Learner Participation in AI-Assisted Introductory Calculus: A Principle of Non-Substitutive Teaching

Wanhong HUANG
huangwanhong@serendip.ngo

Abstract

AI assistance can produce a correct mathematical solution while leaving open what the learner has understood, checked, or become able to extend. This paper develops a proposed principle of non-substitutive teaching for introductory calculus: assistance should sustain accessible opportunities for the learner participation that a justified educational purpose requires. The principle permits direct explanation, worked solutions, and continuing tools. It also subjects question-led teaching to scrutiny where prompts conceal authority or impose avoidable difficulty. Comparisons with scaffolding, intellectual emancipation, educational purpose, and a contemporary reconstruction of *buyan zhi jiao* clarify the philosophical commitments. A derivative argument is developed from its domains and assumptions through algebraic and limit justifications, then used in hypothetical teaching contrasts. Two recent AI tutoring studies illustrate the importance of distinguishing instructional packages and assessment horizons. The paper separates mathematical validity, observed participation, acquired competence, and normative adequacy. Its contribution is a structured framework for criticism and instructional design; empirical validation of the proposed principle, wider applicability, and longer-term formation remain open.

Keywords: AI-assisted learning; introductory calculus; non-substitutive teaching; learner participation; scaffolding; educational purpose

Research Purpose and Scope

This note states the purpose of circulation, the status of the argument, and the role of research assistance. The paper presents current insights for criticism and wider discussion in public and scholarly settings. It makes no claim to conceptual priority. Similar concepts and arguments may already exist in other disciplines, languages, and intellectual traditions; further comparison may alter the terminology, scope, or contribution.

Definitions organize a bounded inquiry. Conditional formal results carry their stated assumptions, and illustrative cases support conceptual analysis. Empirical validation, wider applicability, and the adequacy of proposed normative premises remain open where the main text identifies those obligations. Relational analysis supplies a method of investigation while stronger ontological questions remain suspended.

AI assistance contributed to source discovery and pre-use verification, argumentative development, formal checking, organization, and drafting. The accompanying project records distinguish retained research insights, newly proposed analysis, and verified literature. The manuscript is prepared for the author's review; circulation and submission decisions remain with the author. Critical responses concerning antecedents, assumptions, counterexamples, and alternative formulations are welcome.

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1 Introduction

This paper examines learner participation in AI-assisted introductory calculus. It develops a proposed teaching principle, places that principle in conversation with selected philosophical and educational antecedents, and tests its distinctions through a fully explained mathematical example. The discussion combines conceptual analysis, conditional derivation, hypothetical teaching contrasts, and a bounded comparison of empirical studies.

A learner studying differentiation can obtain the expression $2x$ immediately after submitting x^2 to an assistant. That exchange can serve several purposes. It can supply a value needed in another calculation, confirm an independently completed exercise, initiate an investigation of rates of change, or terminate an assignment before the relevant reasoning becomes intelligible. The visible answer can remain identical across these uses. An educational judgment therefore needs to identify the activity being supported, the learner's role within it, and the further evidence appropriate to the intended outcome.

The retained insight developed here concerns the difference between delivering a product and supporting the formation of a participant. A learner's subsequent possibilities may depend on encounters with examples, mistakes, objects, teachers, peers, texts, and tools. The acquisition of one procedure may open further questions while also stabilizing a narrow habit. Such possibilities motivate investigation of formation; they do not establish that every interaction enlarges a learner's capacities or that a larger repertoire is always desirable. The present scope is an introductory mathematical task whose reasons can be examined in detail.

The proposed principle of non-substitutive teaching asks assistance to preserve meaningful opportunities for the learner acts constitutive of a justified educational purpose. The expression names a constraint on the organization of assistance. A teacher may state a definition, demonstrate a complete argument, repair notation, or provide a calculator while satisfying that constraint. The relevant inquiry concerns which acts remain available to the learner and how the learner can take them up. Conversely, a sequence of leading questions may leave little room for interpreting the problem or challenging the teacher's route. The surface form of the interaction provides limited evidence about its educational character.

This proposal carries a normative commitment that requires defense. If an institution has justified an aim of helping learners understand a derivative, that aim gives it a reason to organize opportunities for understanding. Further considerations govern which burdens it may impose and which methods are appropriate. Learner purposes, accessibility, time, and legitimate institutional responsibilities can change the balance. The argument consequently proceeds through explicit premises and defeasible claims. Its scope is a bounded educational relationship, with participation and burdens assessed for a justified purpose.

The paper also preserves an epistemic separation. A sound proof establishes a mathematical result under stated definitions and assumptions. Observation of a learner reproducing that proof supports an educational inference only through further assumptions about the task, assistance, interpretation, and alternatives. A measured improvement on an immediate test concerns that outcome and horizon. A judgment about enduring formation requires additional evidence. Keeping these warrants separate allows a teacher to value useful assistance without treating every completed exercise as demonstrated understanding.

The analysis begins with learning targets and antecedent debates. A separate section develops a contemporary reading of 不言之教, commonly rendered as teaching without words, while retaining the difference between historical interpretation and present proposal. The central sections specify participation and derive an elementary result from the difference quotient. Hypothetical interactions then show how the same mathematical material can support different activities. The later sections compare empirical designs, develop an assessment approach, examine educational authority, and state objections and open research questions. The objective is a precise contribution to public and scholarly discussion of educational assistance, with its assumptions and revision conditions available for criticism.

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2 Educational Targets and Conditions of Assistance

This section defines the objects of educational assessment used in the argument. It separates products, performances, competencies, and longer-term formation, then specifies how assistance enters each target. These distinctions establish the scope of the proposed teaching principle and prevent a change in task conditions from silently changing the meaning of success.

2.1 Products, performances, and competence

The first distinction concerns what has been produced and what a person can do. A written derivative, a spoken explanation, and a completed worksheet are products of activities. Their correctness and their evidential significance for a learner require different inquiries.

Definition 2.1 (Assisted mathematical product)**.** An assisted mathematical product is an expression, explanation, diagram, or solution generated through an activity involving a learner and specified external support. Its attribution records the contributions relevant to the present inquiry.

The definition includes a jointly discussed proof and a formula copied from a display. It also includes work composed through speech recognition by a learner who supplies the mathematical reasoning. These cases differ in the distribution of epistemic acts. A generic label such as AI-assisted conceals that distribution. The examiner needs to know whether the assistant supplied the conclusion, corrected a transcription, selected a method, checked an algebraic operation, or generated the entire explanation. More than one contribution can occur in the same exchange.

Performance describes success in an activity under its actual conditions. Competence concerns a person's capacity relative to a specified family of activities and support conditions. A learner may competently use an accessible symbolic editor, competently evaluate a generated derivation, and remain unable to reproduce the same derivation unaided. These descriptions are compatible. Their educational significance depends on the purpose of the course and the situations for which it prepares the learner.

Definition 2.2 (Target-relative mathematical competence)**.** A target-relative mathematical competence is a capacity to carry out specified mathematical acts across a declared family of tasks, under declared support conditions and within a relevant horizon. Evidence for that capacity must identify the acts and conditions to which the inference applies.

This working definition leaves several matters open. A capacity can vary with fatigue, familiarity, context, or available time. An assessment supplies a sample of activities with a limited evidential scope. Competence in explaining a derivative at one point differs from competence in diagnosing an erroneous general rule. The task family therefore needs enough specification to make the inference intelligible, without pretending that a finite assessment captures every future situation.

