possible dissipation judgment compares constitution states at two stages. Such a judgment has no automatic interpretation as a loss in a common numerical unit; its direction and meaning depend on the modality and on whose perspective enters the criterion. In the absence

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Table 5: Proposed dynamical modes of heterogeneous value-supporting structures. Each mode requires modality-specific criteria, admissible causal histories, and evidential provenance.

Mode	**Structural description**	**Additional condition**
Transmission	Correspondence between supports at successive histories	Verified access, transfer, or participation rule
Conversion	Relational production of a different modality state	Contextual cross-modality interface
Combination	Several supported states jointly participate in an output	Joint-input enabling conditions
Reproduction	Later configurations sustain comparable generative processes	Specified continuation or regenerative capacity
Fragmentation	Selected support separates or loses operative integration	Declared activity and support protocol
Dissipation	Recognized state or capacity declines under an evaluation	Modality-specific persistence or decay criterion
Capture	Participation, recognition, or control becomes asymmetrically concentrated	Identified claims, beneficiaries, and power mechanism

of a relevant ordering, analysts can record specific lost operations, available pathways, or observed manifestations instead.

A regenerative transformation concerns the restoration or extension of activity and the conditions supporting later formation. For example, replacing a destroyed physical resource and rebuilding a cooperative capacity can require different histories, even when the immediately visible outputs coincide. Regeneration may involve new participants, revised practices, or novel couplings and therefore need not reproduce the former support's cell structure. Conversely, a return to a topologically similar pattern supplies limited evidence about substantive restoration. These distinctions prevent topological diagrams of splitting and merging from becoming unsupported claims about economic or ethical improvement.

Capture requires an additional distributional and institutional layer. One participant's control over a conversion or recognition interface may redirect the benefits of jointly generated activity or the authority to classify its outputs. The mechanism involves historically constituted participants, entitlements, and institutional constraints, rather than a distinguished vertex labelled as a controlling entity. A spatially central position or high-degree coupling structure may be a useful descriptive indicator, but a claim of appropriation requires evidence about decision rights, control, and consequential distribution. The analytical significance of capture will be treated more directly in the circulation-order justice discussion of Section 9.

6.6 Recursive Generation and Future Relational Capacities

GR's interest in generativity motivates attention to how historical activity changes the possibilities of further activity. An event may produce an artifact, establish an ability, or alter permissions and dependencies that affect later feasible histories. These effects may extend beyond the immediate beneficiary or the original value modality. A useful dynamical formalism therefore records a distinction between realized manifestations at the present stage and subsequent configurations that become reachable under explicit assumptions.

Let $\mathscr{R}$ denote a declared collection of event transition relations, including feasibility conditions, and let $\mathcal{X}$ be the disjoint union of admissible states across modeled histories. For a state $x \in \mathcal{X}$ and a finite horizon n, Equation 28 defines a simple reachable-state set. A state is included if some extension of at most n admissible event transitions reaches it.

$$\mathrm{Reach}_n(x;\mathscr{R}) = \left\{ y \in \mathcal{X} : \exists\, m \le n,\ x = x_0\, \mathscr{R}_{v_1}\, x_1\, \mathscr{R}_{v_2} \cdots \mathscr{R}_{v_m}\, x_m = y \right\}. \tag{28}$$

The relations in Equation 28 are read with their appropriate historical indices as defined in Equation 24. The notation is a reachability construction, not a measure of expected value or a complete

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account of human possibility. It depends on the modeled event set, horizon, available resources, and transition rules. A larger reachable set under one model need not signify a normatively better world: the added possibilities might include harms, exploitation, or irreversible loss.

For a specific modality, one may apply the constitution relation $\mathcal{G}_\alpha$ to candidate supports in each reachable state and obtain a set of possible recognized or constituted states. That derived set could be used to test whether an earlier activity enables later capabilities absent from a comparison history. However, the observational protocol and its errors must be made explicit. A newly visible possibility may reflect improved recognition rather than newly generated capacity. Likewise, future options may remain formally reachable while becoming prohibitively costly or unlikely in practice. Stronger models could incorporate durations, resource bounds, uncertainty, or stochastic transition kernels, with each addition justified by the intended application.

There is an analytical difference between *structural recurrence* and *generative recurrence*. Repeated motifs across scales may support studies of geometrical self-similarity, while the reproduction of productive conditions is a dynamical property of the transition system. Identically shaped support complexes can have different reachable futures because their active couplings, constraints, and institutional conditions differ. A historically sustained capacity may also rely on changing shapes. This distinction allows multiscale or fractal descriptors to be explored without inferring a universal law of recursive value reproduction from a finite diagram.

The literature on path-dependent processes and increasing returns provides contextual motivations for examining how early events can constrain or expand later options [5, 1]. These accounts do not establish the proposed GR dynamics; their relevance is the more bounded observation that histories can matter beyond endpoint descriptions. The reachability structure gives a place to express that dependence mathematically once the enabling conditions have been specified.

6.7 Dynamical Constraints and Feasible Transformations

A causal partial order describes required precedence among events, while a feasible transformation also depends on the state of the system when the event is considered. Feasibility may involve scarce resources, temporal deadlines, legal authorization, technical compatibility, and other context-dependent conditions. These conditions must be evaluated on the current historical trajectory. When a transformation creates a new coupling or disables an earlier capacity, an event that was previously executable can become inaccessible, and a previously unavailable event can become feasible. The event-order problem is consequently richer than topologically sorting a fixed dependency graph.

For a finite fixed-event approximation, let $\mathrm{Enabled}(H)$ be the set of causally available next events after a down-closed history H. Let $P_v(x_H, H)$ be a declared admissibility predicate for $v \in \mathrm{Enabled}(H)$. Equation 29 specifies a feasible annotated history as a sequence of event-state pairs satisfying causal precedence, the predicate, and the transition relation at every stage.

$$\begin{aligned}\Gamma_n(x_0) = \big\{(v_1, x_1, \ldots, v_n, x_n) : H_i = H_0 \cup \{v_1, \ldots, v_i\},\\ v_i \in \mathrm{Enabled}(H_{i-1}), \quad P_{v_i}(x_{i-1}, H_{i-1}) = 1,\\ (x_{i-1}, x_i) \in \mathscr{R}_{v_i}^{H_{i-1}} \text{ for } 1 \le i \le n\big\}.\end{aligned} \tag{29}$$

In Equation 29, the indices refer to a chosen finite approximation and the states belong to the corresponding $\mathscr{X}_{H_i}$. If enabled events, optional actions, or the universe of event types evolve, the model should extend the state and event-generation rules; a single fixed poset is then insufficient. Feasibility does not itself confer ethical permission, and the space of possible histories may include outcomes requiring independent normative assessment. Resource commitments and institutional obligations may be incorporated as explicit predicates without claiming that their observed implementations are just.

Even where two event orders are both causally and practically feasible, their transformations can be order sensitive. Suppose events a and b are mutually incomparable, both can occur from a state x, and deterministic partial transition maps are defined along both two-step histories. The two resulting states

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are given in Equation 30.

$$x_{ab} = T_b^{H\cup\{a\}}\big(T_a^H(x)\big), \qquad x_{ba} = T_a^{H\cup\{b\}}\big(T_b^H(x)\big). \tag{30}$$

Equation 30 distinguishes event order from cell orientation. Its two outcomes may differ in active couplings, candidate value supports, recognized states, or later feasible opportunities; they may also be identical. Where both outcomes coincide on a chosen observable, the intervening trajectories can still differ in duration, access, or deprivation. In particular, equal final cell counts or homology groups do not establish equality of modality-specific value states. A linear noncommutator would be meaningful only after a common linear state space and compatible operators have been introduced; the composition comparison above works with general partial event maps.

The dynamical layer is now sufficiently explicit to state the formal domain of later circulation-order analysis. The ambient complex supplies the incidence structure; historical realizations retain the event record; states describe operative couplings and relevant encodings; support-selection and constitution protocols identify value-related configurations; and event relations generate feasible historical trajectories. Section 7 will investigate which parts of these trajectories are represented by chains, boundaries, homology classes, and quotient operations, together with the information such operations preserve or discard. Section 8 will supply deliberately simplified computational examples, and Section 9 will examine the independent ethical criteria needed to compare circulation orders. The present section claims a coherent representational interface among these tasks, while leaving any empirically calibrated transition law and complete theory of justice for subsequent work.

7 Topological Representation of Circulation Histories

This section specifies how event histories generated by the relational dynamics of Section 6 can be examined using cellular chains, boundaries, homology, quotient operations, and topological reduction. Its analytical object is the distinction between an actual ordered history, the event–coupling–primitive complex representing its historical support, and a separate complex constructed to compare alternative executions. A mathematical reduction is useful only to the extent that it preserves the properties relevant to the inquiry. Accordingly, the section first defines projections from event-state trajectories into topological representations, then studies holes and quotient equivalence, and finally examines order-sensitive outcomes, Morse reduction, and structural changes across histories. The arguments are mathematical constructions and conditional illustrations; they do not assign moral status to a homology class or assume that every event transfers a value.

7.1 Event Sequences and Circulation Orders

An *event history* in the present model consists of a sequence of event occurrences and their associated admissible decorated states. For a fixed finite event set, the feasible histories are those described by Equation 29. An event occurrence is a vertex of the ambient GR complex K; its appearance in a history records an occurrence with a position in causal order. The same named social entity can participate in many such occurrences because entity identity is constituted through historical subcomplexes, as described by Equation 16. The sequence may therefore contain observation, communication, conflict, waiting, or repair events in addition to events through which a value-supporting structure changes.

The causal relation $\prec$ determines which linear orderings are admissible before practical constraints are considered. An incomparable pair (a, b) permits a possible exchange of order in the abstract event poset, but the pair need not be executable in both orders from a given state. A resource commitment after a may disable b; a subsequent institutional decision may also change the meaning of the earlier event. Thus, the admissible sequence set should be filtered by the enabling predicates P_v and transition relations $\mathscr{R}_v^H$ of Section 6. A single topological sort supplies a proposed ordering, not an exhaustive account of feasible trajectories.

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The phrase *circulation order* refers here to an event order viewed through a declared value-related observation protocol. Let Γ_{feas} contain annotated histories of the chosen finite model, and let C_α extract a modality-specific circulation trace of the kind described by Equation 26. The projection in Equation 31 records this derived relation.

$$\mathsf{C}_\alpha : \Gamma_{\text{feas}} \rightharpoonup \text{Trace}_\alpha, \qquad \gamma \longmapsto \mathsf{C}_\alpha(\gamma). \tag{31}$$

Equation 31 is partial because the necessary support identifications or value observations may be unavailable. The codomain Trace_α is model-dependent and may include typed transmissions, constitutive changes, timestamps, evidence records, and uncertainty. It can consequently record circulation as transmission or reconstitution of value-bearing structures without supposing that a fixed amount moves through the vertices of K. Distinct histories may produce the same observed trace, while apparently identical transfers can participate in different future-generating histories. The relation between histories and observables must therefore be kept explicit.

7.2 Cellular Chains and Historical Path Representations

A cellular 1-chain in K is a linear combination of coupling edges, rather than a sequence of event transformations. The operator ∂_1 records its net event-vertex incidence. For a traversed edge path p, Equation 5 gives $\partial_1 c_p = v_{\text{last}} - v_{\text{first}}$. Neither the order of repeated traversals nor the times at which they occurred are retained by this chain. In particular, cancellation of coefficients at an intermediate vertex expresses an incidence identity, not the disappearance of an event from the social or historical record.

Some circulation histories contain a coupling path that can be explicitly identified with the relevant transfer or communicative propagation. Only for such histories, and only under a declared path-selection protocol r, do we introduce a partial map $r : \Gamma_{\text{feas}} \rightharpoonup \text{Path}(K_1)$ and its composition with the chain assignment. The resulting representation is stated in Equation 32.

$$\mathsf{P}_r : \Gamma_{\text{feas}} \rightharpoonup C_1(K;\mathbb{F}), \qquad \mathsf{P}_r(\gamma) = c_{r(\gamma)}. \tag{32}$$

The map P_r in Equation 32 is not canonical. A sequence of causally ordered events need not follow adjacent coupling edges, and a value-supporting subcomplex may extend across many couplings and faces. An edge path may therefore be an appropriate representation for a particular transmission mechanism while omitting the distributed generative process that makes the transmitted capacity possible. Even when r exists, comparing $\mathsf{P}_r(\gamma)$ and $\mathsf{P}_r(\gamma')$ can erase precisely the timing and contextual data required by an assessment of circulation order.

A different use of cellular chains arises from the recorded historical subcomplex K_H of Equation 15. Its chain groups and incidence matrices can be computed independently of whether a given circulation path is identifiable. This permits topological comparison of supports, interfaces, and faces while keeping history order in the trajectory data. For computational purposes, each layer should have separate identifiers: event occurrence IDs, coupling-cell IDs, primitive face IDs, and ordered transition IDs. A chain coefficient should not be used as a substitute for a full event provenance record.

7.3 Boundary Operators and Circulation Structures

The identity $\partial_1\partial_2 = 0$ of Equation 3 gives a formal relation between primitive faces and the couplings along their attaching boundaries. Its content depends on incidence and chosen orientation. For a 2-chain F formed from primitive GR faces, $\partial_2 F$ is an edge chain whose vertex boundary vanishes. This identity does not state that value has been produced or conserved, since there is as yet no scalar conserved quantity or flow law assigned to the edges. Value modalities can be incommensurable, temporally extended, or constituted across overlapping subcomplexes.

It is useful to distinguish *structural closure* from *flow balance*. A 1-cycle $z \in \ker \partial_1$ is structurally closed in the cellular sense. A value-flow model would instead need typed quantities on admissible couplings, a transformation rule at events, and a compatibility or balance condition. Such a rule may use a

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boundary matrix, but its coefficients and source terms derive from the chosen modality and dynamics. Where a process creates information or reproduces a capacity, ordinary conservation of a material quantity may be an inappropriate assumption. The boundary matrix specifies the incidence scaffold on which a balance equation could be formulated; it does not supply the equation's substantive meaning.

A further distinction concerns the boundary of a value-supporting subcomplex. A support $L_\alpha \subseteq K_H$ may have a nonempty topological or combinatorial interface with its surroundings. This interface can support transfers, dependencies, or constraints on future changes, but the cellular boundary of an arbitrary sum of 2-faces need not coincide with the full set of couplings connecting the selected support to the ambient complex. One may use an attaching-boundary chain, an incident-edge cut, or an explicitly defined boundary operator on a manifold-like realization according to the analytical question. These constructions should not be conflated merely because each uses the word *boundary*.

Accordingly, a description of value circulation based on ∂_2 alone is incomplete. The operator becomes relevant to circulation when the encoding and transition protocols establish a relationship between a primitive face's boundary couplings and a change in a value-supporting configuration. Even in that case, the resulting relationship is a substantive modeling hypothesis. The incidence identity itself remains mathematically valid independently of the hypothesis.

7.4 Homological Features and Value-Supporting Holes

Let $L_\alpha \subseteq K_H$ be an identified support for modality α under the protocol of Section 5. Its first homology $H_1(L_\alpha; \mathbb{F})$ records first-dimensional cycles modulo face boundaries. A nontrivial class identifies a loop of couplings not filled by a 2-chain *within that support*. It does not imply the existence of an unmet obligation, a blocked circulation path, or an unjust distribution. Figure 3 displays precisely the distinction between filled and holed supports in a simple square-cell model.

The ambient embedding supplies additional information. An unfilled cycle in L_α may bound a face or collection of faces in K_H that the modality-specific identification protocol has excluded. The inclusion $i : L_\alpha \hookrightarrow K_H$ induces the homology map in Equation 33.

$$i_* : H_1(L_\alpha; \mathbb{F}) \longrightarrow H_1(K_H; \mathbb{F}), \qquad [z] \longmapsto [i_\# z]. \tag{33}$$

A nonzero class in $\ker i_*$ represents a support-intrinsic cycle that becomes a boundary in the ambient historical complex. A class surviving under i_* is nontrivial in the ambient homology as well. These are precise topological distinctions: the map may be informative about the selected scope of a value-supporting structure, but its semantic meaning depends on why the relevant faces were included or excluded and whether those choices are empirically defensible. A support-intrinsic hole can also disappear when the support is enlarged or when the identification protocol is revised; neither operation alone establishes a social repair.

Higher-dimensional face structures deserve separate attention. Because the baseline K is two-dimensional, its group $H_2(K; \mathbb{F})$ is the kernel of ∂_2; a nonzero class corresponds to a closed 2-cycle of primitive faces. Such a structure could participate in the formation of a value modality under an appropriate recognition rule, but topology by itself does not establish such participation. The same caution applies to persistent holes across observational scales: mathematical persistence may indicate robustness to a filtration, while the causal and ethical significance remains an independent question [4, 9].

Three separate topological domains can thus be relevant: the ambient event–coupling–primitive complex K, a modality-supporting subcomplex L_α, and a *derived history-comparison complex* built from admissible transitions. A hole in one domain need not determine a hole in the others. The third domain is introduced next because it is better suited to a comparison of event orders than an unqualified interpretation of $H_1(K)$.

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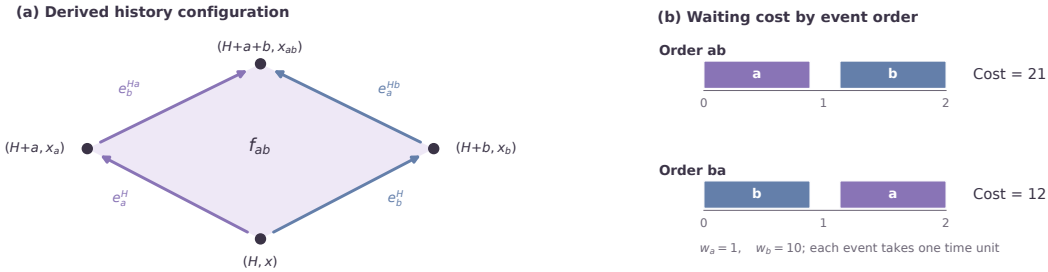

Figure 7: Derived configuration diamond for two possible event orders. The diagram vertices denote completed-event and decorated-state configurations, while the arrows denote state transitions; they are distinct from the event vertices and coupling edges of the ambient GR complex. Panel (a) gives two routes between a common initial and final configuration and a possible derived 2-cell f_{ab} specifying an admissible interchange. Panel (b) annotates one synthetic waiting-cost example: the same final allocation permits order costs of 21 and 12. Attaching the derived face identifies an algebraic difference of paths without asserting their equality under a justice evaluation.

7.5 Quotient Constructions and Historical Equivalence

A history-comparison complex has different vertices from the ambient GR complex. For a finite discretized transition model, let D_0 consist of selected reachable pairs (H, x_H), where H is a completed-event configuration and x_H an admissible decorated state. An oriented 1-cell of D denotes an enabled event transition between two such pairs. Parallel transition edges may be necessary if two distinct event occurrences or actions produce the same state endpoints. These state-configuration vertices are derived analytical objects; the vertices of K remain events. Methods in directed topology and the mathematics of concurrent processes motivate this separation of state paths from their underlying event-occurrence structures [10, 12].

Consider two incomparable events a and b from an initial configuration H, and assume that both two-event orders are feasible. In general, Equation 30 yields distinct outcomes x_{ab} and x_{ba}. Two paths in D then end at different decorated-state nodes, even though both have the completed-event set $H \cup \{a, b\}$. A diamond with common final vertex requires an additional condition, such as equality of decorated outcomes or an explicitly justified equivalence relation on those outcomes. Equality of completed-event sets alone supplies no such identification.

Under an exact common-endpoint condition, choose the transition edges $e_a^H, e_b^{Ha}, e_b^H, e_a^{Hb}$ in D and attach an optional 2-cell f_{ab} representing the specified admissible interchange. The difference between the two oriented routes is a cycle, and its cellular filling is stated in Equation 34.

$$\partial_2^D f_{ab} = e_a^H + e_b^{Ha} - e_b^H - e_a^{Hb}, \qquad \partial_1^D \partial_2^D f_{ab} = 0. \tag{34}$$

Equation 34 concerns the boundary operator in the *derived* complex D. Its square f_{ab} records a selected relation between paths through configuration space and is not a new primitive face of K. We may choose to attach a face only for swaps certified under a named independence, observational-equivalence, or other interchange protocol. In particular, the existence of both orders does not force us to attach that face. Figure 7 illustrates the construction and its subsequent evaluation by a history-sensitive cost.

A second possible construction identifies historical paths through a quotient. Let $\sim_{\mathscr{A}}$ be an equivalence generated by a declared class $\mathscr{A}$ of certified local path interchanges. The quotient $\Gamma_{\text{feas}}/\sim_{\mathscr{A}}$ can simplify a history comparison when the discarded order distinctions are irrelevant to the property being studied. Proposition 3.1 provides the precise test: an assessment J can be evaluated on the quotient if and only if it is constant on the equivalence classes. A stronger equivalence can be convenient for structural classification while failing to preserve urgency, waiting times, access histories, or generative consequences. The quotient criterion is therefore a property-specific restriction rather than a general license to erase historical order.

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7.6 Order-Dependent Transformations and Noncommutativity

The dynamics of Section 6 allow a direct comparison between two event orders. When deterministic partial maps T_a^H and T_b^H are defined, Equation 30 distinguishes the compositions with order ab and ba. If the two compositions return different states, the history is order sensitive at the level of the chosen state representation. This may reflect changes in active couplings, value-supporting subcomplexes, transformed capacities, or the feasible continuations. For general partial maps acting between different history-indexed state spaces, it is better to compare these compositions explicitly than to write an unqualified linear-operator commutator.

Even identical final decorated states do not imply indistinguishable histories. Consider a synthetic case in which both a and b are one-unit allocation operations, a meets a need associated with weight $w_a = 1$, and b meets a more urgent need with weight $w_b = 10$. Each need incurs its specified waiting cost until its allocation completes; both allocations can be made in either order, and both orders reach the same final allocation. The cumulative weighted completion-time costs are stated in Equation 35.

$$\mathcal{L}(ab) = w_a(1) + w_b(2) = 21, \qquad \mathcal{L}(ba) = w_b(1) + w_a(2) = 12. \tag{35}$$

The quantities in Equation 35 are stipulated illustrative weights; no claim of a universal numerical conversion between types of need is made. The example isolates temporal dependence while fixing endpoint allocations. Its numbers could be replaced by modality-specific ordinal or partially ordered outcomes where scalar assessment is ethically or empirically inappropriate. Figure 7(b) displays the unequal waiting consequences.

When a common-endpoint diamond f_{ab} is attached in D, a real-valued additive assessment can be represented by a 1-cochain $\omega \in C^1(D;\mathbb{R})$ whose edge weights encode stage-specific costs. Define $c_{ab} = e_a^H + e_b^{Ha}$ and $c_{ba} = e_b^H + e_a^{Hb}$. The discrete Stokes identity of Equation 8, applied to Equation 34, gives the local order difference in Equation 36.

$$\langle \delta^1\omega, f_{ab}\rangle = \langle \omega, c_{ab} - c_{ba}\rangle = \mathcal{L}(ab) - \mathcal{L}(ba) = 9. \tag{36}$$

This equality is an algebraic accounting identity for the specified cost cochain. In the example, the first transition interval accrues both participants' waiting costs, whereas the second interval accrues only the still-unmet need. The existence of f_{ab} does not require $\delta^1\omega$ to vanish: the cell specifies a topological relation between the paths, and the nonzero coboundary measures sensitivity of the selected additive observable to their interchange. If $\delta^1\omega = 0$, such differences vanish on face boundaries, while a closed cochain may still distinguish pathways belonging to distinct global homology classes. These observations concern a model-specific cost and should not be interpreted as an independent proof of justice or injustice.

This treatment also clarifies the possible significance of a hole in the derived configuration complex. Without a filling 2-cell, the difference between two common-endpoint paths may represent a nontrivial class in $H_1(D)$, provided it is not the boundary of some other 2-chain. The class indicates an unfilled path-difference relation under the chosen interchange protocol. It need not signal a materially blocked value transfer; both paths can be feasible and may even have equal outcomes. A justice-relevant distinction requires a separate history evaluation and evidence about the consequences of the selected order.

7.7 Morse Reduction and Preservation of Historical Information

Discrete Morse theory provides homology-preserving simplifications of finite regular CW complexes under the conditions described in Subsection 3.6. A discrete gradient matching can replace many cells by a smaller collection of critical generators, with associated chain maps and a chain homotopy. Such a reduction can be useful for the ambient K, for a selected L_α, or for a finite history-comparison complex D [8, 13]. The analytical meaning differs among these applications: reducing the ambient structure studies primitive incidence, reducing a support studies its topology, and reducing D studies the topology of permitted transition comparisons.

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Homology preservation is weaker than preserving event-order information. A contractible derived diamond can be collapsed or represented by a Morse complex with the same trivial first homology, even though its two directed routes have distinct intermediate states and, as in Equation 35, different waiting consequences. A Morse reduction does not automatically transfer such weights, chronology, modality labels, permissions, or historical provenance. Nor does the reduced Morse chain complex necessarily have a literal subcomplex realization whose cells retain their original event or face semantics.

Let $R : \Gamma_{\text{feas}} \to Y$ denote a proposed summary extracted after any computational reduction, possibly incorporating retained annotations as well as a reduced complex. Let $J : \Gamma_{\text{feas}} \to Q$ be a target assessment. The preservation requirement is the existence of a map $\bar{J} : Y \to Q$ with the factorization given in Equation 37.

$$J = \bar{J} \circ R. \tag{37}$$

Equation 37 requires J to agree on every fiber of R, just as Proposition 3.1 requires for a quotient. The condition is easy to state and can be difficult to satisfy when assessments depend on temporal intervals, evolving subjects, or historical obligations. If it fails, the reduced representation must retain more information or be used only for a narrower question. For example, a reduction sufficient to compute β_1 may be inadequate to compare a sequence of care allocations. The kind of information preserved should be certified at the level of the question rather than inferred from the size of the reduced chain complex.

A practical alternative is to treat Morse simplification as a topological preprocessing step and retain a linked history ledger. Each reduced generator can carry references to original cells and to the state transitions through which relevant supports were selected. An assessment then acts on the linked representation rather than on the reduced homology alone. This is a proposed data-design strategy, whose effectiveness would need verification for each operational model. It also respects the distinction between historical reconstruction and the structure of the events as originally modeled.

7.8 Topological Changes Across Circulation Histories

An admissible history γ generates a sequence of recorded complexes K_{H_i}, decorated states x_i, and modality-specific selected supports $L_{\alpha,i}$ where the corresponding identification protocols apply. The nestedness $K_{H_i} \subseteq K_{H_{i+1}}$ belongs to the baseline historical-record assumption of Section 4. Its inclusion maps induce homomorphisms in ordinary homology. This permits a persistent analysis of the *recorded* complexes along an ordered history, with all the qualifications concerning observational resolution and retrospective revision noted earlier [4, 9].

The active annotations and selected value supports may change nonmonotonically. Therefore, the inclusion-induced maps on $H_p(L_{\alpha,i})$ generally do not exist unless the supports are nested or a further correspondence is defined. A comparison can instead retain the ambient inclusion, the protocol-specific support-update correspondence $\mathscr{U}^H_{\alpha,v}$ of Equation 25, and a record of which support cells became active, inactive, or reassigned. Such information can differentiate a physically altered relational capacity from an interpretive revision of the support-selection rule.

For two feasible orders γ_1, γ_2 with the same completed-event set, one possible structural comparison is the collection of their support invariants, incidences, interface data, and later reachable histories. An illustrative topological signature may be assembled from the Betti numbers, Euler characteristic, cell counts, and ambient-support inclusion maps. Matching signatures do not prove equal support configurations or equal consequences. Conversely, differing Betti numbers do not justify interpreting one history as more just or more valuable. The signatures become analytically informative only through the declared constitutive and dynamical mappings.

Table 6 summarizes the role and preservation limits of the representations introduced in this section. The result is a layered account rather than a single universal reduction: topological invariants describe selected structural properties, annotated histories encode admissible transformations and order, and modality-specific constitution maps connect those structures to heterogeneous values. These constructions prepare a computational illustration in Section 8 and the independent normative analysis of

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Table 6: Distinct representational levels in the topological analysis of circulation histories. Preservation statements depend on the specified maps, protocols, and observables.

Representation	Analytical content	Typical information requiring separate retention
Ambient K and recorded K_H	Primitive event, coupling and face incidence; historical cell inclusion	Active status, modality identification, actual order and time
Value support L_α	Selected morphology, overlap and ambient interfaces	Constitutive evidence, recognition rules, magnitude and later effects
Annotated trajectory γ	Event order, enabling conditions and state transitions	Unobserved causal factors and normative interpretation
Derived history complex D	Reachable state configurations and selected local path interchanges	Historical significance of a proposed face or quotient
Reduced or quotient representation	Invariants or equivalence classes selected by the reduction	Any cost, responsibility or provenance data not proved invariant

circulation-order justice in Section 9. They do not establish a direct correspondence between a topological hole and a claim of injustice.

8 Illustrative Models and Computational Experiments

This section constructs reproducible finite examples for the formal distinctions introduced in Sections 4–7. Its objective is to determine which structural and order-sensitive properties can be calculated from a declared two-dimensional cell complex, a modality-specific support rule, and a simple event-transition system. The experiments use synthetic cubical complexes, explicit boundary matrices over the field $\mathbb{F}_2$, deterministic event updates, and one seeded stochastic variation. Their labels denote mathematical possibilities rather than measurements of social activity. The seven subsections establish a minimal model, compare support morphologies, follow the formation of a cycle, evaluate alternative event orders, demonstrate typed support combination, examine a multiscale construction, and report sensitivity to two distinct kinds of modeling assumptions. The accompanying source package contains the generating program, numerical output files, and vector figures; Appendix D specifies the run parameters and reproducibility checks.

8.1 Minimal Event–Coupling–Face Complex

A finite cubical two-complex provides a convenient computational realization of the proposed event–coupling–primitive ontology. Consider an ambient rectangular grid of unit squares. Each lattice vertex receives a distinct synthetic event identifier, each grid edge is an oriented coupling, and each square is a primitive face. The coordinates are a drawing convention: equal Euclidean distances or common square areas have no presumed social, causal, or economic interpretation. A particular selected support consists of a collection F of square faces together with their boundary edges and vertices, giving a genuine cellular subcomplex $L(F) \subseteq K$. The construction can be embedded in any sufficiently large ambient grid and does not require identifying a social entity with a lattice vertex.

For each $L(F)$, let n_p be its number of p-cells, and let B_1 and B_2 denote the cellular boundary matrices computed over $\mathbb{F}_2$. Working over this coefficient field removes orientation signs from the numerical matrices while preserving the ordinary incidence information required for the present examples. The ranks $r_p = \operatorname{rank}_{\mathbb{F}_2} B_p$ determine the Betti numbers through the finite-dimensional identities in Equation 38.

$$\beta_0 = n_0 - r_1, \qquad \beta_1 = n_1 - r_1 - r_2, \qquad \beta_2 = n_2 - r_2. \tag{38}$$

The implementation constructs edges once per geometric incidence, computes boundary columns for the closed boundary of each selected square, and checks $B_1 B_2 = 0$ over $\mathbb{F}_2$. It additionally checks that the alternating sum of cell counts agrees with the alternating sum of Betti numbers. These are internal

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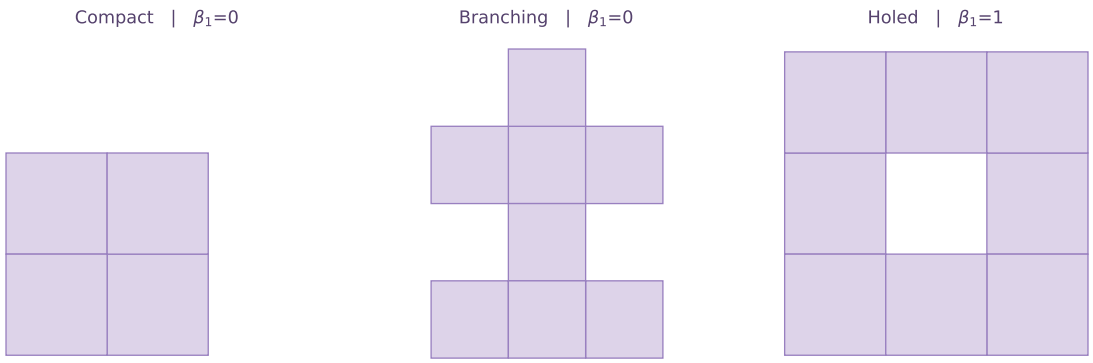

Figure 8: Three synthetic value-supporting cubical subcomplexes. The compact and branching patterns are connected and have trivial first homology; the unfilled ring has one independent first-homology class. Cell shape is a visualization choice, and the labels do not constitute a ranking of value magnitude or social benefit.

Table 7: Computed cellular counts, mod-2 boundary ranks, and Betti numbers for four synthetic support complexes. The filled 3×3 block is included to exhibit the effect of attaching its central face.

Support	n_0	n_1	n_2	r_1	r_2	β_0	β_1	χ
Compact 2×2	9	12	4	8	4	1	0	1
Branching	18	25	8	17	8	1	0	1
Ring 3×3	16	24	8	15	8	1	1	0
Filled 3×3	16	24	9	15	9	1	0	1

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consistency tests for the finite examples, not tests of the GR interpretation. Over $\mathbb{Z}$, the same oriented cells can be assigned signed incidences for analyses requiring directional chain coefficients; the present rank calculations deliberately concern topological support alone.

We distinguish three computational layers. The *ambient complex* supplies possible incidences and primitive faces; an *active modality support* selects the faces currently recognized by an illustrative support protocol; and an *event ledger* records the operations producing successive states. A topology computed from the selected support therefore characterizes that protocol's output. A different recognition protocol can select a different subcomplex from the same ambient complex, even when the event ledger is unchanged. For the order experiments below, a fixed preliminary support is present before either allocation event occurs; subsequent events may activate an additional face without requiring the historical ambient record to lose any previously recorded cell.

8.2 Compact, Branching, and Holed Value Morphologies

The first calculation compares three selected supports whose faces form, respectively, a 2×2 filled block, a connected eight-square branching motif, and the eight-square boundary of a 3×3 block with its central square omitted. The motifs are combinatorial constructions: their names describe observable morphology and do not identify particular modalities with universally fixed shapes. Figure 8 displays the selected faces with the computed first Betti numbers. Table 7 reports the actual cell counts and boundary ranks used in Equation 38.

For the ring, the count $n_0 - n_1 + n_2 = 0$ is consistent with $\beta_0 = 1$, $\beta_1 = 1$, and $\beta_2 = 0$. Filling the center increases n_2 and the rank of B_2 by one, leaving the number of vertices and edges unchanged. Consequently, the first Betti number decreases from one to zero. The comparison establishes a combinatorial relation between primitive-face inclusion and a topological invariant. It does not assign a semantic meaning to the absent face. Depending on the constitution protocol, an unfilled region may be important, incidental, or unrelated to the recognized value modality. Moreover, the branching and compact

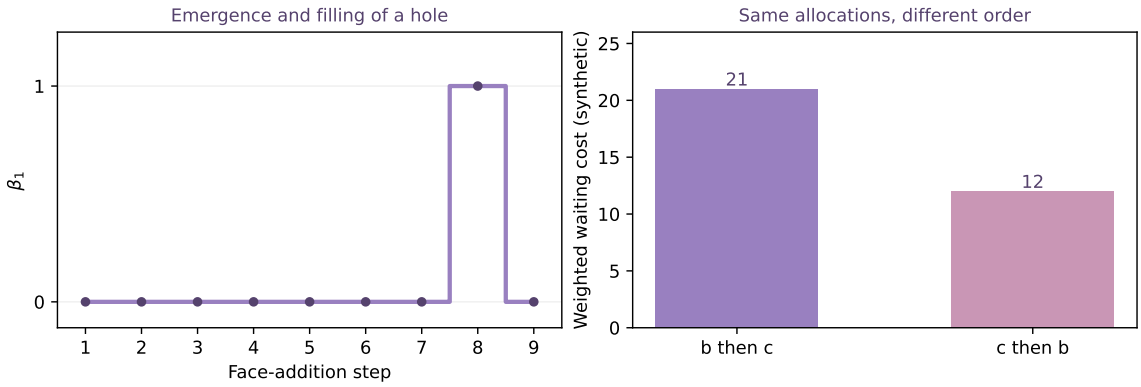

Figure 9: Two independent synthetic calculations. Left: the first Betti number of nine nested face supports, exhibiting an unfilled cycle followed by its filling. Right: weighted waiting costs of two allocation orders with equal final deliveries and different completion times. The two panels display distinct structural and evaluative observables rather than a causal relationship between a hole and deprivation.

supports have identical Betti numbers despite having different incidence structures and extents. Betti numbers therefore supply only part of the morphological description developed in Section 5.

8.3 Historical Formation of Heterogeneous Values

The second calculation follows a nine-step nested sequence of selected square-face sets within a 3×3 ambient block. Faces are added successively around the outer ring, beginning at its upper-left corner and proceeding along the perimeter. The ninth step activates the central square. Each selected set includes its boundary cells. Because the sequence is nested, inclusion maps between consecutive supports are available and its first-homology evolution can be read as a simple finite filtration. This particular growth rule is chosen to isolate a structural transition; social or economic value formation may involve deactivation, recognition changes, and non-nested histories as described in Section 6.

The computed sequence of first Betti numbers is recorded in Equation 39.

$$\big(\beta_1(L_1), \ldots, \beta_1(L_9)\big) = (0, 0, 0, 0, 0, 0, 0, 1, 0). \tag{39}$$

At step eight the closing perimeter completes a one-dimensional cycle; at step nine its central filling makes that cycle a boundary of a face. Figure 9 plots the computed homology sequence alongside a separate order-cost calculation introduced below. In persistent-homology language, the selected one-dimensional class appears at the eighth recorded threshold and is killed at the ninth, subject to the chosen indexing convention. The calculation does not demonstrate a social value emerging at step eight. A protocol that recognizes a ring-shaped configuration could register a structural condition at that step, while another modality could require information unrelated to the hole. Historical constitution and observation are therefore represented by distinct layers even in this small example.

The complete event record also requires a distinction between a recorded event and the activation of a face in a selected modality support. Adding a face in the numerical sequence stands for updating a specified representation, which may follow an actual coupling change or a revised recognition rule. Without evidence that selects between these interpretations, the computed change is attributable only to the declared model operation. The illustrated history thereby separates causal occurrence, support identification, and the appearance of a measured topological property.

8.4 Alternative Circulation Orders and Structural Consequences

The third experiment introduces two synthetic allocation events, b and c, which deliver one unit of an abstract resource to two historically identified recipients, B and C. The recipient labels denote derived participants in the model; the primitive vertices remain event occurrences. A preliminary observation event

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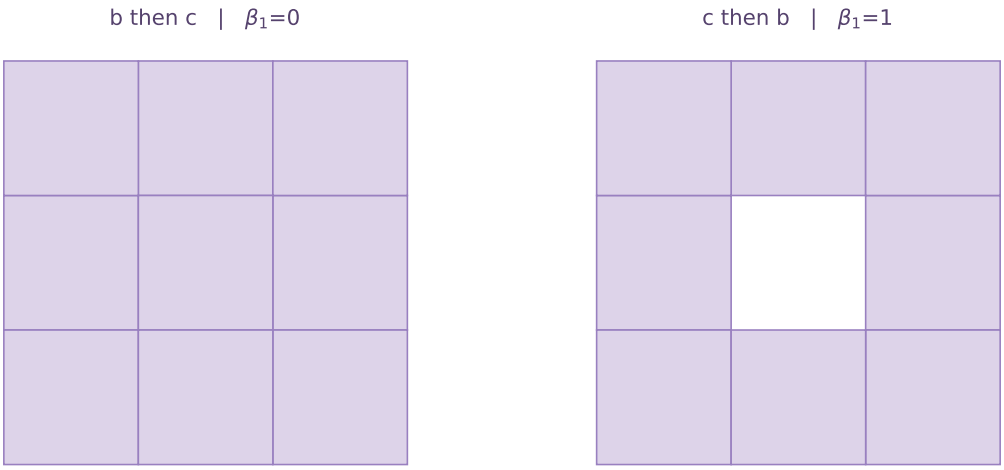

Figure 10: Final selected value-supporting complexes after two feasible synthetic event orders. Completing b before c activates the middle face under the declared contextual rule; completing c before b leaves the ring unfilled. Both histories have the same allocation totals. The face update is an assigned model mechanism and the difference in β_1 has no intrinsic normative sign.

o records contextual information and makes no allocation. The admissible executions are $\gamma_{bc} = (o, b, c)$ and $\gamma_{cb} = (o, c, b)$. Both produce the completed-delivery set $\{B, C\}$, and each allocation occupies one time unit after o. The example uses a fixed event-order comparison rather than an optimization over all possible social relationships.

For an illustrative urgency-sensitive accounting rule, take per-unit waiting weights $w_B = 1$ and $w_C = 10$. The participants begin waiting at the first allocation interval; the relevant completion times are $(t_B, t_C) = (1, 2)$ for γ_{bc} and $(2, 1)$ for γ_{cb}. Equation 40 gives the declared additive cost and its order difference.

$$\begin{aligned} W(\gamma) &= w_B t_B(\gamma) + w_C t_C(\gamma), \\ W(\gamma_{bc}) &= 21, \qquad W(\gamma_{cb}) = 12, \\ W(\gamma_{bc}) - W(\gamma_{cb}) &= w_C - w_B = 9. \end{aligned} \tag{40}$$

The right panel of Figure 9 shows the calculated values from Equation 40. This is an invented evaluation rule, useful for exhibiting order sensitivity. The selection of weights requires independent justification in an applied case, especially when obligations, rights, and heterogeneous needs are not reducible to one numerical metric. The earlier derived history diamond of Section 7 represents local execution comparisons; its possible filling faces should not be confused with the primitive faces of the present ambient GR complex.