2.2 Formation, retention, and transfer

The second distinction concerns change over time. Formation refers here to the development of a learner's ways of attending, reasoning, questioning, and participating. Retention and transfer supply particular possible evidence about that development, each with its own target.

A learner who recalls a proof the following day exhibits a different achievement from one who adapts it to a new function. Transfer itself varies. Replacing x^2 with $x^2 + 7$ tests one relation; examining $|x|$ at zero introduces a different structural issue. Both can be valuable, but their difficulty and significance should be established through the mathematical changes they contain. Calling a task novel supplies insufficient information about the underlying demand.

The longer-term concern includes the learner's ability to recognize a worthwhile question, seek appropriate help, revise an accepted explanation, and participate in inquiry with others. These capacities can develop together or unevenly. A student might become more accurate at differentiation while becoming less willing to disclose uncertainty. Another might become better at asking for assistance before

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acquiring fluent algebra. A comprehensive account should preserve these differences and their separate evidential requirements.

Claim 2.3 (Distinct educational warrants)**.** The correctness of an assisted product, success in a joint activity, retention under later conditions, transfer to a specified variation, and the defensibility of an educational purpose require distinguishable warrants. Evidence for one can contribute to another when the connecting assumptions are stated and supported.

The claim follows from the differences in the objects assessed. A correct expression can be supplied by a tool while a learner remains uncertain about its interpretation. A learner may understand a previously supplied proof and later explain it successfully. An accurate explanation may concern an aim that a learner reasonably wishes to postpone. These possibilities establish the need for separate questions. They leave the strength of particular empirical connections open.

2.3 Supported competence and practical reliance

The third distinction concerns which supports belong to the target activity. Educational independence can mean independence from a particular prompt, from an answer source, or from a person making a judgment on the learner's behalf. It can also mean competent participation within a reliable network of assistance.

Requiring every learner to dispense with notation, reference materials, peers, or accessible interfaces would transform the target into an unusually demanding activity. Such a transformation needs justification. A course may reasonably assess a basic algebraic manipulation without an answer engine because that manipulation supports later checking. The same course may reasonably preserve speech input or enlarged displays because handwriting and visual acuity are incidental to the mathematical target. These decisions require attention to actual functional demands and to the learner's circumstances.

Responsible reliance is itself a possible educational aim. A learner can identify which result was supplied, understand the conditions under which it applies, check a consequential step, and recognize when expert assistance is required. That achievement differs from personally deriving every result. The proposed principle permits both aims while requiring the assessment to distinguish them. A learner who supplies the relevant judgment through continuing assistance may participate more substantively than a learner who writes every symbol while following an opaque script.

3 Antecedents in Tutoring and Educational Philosophy

This section locates the proposal through three selected antecedents: scaffolding, intellectual emancipation, and the purposes of education. The comparison identifies specific similarities and disagreements while retaining the present inquiry's narrower focus. These sources provide conceptual challenges and historical context; their relevance does not establish equivalence with the proposed principle.

3.1 Scaffolding and the learner's activity

The tutoring literature supplies a direct precedent for treating assistance as an organization of learner activity. The comparison here concerns the established functions of support and the interpretation of a learner's response.

Wood, Bruner, and Ross studied tutoring with thirty children aged three to five who worked on a block-pyramid construction task. Their discussion identifies functions including simplifying the task, marking relevant features, maintaining direction, controlling frustration, and demonstrating solutions. It also gives the tutor a model of the task and a model of the learner's performance, both of which inform the interaction. Demonstration can complete or clarify a partially attempted solution. These features make their account a substantial antecedent to learner-sensitive assistance [9].

The present principle adds an explicit question about which participation a defended educational purpose calls for. Simplifying a task can preserve an important learner act while removing irrelevant

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burdens. It can also eliminate precisely the discrimination the lesson intended to develop. The difference depends on the target and the learner's preparation. The observed success of a tutoring move therefore requires interpretation before it becomes a recommendation for a different activity or population.

There is also a question about direction. A tutor can help a learner persist with an agreed objective, but can also treat a legitimate change in purpose as distraction. A mathematical task has constraints that the learner must respect if the original result is sought. Whether that task deserves the learner's present effort is a further educational matter. Bringing this distinction into the account expands the space for criticism while preserving the practical importance of knowledgeable guidance.

3.2 Intellectual emancipation and common objects

The philosophical issue of authority concerns the relation between the learner's judgment and the teacher's judgment. Rancière supplies a demanding comparison because his account addresses intellectual subordination alongside the question of instructional efficiency.

In the first chapter of *The Ignorant Schoolmaster*, Rancière distinguishes the relation between wills from the relation between intelligences. His account of Jacotot gives the common book a role in the learner's activity while questioning the subordination of one intelligence to another. The argument also distinguishes its concern with emancipation from the comparison of pedagogical methods by speed [7]. The present paper treats this as a philosophical argument and narrated episode; it supplies no independent historical verification or controlled replication of the episode.

The comparison makes the object of inquiry significant. A learner should be able to examine the algebra, the diagram, or the source text through which a claim is justified. Teacher approval can support confidence, but an educational arrangement should also enable the learner to identify reasons that another participant could inspect. This offers a practical interpretation of intellectual participation without adopting every premise of Rancière's account.

The differences matter. The proposed principle permits explicit explanation by someone with greater current knowledge. It also accepts legitimate instructional direction, subject to limits and correction. Equality of standing in inquiry can coexist with unequal expertise. A novice can have a valid objection to one step while needing extensive instruction elsewhere. The educational challenge is to make expertise available without treating the teacher's authority as a substitute for the mathematical reason the learner is trying to understand.

3.3 Educational purpose and subject formation

The third comparison concerns the values embedded in a learning objective. Improved performance is educationally significant through an account of what the activity is intended to achieve and why that achievement matters.

Biesta distinguishes qualification, socialisation, and subjectification when discussing the purposes of education. He treats these functions as interacting and discusses mathematics explicitly, including its knowledge, social significance, and possibilities for a learner's more considered relation to established practices. His account also separates the effectiveness of a process from the desirability of its ends [3]. These arguments constrain any suggestion that concern for formation or the plural purposes of mathematics begins with AI.

The participation distinctions developed below serve a different local purpose. They identify acts through which a learner can take part in a mathematical inquiry: carrying out steps, helping direct the inquiry, examining reasons, and developing a revisable relation to the activity. They form neither a translation of Biesta's categories nor an exhaustive theory of subjectivity. Their adequacy depends on whether they help distinguish arrangements that would otherwise be misleadingly grouped together.

The resulting epistemological position is modest. Mathematical claims remain answerable to their definitions and arguments. Interpretations of learner activity remain answerable to appropriately designed evidence. Judgments about the purpose and burden of education remain answerable to explicit

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values and affected participants. A relational account can investigate how these practices develop together while retaining the distinct standards by which their claims are assessed. That separation supports criticism from the very traditions with which the paper seeks conversation.

4 Instructional Restraint and the Reconstruction of 不言之教

This section develops a contemporary educational interpretation of 不言之教. It first identifies the relevant classical passages and a commentarial comparison, then states the modern reconstruction and its limits. The method keeps textual evidence, interpretive judgment, and the proposed application to AI assistance distinct.

4.1 Textual and commentarial context

The historical comparison begins with the received text of the *Dao De Jing*. Chapters 2 and 43 connect the phrase to a wider discussion of non-action and accomplishment, which constrains a reading based solely on classroom silence.