The same execution system receives a second, purely structural update rule. Initially, its selected value support consists of the eight-square ring. If event c occurs after b, the specified coupling context permits activation of the central face; if c precedes b, that activation is absent under this particular rule. At a bridge-activation parameter of one, the final support of γ_{bc} is the filled block, whereas that of γ_{cb} remains the ring. This context dependence is deliberately imposed as a minimal example of the history-dependent transition relations of Equation 24; it is not a claim that a real allocation event typically produces a face of this kind.

Figure 10 and Table 8 report the two outcomes. Both completed histories have delivered the same resource units, but the selected support homologies differ. In the model, the order with smaller waiting cost retains the hole, while the order with greater waiting cost fills it. This directly illustrates that the sign of a topological difference and the ordering of a particular normative cost are independent outputs of the chosen rules.

An additional repair or review event could be defined to activate the missing central face after either

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Table 8: Computed comparison of the two allocation orders. The observation event occurs before both allocations and is omitted from the completion-time clock.

History	t_B	t_C	Weighted waiting cost	Final β_1
o, b, c	1	2	21	0
o, c, b	2	1	12	1

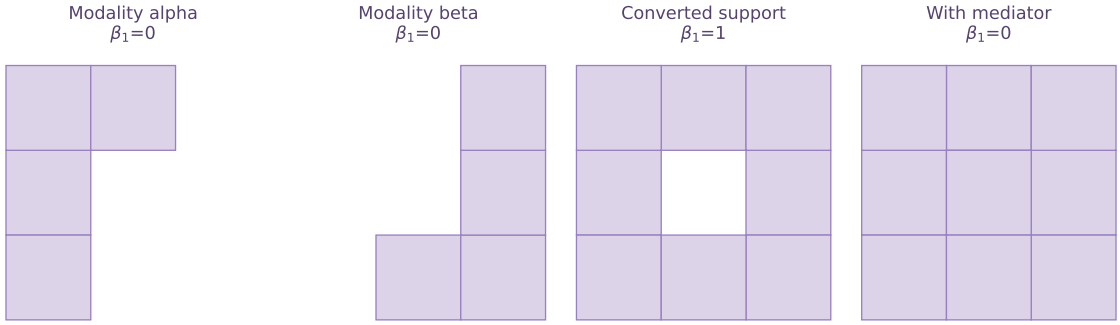

Figure 11: Synthetic typed support combination. Two four-face subcomplexes, each with trivial H_1, are combined by a declared event rule to produce an eight-face ring with $\beta_1 = 1$. Activation of a further primitive face yields a nine-face filled support with $\beta_1 = 0$. These outputs concern the combinatorial consequences of the selected rule, not a numerical equivalence among heterogeneous value types.

history, causing both final supports to agree while preserving their distinct waiting trajectories. Such an extension would separate long-run structural convergence from historical equivalence even more sharply. It is not included in the reported numerical runs, whose scope is the three-event histories and their explicitly selected final supports.

8.5 Value Conversion Across Distinct Morphologies

The fourth experiment uses two different typed supports with four primitive faces each. The selected α-support occupies a left and upper arc of a 3×3 block, and the β-support occupies the complementary lower and right arc. Each has $\beta_1 = 0$ as a closed subcomplex. A declared combination event m is enabled when both typed supports are available. Its illustrative output rule selects their union as the support of a third modality ζ. Equation 41 specifies the simplified rule; it should be read as a protocol for support formation, with the semantic map between modalities remaining to be developed.

$$\mathscr{T}_m(L_\alpha, L_\beta) = L_\zeta := L_\alpha \cup L_\beta, \qquad \beta_1(L_\alpha) = \beta_1(L_\beta) = 0, \quad \beta_1(L_\zeta) = 1. \tag{41}$$

The computation in Equation 41 is an elementary instance of higher-order structural formation: two supports without independent first-homology classes together support a closed one-cycle. A further designated mediating face fills the interior and changes the first Betti number of the output to zero. Figure 11 shows the two inputs, their composed support, and the mediated configuration. The two input modalities are typed mathematical objects, and the composition rule is imposed. Their combination therefore illustrates possible changes in morphology without establishing a conversion rate, a conservation law, or a social interpretation for the cycle.

The experiment also indicates why the topology of a value-supporting complex can depend on more than one modality. Modality labels and constitution maps must specify whether a combined support can be recognized as a new value configuration, whether it continues an earlier one, and which relevant dependencies are preserved. A union operation on selected face sets cannot supply those interpretations by itself. This limited construction is nevertheless a reproducible demonstration that local composition can change global topology, including the creation and subsequent filling of a hole.

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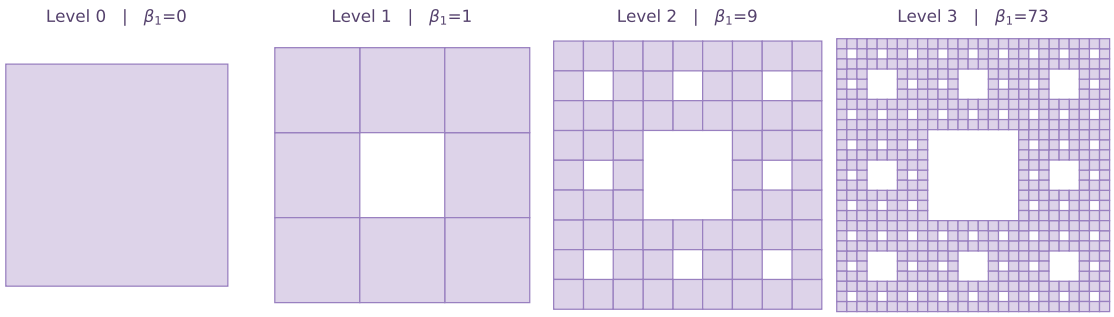

Figure 12: Four finite stages of a recursively substituted cubical support. Face counts grow as 8^k, and the first Betti numbers are 0, 1, 9, and 73. Repeated shapes are generated by an explicit mathematical rule; the illustration neither identifies a real value modality as fractal nor equates the number of holes with the scale of its value.

8.6 Topological Evolution and Multiscale Observables

A fifth calculation examines the geometric variation mentioned in Subsection 5.6. Begin with a single filled square. At each successive level, replace each filled square by a 3×3 square array with its middle face omitted. Repeated substitution produces finite approximations to a classical carpet-like self-similar motif. The implementation constructs the face masks with repeated Kronecker products of the 3×3 eight-face mask, closes the selected face sets under cellular boundaries, and calculates Betti numbers independently at each level. The construction uses the same cubical incidence method as the earlier experiments.

For levels $k = 0, 1, 2, 3$, the computed face counts are 1, 8, 64, and 512, while the first Betti numbers are 0, 1, 9, and 73. For these levels the observed identities agree with the exact replacement-rule counts stated in Equation 42.

$$n_2(k) = 8^k, \qquad \beta_1(k) = \sum_{j=0}^{k-1} 8^j = \frac{8^k - 1}{7}. \tag{42}$$

The recurrence in Equation 42 follows because the construction retains one connected support, introduces one central complementary region in each previously occupied cell, and preserves the existing complementary regions under replacement. The finite grid computations independently verify the formula through level three. Figure 12 visualizes these four levels. For the idealized limiting geometric set, the substitution pattern suggests the conventional similarity dimension $\log 8 / \log 3$; this dimension is a property of the mathematical construction under its limiting interpretation and has not been inferred from data about social value.

The example clarifies the distinction between a repeated geometric motif and a mechanism that reproduces the capability to generate value. The substitution algorithm guarantees a selected pattern; generative recurrence in a social or economic system would instead require explicit transition laws, enabling relations, and evidence about resulting capacities. Moreover, a coarse-graining operation that erases small holes could lower measured Betti numbers even when the underlying history remains unchanged. The resolution and support-selection protocol should therefore accompany any proposed multiscale observable.

8.7 Sensitivity to Coupling, Ordering, and Recognition Rules

The final experiment varies two independent parameters of the order example. First, let $p \in [0, 1]$ denote the synthetic probability that the context-enabled central face becomes active when c follows b. Under the declared Bernoulli activation rule, the b, c history has $\beta_1 = 0$ when activation succeeds and $\beta_1 = 1$

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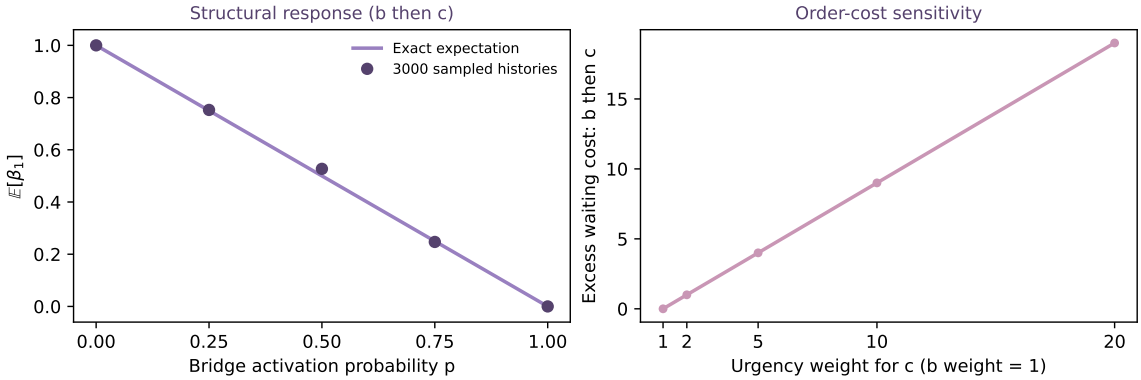

Figure 13: Sensitivity of two separate observables in the synthetic circulation model. Left: exact expected first Betti number under Bernoulli activation with seeded Monte Carlo sample means from 3,000 runs per parameter value. Right: the difference in weighted waiting cost under varying urgency assigned to C while B's weight remains one. Neither panel defines a universal relationship between topology and justice.

otherwise. Equation 43 gives the exact expectation, with no statistical approximation required.

$$\mathbb{E}[\beta_1(L_{bc}) \mid p] = 1 - p, \qquad \mathbb{E}[\beta_1(L_{cb}) \mid p] = 1. \tag{43}$$

A seeded Monte Carlo run evaluates 3,000 independent synthetic b, c histories at each of $p = 0, 0.25, 0.5, 0.75, 1$. The resulting sample means are 1.0000, 0.7527, 0.5270, 0.2470, and 0.0000, compared with the exact means 1, 0.75, 0.5, 0.25, and 0. The discrepancy at intermediate values is sampling variation under the declared randomization. These experiments illustrate sensitivity to an assumed coupling or face-activation mechanism; they estimate no parameter of any actual society.

Second, hold $w_B = 1$ and vary w_C through 1, 2, 5, 10, and 20. Equation 40 shows that the excess waiting cost of the b, c order is exactly $w_C - 1$. This changes the normative-cost output without changing the face-activation model. Figure 13 juxtaposes both parameter families and thereby distinguishes the topological sensitivity of a support-update rule from the evaluative sensitivity of a chosen urgency weighting.

A third sensitivity concerns recognition and measurement rather than dynamics. If the central face is activated in the underlying modeled state but a support-selection protocol omits it, the selected output remains an eight-face ring and has $\beta_1 = 1$. If the protocol includes it, the selected output is the nine-face block and has $\beta_1 = 0$. These two observations are calculated from the same underlying filled state with different face-selection rules. The discrepancy would be epistemic in this construction, whereas the randomized bridge rule changes the operative state specified by the model. A valid empirical investigation would need to distinguish changes in observations from changes in relations before drawing a conclusion about value formation or loss.

The numerical results establish several limited mathematical findings. First, computable morphology distinguishes finite selected supports even when their cell counts or Betti numbers partly agree. Second, a nested history can produce the birth and filling of a first-homology class. Third, declared history-dependent event rules can separate endpoint allocation equality, interim waiting costs, and final support topology. Fourth, combining typed supports and iterating a multiscale rule can change global topological invariants. Every result depends on the explicit finite construction, the selected support protocol, and the stated cost or transition parameters. The source files specify these dependencies and permit independent reproduction. Claims about human needs, justice, institutions, and actual generative capacities require additional social evidence, conceptual justification, and a normative account developed separately in Section 9.

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9 Justice of Heterogeneous Value Circulation Order

This section develops a preliminary normative interface for the event histories and heterogeneous value-supporting structures constructed in Sections 6–7. Its objective is to specify which features of circulation order may enter a judgment of justice, how a comparison can remain meaningful under partially comparable values, and which questions require independent moral, legal, and empirical justification. The method combines explicitly defined history observables with a plural evaluative relation; it does not infer a moral ranking from cell counts, topological invariants, or computed costs. The discussion proceeds from normative admissibility and temporally distributed needs through obligations, historical power, and subsequent generative possibilities, before examining partial comparisons and the interpretive role of topology. The synthetic executions of Section 8 serve as fixed examples rather than evidence about actual social arrangements.

9.1 Historical Order and Normative Assessment

Let Γ_{feas} denote the technically and causally feasible event-state histories introduced in Equation 29. A history $\gamma \in \Gamma_{\text{feas}}$ includes event occurrences, a precedence-respecting execution, relevant state transitions, and records of the changing supports of candidate values. Its events are vertices in the GR complex; people, organizations, and other putative bearers of claims are identified, if at all, through historically supported entity-constitution procedures. Consequently, claims about a beneficiary in the following analysis refer to an identified trajectory or a role in an illustrative protocol, rather than to a primitive event vertex.

Technical feasibility and normative admissibility perform distinct tasks. A possible transfer can satisfy every resource and causal constraint while breaching a right, violating a prior undertaking, or exposing a vulnerable participant to avoidable harm. Let Θ be a declared family of normative interpretations, each supplying its evidential standard, scope of standing, recognized claims, and rules for assessing conflicts. For $\theta \in \Theta$, let $\mathsf{A}_\theta(\gamma)$ denote satisfaction of those histories' specified mandatory constraints. Equation 44 defines the corresponding restricted domain.

$$\Gamma_\theta^{\text{adm}} = \{\gamma \in \Gamma_{\text{feas}} : \mathsf{A}_\theta(\gamma) = 1\}. \tag{44}$$

In Equation 44, admissibility records a conclusion under a declared normative interpretation. The symbol A_θ is an input to ethical analysis, not an algorithm that establishes which rules are justified. Where an alleged entitlement, beneficiary, or relevant legal norm is disputed, several admissibility domains may coexist. An empty domain would identify inconsistency among the specified constraints and modeled feasibility conditions, and would motivate examination of the model and the obligations rather than automatic relaxation of the latter.

A *justice assessment* additionally addresses differences among histories that meet the applicable mandatory constraints. A convenient notation is a family $\mathcal{J}_\theta$ taking annotated histories into an ordered space $(\mathcal{Q}_\theta, \succeq_\theta)$, with $q \succeq_\theta q'$ interpreted as the judgment that q is at least as acceptable as q' within that interpretation. This permits a preorder or partial order whose members are themselves structured profiles. The source of the ordering can be ethical argument, institutional rules, participatory deliberation, or a restricted decision protocol; the topological formalism supplies a history representation to which that ordering can be applied. Comparative and realization-sensitive approaches to justice supply one relevant intellectual precedent for evaluating feasible social arrangements while preserving plural evaluative considerations [14].

Three distinct questions then arise. *Attribution* determines which historically emergent individuals or groups bear a relevant interest. *Assessment* determines which features of their trajectories and surrounding relations count normatively. *Comparison* determines which of the resulting profiles can be ordered. The identification protocol and the resulting moral standing need not coincide: failure to observe a person's contribution or need under a particular data protocol does not establish that no justified claim exists. Historical records, recognized values, and justified value judgments should therefore remain separate fields in a usable implementation.

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9.2 Heterogeneous Needs, Urgency, and Temporal Deprivation

A need can be described through a modality-specific condition and a time interval over which satisfaction matters. Let $\mathcal{B}_\theta$ denote the set of recognized claim-bearing histories or roles for a chosen interpretation; its elements are derived from relational records and may be revised when the evidence changes. For $a \in \mathcal{B}_\theta$ and modality α, let $n^\theta_{a\alpha}(t;\gamma)$ describe the applicable need specification, and let $r^\theta_{a\alpha}(t;\gamma)$ describe access to a resource or capability relevant to it. The two descriptors may be qualitative, interval-valued, or categorical. For modalities where a defensible nonnegative loss scale is available, define a local deprivation intensity $d^\theta_{a\alpha}(t;\gamma)$. The optional time-aggregated assessment in Equation 45 records its dependence on an explicitly chosen horizon and measure.

$$D^\theta_{a\alpha}(\gamma;[t_0,t_1]) = \int_{t_0}^{t_1} d^\theta_{a\alpha}(t;\gamma)\,\mathrm{d}\mu_\theta(t), \qquad d^\theta_{a\alpha} \geq 0. \tag{45}$$

Equation 45 applies only when the intensity, horizon, and measure are meaningfully specified. It neither presupposes a single scale across value modalities nor treats every unmet need as smoothly accumulated damage. A deadline-sensitive obligation may instead require a threshold indicator, a duration above a risk level, or a record of whether an irreversible event occurred. For trust, dignity, care, or knowledge, an observer may be able to identify a consequential absence without producing a justified numerical intensity. Such modalities require their own assessments and uncertainty statements.

Urgency concerns the dependence of consequences on timing. A socially meaningful urgency relation can be represented by a claim-specific deadline, a time-varying risk, or a precedence constraint supported by medical, practical, legal, or contextual reasons. Waiting is therefore assessed relative to the claimant's history and available alternatives. A priority convention based solely on the sequence of requests may exclude relevant differences in severity, exposure, and the relation between early access and later capacity. Equally, an urgency-based exception may create obligations toward those deferred. These considerations are compatible with ethical accounts that treat the organization of care and the distribution of responsibility as political and institutional matters [15].

The synthetic comparison in Equation 35 illustrates a narrow, testable temporal claim. Both possible orders complete two unit-duration allocations. With declared weights $w_B = 1$ and $w_C = 10$, the order B, C has cumulative weighted completion-time cost 21 while C, B has cost 12. The difference is 9, and the nominal final allocation is identical. The weight assignment is an assumption of the toy model: it cannot independently establish C's moral priority, the legitimacy of either allocation, or the appropriate treatment of longer-term effects. The outcome demonstrates that a time-sensitive assessment can distinguish endpoint-equivalent histories, leaving the justificatory interpretation of its weights open.

9.3 Circulation Histories and Distributional Consequences

A distribution recorded at a final event can omit intermediate access, timing of loss, and the conditions through which the distributed resources were constituted. In the present model, a history-specific distributional record $\mathsf{R}_\theta(\gamma)$ should therefore retain at least: the recognized bearer of an allocation or contribution, the value modality, the event interval, the structure supporting the allocated value, the relevant evidential status, and any ensuing restriction on future access. These fields are properties of an interpreted history, not components of the ambient complex's boundary operator.

The difference between cumulative receipts and viable use is significant. One allocation may be recorded as completed when a resource is delivered, while its practical benefit depends on whether earlier events preserved the claimant's opportunity to use it. A course made available after the learner's eligibility ends, or care delivered after a preventable deterioration, may be counted in an endpoint inventory yet fail to meet the relevant need. Conversely, an early enabling intervention can contribute to the formation of additional capacities not enumerated in a transfer ledger. The possibility of path-dependent outcomes provides a conceptual reason to retain histories even when their endpoints or aggregate resource quantities agree [5, 1]; it does not import a specific increasing-returns law into the present model.

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Table 9: Analytical dimensions for comparing justice-relevant circulation histories. Each row specifies possible information to retain and a separate interpretive requirement. The table supplies a modeling interface rather than a universal priority rule.

Dimension	History-sensitive descriptor	Normative question or prerequisite
Timing and urgency	Completion times, delay intervals, irreversible thresholds	Evidence for severity and priority under the circumstances
Distribution and access	Modality-specific receipts, available use, access restrictions	Standing, entitlement, relevant baseline and opportunity to use
Obligations and rights	Undertakings, responsibility chains, constraint violations	Authority and scope of the applicable norm
Conversion and power	Control of exchange interfaces, terms, refusal and exit options	Legitimacy, dependence and avoidable imposition
Generative consequences	Reachable subsequent supports, capacities, and feasible paths	Which future options merit protection or support
Recognition and provenance	Observation protocol, missing records, contested attribution	Reliability, representation and potential epistemic exclusion

Distributional descriptions should also attend to the constitution of the claimant. The entity-recognition relation of Section 4 permits several candidate historical supports to contribute to the identity of one organization or person as represented in the model. A person's present capacities and vulnerabilities may depend on earlier education, labor, institutional exclusions, or supportive relations. An allocation table indexed by supposedly fixed and independent agents will then be a useful projection for some tasks but an incomplete account of the generative history. Such a projection may nonetheless be preferable for a narrowly stated empirical or legal inquiry; the appropriate degree of detail is determined by the object of assessment.

Table 9 separates descriptors from normative questions. Its rows should be read together where the relations make them interdependent. A delay might alter the capacity to exercise a right; a conversion interface might determine who can document a contribution; and an institutional practice might change the set of future feasible histories. These interactions explain why additive summation across heterogeneous descriptors requires further justification rather than following from the existence of a vector of measurements.

9.4 Relational Dependencies, Obligations, and Power

The justice of an event order also depends on who may determine that order, which alternatives are open to affected participants, and how obligations persist as value-supporting structures transform. An obligation may concern delivery, reciprocity, noninterference, attribution, care, or repair. In the GR representation, such relations can be recorded by annotations on histories and couplings, while their bearers are identified through the relevant entity-constitution protocol. An obligation survives a topological operation only if the retained data permit its provenance and applicable force to be reconstructed. A quotient of histories that identifies identical endpoint allocations may therefore discard a wrongful delay or a broken undertaking.

An illustrative exchange of money and labor clarifies the point. A transaction can have a common nominal price under two histories while differing in the sequence of payments, the ability to withdraw, exposure to uncertainty, the availability of safety protections, and opportunities to challenge the terms. Those differences affect the interpretation of consent and the distribution of risk. The question of justice cannot be resolved by equality of monetary endpoints or by the similarity of the topology of the value supports. The particular law, organizational practice, bargaining arrangement, and relevant accounts of legitimate consent must be specified.

Power can be studied through the control of feasible transformations. Consider two history-indexed states with identical total labeled supports. In one state, a participant can access several conversion mech-

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anisms and exit alternatives; in the other, a single gatekeeper controls the only admissible conversion interface. Their raw topology may coincide, yet their sets of reachable future configurations can differ under the enabling predicates and event transitions defined in Section 6. The gatekeeper can influence the circulation order by delaying, bundling, conditioning, or withholding access, even without extracting an explicitly priced resource. Within this paper, *control of a conversion interface* denotes a relational capacity specified by such predicates and available transitions. Its classification as coercive, justified, exploitative, or protective requires institutional and normative interpretation.

Historical asymmetries are relevant to both need and valuation. Those who control record-keeping may influence which contributions enter the recognized history, which modalities count as valuable, and whose waiting or loss is visible. Commensuration procedures have their own social and political consequences [7], and recognition protocols should be treated as possible sites of selection and bias. This is especially important where labor, care, ecological contributions, and expressive activity circulate under different measures of social value [6]. The proposed formalism can record disputed identification and conversion rules; its mathematical notation does not authorize a particular hierarchy among them.

9.5 Generative Capacities and Future Possibilities

A circulation history may change the conditions for producing subsequent values. Let x_γ be the final decorated state of a feasible finite history γ, and let $\mathsf{Reach}_h(x_\gamma)$ be the set of subsequent decorated states reachable within a declared horizon h under specified event rules. Section 6 introduced reachable possibilities in Equation 28. For justice analysis, the meaning of an option depends on who can exercise it, how reliably it can be sustained, and whether its realization would respect relevant rights and obligations. The set of reachable states is thus a descriptive input to an assessment rather than a direct measure of moral progress.

Consider two synthetic orders that yield the same nominal allocation. The first supplies a learning resource before a qualification deadline and permits a subsequent enrollment event. The second supplies the same resource after the deadline, leaving that enrollment event disabled. The final transfer ledger may agree, whereas the reachable-event sets differ. A corresponding distinction could arise for therapeutic access, rebuilding trust after an injury, or a repair activity that preserves future social cooperation. These examples concern stipulated enabling rules, and each would require a separate operational and normative account in an empirical application.

It may seem attractive to rank histories by the number or measured volume of reachable configurations. Such a ranking would be difficult to defend without further conditions. The size of a state space depends on the resolution of the model; it can count frivolous, harmful, inaccessible, and valuable possibilities together. A seemingly expansive history may also impose costs or constraints on others. One defensible limited comparison identifies a named set G_θ of permissible future opportunities, defines a projection Π_θ from reachable states to that set, and tests inclusion among the projected opportunities. Equation 46 records this restricted comparison.

$$\mathsf{O}_\theta(\gamma; h) := \Pi_\theta\big(\mathsf{Reach}_h(x_\gamma)\big) \cap G_\theta, \qquad \mathsf{O}_\theta(\gamma_1; h) \supseteq \mathsf{O}_\theta(\gamma_2; h). \tag{46}$$

The inclusion in Equation 46 establishes only that the first history provides at least the opportunities specified in the model's chosen projection; it does not give a complete justice ranking. The opportunity set, causal horizon, participant capabilities, reliability of access, and effects on others remain part of its interpretation. In some contexts an option's quality, security, and accessibility may matter more than set inclusion. The representation is therefore consistent with historical and generative assessment while keeping substantive judgments visible rather than embedding them unnoticed in geometry.

9.6 Partial Comparability and Justice-Oriented Orderings

A plurality of value modalities and normative considerations suggests a family of partial comparisons. For each $\theta \in \Theta$, define a structured assessment profile $q_\theta(\gamma)$ containing those descriptors of Table 9

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for which there is adequate standing, evidence, and interpretive justification. Its comparison relation $\succeq_\theta$ can prioritize mandatory claims, use Pareto comparisons across commensurable subdomains, or leave some alternatives incomparable. Equation 47 gives a conservative comparison across a declared set of permissible interpretations with a shared account of the compared histories.

$$\gamma_1 \succeq^{\text{rob}} \gamma_2 \quad \Longleftrightarrow \quad \left[\gamma_1, \gamma_2 \in \Gamma_\theta^{\text{adm}} \text{ and } q_\theta(\gamma_1) \succeq_\theta q_\theta(\gamma_2)\right] \quad \text{for every } \theta \in \Theta. \tag{47}$$

The universal comparison in Equation 47 is deliberately restrictive. A failure to establish its right-hand side can reflect incompatible criteria, insufficient evidence, or contested standing, and does not entail an ethical tie. The relation is reflexive and transitive if the domain is shared and each underlying $\succeq_\theta$ has those properties; it may lack antisymmetry when distinct histories receive equivalent assessments. Different interpretations may also disagree on whether an act is admissible, leaving robust comparison unavailable. These are useful outputs of the analysis because they identify where additional deliberation or evidence is needed.

Within a narrower domain, further comparisons are possible. When obligations are agreed to have priority, an admissibility filter can precede evaluation of residual distributional harms. When several deprivation dimensions have contextually legitimate scales, Pareto improvement avoids introducing exchange rates between them. A lexicographic rule is appropriate only if a justified priority ordering has been supplied; a weighted sum requires even stronger commensuration assumptions. These alternatives embody different normative commitments and should be made explicit. They also distinguish the act of modeling the evaluative domain from selecting a social decision procedure.

The difference between a strict ranking and a traceable justification matters for institutional use. A system may report that one order satisfies an agreed deadline constraint, that two orders are incomparable under current evidence, or that changing one priority reverses the ranking. Such a report is analytically informative without claiming that a formula has resolved the underlying moral disagreement. The presentation should preserve supporting evidence, alternative interpretations, and the rationale for the priority rules. This follows the paper's commitment to distinguish a formal representation from normative authority and to treat bias in recognizing and communicating value as part of the inquiry.

9.7 Topological Information and Normative Evaluation

The topological structures of Sections 3 and 7 can help organize and compare circulation histories. The ambient event complex K, a selected value-supporting subcomplex L_α, and a derived history-comparison complex D are different objects. Their homology groups capture distinct properties of the chosen incidence or interchange relations. A first-homology class in L_α describes an unfilled cycle in a selected support under its primitive face specification. An unfilled path-difference cycle in D may describe an interchange relation absent from a declared history-equivalence protocol. Neither invariant carries an intrinsic judgment of benefit, deprivation, or wrong.

A topology-sensitive normative question requires a connecting mechanism. For example, a support hole might matter if a specified face-coupling transformation requires a filling relation and a particular need remains unsatisfied as a result. The argument would then require a verified support-selection rule, an admissible-event mechanism, a demonstrated causal effect on the claim-bearing history, and an independently defended normative criterion. In a different model, the same Betti number could coexist with beneficial redundancy, a stable circulation pathway, or a purely descriptive geometric gap. Figure 9 and Figure 13 show how these distinctions can be isolated in synthetic examples.

The derived order diamond of Figure 7 provides a sharper local analysis. Attaching an interchange face relates two edge paths with common endpoints, while Equation 36 evaluates their difference for one additive cochain. In that selected model the coboundary measures the sensitivity of waiting cost to a local order interchange. A general justice evaluation can depend on nonlinear thresholds, identity-specific obligations, disputed recognition, and future opportunity sets; a single real-valued cochain may be insufficient. The criterion of Proposition 3.1 consequently remains decisive for reductions: the required evaluative observable must be constant on classes identified by the quotient. Homology preservation by

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a Morse reduction, considered in Subsection 7.7, is a separate guarantee and does not imply preservation of normative assessments.

The comparative model in Section 8 is especially instructive. An order with a lower stipulated waiting cost may retain a first-homology class while the alternative order activates a face that fills the corresponding support hole. The two observables vary independently under the chosen parameters. Therefore, a statistical association between topological features and justice-related outcomes, if eventually observed, would require identification of relevant mechanisms and alternative explanations. Topological sensitivity to recognition protocols could otherwise be mistaken for a change in actual value-supporting relations.

For subsequent inquiry, three classes of formal question remain distinct. *Structural* questions concern the morphology and topological transformations of value supports. *Historical* questions concern feasibility, causal order, evidence, and the constitution of entities and values. *Normative* questions concern standing, admissibility, obligations, need, and the comparison of histories. The present formalism connects these classes through explicit projections and declared evaluation rules while preserving their individual assumptions. It supports analysis of the justice of heterogeneous value circulation order as an inquiry into historically situated choices and their generative consequences. The empirical identification of its variables and the justification of its normative ordering remain tasks for specialized applications and further argument.

10 Mathematical Scope and Interpretive Considerations

The event–coupling–face formalism offers a common representation for historical relational configurations, heterogeneous value-supporting structures, and order-sensitive processes of circulation. Its mathematical usefulness depends on specifying the connection between a chosen complex and the phenomenon under examination. This section examines the levels of claim supported by the formalism, the semantics of its topological observables, information preservation under reductions, epistemic uncertainty in the recognition of value, the requirements of empirical operationalization, and possible extensions of the underlying cell complex. The method is an internal assessment of definitions and inferential dependencies, informed by the preceding synthetic constructions. It seeks to identify conditions under which the formal representation may support further inquiry while leaving ontological, empirical, and normative questions appropriately open.

10.1 Structural Formalization and Ontological Suspension

The primitive modeling decisions are specified in Section 4: a vertex denotes an event, an edge a coupling, and a two-cell a primitive higher-dimensional relation whose particular semantics require a modeling convention. The cellular incidences, boundary operators, and representation labels then determine a formal object. The use of a two-complex is an analytical proposal about which distinctions to record; the existence of a useful cellular description does not establish that a social process is, in a literal physical sense, a quantum spin foam. In the physical literature, spin foams supply mathematical structures for histories of spin-network states and their amplitudes [2]. The present paper borrows certain combinatorial and representational possibilities without importing a physical probability law, an identification of social value with spin, or a theory of quantum social interaction.

The meaning of an individual modeling decision can be described at three levels. At the *combinatorial level*, a finite cell complex has vertices, incidence relations, and a well-defined cellular chain complex once orientations and coefficients are chosen. At the *representational level*, a research protocol identifies recorded occurrences as events, encodes relevant couplings, and decides which higher-dimensional primitive relations warrant faces. At the *interpretive level*, a historically selected subcomplex is associated with an emergent value or entity according to separately declared criteria. Only the first level is fixed by the relevant mathematical definitions; the latter two involve substantive choices about evidence and interpretation. The same distinction applies to the causal partial order, history-realization map in Equation 15, value-constitution relation in Equation 19, and event-transition relation in Equation 24.

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Table 10: Claim levels in the proposed relational-spacetime formalism and their requirements.

Claim level	Formal or evidential object	Requirement for interpretation
Mathematical	Cellular boundaries, homology, order relations, and valid deductions	Declared definitions, hypotheses, coefficients, and correct calculations
Representational	Assignment of events, couplings, faces, time labels, and states	Documented coding rules and an adequate record of the setting
Constitutive	Identification of entities and heterogeneous values through historical supports	Defensible modality-specific and entity-specific criteria
Empirical	Mechanisms, parameters, and predictions of observed processes	Appropriate observations, identification assumptions, and validation
Normative	Standing, obligations, acceptable histories, and justice comparisons	Independent ethical or legal justification with recorded disagreement

The analytical layers and their respective evidential requirements are summarized in Table 10. The table is also a guide to the scope of claims that may legitimately be derived from the manuscript's synthetic examples.

A mathematically valid statement may remain conditional on the representation. For example, a computation can establish that a selected subcomplex has first Betti number one over a specified field. Whether the selected cells constitute the support of a recognized value, whether its hole affects the feasibility of an exchange, and whether any resulting deprivation is unjust require additional arguments. Conversely, an empirical observation of deprivation may motivate a search for structural mechanisms without requiring an initially successful topological representation. The distinction allows the formalism to coexist with historical investigation, qualitative interpretation, and institution-specific methods.

The paper therefore takes an ontologically suspended stance. A historical relational account motivates the selection of mathematical objects, while the ontology and the representation remain open to revision. Such restraint is compatible with the historical development of economic categories: the categories used to identify and compare forms of value may themselves be products of social practices. Work on commensuration, for example, describes how the construction of common metrics constitutes a consequential social process [7]. Within the present model, a common metric is an additional institutional or analytical operation, not an automatic consequence of assembling multiple modalities in one complex.

10.2 Topological Invariants and Value Semantics

Topology identifies structural features that are invariant under the selected notion of equivalence. For a finite complex with a specified coefficient field, Betti numbers measure ranks of homology groups; under the usual conventions the Euler characteristic alternates the counts of cells and also the Betti numbers. These properties follow from the chain-complex construction reviewed in Section 3. They remain distinct from the annotations and interpretation of the complex as an event history or a value-supporting structure. In particular, two supports can share Betti numbers while differing in edge couplings, geometric extent, temporal order, value modalities, recognition conditions, and reachable future configurations.

A useful distinction separates *intrinsic support topology* from the *embedding or inclusion relation*. The homology of a candidate value support L_α concerns its cellular structure. Its inclusion into the ambient recorded complex K_H provides additional information about which events and couplings the support shares with surrounding configurations. An inclusion-induced map on homology, when the underlying inclusion is a cellular map, can determine whether a cycle in L_α remains nontrivial in K_H or becomes a boundary there. This distinction may be relevant when a support has a closed one-dimensional coupling pattern that is filled by a primitive face available in the ambient complex but absent from the selected value support. Whether that distinction participates in an actual value transformation depends on the role assigned to the face and on the applicable transition rules.

The interpretation of a hole is therefore conditional on the definitions of cells, labels, and admissible transformations. A nontrivial class in $H_1(L_\alpha)$ represents a closed cellular one-chain that is not a boundary in L_α under the chosen coefficients. Such a class may be used to locate structural differences

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between candidate supports. It establishes neither a deficient resource nor the existence of a circulation route, because actual circulation additionally requires compatible value states, feasible transformations, and a causal history. A history-comparison complex introduced to represent interchanges among event orders has a different face semantics from the primitive faces of K. Conflating their homology groups would merge descriptions with different domains of interpretation.

Geometric weights and modality labels add further qualifications. Length, area, and duration require conventions for measurement and often change under a different resolution or time discretization. Similarly, an algebraic representation or phase attached to a primitive face can participate in a coupling rule while having no independently established economic meaning. Multiple value modalities can share portions of a support, as in Equation 22, without acquiring a common magnitude. A model that uses a vector of descriptors can keep these distinctions visible, while any aggregation must specify which components may be compared and why.

The computed examples in Section 8 provide a finite counterexample to the inference from topology to normative ranking. In one stipulated order, lower cumulative waiting cost accompanies a surviving support hole; in another, higher waiting cost accompanies the filling of the corresponding cycle. The result is internal to its synthetic rules. It demonstrates that a homological change and the specified temporal evaluation are separate observables, even when both respond to event order. The design of a causal explanation linking them is a further research task. Persistent topology and multiscale descriptors likewise require a documented filtration, rather than a presumption that persistence itself measures lasting social benefit [9, 4].

10.3 Coarse-Graining and Historical Information Preservation

Relational spacetime may contain many recorded events and primitive relations, while practical inquiry often operates with aggregations such as projects, institutions, periods, or regional systems. A reduction can compress the original data into a more manageable representation. Its adequacy is relative to the questions being asked. Homology-preserving simplification, history-preserving compression, preservation of feasible transitions, and preservation of a normative assessment are distinct requirements. Discrete Morse theory supplies methods for reducing cell complexes with controlled homological properties [8, 13]; a chosen Morse matching supplies no general guarantee that event timestamps, rights-relevant evidence, or modality-specific recognition data survive unchanged.

At the chain level, a morphism between chain complexes is represented by degree-indexed maps $q_p : C_p(K) \to C_p(K')$ that commute with boundaries. Equation 48 states the required relation.

$$\partial'_p \circ q_p = q_{p-1} \circ \partial_p \qquad (p = 1, 2). \tag{48}$$

A family satisfying Equation 48 induces maps on homology. An arbitrary grouping of cells, however, need not define such a family, and an induced homology map need not be an isomorphism. Even if it is an isomorphism, histories whose distinctions concern order, timing, or recognized claims can become indistinguishable under the reduction. A reduced complex should therefore be accompanied by metadata about the projection and by a clear statement of which information the construction preserves.

An assessment-oriented reduction has a separate criterion. Let $r : \Gamma \to \overline{\Gamma}$ map detailed histories into reduced histories. For an assessment $\mathcal{J}_\theta : \Gamma \to \mathcal{Q}_\theta$, exact recovery from the reduced representation is possible when there is a function $\overline{\mathcal{J}}_\theta$ with the factorization in Equation 49.

$$\mathcal{J}_\theta = \overline{\mathcal{J}}_\theta \circ r. \tag{49}$$

The condition is equivalent to requiring $\mathcal{J}_\theta$ to be constant on each fiber of r. It is a criterion for preservation of the assessment as defined, rather than an endorsement of the assessment's ethical legitimacy. In particular, when an order-sensitive deprivation record depends on the time at which a need is met, a reduction that retains only final resource totals may violate Equation 49. This relates the information-theoretic motivation for a quotient to the interpretive restrictions established in Proposition 3.1 and Subsection 7.7.

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In a bounded scalar approximation, the loss of exact preservation can be quantified. Assume a justified real-valued assessment L_θ on a specified finite history domain, a reduced estimate $\overline{L}_\theta$, and a stated uniform error bound. Equation 50 records this assumption.

$$\sup_{\gamma\in\Gamma}\left|L_\theta(\gamma) - \overline{L}_\theta(r(\gamma))\right| \le \varepsilon. \tag{50}$$

For two histories γ_1, γ_2, the difference between their actual assessment gap and their reduced assessment gap is then at most 2ε by the triangle inequality. Consequently, a reduced strict ordering whose gap exceeds 2ε has the same sign as the unreduced ordering, provided the bound is valid for both histories. This conclusion follows from the approximation assumption; establishing a credible bound in a social application is a separate evidential problem. For partially ordered justice profiles, analogous preservation would require a stated margin or stability criterion appropriate to the chosen order, and some comparisons may remain unresolved.

A useful implementation can therefore retain two linked records: an aggregated representation for comparison or visualization and an auditable event-level history for provenance-sensitive reconstruction. Effective models may also retain sufficient summaries of deadlines, unmet needs, and relevant conversion interfaces. The summary cannot be declared sufficient solely because it reproduces the Betti numbers of its source. The appropriate preservation criterion should follow from the intended analytical use, particularly when the model is used to inform decisions that affect identifiable participants.

10.4 Epistemic Limits of Value Recognition

The historical support of an emergent value is rarely given as an unambiguous observable. Events may be undocumented; couplings may be inferred from incomplete traces; institutional categories may record certain contributions while overlooking others; and a value may become recognizable only after several subsequent events. The selection of a support therefore involves epistemic conditions alongside topology. Section 5 introduced a recognition relation separate from the proposed constitution of value. For empirical work, this separation should be preserved in the database, identification procedure, and presentation of model outputs.

One representation of uncertainty is to retain a family $\mathscr{K}(\mathcal{O};\kappa)$ of candidate event–coupling–face reconstructions compatible with an observation record $\mathcal{O}$ and documented coding protocol κ. Different elements can reflect alternative readings of one reported interaction, incomplete evidence about event precedence, or contested choices of primitive faces. A value-constitution protocol $\mathcal{G}_\alpha$ may in turn associate several candidate value-supporting complexes with the same reconstruction. If an alleged support hole occurs in only some members of the candidate family, its status should be reported as reconstruction-dependent. The notation describes a set of admissible interpretations; it does not assume a probability measure on them.