Chapter 2 includes 行不言之教, alongside actions through which things develop and accomplishment remains unclaimed. In the consulted transcription, Wang Bi's commentary includes 自然已足，為則敗也 and later 因物而用，功自彼成，故不居也. Chapter 43 places 不言之教 beside 無為之益 [5]. These passages locate the phrase within a discussion of how activity and accomplishment proceed. The digital transcription provides a received textual and commentarial resource; questions about textual variants and historical authorship remain outside the present examination.

Legge translates the relevant part of chapter 2 in terms of conveying instruction without speech, and chapter 43 in terms of teaching without words together with non-action [6]. This is one verified translation. The present reconstruction draws attention to room for another's activity and to restraint in claiming its accomplishment. That interpretation is proposed for contemporary discussion, with a more extensive comparison of translations and commentaries remaining open.

4.2 Restraint within an active teaching relation

The modern proposal concerns how a teacher organizes assistance. It treats restraint as a selective decision concerning an intervention and a learner's activity, with the teacher remaining responsible for the conditions in which that decision operates.

Definition 4.1 (Instructional restraint)**.** Instructional restraint is a deliberate limitation, postponement, or redirection of an otherwise available intervention, adopted for a stated educational purpose and accompanied by attention to the learner's access, preparation, and opportunity for further assistance.

A teacher can exercise restraint by allowing time to inspect a diagram, by inviting a learner to evaluate a supplied proof, or by refraining from replacing an intelligible alternative method with a preferred one. Each case can involve considerable speech and preparation. The relevant contrast concerns the effect of an intervention on the learner's activity. Counting words or pauses would fail to capture that relation.

The reconstruction has an affirmative institutional dimension. The teacher may need to choose a tractable example, supply missing notation, ensure that a learner can access the material, and remain available when difficulty becomes unproductive. An AI assistant may help generate variants or convert an explanation into an accessible format. These actions establish conditions within which a limited restraint becomes useful. Silence without those conditions can leave a learner with an inaccessible task and little means to identify the difficulty.

The proposal also concerns accomplishment. When a learner develops an objection through a discussion, the teacher's contribution can be acknowledged without treating the objection as the teacher's achievement alone. The converse also matters: substantial assistance should remain visible when it supplied an argument. Accurate attribution supports a clear account of participation. It need not divide an inseparable learning encounter into artificial percentages of intellectual ownership.

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4.3 Interpretive limits and counterconditions

The reconstruction is evaluated through contemporary reasons and counterexamples. Its historical inspiration supplies a way of asking about instruction; it leaves the adequacy of the resulting educational proposal open to independent criticism.

A learner who lacks the meaning of a symbol may need a direct explanation. A teacher who continually withholds that explanation can transform a manageable difficulty into an opaque test of persistence. A learner who requests a complete worked example may be selecting an effective way to begin. An assistant that mechanically refuses the request would need a stronger justification than the presence of the word teaching in its instructions.

The same applies to errors. A misconception that blocks the activity can warrant immediate correction. A deceptive or harmful premise can warrant explicit challenge. Restraint should be judged relative to the educational purpose and the consequences of waiting. Its value depends on what it enables, what it burdens, and whether a less restrictive intervention would serve the purpose better.

There is therefore an important distance between the proposed principle and a literal rule of verbal abstention. The paper preserves that distance as part of its interpretation. It also suspends stronger metaphysical commitments that might accompany a complete account of the classical text. The modern argument concerns accountable assistance among learners, teachers, tools, and objects. Its credibility rests on the quality of that argument and its eventual empirical examination, while historical scholarship can refine the comparison that helped formulate it.

5 A Principle of Non-Substitutive Teaching

This section states the proposed principle and develops the participation distinctions needed to apply it. It then derives a limited teaching obligation from explicit normative premises. The analysis separates opportunities offered by an arrangement, the learner's actual activity, and the competence that may develop through that activity.

5.1 Participation within a mathematical inquiry

Participation concerns the learner's role in an activity with a declared purpose. Table 1 gives four overlapping lenses for examining that role. The table organizes possible observations; it supplies neither a psychometric scale nor a sequence through which every learner must pass.

Executional participation concerns carrying out relevant operations. Its significance depends on the target. Expanding $(a+h)^2$ can be central when a learner studies algebraic structure and incidental when a learner already commands that structure and is investigating the meaning of a limit. Automation can therefore support one purpose while displacing an act another purpose calls for. The same interface action can have different educational roles.

Directional participation concerns some influence over the route or object of inquiry. A learner may suggest changing the function, question an assumption, or request a different representation. The influence must be meaningful in context. An assistant that offers several cosmetically different explanations while enforcing a single unexamined interpretation supplies a narrow form of choice. A teacher who accepts a reasoned alternative can make the inquiry responsive even within a fixed curriculum.

Reflective participation concerns the assessment of reasons, assumptions, and limitations. A learner can explain why a cancellation requires a nonzero quantity, distinguish a finite difference from a derivative, or identify which part of an argument fails for another function. These acts are particularly relevant to the present example. Their observation still requires interpretation: reproducing the words of a reason and using that reason appropriately across a variation provide different evidence.

Subject-formative participation concerns how a learner comes to inhabit the activity: the capacity to treat confusion as examinable, seek help responsibly, revise a commitment, and pursue a question beyond an assigned answer. These are proposed objects of investigation. Their value and durability need further

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Table 1: Analytic lenses for learner participation in a derivative inquiry

Participation lens	Mathematical activity	Interpretive caution
Executional	Expanding a square, selecting an increment, checking a cancellation	Correct steps can follow an opaque script.
Directional	Selecting a relevant variation or proposing a different route	Selecting a menu item can leave the inquiry unchanged.
Reflective	Explaining a condition, finding an error, assessing a reason	Fluent explanation can reproduce supplied language.
Subject-formative	Developing a revisable relation to inquiry, help, error, and future questions	A single task gives limited evidence about durable formation.

argument and evidence. The term supplies no license to infer a transformed subject from a successful classroom exchange.

5.2 Purpose-relative assistance

The principle connects these possible acts to a justified aim. Its scope is local enough to guide a decision while remaining open to disagreement about the educational purpose itself.

Claim 5.1 (Non-substitutive teaching principle)**.** Where an educational purpose is justified, assistance should preserve or restore meaningful and accessible opportunities for the learner acts constitutive of that purpose, with the form and burden of participation responsive to the learner's circumstances and to legitimate competing aims.

An opportunity is meaningful when the learner has a practicable way to undertake the relevant act and the activity can respond to its outcome. Inviting a challenge while ignoring every challenge would fail this condition. Providing a proof in an inaccessible form would also fail it. The condition leaves room for supported performance and for a learner to decline a particular opportunity. An arrangement can make participation available without causing uptake or guaranteeing learning.

The word constitutive requires care. If the aim is understanding why the derivative of a square has its stated value, some grasp of the argument belongs to that aim. Merely obtaining the expression serves a narrower product aim. Yet the learner need not personally perform every operation used to reach understanding. A teacher can provide the algebraic expansion and ask the learner to examine the limiting argument. The task must identify which part of the reasoning the learner is currently learning to command.

5.3 Normative derivation and scope

The normative argument proceeds from a defended educational relationship to a limited reason for action. This order prevents a fact about task performance from carrying an unstated conclusion about what teachers must do.

Assumption 5.2 (Educational purpose and participation)**.** A justified educational purpose gives relevant educators a reason to sustain the learner acts through which that purpose can be achieved. Learners have standing in the interpretation and revision of their participation, within legitimate institutional responsibilities. The burdens of assistance and practice require proportionate comparison with feasible alternatives.