This uncertainty has at least three distinguishable sources. *Observational incompleteness* concerns events, couplings, or records that are unavailable. *Interpretive plurality* concerns defensible differences in the coding of value, entity continuity, or causal connection. *Representational selectivity* concerns effects of the chosen vocabulary, data structure, thresholds, or visual display. These sources can interact. A visualization showing a disconnected support may emphasize apparent absence when the underlying record in fact contains an unresolved classification or an observation gap. Conversely, a coarse institutional label may join activities that bear different histories and obligations. A reliable display should allow users to examine the evidence and assumptions behind significant cells, classifications, and inferred relations.

Recognition can also have causal consequences. If an institution recognizes certain activities as productive and treats others as peripheral, that judgment may shape future access to funding, time, training, collaboration, or public standing. Such mechanisms need empirical support in each case, yet the formal architecture accommodates their possibility by representing recognition as part of a history that can affect later coupling and transition rules. The present framework thereby distinguishes an observer's report about a value from an event in which the report itself changes a social configuration. Historical records

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and interpretive systems may both enter the subsequent modeled history, while their evidential status remains separately annotated.

The normative consequences of epistemic asymmetry are equally important. A claimant's capacity to have a contribution or need documented may differ from another's. Institutional classifications can therefore influence whose histories enter the feasible comparison set and which deprivations are recognized. The formalism should support disagreements and versioned corrections rather than issuing a single hidden classification as if it were definitive. The concern is related to the socially consequential character of measurement and classification discussed in the literature on commensuration [7]. It is also consistent with the claim in Section 9 that normative standing cannot be inferred solely from successful observation or formal identification.

An epistemically cautious analysis should report sensitivity across reasonable reconstruction protocols, retain uncertainty about dates and couplings, preserve the source of contested interpretations, and distinguish missing evidence from evidenced absence. These practices do not settle disputes about the correct account of value, but they make the dependence of mathematical outputs on interpretive choices more accessible to criticism and revision.

10.5 Formal Identifiability and Empirical Operationalization

A proposed structural formalism becomes empirically informative when its objects can be related to an observation protocol and its mechanisms are stated so that they can be examined against evidence. A first operational study would define an observation boundary, a unit and interval of event recording, rules for assigning couplings and faces, and criteria for identifying candidate entities and value modalities. It would also distinguish observed events from inferred or simulated ones. The event-based ontology does not prescribe a unique empirical sampling unit: a transaction, a meeting, an episode of care, a learning activity, or a machine-state transition can each have a different defensible resolution. The selected resolution affects which features of the resulting complex can be compared.

An observational design should specify what the data allow the analyst to recover. For example, an institutional event log might document a completed transfer while omitting prior informal labor or the deadline after which a transfer ceases to be useful. A survey might record recognized trust while omitting the interactions through which that trust was formed. An ethnographic account may preserve temporal dependencies that are absent from administrative logs while supporting less precise event times. Combining records requires rules for aligning event descriptions, provenance, and uncertainty across modalities. It also requires attention to consent, confidentiality, and the consequences of classifying people's actions as economically or morally valuable.

Formal identifiability is distinct from a successful fit to a sample. Let $\theta \in \Theta_{\mathrm{dyn}}$ denote a parameterization of event-transition, coupling, and recognition rules, and let $\mathcal{P}_\theta^{\mathrm{obs}}$ be the resulting probability law over a clearly specified set of observables, whenever such a stochastic model has been justified. A global identifiability criterion, relative to a declared equivalence $\sim$, is stated in Equation 51.

$$\mathcal{P}_{\theta_1}^{\mathrm{obs}} = \mathcal{P}_{\theta_2}^{\mathrm{obs}} \quad \Longrightarrow \quad \theta_1 \sim \theta_2. \tag{51}$$

Equation 51 asks whether observationally indistinguishable parameterizations can differ in a way relevant to the claims of the model. A deterministic design can formulate the corresponding condition using an observation map rather than a probability law. Distinct support-selection or transformation mechanisms may generate the same cell counts, similar homology, or identical final allocations. Without additional observations or assumptions, such mechanisms may be observationally equivalent. A goodness-of-fit statistic, including an exact match to synthetic outputs, does not resolve that ambiguity.

A feasible validation sequence would begin with transparent descriptive reconstruction of a bounded case; assess sensitivity to alternative resolutions and face semantics; test local hypotheses about event order and the availability of particular conversions; and evaluate whether the model predicts held-out historical features or distinguishes plausible counterfactual mechanisms. Predictive success, when observed, would support the tested operational model within its domain, not prove that value is fundamentally topological. Normative evaluation requires an additional protocol for determining whose interests

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count, which harms or capabilities are relevant, and which comparisons are ethically defensible. In high-stakes settings, the analysis should preserve the ability to contest classifications and correct source records.

The results of Section 8 constitute a computational consistency and illustration exercise. Their boundary-rank checks, controlled order changes, and stochastic sensitivity calculation show that the specified algorithms behave as described for the constructed finite complexes. The generated samples originate from declared rules rather than social observations. External validity, parameter estimation, and causal attribution remain open. A future empirical pilot should disclose this separation in its methods, code, data dictionary, and reporting of uncertainty.

10.6 Extensions to Higher-Dimensional and Dynamical Complexes

A two-dimensional complex is sufficient to express event vertices, coupling edges, primitive faces, and first- and second-dimensional cellular information. Some relational phenomena may call for higher-dimensional cells, additional coefficient structures, or complexes that change with the modeled history. Extensions should be selected in response to an explicit representational task, since additional dimensions create both expressive possibilities and new identification obligations.

A three-dimensional cellular extension would introduce $C_3(K)$ and a boundary map $\partial_3 : C_3(K) \to C_2(K)$ satisfying $\partial_2 \partial_3 = 0$. Three-cells could, under a chosen semantics, record relations among primitive faces or coherence conditions among different local configurations. They would alter the interpretation of H_2, which would then quotient closed two-chains by boundaries of three-cells. The dimensional increase therefore changes which closed face structures are counted as homologically nontrivial. It does not automatically capture more social reality: a new cell requires evidence or an explicit formal rule that gives its boundary a substantive interpretation. The same caution applies to simplicial, cubical, and more general CW representations.

A dynamically evolving complex requires a second distinction. An *accumulating record* may add previously occurred events and their documented relations, while an *active configuration* may gain and lose couplings and supports. The former can sometimes be represented as a filtration of included subcomplexes; the latter may need non-monotone correspondences or a sequence of complexes linked through common refinements. Persistent homology is useful for filtrations, provided the filtration parameter is defined and its relationship to event time, measurement resolution, or a threshold is declared [9]. For changing active relations, a suitable non-monotone persistence construction may be considered, but its maps and interpretation would need to be specified separately. A value's apparent disappearance under a different recognition protocol must likewise be distinguished from disappearance caused by historical events.

Other mathematical directions include cellular sheaf or cosheaf coefficients for typed local data, categories of transformations for composition and compatibility, directed-topological methods for non-reversible execution histories, and multiscale effective models. Directed topology is relevant insofar as equivalence of execution paths is intended to respect an explicit direction of permissible change [10]. For heterogeneous value, a coefficient system could record type-dependent local information, while non-linear and history-sensitive value constitution may require additional maps beyond ordinary linear chain complexes. An extended construction would need to state whether it represents a physical mechanism, a data schema, a normative interface, or a formal analogy; these roles may coexist but do not follow from one another.

The most useful criterion for an extension is whether it resolves a concrete limitation of the current model. Higher-dimensional coherence may improve the expression of interacting primitive relations; enriched coefficients may improve the representation of heterogeneous modalities; directed history complexes may preserve causal order; and empirically constrained transition rules may clarify the mechanisms by which value-supporting configurations evolve. Each extension also changes the assumptions needed for an interpretation. The present two-complex therefore serves as a revisable starting point whose chief result is an explicit separation of relational structure, historically constituted value, admis-

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sible circulation history, and justice-sensitive assessment. Subsequent work can strengthen the links among these levels through formal derivation, reproducible modeling, and context-specific evidence.

11 Conclusion

The present study has developed a preliminary topological formalism for representing heterogeneous value circulation in a historically constituted relational spacetime. The inquiry began from a representational difficulty: socially consequential values often develop across extended sequences of activity, and their circulation can depend on the timing, context, and transformation of those activities. A recorded transfer may be localized in time, while the capacities enabling it, the obligations accompanying it, and the possibilities it generates have longer histories. A formal treatment of circulation order consequently requires a description of relational structure and historical processes that can retain these distinctions. The model presented here is a proposal for such a description; its intended contribution is an explicit organization of formal objects, their interpretive roles, and the conditions under which comparisons may be meaningful.

The first component is an oriented two-dimensional event–coupling–primitive complex. Vertices represent event occurrences, edges represent couplings among events, and faces provide higher-order primitive structures whose specific semantics are supplied by a declared modeling protocol. The complex accommodates diverse events, including occurrences that have no direct exchange or allocation interpretation. Historically emergent entities are identified, where possible, through relations to subcomplexes and recorded trajectories. Section 4 distinguishes this ambient structure from selected historical supports, active configurations, and the additional causal order needed to construct feasible event histories. These distinctions allow historical record and present activity to be represented separately, while retaining the possibility that changes in coupling, observation, or interpretation affect subsequent identifications.

The second component concerns the constitution and morphology of heterogeneous values. Section 5 treats candidate values as potentially supported by extended, overlapping, and modality-dependent relational subcomplexes. A constituent face can carry a primitive encoding without representing a complete value. Compact, branching, and holed supports exhibit differences in cellular organization; further descriptors record structural extent, temporal duration, geometric weighting, and scale-dependent features. These descriptors are informative only in relation to the selected support and its constitution protocol. The magnitude, social significance, or ethical standing of a value requires criteria beyond its cell count, area, Betti numbers, or apparent geometric complexity. The proposed multiscale constructions also allow recursive organization to be studied without presupposing exact fractal structure as a general property of value.

The third component is a history-sensitive account of generative dynamics. Event transitions and support-update correspondences in Section 6 describe candidate changes in locally coupled configurations, including transmission, conversion, combination, fragmentation, dissipation, and regeneration. Circulation may therefore be represented through the changing supports and capacities associated with successive events, as well as through conventional transfer records. Feasible histories additionally depend on causal precedence and declared resource or transformation constraints. The possibility that an earlier event changes the set of feasible later events is central to the longer-term inquiry into generativity: circulation order may modify the conditions of further value formation. The manuscript proposes mathematical interfaces for expressing that possibility while leaving the selection and calibration of transition laws to subsequent formal or empirical work.

The fourth component separates topological reduction from historical and normative comparison. The cellular chain complex provides boundary operators and homology groups for the ambient complex and its value-supporting subcomplexes. A distinct, derived history-comparison complex can represent admissible transitions between decorated configurations and possible local interchanges of event order. The difference between two paths in that derived complex may be a boundary of an explicitly attached face, while their interim costs or consequences remain unequal. Likewise, a quotient identifies histories

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for a particular analytical purpose only when the properties to be evaluated are preserved by its equivalence relation. Discrete Morse simplification, homological invariants, and coarse-graining accordingly serve as conditional reduction tools. Their suitability depends on which causal, modal, and historical information the analysis must retain. A topological hole, in particular, specifies a feature of a chosen complex and coefficient system; its meaning for value circulation depends on the primitive and support semantics established in the model.

The computational exercises in Section 8 make several of these distinctions explicit. Finite cubical examples exhibit compact, branching, and ring-shaped supports and verify their cellular boundary and homology calculations. An illustrative filtration produces a one-dimensional class that appears when an outer ring closes and disappears after its center is filled. A separate pair of allocation orders yields the same completed deliveries with weighted waiting costs of 21 and 12 under stipulated assumptions. Additional synthetic rules link circulation order to different final support topologies and explore sensitivity to coupling and recognition decisions. These calculations demonstrate the internal behavior of the specified examples and the distinguishability of selected structural and temporal observables. Their numerical outcomes do not establish empirical regularities in social value formation, causal effects in observed institutions, or ethically justified allocation priorities.

The normative extension in Section 9 treats circulation-order justice as an assessment of admissible histories under explicitly declared ethical interpretations. Its objects include historically identified claim bearers, modality-specific needs, temporal deprivation, obligations, conversion power, and the future opportunities made reachable by different trajectories. Under such a construction, technically feasible histories can be distinguished from normatively admissible ones, and justice comparisons can remain partial when relevant values or claims are incompletely commensurable. Endpoint equivalence, equality of certain topological descriptors, and equality under a justice assessment are therefore different relations. The model is designed to make possible consequences and evidential dependencies visible for ethical argument; it leaves the justification of rights, weights, standing, and permissible trade-offs to the relevant normative and institutional inquiry.

Several tasks follow from the limits examined in Section 10. A more complete theory would need defensible semantics for primitive faces and the coupling data attached to them; coherent procedures for identifying value- and entity-supporting subcomplexes across histories; mathematically specified transformation rules, including cases in which the complex itself evolves; and criteria for which historical distinctions may survive quotienting or coarse-graining. Empirical applications would further require documented event reconstruction, uncertainty in support selection and recognition, tests of model identifiability, and comparisons against alternative explanations. Normative applications would need to make explicit the ethical standing of affected entities and the reasons supporting assessments of urgency, obligation, loss, and future capability. These lines of work may be pursued through formal derivation, computational modeling, historical interpretation, and participatory or empirical investigation, according to the problem at hand.

The resulting formalism provides a common vocabulary for an inquiry that begins with historically distributed events and couplings, considers the relational constitution of heterogeneous values, and compares the consequences of different feasible circulation histories. Its central analytical proposal is that the morphology and trajectory of value-supporting structures may carry information about generation, circulation, and future possibilities that is absent from an endpoint-only allocation record. Whether those additional descriptions yield explanatory, predictive, or normative advantages remains an open and testable matter. The present contribution is the explicit framework and synthetic demonstrations from which those questions can be investigated with greater mathematical and interpretive precision.

A Notation and Mathematical Conventions

This appendix consolidates the mathematical symbols and interpretive conventions used in Sections 3–10. Its purpose is to make the domains of the formal operations explicit, distinguish objects that share a convenient diagrammatic presentation, and state the assumptions under which a topological expression

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Table 11: Cellular, homological, and coefficient conventions used in the manuscript.

Symbol	Mathematical role	Interpretive qualification
K, K_p	Ambient finite oriented 2-complex and its p-cells	K_0 events, K_1 couplings, K_2 primitive faces
$\Lambda = (\Lambda_0, \Lambda_1, \Lambda_2)$	Annotations or local encodings by cell dimension	Labels may be measured, inferred, posited, or interpreted
$C_p(K;\mathbb{F})$, ∂_p	Cellular chains and signed boundary operators	Orientation is algebraic, separate from causal order
$[f : e]$, B_p	Cellular incidence coefficient and boundary matrix	Matrix entries depend on chosen bases and field
Z_p, B_p^{bdry}, H_p	Cycles, boundaries, and homology groups	B_p^{bdry} distinguishes boundaries from boundary matrices B_p
β_p, χ	Betti numbers and Euler characteristic	Topological descriptors with no intrinsic normative ranking
δ^p, ω	Cochain coboundary and an optional p-cochain	The interpretation of a weighted evaluation is additional
$\mathrm{Sub}(K)$	Cellular subcomplexes of the ambient complex	Included cells are closed under cellular boundaries

has its usual algebraic meaning. The definitions and choices listed here summarize the working construction; they introduce neither an additional social ontology nor a new criterion of justice. In an application, the observation protocol, value modalities, historical resolution, and admissibility rules must still be supplied. Tables 11, 12, and 13 summarize the cellular, historical, and evaluative notation, respectively.

A.1 Cellular Structure, Coefficients, and Orientation

The baseline K is a finite, oriented, regular two-dimensional CW complex. The notation K_p denotes its set of p-dimensional cells, for $p = 0, 1, 2$. Vertices $v \in K_0$ are events; edges $e \in K_1$ are couplings; faces $f \in K_2$ are primitive two-dimensional cells. This is a proposed GR interpretation of a mathematical cell complex. The ontology does not equate an event with an emergent entity or require that an event involve a value transfer. The primitive semantics of faces are left model-dependent, as discussed in Subsection 4.2.

Unless another coefficient system is specified, $C_p(K;\mathbb{F})$ denotes the cellular p-chain space over a field $\mathbb{F}$, and ∂_p denotes its cellular boundary map. The corresponding space over $\mathbb{Z}$ is a free abelian chain group. An oriented p-cell supplies one basis generator, and reversing the chosen orientation reverses that generator's sign over characteristic other than two. Incidence numbers $[f : e]$ are integers whose signs depend on chosen orientations. The algebraic identity $\partial_1\partial_2 = 0$ is structural; its validity does not require the coupling orientation to agree with an event's causal precedence. Equation 3 introduces the chain complex, and Equation 13 specifies the face boundary.

A rank calculation over $\mathbb{F}_2$ (also denoted $\mathrm{GF}(2)$) discards orientation signs in its matrix representation because $+1 = -1$ in that field. Integer or real coefficients retain the sign conventions. The experiments of Section 8 use $\mathbb{F}_2$ for boundary ranks and Betti numbers, whereas the signed order-coboundary example of Subsection 7.6 uses real-valued cochains. These are separate calculations with different coefficient assumptions; a numerical equivalence between them is not asserted.

Table 11 gathers the principal cellular symbols. In a finite complex, $\beta_p(K;\mathbb{F})$ is the dimension of $H_p(K;\mathbb{F})$, and the Euler characteristic $\chi(K)$ can be obtained from cell counts or the alternating sum of Betti numbers. Betti numbers depend on the coefficient field in the possible presence of torsion, while the Euler characteristic does not. A homology class describes an equivalence class of cycles modulo boundaries of higher-dimensional cells, under the selected complex and coefficients.

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Table 12: Historical and derived-configuration notation with domain distinctions.

Symbol	Domain and function
$E, \prec, \preceq$	Modeled event set and strict/reflexive precedence relations
$\mathscr{J}(E, \prec), H$	Downward-closed event configurations and one selected historical configuration
$\mathscr{K}, H \mapsto K_H$	Record-realization rule associating H with a cellular subcomplex of K
A_H, τ	Active-cell selection and optional event-time annotations
$\mathscr{E}_\kappa(H)$	Set of candidate entity supports with identification data under protocol κ
$\Gamma_{\text{feas}}, \Gamma_n(x_0)$	Feasible histories and finite state-bearing trajectories under the declared transition system
D	Separate history-comparison complex; its vertices are execution configurations
$\mathsf{P}_r, \mathsf{C}_\alpha$	Partial projections from histories to cellular chains and to value-circulation traces
$L_\alpha \hookrightarrow K_H$	Inclusion of one value support in its recorded historical ambient complex

Some conventional mathematical notation is overloaded across disciplines. In this manuscript, B_p designates a matrix representation of ∂_p in calculations, while $\operatorname{im} \partial_{p+1}$ denotes the group or space of cellular boundaries. The latter may be written B_p^{bdry} where a distinction is needed. The cochain coboundary δ^p is dual to ∂_{p+1} with the usual pairing convention; in the real-valued calculation of Equation 36, it is used on a separately constructed history-comparison complex.

A.2 Events, Causal Histories, and Recorded Configurations

An event set $E \subseteq K_0$ carries a strict partial order $\prec$ when the model specifies causal or precedence dependencies. In the reflexive convention $v \preceq w$ means either $v = w$ or $v \prec w$. Two events can be incomparable because the model does not require an order; incomparability does not itself establish simultaneity or causal independence. A linear extension is a total ordering consistent with the given partial order. The choice of an algebraic orientation for an edge or face places no constraint on this event order.

The symbol H designates a finite downward-closed subset of E, and $\mathscr{J}(E, \prec)$ is the collection of such subsets. The recorded historical subcomplex $K_H = \mathscr{K}(H)$ is defined by a history-realization assignment, described in Equation 15. The intended conditions are cellular closure, $(K_H)_0 = H$, and monotonicity under enlargement of recorded histories. These conditions require a selection rule for couplings and primitive faces; they are not implied by the event subset alone. When the observational reconstruction is incomplete, the meaning of K_H must be qualified by the evidence and provenance attached to the assignment.

The selection A_H records operative or active cells at history H. Activity can decline or cease even as the recorded historical complex grows. An active-cell selection need not be a subcomplex unless closure under boundaries has been verified. The symbol τ denotes temporal annotations, such as timestamps or event intervals, when available; it is not identified with $\prec$ or an edge's orientation. The causal-past and accessibility operations appear in Equation 17.

Table 12 distinguishes the representations used for historical and circulation-order analysis. Its entries prevent the ambient event complex from being conflated with the derived execution or comparison diagram of Sections 3 and 7.

There are two uses of "history" which are related but mathematically distinct. A down-closed event configuration H describes what has occurred according to an event inventory. A trajectory $\gamma \in \Gamma_{\text{feas}}$ additionally records a permissible order and, where specified, intermediate states and event-transition data. Distinct trajectories may therefore have the same completed-event set. The projection P_r in Equation 32 requires a separately declared routing rule r to turn an event trajectory into a path on a selected cellular graph. A cellular 1-chain is a linear combination of oriented edges; it is not, by definition, a chronologically ordered event list.

In particular, D denotes a *derived* history-comparison complex whose vertices encode completed-event configurations rather than GR events themselves. A face of D may encode a stipulated permissible

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interchange between two event orders. Such an interchange face is different from a primitive face of K, even if both are drawn as polygons. The boundary identity of Equation 34 belongs to D. It supplies no general correspondence between the topology of D and that of the ambient K.

A.3 Value Modalities, Supports, and Recognition

A modality index α denotes a contextually specified kind of value. The index is a type designation, not automatically an element of a globally commensurable numerical scale. The selected support $L_\alpha \subseteq K_H$ is a candidate historical subcomplex contributing to the constitution of value of modality α. Supports of distinct modalities may overlap and may have different event spans, connectivity, or cell counts. Their identification requires selection criteria and cannot be recovered uniquely from homology alone.

The set $\mathscr{S}_{\alpha,\kappa_\alpha}(H)$ of Equation 18 lists admissible pairs of candidate supports and associated support data for a declared protocol κ_α. The constitution relation $\mathcal{G}_\alpha$ of Equation 19 maps admissible, decorated historical inputs into candidate value states $\mathfrak{V}_\alpha$. Its set-valued arrow allows undecided or multiple defensible identifications. The observer-relative recognition relation $\mathcal{R}_{\alpha,r}$, defined in Equation 23, concerns the available evidence and interpretation protocol of observer r. Constitution, recognition, and manifestation occupy different analytical roles; an observed manifestation may inform recognition and may itself add new events to a history.

The morphology descriptor $\mathfrak{M}_\alpha(L)$ in Equation 20 contains selected topological, geometric, and scale-dependent measurements. The optional extent descriptor $\mathrm{Ext}(L;\tau)$ in Equation 21 counts cells and records a temporal span under a specified time annotation. A weighted measure $\mu_\alpha(L)$ or an estimated dimension $D_\alpha(L)$ requires an independently declared metric, covering procedure, or measurement rule. A higher cell count or Betti number is not a definition of greater value magnitude. Where supports overlap, O_c denotes a selected, cell-count-based overlap index as in Equation 22.

A.4 Transformation Relations and Comparison of Histories

The symbol $\mathscr{X}_H$ denotes the admissible decorated states at a recorded historical configuration H. A local event-transition relation $\mathscr{R}_v^H \subseteq \mathscr{X}_H \times \mathscr{X}_{H'}$ links such states under a stated enabling and feasibility protocol. The support-update correspondence $\mathscr{U}_{\alpha,v}^H$ relates selected value supports across that event. A conversion relation $\mathcal{C}_{I,J}^\eta$ relates products of potentially distinct value-state spaces under contextual conditions η. The notation is relational because unique outputs, reversibility, linearity, and conservation are not assumed in the general proposal. Equation 29 describes a finite operational model with enabling, permission, and transition checks; the event-order composition in Equation 30 illustrates conditional dependence on historical order.

The reachable-state construction $\mathrm{Reach}_n(x;\mathscr{R})$ is conditional on a horizon and available transformations. It describes specified further possibilities rather than a universal magnitude of generativity. The justice-related notation of Section 9 is likewise indexed by a normative interpretation θ. The subset $\Gamma_\theta^{\mathrm{adm}}$ contains histories that pass the corresponding normative admissibility conditions; it may be strictly smaller than the technically feasible set Γ_{feas}. A component $D_{a\alpha}^\theta$ denotes a conditional temporal-deprivation measure for a designated claim-bearing position a and modality α, and $\mathsf{O}_\theta(\gamma;h)$ is an interpretation-indexed projection of permissible future opportunities. The notation a here is a position or claimant designation within an applied normative description, not a GR vertex.

Table 13 records the remaining operators. Their domains should be declared afresh whenever a particular empirical or normative application replaces these schematic constructions with measured or estimated quantities.

A symbol such as q, R, or r can denote a quotient, a topological simplification, or a map on histories in different portions of the manuscript. The applicable domain and codomain govern its interpretation. A cellular reduction that preserves homology need not preserve event timestamps, interpreted needs, or the evaluative order of histories. The factorization criteria in Proposition 3.1 and Equation 49 express exact preservation of a specified assessment; Equation 50 instead addresses a bounded approximation for a declared real-valued loss. Neither condition follows from homology preservation alone.

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Table 13: Transformation, reduction, and normative-assessment notation.

Symbol	Specified use
$\mathscr{X}_H, \mathscr{R}_v^H$	Historical state space and event-triggered transition relation
$\mathscr{W}_{\alpha,v}^H, \mathcal{C}_{I,J}^{\eta}$	Value-support update and context-dependent conversion relation
Reach_n	States reachable under explicitly specified transition rules and horizon
$\Gamma_\theta^{\mathrm{adm}}$	Histories passing the normative admissibility criterion associated with θ
$D_{a\alpha}^{\theta}, \mathsf{O}_\theta$	Conditional deprivation integral and permissible opportunity projection
$q_\theta, \succeq_\theta, \succeq^{\mathrm{rob}}$	Normative profile, comparison relation, and robust comparison across interpretations
R, r, q	Context-dependent reduction or quotient map; domain and target specified locally
$\overline{J}, \overline{\mathcal{J}}_\theta$	Assessments on a reduced space when factorization is justified

A.5 Scope of Mathematical and Normative Claims

Four claims should remain separate during formal interpretation. First, a statement about a cellular boundary, homology group, or chain map is a theorem or computation under explicit algebraic assumptions. Second, labeling a selected subcomplex as an entity or value support is a model-mediated identification under a stated historical and interpretive protocol. Third, assigning feasible event transitions or identifying a causal effect requires specified mechanisms and, for empirical claims, evidence. Fourth, classifying or comparing histories as just requires declared normative standing and criteria. The model is designed to relate these operations, while retaining their distinct conditions of validity.

Expressions such as "hole," "circulation," "face state," and "generation" can otherwise appear to carry more than one meaning. Throughout the manuscript, a hole means a topological feature of a specified complex with specified coefficients and attaching cells. A circulation trace is a selected representation of a feasible history or of changes in value-supporting configurations. A face state is local encoding data whose social semantics are additional. Generation refers to changes in constituted values or reachable capabilities under a proposed transformation model. None of these definitions supplies an empirical mechanism or normative verdict without the appropriate connecting assumptions. The planned subsequent appendices will address explicit boundary calculations, historical construction protocols, reproducibility specifications, and possible extensions under these conventions.

B Cellular Boundary Matrices and Homology Calculations

This appendix supplies exact chain-level calculations for the finite examples used to motivate the topological formalism. Its objective is to make the choice of cells, orientations, coefficient fields, and attaching data reproducible, and to distinguish the ambient relational complex from a selected value-supporting subcomplex and a separate history-comparison complex. The calculations comprise a triangular cycle, a ring included in a filled cubical complex, a closed two-dimensional shell, and a diamond of event-order configurations. They establish mathematical relations among boundary ranks and homology groups. Interpreting a selected subcomplex as a constituted value, or evaluating an order as just, remains a separate model-dependent operation. The computations are independently reproduced by the source-package script `experiments/run_appendix_b_homology.py`.

B.1 Cellular Bases and Rank Conventions

Let K be a finite regular two-complex, with its cells ordered in a declared basis and with an algebraic orientation chosen for each positive-dimensional cell. Over a coefficient field $\mathbb{F}$, the corresponding boundary matrices are $B_1 \in \mathbb{F}^{n_0 \times n_1}$ and $B_2 \in \mathbb{F}^{n_1 \times n_2}$, where $n_p = |K_p|$. These matrices represent the boundary maps introduced in Equation 3. A zero-column matrix represents an absent group of two-cells.

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Table 14: Exact cellular counts, rational boundary ranks, and Betti numbers for the appendix examples. Identical Betti numbers were obtained over $\mathbb{F}_2$ by an independent elimination routine.

Complex	n_0	n_1	n_2	r_1	r_2	β_0	β_1	β_2
Unfilled triangle	3	3	0	2	0	1	1	0
Filled triangle	3	3	1	2	1	1	0	0
Eight-face ring	16	24	8	15	8	1	1	0
Nine-face filling	16	24	9	15	9	1	0	0
Tetrahedral shell	4	6	4	3	3	1	0	1
Diamond without face	4	4	0	3	0	1	1	0
Diamond with face	4	4	1	3	1	1	0	0

Equation 52 gives the Betti numbers in terms of exact matrix ranks.

$$\begin{aligned} \beta_0(K;\mathbb{F}) &= n_0 - \operatorname{rank}_{\mathbb{F}}(B_1), \\ \beta_1(K;\mathbb{F}) &= n_1 - \operatorname{rank}_{\mathbb{F}}(B_1) - \operatorname{rank}_{\mathbb{F}}(B_2), \\ \beta_2(K;\mathbb{F}) &= n_2 - \operatorname{rank}_{\mathbb{F}}(B_2). \end{aligned} \tag{52}$$

Equation 52 follows from rank–nullity and the definition of homology in Equation 6; it assumes that the complex has no three-dimensional cells. The matrix identity $B_1 B_2 = 0$ is checked exactly for each example. We use $\mathbb{Q}$ for signed incidence and exact rank calculations and repeat the ranks over $\mathbb{F}_2$. For these particular complexes the resulting Betti numbers agree. Such agreement is an outcome of the selected examples and does not establish field-independent homology for arbitrary complexes. The algebraic orientations are distinct from event precedence and from the direction of a value transformation.

B.2 Triangular Cycle and Primitive-Face Attachment

Take three event vertices v_0, v_1, v_2 and coupling edges e_{01}, e_{12}, e_{20}, oriented along the indicated ordered pairs. The three-edge loop has the first boundary matrix stated in Equation 4. With no primitive face, $C_2 = 0$; consequently rank $B_1 = 2$ and rank $B_2 = 0$. The generator $e_{01} + e_{12} + e_{20}$ lies in $\ker B_1$ and represents a nonzero first-homology class. Equation 53 records the Betti vectors of the two complexes.

$$\begin{aligned} (\beta_0, \beta_1, \beta_2)_{\text{unfilled}} &= (1, 1, 0), \\ (\beta_0, \beta_1, \beta_2)_{\text{filled}} &= (1, 0, 0). \end{aligned} \tag{53}$$

For the filled triangle, attach a primitive face f whose oriented boundary is $e_{01} + e_{12} + e_{20}$. The resulting column $B_2 = (1, 1, 1)^\top$ spans $\ker B_1$, and the cycle is a cellular boundary. The attachment does not remove an edge or a vertex; it changes the available two-dimensional filling data. This is an elementary example of how face selection affects the topology of a value-supporting candidate. No inference about value magnitude, circulation feasibility, or an unmet obligation follows from Equation 53 alone.

B.3 Cubical Ring, Filled Complex, and Induced Homology

Consider the regular cubical cell structure of a 3×3 block of unit squares. Let K_{disk} contain all nine square faces and their boundary cells, and let $L_{\text{ring}} \subset K_{\text{disk}}$ contain the eight outer faces and all their incident edges and vertices. Both have the same 16 vertices and 24 edges; the inner four edges remain in the ring because they bound the surrounding faces. An orientation is fixed by ordering vertices by planar coordinates, orienting edges from the lexicographically smaller to the larger endpoint, and orienting each square counterclockwise. The exact counts and ranks are collected in Table 14.

The ring has rank $B_1 = 15$ and rank $B_2 = 8$. Its nine-dimensional cycle space thus contains one class outside the eight-dimensional boundary space. Attaching the ninth face raises rank B_2 to nine, so the

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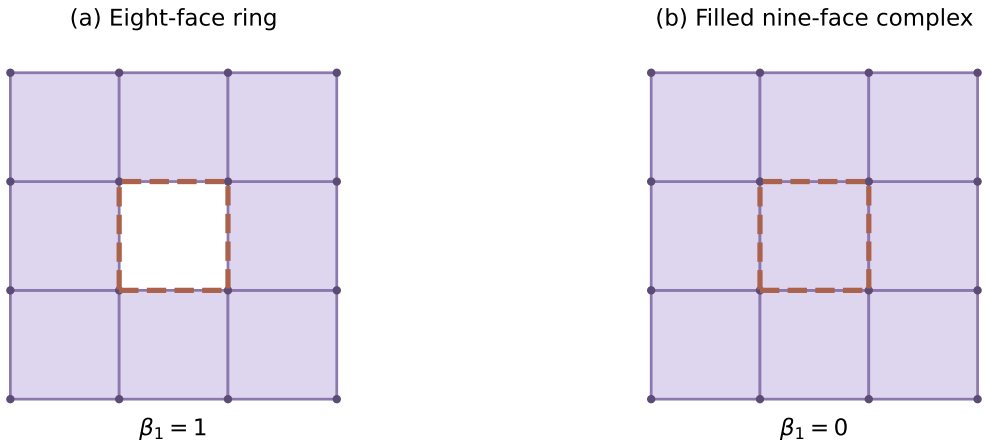

Figure 14: An eight-face cubical subcomplex included in a nine-face filled complex. Both retain the same one-skeleton. The dashed inner cycle is nontrivial in the first homology of the eight-face ring and is the boundary of the additional central face in the filled complex. Vertex dots denote abstract GR events in this illustrative incidence structure; no empirical circulation or value magnitude is encoded.

filled complex has trivial first homology. Figure 14 exhibits the shared one-skeleton and the additional filling explicitly. This comparison also verifies Euler's identity: the ring has $\chi = 16 - 24 + 8 = 0$, whereas the filled complex has $\chi = 16 - 24 + 9 = 1$.

The inclusion $i : L_{\text{ring}} \hookrightarrow K_{\text{disk}}$ induces a homomorphism on first homology. Equation 54 records this map.

$$i_* : H_1(L_{\text{ring}}; \mathbb{Q}) \cong \mathbb{Q} \longrightarrow H_1(K_{\text{disk}}; \mathbb{Q}) = 0. \tag{54}$$

The generator represented by the inner four-edge cycle is carried to zero by Equation 54, because that cycle is the boundary of the additional central face in the ambient filled complex. Thus a value-supporting candidate subcomplex may have a nontrivial homology class that becomes trivial in the larger recorded spacetime. This is a statement about the inclusion and its selected cells. Whether the added face participates in the constitution or recognition of a particular value modality requires an explicit constitution or observation protocol, as specified in Section 5.

B.4 Closed Two-Dimensional Shell and Second Homology

A closed collection of faces can support nontrivial second homology even in a strictly two-dimensional complex. Consider the boundary of a tetrahedron, with vertices $0, 1, 2, 3$, edges $01, 02, 03, 12, 13, 23$, and oriented triangular faces $012, 013, 023, 123$, with orientations induced by increasing vertex order. The six-by-four matrix B_2 is displayed in Equation 55.

$$B_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad B_2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0. \tag{55}$$

The first boundary matrix has rank three, and B_2 also has rank three. Equation 52 yields $(\beta_0, \beta_1, \beta_2) = (1, 0, 1)$, with Euler characteristic two. The signed sum of the four faces in Equation 55 is a two-cycle. Because the complex contains no three-cells, this nonzero two-cycle cannot be a cellular boundary of a three-chain within the specified complex. This example distinguishes second-dimensional closed face configurations from first-dimensional holes. It does not require identifying the

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collection with a specific economic value, and it does not treat an abstract cell complex as a physical spacetime realization.

B.5 Derived Event-Order Diamond and Coboundary Calculation

The history-order diagram in Section 7 is a different cellular construction. Its vertices S_0, S_b, S_c, S_{bc} are completed-event configurations, not primitive GR events. Give the diagram the ordered edges d_{0b}, d_{bF}, d_{0c}, and d_{cF}, each oriented toward the next configuration. In these bases, Equation 56 gives its matrices when an interchange face is included.

$$B_1^D = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad B_2^D = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad B_1^D B_2^D = 0. \tag{56}$$

The two paths $d_{0b} + d_{bF}$ and $d_{0c} + d_{cF}$ have identical endpoint boundaries. Their difference is the column of B_2^D. Without an interchange face, the diamond has $\beta_1 = 1$; with one, $\beta_1 = 0$. Attaching this face expresses a selected relation among possible executions. It adds no primitive face to the ambient GR spacetime K unless a separate modeling rule expressly establishes such a correspondence.

For completeness, the weighted waiting-cost example of Subsection 7.6 can be checked at the cochain level. Let the costs assigned to the four directed execution steps be the real-valued one-cochain $\omega = (11, 10, 11, 1)$ in the edge order of Equation 56. These values represent the specified cumulative interval costs when the first waiting period involves both recipients, and the second involves only the recipient still waiting. The two path totals are 21 and 12. Equation 57 verifies their difference by the dual boundary pairing.

$$\langle \delta\omega, f_D \rangle = \langle \omega, \partial_2 f_D \rangle = (11 + 10) - (11 + 1) = 9. \tag{57}$$

The comparison in Equation 57 is over real-valued cochains; the preceding $\mathbb{F}_2$ ranks do not encode signed or weighted harm. A nonzero coboundary establishes the path sensitivity of the *selected additive evaluation*, not a general moral ranking of the two histories. The evaluation depends on the imposed urgency weights, feasible steps, and time discretization. The derived interchange face and the primitive value-bearing faces of K remain semantically distinct.

B.6 Verification Protocol and Interpretive Boundaries

The supplied script constructs boundary matrices from declared oriented simplicial and cubical incidences, checks $B_1 B_2 = 0$ exactly, computes ranks over $\mathbb{Q}$ using exact symbolic arithmetic, and recomputes ranks over $\mathbb{F}_2$ using binary elimination. It verifies the Euler identities, the displayed tetrahedral two-cycle, the ring's inner cycle and its filling in the ambient complex, and the numerical coboundary in Equation 57. Its machine-readable outputs are `experiments/appendix_b_results.json` and `experiments/appendix_b_results.csv`. Each case is a deterministic constructed example rather than an empirical observation; no random seed or statistical uncertainty is involved.

The calculations support three delimited uses in the main text. First, cellular attachment data can distinguish an unfilled edge cycle from a filled one with the same one-skeleton. Second, the homology of a selected support can differ from that of its ambient recorded complex, and this distinction is expressed by an induced map under inclusion. Third, homology on a derived execution complex and real-valued cochain evaluations can describe selected order relations without identifying those relations with ambient primitive faces. Identifying a value modality, determining its economic significance, establishing historical causation, and comparing justice-sensitive consequences require additional rules and evidence beyond the calculations of this appendix.

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C Formal Constructions of Historical Subcomplexes

This appendix gives explicit finite constructions for the history-realization assignment introduced in Equation 15. Its role is to connect the event-based ontology of Section 4 to the value-support and dynamical operations of Sections 5 and 6. The constructions specify how completed-event configurations, recorded primitive cells, active couplings, and candidate supports can be represented without reducing an entity or a value to an event vertex. They also establish conditions under which accumulation of historical records defines an inclusion filtration. The example and the companion calculation are synthetic; their semantic annotations illustrate choices made by a protocol and are not empirical classifications. Figure 15 and Table 15 provide the concrete reference structure.

C.1 Event Configurations and Admissible Historical Prefixes

Fix the modeled event inventory $E = K_0$ of an ambient finite regular two-dimensional CW complex K, and a strict partial order $\prec$ on E. The order represents stated precedence dependencies, rather than the orientations chosen for the cellular edges. A *completed-event configuration* H is an order ideal: whenever $v \in H$ and $w \prec v$, the predecessor w also lies in H. Write $\mathscr{I}(E, \prec)$ for the set of such ideals, including the empty configuration. The topology of K places no independent restriction on the order ideals beyond the compatibility rules explicitly introduced by the model.

A finite event execution is a sequence $\gamma = (v_1, \dots, v_n)$ of distinct events whose prefixes $H_i = \{v_1, \dots, v_i\}$ are ideals. Equivalently, each newly appended event is enabled by the declared order. This definition permits several executions with the same terminal set H_n, because concurrent or incomparable events may be ordered differently. A more restrictive feasible execution must also satisfy the state and permission checks in Equation 29. Hence an admissible prefix in the order-theoretic sense is a necessary structural input, rather than a declaration of technical or normative feasibility.

For a fixed event inventory, a history prefix retains the events that have occurred, whereas the evidence available to an observer may arrive later. It is useful to distinguish an *occurrence-indexed* record from an *observation-indexed* reconstruction: a source may subsequently document a coupling involving earlier events, and a researcher's revised interpretation may assign a different primitive-face structure to the same reported episode. The construction below operates on one explicitly selected version of that reconstruction. A change of version can be compared by another map, without being mistaken for an event that occurred at the time of revision.

C.2 Cell Prerequisites and the Historical Realization Map

A history-realization map needs a rule for including edges and faces after the relevant events have occurred. The following finite construction supplies such a rule. For each cell $\sigma \in K$, assign a *prerequisite set* $W_\sigma \subseteq E$; this set specifies which completed events suffice for the selected recording protocol to include σ. We require $W_v = \{v\}$ for each event vertex and $W_\tau \subseteq W_\sigma$ whenever τ is a cell in the closure of σ. In particular, the endpoint events of an edge, and all vertices in a face boundary, belong to the relevant cell's prerequisite set. The sets may contain additional events associated with documentary confirmation or a chosen reconstruction threshold. These prerequisites are formal recording conditions and should not automatically be read as causal mechanisms.