First, specify and defend the purpose. Understanding a derivative argument is a possible aim; its role in a course and its relevance to the learner require an explanation. Second, identify the learner acts that the aim contains or requires. For this example, distinguishing a nonzero quotient from its limit is one such act. Third, identify a mechanism by which the proposed arrangement makes that act accessible or

systematically displaces it. This step needs evidence about the arrangement and the learner, beyond the teacher's intention.

Fourth, determine who can alter the relevant conditions and what responsibilities that person or institution has undertaken. A classroom teacher, a platform designer, and a peer have different roles. Fifth, compare feasible changes, including direct instruction, a worked example, a pause, a simpler prerequisite task, and continuing accommodation. Account for time, frustration, access, and the learner's own purposes. Sixth, state the resulting defeasible reason to choose or revise the arrangement. A final obligation requires the relevant competing claims to be resolved.

The argument supports a bounded conclusion. When a feasible adjustment materially restores participation required by a justified aim at an acceptable burden, an educator with the relevant responsibility has a reason to make that adjustment. The conclusion can favor more explanation or less intervention. It cannot be read from the number of answers supplied, the number of questions asked, or the mere presence of AI.

6 A Worked Derivative Argument

This section supplies the mathematical object used in the teaching comparison. It defines the symbols and domains, derives the derivative of a square, and examines two extensions and a counterexample. Each inferential step is made explicit so that later claims about learner participation refer to identifiable mathematical acts through precise requirements for understanding.

6.1 Domains, increments, and the target limit

The mathematical setting is real elementary calculus. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, let $a \in \mathbb{R}$ be a fixed point, and let $h \in \mathbb{R} \setminus \{0\}$ be an increment. The two points a and $a + h$ then determine an average change per unit increment.

The standard difference-quotient and derivative definitions provide the starting point [8]. Write $Q_f(a, h)$ for the quotient and define the derivative at a when the indicated finite two-sided limit exists:

$$Q_f(a,h) = \frac{f(a+h) - f(a)}{h}, \qquad f'(a) = \lim_{h \to 0} Q_f(a,h). \tag{6.1}$$

Equation 6.1 distinguishes two objects. The quotient is evaluated at a nonzero increment. The derivative is a limit of those quotient values as the increment approaches zero through permitted values. Substituting zero into the original denominator would leave the quotient undefined. The definition asks about nearby values and their convergence.

Assumption 6.1 (Elementary real operations)**.** The argument uses real addition and multiplication, their distributive laws, cancellation by a nonzero factor, and the epsilon–delta definition of a finite two-sided limit. The point a remains fixed while h varies.

Stating the assumptions separates prerequisite knowledge from the new object of instruction. A learner who cannot expand a square may need algebraic support before examining convergence. A learner who can expand it may still need help understanding which variable is fixed. Both difficulties can produce an incorrect answer, but they call for different interventions.

6.2 Expansion, cancellation, and convergence

The derivation first simplifies the quotient on its permitted domain and then establishes its limit. This order makes the role of the nonzero increment explicit and supplies a precise target for explanation or error diagnosis.

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Substitute the specified function into the numerator. Distributivity gives $(a+h)^2 = a^2 + 2ah + h^2$. Subtracting a^2 leaves $2ah + h^2 = h(2a + h)$. Because $h \neq 0$, division by that common factor is permitted. The complete sequence is

$$Q_f(a,h) = \frac{(a+h)^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h, \qquad h \neq 0. \tag{6.2}$$

Equation 6.2 is an equality on the punctured increment domain. The expression $2a + h$ is also defined at zero, but the original quotient is not. Their equality at nonzero increments is sufficient for the limiting argument.

Proposition 6.2 (Derivative of the square function)**.** *For every fixed real point a, the derivative of $f(x) = x^2$ exists and equals $2a$.*

Proof. Let $\varepsilon > 0$ be arbitrary and choose $\delta = \varepsilon$. For an increment satisfying $0 < |h| < \delta$, Equation 6.2 gives $|Q_f(a,h) - 2a| = |h| < \varepsilon$. This is the epsilon–delta condition for convergence to $2a$. Equation 6.1 therefore gives $f'(a) = 2a$. The argument applies to each fixed real a. □

The proof separates the algebraic and limiting warrants. Cancellation gives a simpler expression for each nonzero increment. The bound then ensures that the quotient can be made as close to $2a$ as any prescribed positive tolerance requires. The numerical expression alone leaves these two warrants unstated.

6.3 Local approximation and nearby functions

The next comparisons identify aspects of the argument that survive a change of target. They supply mathematically interpretable variations for teaching, while their educational difficulty remains a question for learner-sensitive investigation.

The expansion also provides an exact decomposition of the changed value:

$$f(a+h) = a^2 + 2ah + h^2, \qquad \frac{|f(a+h) - f(a) - 2ah|}{|h|} = |h| \quad (h \neq 0). \tag{6.3}$$

Equation 6.3 identifies $2ah$ as the linear change and h^2 as the residual. The absolute residual tends to zero; its size relative to $|h|$ also tends to zero. The latter comparison explains the first-order approximation. A small absolute error by itself would be a weaker observation because many expressions can have small errors near a fixed point without giving the correct first-order coefficient.

For $g(x) = x^2 + c$ with a fixed real constant c, subtraction removes the same constant from $g(a + h) - g(a)$. The quotient is again $2a + h$, so the previous convergence proof applies. A learner can use this variation to explain which part of the function changes the output level and which part changes the local rate. The same explanation would need revision if the added term varied with x.

For $r(x) = x^3$, distributive expansion gives $(a + h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$. Subtracting a^3, factoring h, and using $h \neq 0$ yields

$$Q_r(a,h) = 3a^2 + 3ah + h^2. \tag{6.4}$$

Equation 6.4 suggests the candidate limit $3a^2$, which can be checked directly. Given $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon/(3|a| + 1)\}$. If $0 < |h| < \delta$, then $|3ah + h^2| \leq (3|a| + |h|)|h| < (3|a| + 1)\delta \leq \varepsilon$. Thus the cube's derivative at a equals $3a^2$. The denominator $3|a| + 1$ is strictly positive for every real a, including zero, so the choice of δ is valid throughout the stated domain.

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6.4 A counterexample at a corner

The final comparison identifies an assumption that a familiar-looking graph can conceal. Let $v(x) = |x|$ and assess the derivative at zero using the same definition. The quotient is $|h|/h$, which equals 1 for positive h and -1 for negative h.

If a finite two-sided limit existed, its value would have to agree with the limit along positive increments and with the limit along negative increments. Those constant values differ. More explicitly, convergence to a common value L would require both $|1-L| < 1/2$ and $|-1-L| < 1/2$ for sufficiently small increments of the respective signs. The triangle inequality would then give $2 = |1-(-1)| < 1$, a contradiction. Hence the derivative at zero fails to exist under the stated definition.

This counterexample preserves the role of the domain and of two-sided convergence. A finite table of positive increments would miss the negative branch. A visually persuasive tangent sketch would still require interpretation through the definition. The mathematical result is elementary and established; its use here is to specify an object whose reasons, extensions, and failure conditions a learner can investigate with substantial assistance.

7 Hypothetical Arrangements for the Derivative Inquiry

This section compares four hypothetical uses of the derivative argument. The cases hold the mathematical object sufficiently stable to expose differences in participation, assistance, and educational purpose. They establish conceptual possibilities and design questions; they supply no observations about actual learners or estimates of instructional effects.

7.1 A supplied result and a completed task

The first arrangement concerns a learner whose immediate activity ends when a correct expression has been obtained. An assistant supplies the derivative, the learner records it, and a system marks the response correct. The educational interpretation depends on the target that this interaction was intended to serve.

If the learner needed the derivative as a component of a different calculation, the exchange may have fulfilled its purpose. The learner might already understand differentiation and reasonably delegate a routine operation. The use of a supplied expression would then provide little evidence of a learning failure. It would also provide little new evidence about the learner's competence in differentiation.

Suppose, however, that the declared purpose was to understand the difference quotient and its limit. The visible record now leaves the relevant learner acts largely unobserved. It shows that a correct product entered the worksheet. It leaves open whether the learner distinguished $h \neq 0$ from $h = 0$, recognized the algebraic factorization, or understood the convergence argument. A teacher can acknowledge completion of the product while retaining an open assessment of the intended learning.

The normative criticism requires a further step. If the arrangement repeatedly treats the product as sufficient, removes meaningful opportunities to examine its reasons, and has a feasible accessible alternative within the educator's responsibility, the proposed principle supplies a reason for revision. An isolated use of an answer source cannot establish that entire pattern. Prior competence, subsequent discussion, or a different legitimate aim may change the assessment.

7.2 A complete solution used for examination

The second arrangement supplies the full derivation at the outset. The learner then examines the equality between the quotient and $2a + h$, identifies the domain on which that equality holds, and explains how the epsilon–delta bound establishes the limit. The supplied solution becomes an object of inquiry.

The teacher can ask the learner to select a step that needs explanation and can accept a diagram, spoken account, symbolic annotation, or comparison with another function. The learner may discover that the cancellation is clear while the meaning of an arbitrary tolerance remains uncertain. That response

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changes the next instructional act. The teacher might work through a numerical tolerance before returning to the general argument.

This arrangement preserves a substantial contribution by the assistant. The assistant supplied the proof and possibly the examples. Accurate attribution can coexist with meaningful learner participation in understanding and testing those materials. The learner's later performance on a variation provides additional evidence, but a successful variation should be interpreted according to its actual demands. A constant shift and a corner function test different relations.

The case also preserves the possibility of failure. A learner could memorize the teacher's preferred explanations without gaining the intended discrimination. The full solution creates an opportunity for examination; it does not guarantee uptake. The proposed principle concerns how the arrangement responds to such evidence and whether a learner can disclose uncertainty or choose a different route.

7.3 Leading prompts and inaccessible prerequisites

The third arrangement uses only questions and incremental hints. The learner encounters a symbol whose meaning is unclear, while the assistant repeatedly requests the next step and treats each request for explanation as an attempt to avoid effort. The interaction contains many learner responses and little successful mathematical orientation.

This case exposes the limits of a participation count. The learner may type extensively while trying to infer the expected wording. The assistant may control the route through a sequence of narrowly acceptable answers. If a missing definition prevents the learner from interpreting the prompts, more prompts can intensify the obstacle. A direct explanation of the definition could restore the intended activity with less burden.

The teacher should also consider evidence that the learner has a legitimate alternative approach. A student may describe local error through a table or explain a cancellation in words before adopting the expected notation. The response can be mathematically promising even when it falls outside an automated answer pattern. A flexible arrangement allows the representation to change while preserving the target distinction.

Question-led teaching remains a possible method. A well-chosen question can reveal uncertainty, invite a conjecture, or help a learner notice a consequential assumption. The criticism concerns an arrangement that identifies the method's surface form with its educational purpose. The principle requires attention to the learner's actual access and the possibility of changing the intervention.

7.4 Continuing support and substantive judgment

The fourth arrangement concerns a learner who uses speech input and a symbolic editor throughout the lesson and assessment. The learner specifies the reasoning, requests an expansion, evaluates the nonzero condition, and rejects an invalid extension to the absolute-value case. The tools remain part of the activity.

If the intended competence concerns interpreting and checking the derivative argument, those tools may preserve access to the relevant acts. Removing them could introduce demands concerning handwriting, visual presentation, or motor control that the mathematical target never required. The educational assessment should identify what the support supplied and what judgment the learner undertook.

An assistant can still displace a target act within this arrangement. If it silently supplies the learner's error diagnosis, the evidence for that diagnostic competence changes. A careful account therefore examines contributions at the level relevant to the target, and allows judgments about appropriate aid to vary with its actual contribution. The same learner may benefit from different support configurations for different tasks.

Across the four cases, the operative distinction concerns the relationship among purpose, assistance, opportunity, and uptake. Direct answers, complete explanations, questions, and continuing tools each admit more than one educational use. That conclusion supports a design inquiry with explicit criteria and empirical obligations. It leaves room for learners and teachers to revise those criteria when the task or its purpose changes.

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8 Empirical Comparisons of AI Tutoring Designs

This section examines two recent studies that bear on the distinction between assisted task performance and educational outcomes. It identifies the scope of each design and then develops the limits of a cross-study comparison. The studies constrain broad claims about AI assistance while leaving the proposed teaching principle open to direct evaluation.

8.1 High-school mathematics review

The first study is closely related to the paper's mathematical setting. Its design distinguishes an assisted practice period from a subsequent assessment and compares two tutoring packages with a control condition.

Bastani and colleagues studied nearly one thousand students in a Turkish high school through four mathematics review sessions. The base interface and a specialized tutor both improved practice performance, while the base arm subsequently performed worse on immediate closed-resource examinations. The specialized tutor largely mitigated that disadvantage, with a positive examination effect remaining undetected. Its package included teacher-supplied solutions, common-error guidance, and restrictions on supplying complete answers. The study's assignment construction used existing classroom allocation and matched arm assignments under scheduling constraints; the main sample excluded honors classes. The authors distinguish their short-term examination outcome from longer-term learning [2].

The published correction was inspected before use of the article. The notice corrects Osbert Bastani's institutional affiliation following a production error; it states no correction to results or analysis [1]. That observation describes the notice and supplies no independent validation of the findings.

For the present argument, the study demonstrates the value of specifying the assessed occasion and available support. An improved practice product and an improved later assessment are different outcomes. Their divergence within an instructional package merits investigation of the interaction that produced them. It also cautions against treating convenience or immediate success as sufficient evidence for the declared learning aim.

The comparison leaves several mechanisms intertwined. A design can change answer availability, error feedback, the correctness of supplied material, the interpretation of a student response, and the time spent on a step. A conclusion about one component requires a design that distinguishes its contribution from the others. The paper therefore takes the study as motivation for a more precise comparison of assistance conditions, while leaving a universal rule against complete solutions unsupported.

8.2 Customized tutoring in introductory physics

The second study provides a positive comparison with an active classroom setting. Its relevance concerns the possibilities of carefully organized AI assistance and the need to identify the complete instructional package.

Kestin and colleagues report a two-lesson crossover study in Harvard introductory physics during autumn 2023. Of 233 enrolled students, 194 met the stated eligibility conditions. Assignment preserved existing peer groups, and students experienced a customized at-home AI lesson and an in-class active-learning lesson on different topics. The AI condition had higher short-term post-test performance, a shorter median study time, and higher reported engagement and motivation. The design included prepared content, videos, and structured feedback; the comparison therefore concerns those particular lesson arrangements [4].

This evidence matters because the proposed principle should remain open to AI-supported formation. A system can be designed to make relevant reasoning accessible, adapt a pace, and respond to a learner's attempts. The possibility that an AI arrangement performs well on a specified assessment is compatible with concern about other arrangements that displace important learner acts.

The present philosophical argument asks a further question about the interpretation of an outcome. A short-term post-test can support a conclusion about the assessed material and conditions. Longer

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retention, transfer to a changed problem structure, responsible reliance, and a learner's relation to future inquiry require additional observations. Those distinctions apply to human teaching and to AI teaching alike. Their purpose is to preserve the scope of evidence without discounting meaningful measured improvements.