Definition C.1 (Prerequisite-based history realization)**.** For an order ideal $H \in \mathscr{I}(E, \prec)$, the recorded historical complex K_H consists of exactly those cells whose declared prerequisites have occurred. Equation 58 specifies the assignment.

$$K_H = \{\sigma \in K : W_\sigma \subseteq H\}, \qquad \mathscr{K}_W : H \longmapsto K_H. \tag{58}$$

The requirement that all faces of an included cell have prerequisites contained in those of the cell is essential. Merely selecting all cells with incident vertices in H is another possible rule, but it makes an implicit decision that all incident couplings and faces are already documented when their vertices are

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present. Equation 58 makes that decision inspectable and allows a primitive face to be recorded only after an additional modeled event.

Proposition C.2 (Closure and accumulation of prerequisite histories)**.** *Under the stated prerequisite conditions, K_H is a subcomplex of K with zero-skeleton H for every order ideal H. Furthermore, $H \subseteq H'$ implies $K_H \subseteq K_{H'}$.*

Proof. If $\sigma \in K_H$ and τ belongs to its cellular closure, the prerequisite condition gives $W_\tau \subseteq W_\sigma \subseteq H$. Thus $\tau \in K_H$, proving closure. A vertex v lies in K_H exactly when $W_v = \{v\} \subseteq H$, so $(K_H)_0 = H$. Finally, $W_\sigma \subseteq H \subseteq H'$ implies $\sigma \in K_{H'}$. The claim follows. □

Proposition C.2 gives a sufficient condition for the monotone history-realization assumption in Section 4.4. The result depends on a fixed event inventory, ambient complex, and prerequisite protocol. If the observer revises any of these, two successive reconstructed records may cease to be nested. Such a revision is better represented as an explicit relation between recorded versions than silently incorporated into an apparent historical filtration.

C.3 Active Relations and Versioned Observation

The accumulated recorded complex can retain a coupling even after it stops operating. For each completed-event configuration H and observation version ν, let $a_{H,\nu} : K_H \to \mathscr{A}_0$ be an activity annotation, with a declared range such as `inactive`, `active`, and `undetermined`. The active-cell selection $A_{H,\nu}$ then consists of the cells assigned an active state under this protocol. It may fail to be a CW subcomplex: an active edge can have an inactive endpoint under a particular operational convention, and a face's activity need not follow from the statuses of all its boundary couplings. Calculations requiring a subcomplex must therefore use an explicitly constructed boundary-closed selection, and document which additional cells closure introduces.

Documentation also has an epistemic time axis. A revised source record can change the inferred prerequisites, primitive labels, or status of an earlier coupling. Let ν index a reconstruction version; $K_H^{(\nu)}$ denotes the subcomplex produced by that version's admissible cell prerequisites. This formal distinction supports audit trails and competing interpretations. In a historical report, a newly discovered document may change $K_H^{(\nu)}$ even when the modeled completed-event set H stays fixed. The revised topology is then a change in representation, with its possible implications for claims about the underlying history requiring independent assessment.

A minimal provenance record can attach the event inventory and its resolution, source references, the meaning of each coupling, face-annotation conventions, prerequisite sets, observation timestamps, protocol versions, and reasons for subsequent modifications. These data cannot eliminate interpretive bias, but they allow the transformation from documentary materials to a cellular representation to be inspected and contested. Such traceability is particularly important when an institutional observer has authority to recognize or omit the historical activities of another participant.

C.4 Entity and Value-Support Selection

Recorded historical cells provide candidate support for higher-level identifications. They do not automatically constitute either entities or values. Given H and a declared entity protocol κ, the previously defined relation $\mathscr{E}_\kappa(H)$ in Equation 16 selects pairs (L, ξ) with $L \in \mathrm{Sub}(K_H)$ and identity data ξ. A modality-specific value protocol κ_α analogously selects candidate pairs (L_α, η_α) from the space $\mathscr{S}_{\alpha,\kappa_\alpha}(H)$ in Equation 18. The identification of an entity and the constitution of a value can use overlapping cells while imposing different continuity, recognition, and validation requirements.

There are two useful general strategies for constructing these supports. In a *generating-cell protocol*, a specified seed set $S \subseteq K_H$ generates its smallest boundary-closed subcomplex $\mathrm{Cl}_K(S)$. In a *constraint protocol*, a candidate subcomplex is admitted only if it satisfies a declared predicate incorporating relevant labels, provenance, temporal conditions, or participation in specified interactions. Equation 59

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Table 15: Construction layers for an event-based historical reconstruction. Each layer requires a separately declared rule and preserves a different type of information.

Layer	Construction rule	Information preserved or required
Completed events H	Order ideal in $(E, \prec)$	Modeled occurrence and precedence; not a value-transfer ledger
Recorded subcomplex K_H	Cell prerequisites and boundary closure	Historical incidence and primitive attachments; depends on observation protocol
Active relations $A_{H,\nu}$	Time- and version-indexed activity annotations	Currently operative statuses, potentially non-closed as a cell set
Entity support L_E	Identity and continuity protocol κ	Candidate historical individuation, with ambiguity and provenance
Value support L_α	Modality-specific protocol κ_α	Candidate value constitution, potentially overlapping with other supports
History and circulation trace	Ordered transitions and trace protocol π_α	Timing, intermediate states, transmission claims, and their evidential basis

expresses the second strategy schematically, with $\mathrm{Sub}(K_H)$ denoting the family of subcomplexes.

$$\mathscr{L}_\kappa(H) = \{L \in \mathrm{Sub}(K_H) : P_\kappa(L, H, \Lambda) = 1\}. \tag{59}$$

The predicate P_κ in Equation 59 is a declared model choice, not an empirically established classifier. For example, a historical entity protocol might depend on evidence of continuity in practice or legal recognition; a knowledge-value protocol might depend on documentation of an acquired capability; a care-value protocol might include the timing of responsiveness to an identified need. These are distinct applied criteria. A subcomplex with compact geometry or a nontrivial homology class does not, by those features alone, satisfy any particular constitution protocol.

History-dependent supports also need not be nested even though K_H is nested. As further events become available, a protocol may withdraw an earlier interpretation, divide a candidate entity into several continuities, or associate a value with a different configuration. A support correspondence can record which earlier selected structures continue, overlap, split, or merge. An equality test of supports is therefore distinct from an identity-continuity judgment. The latter can carry evidential uncertainty, and multiple candidates can remain defensible under the available records.

Table 15 separates the operations involved in a practical reconstruction. In a single application, the underlying cell inventory may remain fixed while both the active relation mask and the selected value support change as historical observations accumulate.

C.5 A Four-Event Historical Construction

Consider four abstract events $E = \{a, b, c, d\}$ with precedence relations $a \prec d$, $b \prec d$, and $c \prec d$, and no precedence relation among a, b, c. Define a two-dimensional simplicial complex with five edges ab, ac, bc, bd, and cd, and two triangular primitive faces abc and bcd. The edge bc is shared. Assign to each edge the prerequisite set of its endpoints; assign $W_{abc} = W_{bcd} = \{a, b, c, d\}$, so the primitive faces are recorded only after event d. These choices meet the closure conditions because each face prerequisite set contains those of its boundary edges.

Choose the ideals $H_1 = \{a, b\}$, $H_2 = \{a, b, c\}$, and $H_3 = E$. By Equation 58, K_{H_1} is one edge with its endpoints; K_{H_2} is an unfilled triangular one-skeleton; K_{H_3} contains both triangles and all five edges. The Betti numbers, computed over $\mathbb{F}_2$, are summarized by Equation 60.

$$\begin{aligned}(\beta_0, \beta_1, \beta_2)(K_{H_1}) &= (1, 0, 0),\\ (\beta_0, \beta_1, \beta_2)(K_{H_2}) &= (1, 1, 0),\\ (\beta_0, \beta_1, \beta_2)(K_{H_3}) &= (1, 0, 0).\end{aligned} \tag{60}$$

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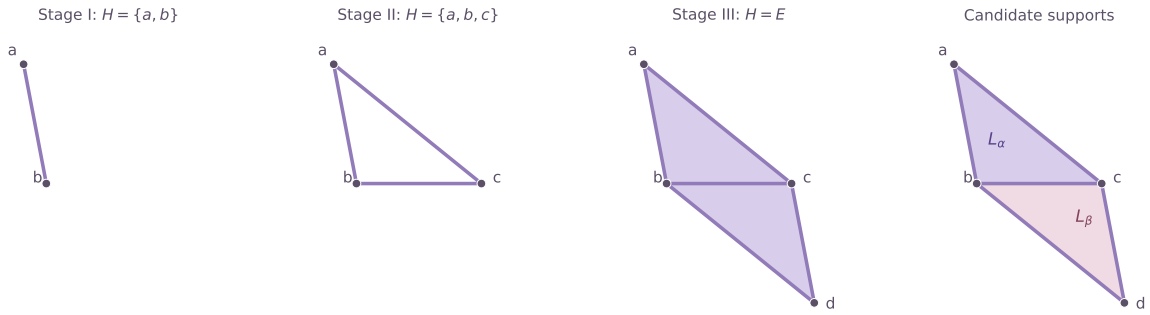

Figure 15: Prerequisite-based construction of recorded historical complexes at three causally admissible event configurations, followed by two illustrative overlapping modality supports in the final complex. Vertices denote events, edges couplings, and filled triangles primitive faces. The shaded regions in the last panel are protocol-dependent candidate value supports sharing an edge. The apparent geometric positions are diagrammatic, and the labels α, β do not establish observed social values.

The temporary first-homology class in Equation 60 is a feature of the chosen recording filtration. Its disappearance after d reflects the inclusion of a primitive face filling the abc cycle. It does not establish that event d created an ethically significant value or repaired a socially meaningful deficiency. The additional bcd face is attached to a second triangle sharing one edge with the first, so the final complex has the homotopy type of a disk.

Figure 15 shows the recorded support at each stage and, in a fourth panel, two candidate modality supports. For illustration only, let L_α be the closed triangular support abc and L_β the closed triangular support bcd. They share the edge bc and its endpoints, while their face sets differ. Assigning them names such as *knowledge-related* and *care-related* would require a genuine constitution protocol and observations in a concrete setting; the diagram records only the combinatorial possibility of distinct overlapping value supports. An activity annotation may simultaneously mark ab inactive at H_3 while keeping it within K_{H_3} as part of the recorded history.

The two incomparable events a and b may occur in either order while leading to the same ideal $\{a, b\}$. Once event c has also occurred, the recorded subcomplex is the same under both orders for the present history-realization map. A state-dependent model can nevertheless assign different interim costs or different subsequent capacities to those executions. This example therefore exhibits a limitation of K_H as a complete description of ordered dynamics: the occurrence set and its associated recorded incidence do not retain the entire sequence or intermediate value-related state. The operational histories of Section 7 supply that additional information.

C.6 Operational Histories, Validation, and Identifiability

A practical implementation can build the historical representation in two passes. First, validate the ambient incidence data and verify the declared event precedence and prerequisite conditions. Second, traverse one enabled event at a time, update the completed-event ideal, and include cells whose prerequisite sets are now contained in that ideal. Additional layers assign active-status annotations, identify candidate entity and value supports, and apply event-transition and circulation-trace protocols. Such a workflow preserves the distinction between a completed event, a recorded coupling or face, and a subsequently recognized value. It also provides a direct consistency check of the history-realization assignment.

The companion Python script `experiments/run_appendix_c_histories.py` constructs the four-event case, verifies closure under cellular faces and accumulation across the three prefixes, computes Betti numbers by exact elimination over $\mathbb{F}_2$, and checks the overlap of the two selected triangular supports. It writes a JSON summary and CSV table under the `experiments/` directory, with filenames beginning `appendix_c_results`. The related vector figure is generated by `figures/scripts/draw_appendix_c_histories.py`. These tests establish internal consistency

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of the declared finite example and can reveal implementation errors; they do not validate the GR interpretation of a cell inventory or a value-constitution rule.

Three practical limitations remain material. An event inventory is constructed from selected evidence and may omit processes that were unrecorded, inaccessible, or historically marginalized. Distinct prerequisite protocols can produce nonisomorphic complexes from the same event descriptions; apparent holes or fillings can consequently be sensitive to recording choices. Finally, distinct circulation histories and value-constitution hypotheses can generate the same available observations. A concrete analysis should therefore provide source provenance, uncertainty and alternative reconstructions, and an explicit account of which historical distinctions are preserved when its representation is simplified. The formal method supports such an audit, while claims of causal mechanism or justice remain conditional on substantive empirical and normative work.

D Illustrative Simulation Specifications and Reproducibility

This appendix specifies the computational models used in Section 8 and the exact symbolic constructions in Appendices B and C. Its purpose is to make the finite examples inspectable, repeatable, and distinguishable from empirical research. The account covers data representations, event-transition rules, parameter settings, expected outputs, validation checks, and execution procedures. All inputs are constructed for illustration; the programs contain neither observed social-network data nor fitted economic, causal, or normative parameters. The correspondence between mathematical cells and GR interpretations remains a proposal external to the computational tests.

D.1 Computational Domains and Data Representations

The main computational experiment uses finite two-dimensional cubical complexes. A selected primitive-face support is a finite set of integer-coordinate unit squares indexed by (i, j). Each selected square includes its four boundary edges and four corner vertices; shared edges and vertices are stored once. Thus, the selected cells form a subcomplex of a sufficiently large ambient square grid. The program constructs B_1 from edge–vertex incidences and B_2 from face–edge incidences, and computes ranks over $\mathbb{F}_2$ by finite elimination. The resulting dimensions and Betti numbers are the cellular quantities defined by Equation 38. Grid positions are plotting coordinates and do not carry calibrated social distances, areas, prices, or durations.

The exact calculations of Appendix B use a separate script with integer signed incidence matrices represented by `SymPy`. That script evaluates ranks over $\mathbb{Q}$ and, independently, after reduction modulo two. The latter calculation checks the coefficient-field agreement in the selected examples; such agreement is an observed property of these particular complexes, not an assertion of universal agreement across coefficient systems. The event-prerequisite construction of Appendix C uses a separate finite simplicial complex with four event vertices, explicit edge and face incidences, and prerequisite sets for recorded cells. Its configurations are represented by sets of completed events. The event vertices in both models identify formal events, not presumed enduring social subjects.

Table 16 records the data contracts relevant to reproduction. A row called *support* identifies a selected cell collection under the model's incidence rules; it does not by itself establish the presence or magnitude of a social value. In particular, labels such as α, β, and *converted* name mathematical support-selection protocols rather than measurements of money, care, knowledge, or other real-world value modalities.

D.2 Declared Morphologies and Support Evolution

The morphology experiment selects four small cubical supports. The *compact* motif is a filled two-by-two block of squares; the *branching* motif is an eight-square branched construction; the *ring* is a three-by-three block with its central square omitted; and the *filled* motif contains all nine squares. Their plotted geometries are stipulated examples. The corresponding first Betti numbers are zero, zero, one,

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Table 16: Data representations used by the three computational construction scripts. The files contain synthetic structural and dynamical output.

Computational object	Stored representation	Intended analytical role
Cubical support	Finite set of integer square coordinates	Cell counts, homology, and support recombination
Boundary matrices	Finite incidence matrices over $\mathbb{F}_2$ or signed matrices over $\mathbb{Z}$	Verify the chain identity and calculate homology
Execution trace	Ordered event labels, step index, delivered-label set, and support update	Contrast sequences sharing a terminal delivery set
Event record	Completed-event ideal with cell prerequisites and activity annotations	Construct closed historical subcomplexes and distinguish active selection
Morphology or sensitivity result	CSV row with parameters, counts, ranks, and diagnostics	Reproduce tables and compare runs
Run summary	JSON with selected parameters and reported synthetic outputs	Machine-readable check of principal numerical results

and zero, respectively, as calculated by the source program. In the ring case, the full boundary cycle surrounds an omitted primitive face. Attaching that face fills the first-homology class in the selected finite complex. The exact signed analogue is developed in Appendix B.

For the sequential support experiment, the first eight face additions traverse the perimeter of the three-by-three square block in a prescribed order; the ninth adds its central face. The first seven stages contain no first-homology class, stage eight has one, and stage nine has none. These results describe how the specified inclusion sequence changes the topology of a recorded support. They do not imply that a social value was created at stage eight or that a perceived deficit was remedied at stage nine. The face-selection rule would need separate historical and interpretive evidence before acquiring either meaning.

The typed-support illustration assigns one four-square path to modality α and another four-square path to modality β. A stipulated union operation produces a ring, while a further selected central face produces a filled support. The union is a geometric operation on supports under a declared conversion convention. It contains no exchange-rate, utility, price, or normatively justified conversion law. Its use in Section 8 demonstrates the possibility that a combination rule alters topology even when each input support has trivial first homology.

D.3 Event Order, Synthetic Waiting Cost, and Support Updates

The order-sensitive experiment uses three event labels: `observe`, `b`, and `c`. The observation occurs first in either recorded execution and only updates an audit flag. Events `b` and `c` are incomparable under the declared prerequisite ordering and complete two allocations, each taking one synthetic time unit. The model therefore compares the sequences $(\texttt{observe}, \texttt{b}, \texttt{c})$ and $(\texttt{observe}, \texttt{c}, \texttt{b})$. The recipients of the hypothetical allocations are roles represented by external labels; neither is identified with a vertex of the cubical complex.

An initially unserved allocation carries a waiting-cost weight w_b or w_c. At each completion step, the code adds the weights of allocations still outstanding immediately before that completion. For the two-step system, this gives the order difference stated in Equation 61.

$$\mathcal{L}(bc) - \mathcal{L}(cb) = w_c - w_b, \tag{61}$$

where $\mathcal{L}$ is a synthetic cumulative waiting-cost functional. Equation 61 follows from $\mathcal{L}(bc) = w_b + 2w_c$ and $\mathcal{L}(cb) = 2w_b + w_c$, assuming both tasks have unit duration and both needs are initially pending. The baseline weights are $w_b = 1$ and $w_c = 10$, which yield 21 and 12, respectively. The weights are declared illustration parameters, not empirically measured need intensity or a validated criterion of justice.

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Table 17: Declared settings and variations for the finite synthetic experiments. All weighting, timing, and activation parameters are illustrative.

Model component	Fixed specification	Variation or output
Cubical supports	Unit squares with boundary closure; $\mathbb{F}_2$ ranks	Compact, branched, ring, and filled motifs
Order experiment	Observation precedes either allocation; each allocation takes one time unit	bc compared with cb
Waiting weights	Baseline $w_b = 1$, $w_c = 10$	$w_c \in \{1, 2, 5, 10, 20\}$
Face activation	Candidate central face after c follows b	$p \in \{0, 0.25, 0.5, 0.75, 1\}$
Stochastic draws	Seed `20261018`; one generator stream	3,000 trials per value of p
Multiscale support	Three-by-three center-omission rule	Levels $k \in \{0, 1, 2, 3\}$
Historical reconstruction	Four event vertices, five edges, two faces	Ideals ab, abc, $abcd$

The initial selected support in both orders is the eight-square ring. If event `c` occurs after `b`, the model additionally attempts to activate its central primitive face. At the default activation probability $p = 1$, order bc produces the filled support with $\beta_1 = 0$, whereas order cb retains the ring with $\beta_1 = 1$. The difference results directly from the stipulated event-dependent support-update rule. Equal completed allocation sets therefore coexist in the example with different waiting costs and support topologies, and neither descriptor logically determines the other. The added face is part of the selected value-supporting complex; it is distinct from the derived interchange face of the event-order diamond analyzed in Section 7.

D.4 Finite Multiscale Construction and Stochastic Variation

The multiscale illustration begins with a single square and recursively replaces each occupied square by a three-by-three block with its center omitted. For levels $k = 0, 1, 2, 3$, the face counts are 8^k and the computed first Betti numbers are $(8^k - 1)/7$. These identities hold for the specified finite construction. Three recursive substitutions provide an example of geometric organization across resolutions; they do not establish fractality of an empirical value structure. The drawings and tabulated results record the finite combinatorial construction alone.

The stochastic extension varies the probability p that the central face is added after the order bc. It uses the five values 0, 0.25, 0.50, 0.75, and 1, with 3,000 independently sampled synthetic trials per parameter setting. A `NumPy` pseudorandom generator is initialized with seed `20261018`, and the script evaluates settings sequentially using the same generator stream. For this two-outcome construction, the exact conditional expectation is reported by Equation 62.

$$\mathbb{E}[\beta_1 \mid bc, p] = 1 - p. \tag{62}$$

Equation 62 follows because the ring retains one first-homology class if the candidate face is not activated and has none if it is activated. The sampled means recorded in `sensitivity_results.csv` are 1, 0.7527, 0.5270, 0.2470, and 0, rounded to four decimals. Their agreement with $1 - p$ is checked using the script's explicitly selected absolute-error tolerance of 0.05. This test is an implementation sanity check and supplies no statistical validation of a social process.