8.3 Design differences and inferential limits

The two studies should be compared through their questions and designs. A single ranking would conceal differences in population, subject, prior instruction, assignment structure, support, and assessment occasion. The present analysis uses them to identify research obligations while leaving pooled effect estimation outside the comparison.

An instructional package can provide accurate problem-specific material and still leave some learner acts unexamined. Another package can preserve reflective participation while producing a modest immediate score because it spends time on a difficult distinction. These are possibilities to test, not reasons to disregard outcomes. A useful investigation would measure both the intended learning and the mechanisms through which the arrangement is expected to support it.

The distinction between a treatment effect and a mechanism is especially important. Assignment to an instructional arrangement can support an estimate of that arrangement's effect under appropriate assumptions. An observed association between particular conversational behavior and performance does not automatically identify the causal contribution of that behavior. Prior preparation may influence both. Learners may also seek more assistance precisely when a task is difficult. A mediation claim needs a design and analysis that address such alternatives.

The comparison also raises a resource question. A customized tutor can depend on substantial work by teachers, designers, and evaluators. A short learner interaction may rest on a longer preparation process. Efficiency therefore needs a declared unit: learner time, educator time, development cost, correction work, or institutional expenditure. A benefit in one unit should remain visible without being silently generalized to every other unit.

The appropriate empirical posture is consequently design-sensitive. The proposed principle predicts neither universal improvement from AI nor universal decline. It directs attention to the target acts, the accessibility of their performance, the conditions of assistance, and the evidence connecting an arrangement to learning. Those are variables for further investigation, with educational purpose remaining an additional normative question.

9 Assessment of Participation and Learning

This section develops an assessment approach suited to the proposed principle. It begins with the inferential limits of a correct product, identifies observations that can distinguish relevant competencies, and outlines a prospective study. The aim is proportionate evidence about declared targets, with assistance and accessibility included in the assessment conditions.

9.1 The evidential role of a correct product

A product provides evidence about its own mathematical properties before it supports an inference about the learner. The first task is therefore to identify which participant or tool supplied the relevant reasoning and which educational claim the observation is being used to support.

Consider two stipulated cases. In one, a learner can justify the nonzero cancellation and the limit but elects to record only $2a$. In the other, a learner records the same expression after receiving it and remains unable to explain either step under the declared conditions. The visible product is identical while the target competence differs. Any assessment rule that receives only that product must assign it the same interpretation in both cases. Such a rule cannot distinguish the stipulated competence difference without additional information.

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Table 2: Assessment targets and evidence limits in the derivative inquiry

Assessment target	Possible observation	Remaining obligation
Domain interpretation	Explain the role of $h \neq 0$ in cancellation	Distinguish usable understanding from repeated wording.
Structural transfer	Compare a constant shift with a cubic function	Identify which new algebra and limit demands were introduced.
Error diagnosis	Reject a claimed two-sided derivative of $\|x\|$ at zero	Check both signs of the increment and the learner's reason.
Retention	Revisit a matched task after a declared delay	Record intervening practice, assistance, and changed conditions.
Responsible reliance	Identify supplied steps and a consequential independent check	Assess the justification of reliance for the actual purpose.

This elementary non-identification argument is deliberately limited. It establishes the insufficiency of one observation across a stated pair of possibilities. It supplies no estimate of how frequently either case occurs. It also leaves open the possibility that prior records or a carefully constructed task provide informative evidence. The educational response should improve the inference through proportionate additional evidence.

9.2 Reason-sensitive variations and support conditions

Additional observations should be selected for the mathematical distinctions they can reveal. Table 2 organizes examples drawn from the derivative argument. Each observation needs a clear account of what support remains available and what alternative explanation could produce the same response.

A reason-sensitive variation changes an assumption or structural feature so that a previously adequate response requires reconsideration. Adding a constant to the square preserves its difference quotient. Replacing the square by an absolute value at zero changes the existence of the two-sided limit. A learner's explanation of the contrast can support an inference about which features the learner treats as relevant.

The assessor should permit alternative expressions of that explanation where the target allows them. A spoken comparison, a symbolic proof, and a carefully interpreted diagram can provide different but useful evidence. The use of a preferred academic vocabulary should not become an unannounced additional condition of mathematical understanding. At the same time, an explanation must remain precise enough for the claim being assessed. A diagram may locate the issue while a limit argument supplies a stronger warrant.

Support conditions should be reported at the level that matters. A notation aid may preserve the target act; a generated explanation may supply it. The same tool can do both in different interactions. An assessment can deliberately test supported competence, unaided competence, or the transition between them. Its conclusion should name the selected target and preserve accommodations unrelated to the intended demand.

9.3 A prospective instructional comparison

A direct empirical study would test the proposed principle through arrangements with explicit mechanisms. The following design is a research proposal, with feasibility, consent, measurement quality, and analysis requiring preparation before implementation.

Participants would first receive an accessible orientation to the mathematical prerequisites. The study would then compare instructional packages with the same intended content and a declared time budget. One package could supply a complete worked solution with structured opportunities to examine its assumptions. Another could use incremental hints with responsive explanation when prerequisites are

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missing. A third could supply the solution for ordinary self-directed use. The aim would be to distinguish these actual packages, not to infer their character from labels such as active or passive.

An independent assessment team would prepare tasks that examine specified mathematical distinctions without merely repeating the teaching prompts. Immediate product accuracy, explanation, error diagnosis, and a delayed variation would be reported separately. The study would document assignment, missing observations, prior preparation, intervening practice, and assistance conditions. Where feasible, assessors would evaluate anonymized responses without knowing the instructional condition. A design involving shared classes or peers would need to account for those dependencies.

Evidence about process would serve a limited role. Conversation records might show a learner identifying a contradiction, requesting a definition, or changing an interpretation. Such records would be interpreted alongside performance and the learner's own account. Their use should be proportionate and consensual, with private exploratory material protected. A complete log is unnecessary for many educational decisions, and recording can itself alter participation.

The study should also seek counterevidence. The worked-solution package might encourage imitation of explanations. The hint package might impose greater burden on learners with particular access needs. The ordinary self-directed package might work well for prepared learners who already know how to interrogate a solution. Such findings would refine the principle's application. They would also help distinguish the normative aim of sustaining participation from an empirical prediction that one interface reliably realizes it.

An adequate report would preserve multiple outcomes and explain any tradeoff. Faster completion, better delayed reasoning, greater frustration, and lower preparation cost can move in different directions. Combining them into one recommendation requires stated educational priorities and an account of who bears the costs. The proposed principle supplies one relevant reason; its role in a final judgment remains open to deliberation.

10 Educational Authority and Shared Conditions

This section examines the authority required to organize learner participation. It separates expertise from control of educational purposes, considers the roles of teachers and AI systems, and identifies conditions that institutions must sustain. The argument treats learner choice and proportionate support as integral to the proposed principle.

10.1 Expertise and the interpretation of difficulty

Instruction often involves an asymmetry of knowledge. A teacher may recognize the relevant definition, anticipate an error, and understand which prerequisite would make a task tractable. That expertise can support the learner's participation when its judgments remain responsive to evidence.

The asymmetry creates an interpretive risk. The teacher may classify a learner's hesitation as avoidance when the difficulty concerns an unfamiliar notation, a different method, or an access barrier. An AI assistant can reproduce a similarly narrow interpretation through its prompts and acceptance rules. The appropriate response is to make revision possible: invite the learner to identify the difficulty, inspect an alternative representation, and test whether the proposed explanation actually addresses it.

A learner's correction need not be accepted merely because it is sincerely offered. Mathematical reasons constrain the inquiry. If a learner proposes cancellation at $h = 0$, the teacher can explain the undefined denominator and demonstrate the valid punctured-domain argument. Respect for participation includes making such correction intelligible. It also includes recognizing a valid alternative when it falls outside the anticipated route.

10.2 Learner purposes and institutional commitments

Educational purposes are shaped by learners, teachers, curricula, and institutions. This subsection identifies the conditions under which a purpose can guide assistance without being treated as an unquestionable

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preference of the teacher.

A course can legitimately specify foundational competencies and assess them. A learner's immediate preference for an answer does not automatically dissolve every such commitment. Conversely, enrollment does not authorize unlimited burdens or every possible intervention in the learner's intellectual life. The institution should explain the relevance of the target, provide accessible means of pursuing it, and permit appropriate routes for challenge or accommodation.

The distinction between an educational relationship and an instrumental service is especially important for an assistant. A user may seek a formula for a practical task, or may explicitly request tutoring. The system should make its instructional mode intelligible and allow appropriate changes. A universal policy that turns every request into a lesson would impose an educational purpose even where the user has selected a different legitimate activity.

Where the learner's purpose conflicts with an institutional requirement, the conflict should be named. A teacher might explain that a particular assessment concerns independent diagnosis of a limit argument because later work depends on that capacity. The learner might identify an unnecessary presentation burden or propose another way to demonstrate the same competence. The result can be a revised arrangement that preserves the purpose while changing the means.

10.3 Teacher labor and the organization of support

The proposed principle depends on shared conditions that extend beyond an individual interaction. This subsection considers preparation, correction, continuity, and access as parts of the educational arrangement whose distribution affects its feasibility.

Someone must select tasks, verify mathematical material, prepare accessible representations, review misleading outputs, and interpret difficult cases. AI assistance may reduce some of this work while adding other forms of evaluation and maintenance. A comparison of efficiency should therefore include the relevant labor and its allocation. A system that produces fluent explanations quickly may still require substantial checking before those explanations can responsibly guide learners.

Continuity matters as well. A learner may develop a useful relation to a tutor, a peer group, or a shared set of materials. Abrupt changes in access can alter the conditions under which a supported competence can be exercised. The institution should identify which supports are expected to continue, which are temporary, and how transitions will be handled. This concern supports preparation and clear expectations; it supplies no general entitlement to preserve every particular service indefinitely.

Shared inquiry also needs a place for objects that can correct the interaction. In calculus, a definition, a counterexample, or an independently checked derivation can expose an error even when both learner and assistant initially accept it. Teachers and systems should make these objects accessible and retain their provenance where relevant. Agreement within a conversation is then one event in the inquiry, with the mathematical warrant available for further inspection.

The resulting account of authority is differentiated. Expertise supports explanation and correction. Institutional responsibility supports the organization of accessible conditions. Learner standing supports meaningful participation and revision. Mathematical objects constrain the validity of conclusions. These roles can reinforce one another, but none supplies a complete substitute for the others. Their coordination is an ongoing practical and normative task.

11 Objections and Boundary Conditions

This section tests the proposed principle against objections concerning instruction, efficiency, paternalism, measurement, and technological adaptation. Each objection identifies a condition under which the principle could be misapplied or would require refinement. The aim is to preserve a defensible local contribution while exposing the limits of its present formulation.

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11.1 The value of direct instruction

The first objection holds that the principle risks discouraging explanations that learners need. A novice may benefit from a clear demonstration precisely because the learner lacks the resources to construct the argument independently.

This objection constrains the interpretation of non-substitution. The principle permits a teacher to supply a complete solution and to perform difficult intermediate operations. Its demand concerns the opportunities relevant to the intended learner achievement. If seeing an explicit model enables the learner to interpret a previously opaque process, direct instruction may be the most appropriate intervention. A prohibition on demonstrations would contradict the purpose-relative formulation.

The remaining empirical question is how different forms of explanation affect particular learners. A proof followed by a meaningful variation might support understanding; the same proof could be memorized mechanically. A sequence of hints could promote insight or accumulate confusion. The principle therefore guides the questions posed to an instructional design while leaving its effectiveness to investigation.

11.2 Efficiency and division of epistemic labor

The second objection concerns the cost of asking learners to participate in every relevant operation. Human inquiry depends on division of labor, and competent practice often includes using results that someone else has established.

The objection succeeds against an expansive interpretation that requires complete personal reconstruction. The present formulation permits targeted delegation. A learner may rely on an algebraic expansion while investigating a limit, or rely on a derivative rule while studying a later application. Responsible reliance can itself become the target, including knowing which assumptions matter and when a result requires further checking.

An educational judgment should also consider opportunity costs. Time spent reconstructing a routine operation can displace a more valuable inquiry, rest, or another responsibility. The justified aim and the learner's preparation determine the appropriate burden. The principle asks for meaningful participation in a bounded purpose, not maximal effort or permanent intellectual self-sufficiency.

11.3 Paternalism and imposed formation

The third objection concerns a teacher's claim to know what kind of subject the learner should become. An appeal to formation can justify extensive control when its desired outcome remains vague or insulated from criticism.

The proposed account responds by making the purpose and the intervention explicit. A teacher should identify the competence or opportunity at issue, explain its relevance, and acknowledge the learner's standing in its interpretation. Broad claims about developing the whole person cannot replace that local justification. The learner may reasonably prefer an instrumental answer, a different topic, or a pause.

Institutional commitments still matter. A course can defend requirements and make their consequences clear. The principle therefore preserves a tension between learner choice and legitimate educational responsibility, with competing claims assessed in context. The quality of the justification, accessibility of the means, and available routes for revision remain central.

11.4 Measurement and the appearance of participation

The fourth objection holds that participation may become another proxy imposed on learners. Required explanations, visible revisions, or a quota of questions could reward performances designed for the assessment while concealing the competence of interest.

This risk applies to every lens proposed in the paper. Their purpose is to identify possible evidence and alternative interpretations. They should remain revisable analytic distinctions, with the assessor responsible for the inference. A learner who quietly understands a proof need not generate a theatrical

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record of discovery. A learner who uses a different expressive mode should have an appropriate way to demonstrate the target competence.

The response also limits data collection. More detailed monitoring can burden or distort the activity. Evidence should be proportionate to the decision and collected with suitable protections. A modest follow-up task may answer a question more appropriately than a complete record of exploratory conversation.

11.5 Adaptive AI and changing capabilities

The final objection concerns technological change. An AI system may increasingly identify misconceptions, generate useful variations, and help learners examine reasons. A principle founded on an assumption of permanently inferior AI tutoring would be unstable.

The present proposal makes a functional distinction among roles and outcomes. It permits a system to support the relevant learner acts whenever evidence warrants that description. It also subjects human teaching to the same questions about purpose, accessibility, and displacement. Claims about AI consciousness or moral personhood remain separate from this analysis of instructional function. Future capabilities can change which arrangements work well while leaving the need to justify targets and interpret evidence intact.

12 Open Questions and a Research Programme

This section identifies the questions that remain after the conceptual and mathematical analysis. It organizes them around operationalization, causal mechanisms, longer-term formation, and institutional design. The proposed programme treats counterevidence and participant criticism as resources for revising the principle and its applications.

12.1 Operationalization across learners and tasks

The first research area concerns the identification of target-constitutive acts. A useful operationalization must preserve the mathematical purpose while allowing variation in preparation, access, representation, and legitimate reliance on tools.