Table 17 identifies the principal fixed and varied quantities. The distinction between a fixed modeling convention and an investigated parameter is relevant because the demonstrated sensitivity is conditional on every remaining rule. In particular, the support-update rule and waiting-cost rule are stipulated independently; the experiment does not estimate any causal effect of a real institutional decision.

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D.5 Validation Tests and Limits of Reproduction

The three calculation scripts assert the chain-complex identity $B_1 B_2 = 0$ for each relevant complex. The cubical experiment checks Euler-characteristic consistency and the expected first Betti numbers for the declared motifs and filtration. Appendix B checks signed rational ranks, ranks over $\mathbb{F}_2$, the homological effect of central-face attachment, the tetrahedral shell's nontrivial H_2, and the order-diamond coboundary calculation. Appendix C checks the closure of recorded subcomplexes, monotonicity along the selected event ideals, specified overlap of two candidate supports, and the first-Betti-number sequence $0, 1, 0$. These are finite formal calculations: they validate implementation consistency against the declared incidence and event rules.

A separate verification command supplied in the source package inspects the generated numerical files and compares their contents with the expected structural and analytic checks. It tests the previously generated section-level tables, the expected waiting-cost outputs, the finite multiscale counts, and the exact reference values for the stochastic mean, using a declared numerical tolerance for Monte Carlo results. This output-level audit supplements the scripts' own assertions. It does not provide a mathematically independent implementation of all cellular calculations, and therefore is not a substitute for external code review.

Exact structural invariants and finite event-order costs should reproduce under an unchanged implementation. The pseudorandom sequence is deterministic for the stated seed and tested software environment, but bit-for-bit equality of generated graphics or random streams need not persist across plotting backends and numerical-library versions. The appendix therefore privileges machine-readable numeric outputs and explicitly reported tolerances over file-level image equality. A successful re-run warrants the limited statement that the specified finite construction has been reproduced; it does not independently confirm the social interpretation of its cells, validate a model of heterogeneous value, or establish normative priority relations.

D.6 Source Files, Execution Sequence, and Archival Outputs

The complete source archive contains three principal calculation programs and two figure-generation scripts for Appendices B and C. Starting in the directory containing `manuscript.tex`, a reader can execute the scripts listed in Table 18 after installing the Python dependencies specified in the accompanying requirements file. The scripts resolve output locations relative to their own source paths and overwrite the corresponding generated files. For manuscript compilation, the same directory supports `make`, which invokes XeLaTeX and Biber in the sequence already used for the main text. The Python experiments can be reproduced independently of manuscript compilation.

The source archive also includes `experiments/REPRODUCIBILITY.md`, which gives the setup, commands, tested local dependency versions, and expected diagnostic results. The generated CSV and JSON files provide the numerical outputs presented in the main text and appendices. Vector PDFs are used for the manuscript figures, and the PNG files provide convenient inspection images. Figure-generating scripts for the earlier conceptual diagrams are retained separately under `figures/scripts/`; they are not part of the Section 8 numerical experiments. Regenerating the same diagrammatic image does not verify the proposed meaning of its labels.

For further empirical study, an additional layer of source provenance would be necessary: event and cell-identification protocols, timestamp uncertainty, alternative reconstructions, observer or institutional context, ethically appropriate handling of social records, and explicit justification for any modality-specific normative weights. The present package keeps the computational and interpretive layers separable so that later researchers can change the assumptions, reproduce the finite consequences, and identify where new evidence is required.

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Table 18: Reproduction commands from the manuscript source directory and representative outputs. Commands are listed in the order recommended for a clean validation run.

Command	Principal products
`python experiments/run_section08_experiments.py`	Four section-level CSV tables, `summary.json`, six vector figures and PNG previews
`python experiments/run_appendix_b_homology.py`	`appendix_b_results.csv` and `appendix_b_results.json`
`python experiments/run_appendix_c_histories.py`	`appendix_c_results.csv` and `appendix_c_results.json`
`python figures/scripts/draw_appendix_b_grid.py`	Ring-inclusion PDF and PNG figure
`python figures/scripts/draw_appendix_c_histories.py`	Historical construction PDF and PNG figure
`python experiments/validate_reproducibility.py`	Output-level assertion checks, printed pass/fail summary
`make`	`manuscript.pdf` with regenerated cross-references and bibliography

E Mathematical Extensions and Open Formal Problems

This appendix identifies mathematical extensions that can be developed from the finite event–coupling–primitive formalism without assigning an empirical or normative meaning to topological features by definition. Its role is to specify the additional data and consistency conditions required by typed encodings, higher-dimensional relations, directed event histories, multiscale changes, and justice-sensitive abstraction. The constructions below are candidate mathematical interfaces. Only results stated under explicit assumptions are treated as deductions; proposed interpretations of money, labor, care, knowledge, and other value modalities require separate argument. The distinction between the event complex K, value-supporting subcomplexes L_α, and the derived history-comparison complex D follows Appendix A.

E.1 Typed Cellular Coefficients and Value Encodings

Section 5 treats value modalities as historically constituted configurations supported by subcomplexes. A possible refinement is to attach typed primitive data to the cells composing those supports. This would allow two otherwise identical subcomplexes to carry different admissible labels and coupling constraints. The refinement must respect the distinction between the primitive data attached to a face and the constituted value of an extended configuration. In particular, a local face label is not assumed to be a complete economic or ethical evaluation.

For a finite oriented regular cell complex K, fix a coefficient field $\mathbb{F}$ and finite-dimensional $\mathbb{F}$-vector spaces W_σ associated with cells σ. For each codimension-one incidence $\tau < \sigma$, suppose a linear map $r_{\sigma\tau} : W_\sigma \to W_\tau$ is specified. These maps are an additional assignment of typed information; the basic cellular structure supplies incidence signs but does not determine $r_{\sigma\tau}$. Under this assignment, a candidate coefficient-aware cellular boundary is given by Equation 63.

$$C_p(K; W) = \bigoplus_{\sigma \in K_p} W_\sigma, \qquad \partial_p^W|_{W_\sigma}(x) = \sum_{\substack{\tau < \sigma \\ \dim \tau = p-1}} [\sigma : \tau] r_{\sigma\tau}(x). \tag{63}$$

Equation 63 is an algebraic construction over the selected field. Money, labor hours, environmental effects, and interpretive recognition cannot be added indiscriminately because they are represented by vectors. A typed direct sum separates coordinates; legitimate conversion among modalities requires independently defined maps with justified domains, units, and interpretive assumptions. Nonlinear or history-dependent constitution is outside the information carried by the linear boundary alone.

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Proposition E.1 (A sufficient condition for a coefficient-aware chain complex)**.** *For a finite oriented regular two-dimensional cell complex, suppose that for each face f and incident vertex v, the composites $r_{ev} \circ r_{fe} : W_f \to W_v$ are equal for all boundary edges e incident to both f and v. Then the maps of Equation 63 satisfy $\partial_1^W \partial_2^W = 0$.*

Proof. Fix $f \in K_2$, $x \in W_f$, and an incident vertex v. The coefficient of v in $\partial_1^W \partial_2^W(x)$ is the sum over incident edges e of $[f : e][e : v](r_{ev} \circ r_{fe})(x)$. By the stated compatibility assumption all such composites equal one map r_{fv}, so the sum equals $\big(\sum_e [f : e][e : v]\big) r_{fv}(x)$. The incidence sum vanishes by the ordinary cellular identity $\partial_1 \partial_2 = 0$. The argument applies to every vertex and face, hence to their direct sums. □

Proposition E.1 gives a sufficient condition, not a necessary characterization of all possible coefficient systems. For a single oriented triangular face with consistent identity maps between equal one-dimensional coefficient spaces, Equation 63 reduces to the ordinary triangular cellular boundary in Appendix B. With incompatible edge or vertex spaces, the requisite maps may fail to exist, or the relevant composites may fail the compatibility condition. Such a failure indicates that the proposed coefficient construction is incomplete; it does not establish that a corresponding real-world value cannot circulate.

The construction can be related to cellular (co)sheaf-like data when the incidence maps satisfy the appropriate functorial requirements, but terminology and covariant or contravariant conventions must be fixed before making that identification. If types are not naturally vector spaces, alternatives could use modules, ordered monoids, categories of typed states, or probabilistic kernels. Their homological treatment cannot be presumed to follow from Equation 63. A valuable early test would compare the ordinary groups $H_p(L_\alpha; \mathbb{F})$ with homology constructed from admissible typed data, under fully specified incidence and coupling rules. A difference between those invariants would be a property of the selected representation and coefficient assignments, with empirical significance requiring independent validation.

E.2 Higher-Dimensional Coherence and Primitive-Face Relations

Our foundational ontology employs a two-dimensional complex: vertices denote events, edges couplings, and faces primitive higher-order structures. That dimensional choice supports a clear minimum model. A possible extension adds cells of dimension three when relations among several primitive faces themselves need to be represented. The added cells would be mathematical descriptors of specified higher-order dependencies, without changing the event interpretation of vertices or requiring that any value be carried by a single face.

For a three-dimensional extension $K^{(3)}$, the second homology group becomes the quotient in Equation 64.

$$H_2(K^{(3)}; \mathbb{F}) = \ker(\partial_2 : C_2 \to C_1) \,/\, \mathrm{im}(\partial_3 : C_3 \to C_2), \qquad \partial_2 \partial_3 = 0. \tag{64}$$

As Equation 64 shows, a closed face combination that represents a nonzero class in a purely two-dimensional complex may become the boundary of an added three-cell. Appendix B computes $\beta_2 = 1$ for the tetrahedral two-shell; attaching its solid three-cell makes that shell homologous to zero in the enlarged complex. The numerical change is an ordinary topological consequence of a specified attachment. Whether a three-cell represents a feasible reconciliation among primitive relations, a consistency constraint, or some other social mechanism is a separate semantic choice.

For a concrete extension, the face boundary of each three-cell would need to be an explicitly declared cellular two-cycle, its historical availability would need to be related to the event-recording protocol, and any face labels would require coherent maps compatible with that attachment. Additional dimensions can also create representational ambiguity. Two distinct higher-dimensional models may have the same event logs and lower-dimensional observable data. Higher-dimensional coherence becomes useful when it expresses a concrete mechanism or supplies a new checkable implication, rather than simply filling a visually salient hole. Existing treatments of cellular homology provide the relevant mathematical background [11, 13].

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E.3 Directed History Categories and Order-Sensitive Operations

A cellular orientation supplies algebraic incidence signs; it does not identify which event orders are causally realizable. Section 7 therefore introduces trajectories on top of an event-precedence relation and constructs a separate history-comparison complex D. One extension would formalize this separation through a category of executions or, with further structure, a directed topological space [10]. The choice is justified when the research question concerns composition and substitution of causal histories, rather than homology of the ambient complex alone.

In a simple finite model, let objects be attainable pairs (H, x) comprising a completed-event ideal H and a state x. Let a generating morphism be one admissible event transition $(H, x) \to (H \cup \{a\}, T_a(x))$, with composition represented by concatenation of compatible histories. Associativity of concatenation is available at the path-description level, although an interpretation as a well-defined state-transforming category requires consistent domains and composite transition rules. Such a category retains differences between histories that a mere final event set forgets. It can also distinguish a recorded event sequence from a later interpretation of that sequence as circulation of a particular value modality.

Suppose two events a and b are incomparable in the precedence relation and both orders are technically admissible from a given state x. A candidate interchange relation becomes *state-preserving* only under the condition stated in Equation 65.

$$T_b\big(T_a(x)\big) = T_a\big(T_b(x)\big) \quad \text{for each state } x \text{ in a declared common domain.} \tag{65}$$

Equation 65 is stronger than the causal incomparability of a and b: temporal independence in a precedence graph does not imply commutation of transformations. Even if the equation holds at final states, intermediate deprivation or obligations may distinguish the histories. A face in the derived complex D could record that a local interchange is allowed under stated rules. It remains a different mathematical object from a primitive value-related face in the ambient complex K.

This extension offers a practical test of historical reduction. A proposed equivalence between executions should explicitly specify the events whose order can be exchanged, the state transitions that must be preserved, and the retained historical observables. Since emergent entities and values may depend on the past, a valid interchange at one observation level may fail to preserve later constitution at another. Directed-history methods would let us express such failures as limitations of the equivalence relation, without inferring their justice consequences in advance.

E.4 Dynamic Complexes, Coarse-Graining, and Multiscale Structure

Appendix C gives a prerequisite-based procedure for building nested recorded subcomplexes K_H from downward-closed event sets H. Those accumulated records support ordinary inclusion maps when histories extend one another. A changing active selection A_H, by contrast, need not itself form a subcomplex or remain nested. Value-supporting complexes $L_\alpha(H)$ may also gain or lose operative cells under changing historical or recognition rules. These cases require distinct temporal constructions.

For the nested record, persistent homology can compare the induced maps on $H_p(K_H)$ as the recorded event history grows; the filtration and its indexing order must be specified. For two nonnested value supports $L_\alpha(H)$ and $L_\alpha(H')$ within a common ambient complex, one possible comparison uses their union U, provided each support and the union are subcomplexes. Equation 66 gives the associated diagram of inclusion-induced maps.

$$H_p\big(L_\alpha(H); \mathbb{F}\big) \longrightarrow H_p(U; \mathbb{F}) \longleftarrow H_p\big(L_\alpha(H'); \mathbb{F}\big), \quad U = L_\alpha(H) \cup L_\alpha(H'). \tag{66}$$

The span in Equation 66 describes a common comparison space. It is not automatically a lineage map identifying an individual value across time. Identity across different supports requires a history-dependent constitution criterion, together with an account of overlap, persistence, fragmentation, or reconstitution. In particular, identical Betti numbers or similar persistence diagrams can occur for different coupling structures and interpretations.

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A separate multiscale operator Q_s could aggregate cells, labels, and historical observations at resolution s, so long as its inputs, outputs, and incidence-preservation properties are declared. Preserving homology is one possible design condition; preserving admissible event sequences, value support provenance, and justice-relevant attributes imposes further conditions. Section 10.3 defines an exact criterion for assessment preservation under reduction. A coarse-grained representation can usefully lower computational cost even when it is adequate only for a restricted set of questions. In that case, the lost information should be described rather than interpreted as absent in the modeled world.

The fractal possibility discussed in Subsection 3.7 is similarly conditional. Repeated organizational patterns or recursive value reproduction need not imply mathematical self-similarity. A finite sequence of constructed patterns can demonstrate finite-range scaling, whereas a fractal-dimension claim requires a specified metric or covering procedure and appropriate evidence of scaling. This distinction is important for heterogeneous modalities: recurrent social relations, visually branching supports, and scaling laws concern different empirical and mathematical properties. Persistent topology and discrete Morse reductions remain candidates where their filtrations and preservation assumptions fit the question [9, 8].

E.5 Historical Quotients and Justice-Sensitive Preservation

The motivation for representing circulation order is to compare feasible histories that may generate different value structures, deprivation patterns, and future capabilities. Quotienting histories can make a large system tractable, but it can erase precisely the information necessary for such comparisons. A meaningful quotient therefore depends on an explicit purpose: preserving endpoints, feasible futures, typed supports, ethical obligations, or a declared assessment. None of these purposes is supplied by cellular homology itself.

Let $q : \Gamma_\theta^{\mathrm{adm}} \to \overline{\Gamma}$ be a surjective history abstraction on a domain already restricted by stated admissibility criteria, and let $\mathcal{J}_\theta : \Gamma_\theta^{\mathrm{adm}} \to \mathcal{Q}_\theta$ be an assessment into an arbitrary set or partially ordered space. The familiar factorization condition is given by Equation 67.

$$\exists\, \overline{\mathcal{J}}_\theta : \overline{\Gamma} \to \mathcal{Q}_\theta \text{ such that } \mathcal{J}_\theta = \overline{\mathcal{J}}_\theta \circ q \quad \Longleftrightarrow \quad q(\gamma) = q(\gamma') \Rightarrow \mathcal{J}_\theta(\gamma) = \mathcal{J}_\theta(\gamma'). \tag{67}$$

The equivalence in Equation 67 follows because a function on equivalence classes is well defined exactly when the original function is constant on the fibers of q. It is an elementary set-theoretic statement. The difficult part is substantive: which historical differences ought to matter to $\mathcal{J}_\theta$, and how much empirical and interpretive information is required to evaluate them? A quotient identifying the two orders of the synthetic delay example in Appendix B cannot preserve that example's waiting-cost assessment, even though the derived history diamond can be filled by a two-cell.

For a family Θ of defensible normative interpretations, preserving assessments for a single chosen θ may be insufficient. A reduction that is robust to declared normative disagreement would need to preserve the relevant comparisons for each $\theta \in \Theta$, or transparently report the interpretations for which preservation fails. Numerical approximation requires a further choice of distortion measure on the assessment space; a partially ordered assessment may not admit a norm or universal scalar distance. It would therefore be misleading to announce a uniform approximation error without first providing the relevant geometry and normative assumptions. This extension places mathematical efficiency and epistemic accountability within the same explicitly qualified design problem.

E.6 Identifiability, Causal Interpretation, and Research Dependencies

A topological description becomes an empirical model only through a documented observation protocol. Identification of event vertices, coupling edges, primitive faces, active relations, and value-supporting structures may depend on the resolution of a record and on judgments about what was relevant or meaningful. Section 10.5 introduces an observational identifiability criterion; the criterion is especially important if competing constitution rules produce the same observed topology. Missing face records may reflect data collection, an incomplete interpretation, or a genuinely absent primitive relation. These alternatives have different meanings and should not be conflated in data analysis.

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Table 19: Open formal problems and the additional requirements for a defensible extension.

Problem	**Required formal specification**	**Principal limitation**
Primitive-face semantics	Incidence interpretation, empirical or model-based assignment, provenance	Shape alone leaves meaning undetermined
Typed value encodings	Local state spaces, incidence maps, conversion domains	Algebraic compatibility does not establish commensurability
Entity and value constitution	Selection protocols, overlaps, historical identification	Supports can be ambiguous or observationally equivalent
Order-dependent dynamics	Causal precedence, enabled transitions, state-update rules	Valid causal sequences may not commute
Topology-changing histories	Temporal comparison maps and observation resolution	Recorded and active structures have different dynamics
History reduction	Preserved observables and normative assessment criteria	Homology preservation is weaker than justice preservation
Empirical and ethical use	Observation design, consent, contestability, counterfactual assumptions	Synthetic consistency gives neither external validity nor moral authority

Counterfactual comparisons of event order add further requirements. A causal claim that one order changed a value-supporting structure would need a model of interventions, a feasible counterfactual ordering, and a defensible relation between the unobserved counterfactual and available evidence. Varying the order in a deterministic synthetic program demonstrates the consequences of that program's transition rules. It does not identify the consequences of reordering events in an actual institution, household, research programme, or care setting. Moreover, an event's occurrence may change whether a later event is feasible, so two idealized orders should not automatically be treated as interchangeable observations of the same social system.

Table 19 summarizes the research dependencies that would make the present mathematical framework more informative. The table is ordered by conceptual dependence rather than by presumed difficulty or priority for resource allocation. In particular, a more expressive coefficient or higher-dimensional topology is useful only if its face semantics, admissible histories, and interpretive purpose are also sufficiently specified.

A defensible first empirical use would select a bounded historical process whose events and timing can be documented with care, construct alternative support-selection rules, and test how topology and order-dependent descriptors change when those rules vary. A first theoretical development could instead define one simple typed coefficient system and a small history category, then prove which reductions preserve selected state and timing attributes. Both routes would improve the mathematical account without requiring a universal metric of heterogeneous value. Normative inquiry could then formulate context-sensitive admissibility and comparison criteria with participation from the affected parties and appropriate attention to rights, obligations, and inequalities in recognition.

The open research programme is therefore a sequence of explicit mathematical and interpretive tasks: clarify primitive semantics; define constitution and transformation of extended values; identify historically feasible circulation orders; establish which abstractions preserve relevant consequences; and justify the conditions for justice comparisons. These tasks maintain the central separation of the paper. Topology can describe the organization and changes of relational supports, history can record their causal and temporal constitution, and normative analysis can evaluate consequences using defended ethical premises. The connections among them remain the object of further formal and empirical inquiry.

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Research Ethics and Source Provenance

Source and model status. Sections 1–11 and Appendices A–E develop an exploratory construction and use cited publications for bounded mathematical and disciplinary background. Established mathematical results are distinguished from proposed GR semantics. No spin-foam formalism for economic value is attributed to the cited quantum-gravity literature. Bibliographic sources included in this draft have been checked against publishers, author pages, or other primary institutional records; see the accompanying verification ledger. Such records do not imply line-by-line scrutiny of every cited publication.

Ethics and verification. The numerical experiments and diagrams of Section 8 are synthetic and are supplied with executable source code. Further illustrations will be explicitly synthetic unless supported by a separately documented empirical protocol. No human-participant data, research-ethics clearance, institutional endorsement, or normative authority is claimed. Justice-related assessments require independent ethical and context-sensitive arguments.

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