Open Problem 12.1 (Participation-sensitive task design)**.** Develop and validate task families that distinguish relevant learner acts under declared support conditions, while allowing multiple accessible forms of response and making alternative explanations of performance explicit.

For introductory calculus, a candidate family could compare a square, a constant shift, a cubic, and a corner function. The mathematical relations among these tasks are explicit, but their educational ordering remains an empirical question. Some learners may understand the corner through a graph before interpreting symbolic limits. Others may require a different prerequisite route. Validation should examine these differences with sequences open to learner-sensitive revision.

The operationalization should also identify what would count against the proposed interpretation. If a learner gives the expected explanation but repeatedly applies it to cases where its conditions fail, the evidence for understanding changes. If a learner succeeds through an unanticipated representation, the assessment may need revision. A credible method must be able to learn from both patterns.

12.2 Mechanisms and comparative interventions

The second research area concerns how assistance affects learning. The conceptual distinction between an available opportunity and its uptake calls for empirical designs that observe both without conflating them.

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Open Problem 12.2 (Mechanism identification in instructional packages)**.** Determine how explanation, response restrictions, feedback accuracy, pacing, and learner control jointly affect specified learning outcomes, using designs that can distinguish package effects from the contribution of individual components.

A component may have different effects depending on another component. Withholding an answer could be useful when responsive explanation is available and burdensome when the system misinterprets a prerequisite difficulty. A complete solution could support reflection when accompanied by a well-chosen variation and encourage premature completion in another setting. Such interactions challenge a simple ranking by amount of assistance.

Research should also distinguish realized learning from the learner's perception of learning, and both from enjoyment or frustration. These outcomes can inform an educational judgment without being treated as interchangeable. The relevant comparison should specify the population, target, assistance conditions, and horizon before interpreting its findings.

12.3 Formation and the conditions of future inquiry

The third research area concerns changes that extend beyond a single assessed competence. The retained discussion emphasizes future possibilities of questioning, interpretation, and participation; these ideas need careful evidence and normative scrutiny.

Open Problem 12.3 (Longer-term epistemic formation)**.** Investigate whether and how particular assistance arrangements affect learners' later recognition of problems, revision of reasons, help-seeking, and participation in shared inquiry, while distinguishing those changes from prior dispositions and changes in available opportunities.

Longitudinal evidence would need to record the changing environment as well as the learner. A person may ask fewer questions because understanding has improved, because a setting discourages uncertainty, or because relevant opportunities have disappeared. The same observable pattern can have different interpretations. A relational account should therefore investigate the conditions through which a capacity can be exercised, together with evidence about the capacity itself.

The desirability of a change also remains open. Specialization can support valuable expertise while narrowing some activities. A learner may reasonably leave a subject after acquiring enough knowledge for a chosen purpose. Formation should remain a field of plural judgments with legitimate specialization and ending included.

12.4 Institutional and interpretive development

The final research area concerns governance of the principle itself. Institutions would need to specify who defines educational targets, who can challenge a support restriction, and how the work of preparation and correction is allocated.

Further philosophical work should examine disagreements about the meaning of participation, the limits of instructional authority, and the relation between common objects and personal judgment. Further historical work should compare the proposed reconstruction of 不言之教 with additional translations, commentaries, and educational interpretations. These inquiries can alter the vocabulary and narrow the contribution.

The programme has an explicit revision condition. If the principle contributes only a new label for already adequate target-sensitive teaching, its value should be stated as synthesis or clarification. A stronger contribution would require showing that it identifies a consequential distinction, improves a decision, or supports a better research design. The present paper offers those possibilities for examination while leaving conceptual priority and general effectiveness open.

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13 Conclusion

This concluding section states the paper's bounded contribution and the obligations that remain. It brings together the proposed principle, the worked mathematical analysis, and the implications for the design and assessment of educational assistance.

The paper has proposed a principle of non-substitutive teaching for AI-assisted introductory calculus. Within a justified educational purpose, assistance should sustain accessible opportunities for the learner acts constitutive of that purpose, with burdens and competing aims assessed proportionately. The principle permits direct explanation, full worked solutions, continuing tools, and legitimate division of epistemic labor. It also requires scrutiny of question-led arrangements when they obscure reasons or prevent the learner from identifying a difficulty.

The derivative example gives that proposal a concrete object. Expansion, cancellation on a nonzero domain, and convergence provide distinct warrants for the derivative of a square. Constant shifts, cubic functions, and a corner case expose which aspects of the reasoning survive a variation. These mathematical results establish the validity of the argument under its assumptions. A learner's acquisition of the relevant competence remains a further question, requiring evidence appropriate to the task and assistance conditions.

The philosophical comparisons locate the proposal within existing discussions of scaffolding, intellectual emancipation, educational purpose, and instructional restraint. The reconstruction of 不言之教 is offered as a contemporary interpretive contribution, with its distance from a complete historical reading made explicit. The empirical comparison likewise preserves the scope of particular tutoring designs and short-term outcomes. It supplies reasons for design-sensitive inquiry while keeping wider judgments about AI in education open.

The practical contribution is a sequence of accountable questions expressed through definitions, assumptions, and limited claims: identify the purpose, specify the learner acts, examine the mechanism of assistance, interpret the evidence, and assess the authority and burdens of the arrangement. Future work must test whether these distinctions improve instruction and evaluation, especially across varied learners and longer horizons. Their present role is to make an important educational concern more precise and more open to criticism.

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References

[1] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman. Correction for bastani et al., generative AI without guardrails can harm learning: Evidence from high school mathematics. *Proceedings of the National Academy of Sciences*, 122(34):e2518204122, 2025.

[2] Hamsa Bastani, Osbert Bastani, Alp Sungu, Haosen Ge, Özge Kabakcı, and Rei Mariman. Generative AI without guardrails can harm learning: Evidence from high school mathematics. *Proceedings of the National Academy of Sciences*, 122(26):e2422633122, 2025.

[3] Gert Biesta. Good education in an age of measurement: On the need to reconnect with the question of purpose in education. *Educational Assessment, Evaluation and Accountability*, 21(1):33–46, 2009.

[4] Greg Kestin, Kelly Miller, Anna Klales, Timothy Milbourne, and Gregorio Ponti. AI tutoring outperforms in-class active learning: An RCT introducing a novel research-based design in an authentic educational setting. *Scientific Reports*, 15:17458, 2025.

[5] Laozi, traditionally attributed. Dao de jing: Wang bi text and commentary. Chapters 2 and 43; Chinese digital transcription, accessed 14 September 2026: `https://zh.wikisource.org/wiki/%E9%81%93%E5%BE%B7%E7%B6%93_(%E7%8E%8B%E5%BC%BC%E6%9C%AC)`.

[6] Laozi, traditionally attributed. The tao teh king, or the tao and its characteristics. Project Gutenberg ebook 216. James Legge translation; chapters 2 and 43. Electronic edition accessed 14 September 2026: `https://www.gutenberg.org/ebooks/216`.

[7] Jacques Rancière. *The Ignorant Schoolmaster: Five Lessons in Intellectual Emancipation*. Stanford University Press, 1991. Translated with an introduction by Kristin Ross; author chapter 1 consulted.

[8] Gilbert Strang and Edwin Herman. *Calculus Volume 1*. OpenStax, 2016. Section 3.1, current online edition accessed 14 September 2026: `https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative`.

[9] David Wood, Jerome S. Bruner, and Gail Ross. The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, 17(2):89–100, 1976.

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