

Modeling the Potential Effects of Historically Formed Relations in the Generative Relational Metamodel

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Abstract

A historically formed relation can retain a potential role in subsequent activity even when its effect remains unrealized. This paper develops a conditional account of that possibility within the generative relational metamodel. It distinguishes candidate histories, retained records, historical queries, and prospective observations under declared continuations. Exact query recovery is characterized by constancy on record fibres; compatible-answer sets represent incomplete recovery and its refinement. Conflict sets identify consistency-restoring retractions, while time-aligned filtering separates later evidence from changes to the past. A joint continuation record preserves both selected historical answers and future observations, with explicit update, finite-refinement, and minimality arguments. Counterexamples show why current-answer storage can omit needed memory and why a finite-valued query can require an unbounded exact record. Potential-effect tests distinguish current difference, dormant efficacy, blocked execution, and test-induced erasure. Under uncertain comparison pairs, possible efficacy, a witness for every compatible pair, and one common witness have different quantifier structures. A finite model demonstrates that sequential testing can destroy a distinction preserved by a family of fresh tests. An authorization-formation illustration separates predictive and historical records and examines a formation-sensitive review. A causal countermodel establishes the limits of observational discrimination for attribution. The philosophical discussion develops historical constitution, relational continuity, potentiality, and evidential responsibility under ontological suspension. The resulting framework coordinates historical and prospective inquiry while retaining their separate assumptions, evidence requirements, and revision conditions.

Keywords: historical formation; potential efficacy; relational modeling; provenance; continuation observations

Note on Scope and Research Status

This working paper develops one branch of a generative relational research program. Its purpose is to make a bounded representation precise enough for criticism and further application. Ontological questions remain suspended. The terms used in the model specify analytical roles and declared interpretations.

Definitions and modeling assumptions precede the conditional propositions that use them. The finite example illustrates their consequences under stipulated inputs. A mathematical witness and an empirically warranted claim have different requirements. The source audits and project notes record the distinction between earlier repository ideas, agent-added development, and verified literature positions.

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Status. This comprehensive revision is in progress for author review. Its formal propositions are conditional on the displayed assumptions. The relational interpretation and authorization example remain revisable modeling proposals. Empirical validation, a broader originality review, and publication sequencing remain open.

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1 Introduction

This paper examines historically formed relations whose possible effects can remain unrealized. It develops an explicit account of the records needed to reconstruct formation and to predict selected consequences under subsequent activity. The method combines a scoped relational interpretation, conditional mathematical derivations, a finite authorization example, and philosophical analysis of their limits. Ontological questions remain suspended throughout.

An authorization can remain unused during an observed period. The absence of activity leaves open whether authorization is present, whether an attempt would succeed, and whether practical conditions would permit the attempt. At the same time, two arrangements with the same current operational effect can have different formation histories. Those differences can matter to an inquiry about participation, contribution, succession, or procedural judgment even when a selected predictive output remains unchanged.

The generative relational program expresses a connection between these temporal directions: a relation is historically generated and can participate in generating further history. The second part concerns a possibility. Requiring an actualized effect would exclude an unused authorization at precisely the point where its prospective role is being investigated. A useful formulation must therefore distinguish formation, current organization, possible continuation, and actual occurrence.

The abstract action framework of Part I supplies a common setting for mechanisms, composition, configurations, and preserved distinctions [1]. A particular social interpretation adds participants, practices, records, and relevant observations. The present paper asks what follows once those additions have been declared and what remains underdetermined by the abstract base.

The first contribution is an exact recovery criterion for historical queries. A query can be recovered from a retained record precisely when it has the same value on every history that the record merges. This elementary result gives a decisive counterexample method and a constructive design rule. An endpoint can be sufficient for computing an effect while omitting the formation route, occurrence count, or sequence needed by another question.

The second contribution extends recovery to uncertainty. A record determines a set of compatible historical answers. Valid refinement on a fixed history domain can only narrow that set. A singleton becomes a warranted answer only under premises connecting the domain and record to the actual case. This distinction prevents exact model results from silently becoming historical evidence.

The third contribution constructs a record for prediction and historical inquiry together. Equality of their joint continuation signatures gives the coarsest update-compatible equivalence retaining both tasks. The proof develops the quotient maps, a finite refinement procedure, and a minimality property among exact deterministic records. The standard finite-word behavior construction supplies the mathematical antecedent [6]; the application coordinates distinct historical and prospective responsibilities.

The authorization example makes these results concrete. Two formation labels produce the same current operational state. A nine-value enriched record retains the most recent formation query. Prediction alone requires three continuation classes, while prediction together with the query requires seven. A new review operation that reads formation history changes the sufficient-state requirement. Every count and separation follows from the displayed transition rules.

Potential efficacy receives a comparative formulation. A positive-length continuation witnesses a possible future difference under a declared observation and admissible comparison. Present difference can be erased by every positive test, while a dormant difference can become visible only after activation. Availability differences require a convention for blocked continuations. These distinctions specify the mathematical content of a witness before its social or causal interpretation is assessed.

The philosophical discussion develops the consequences of this separation. Relational identity differs from behavioral equivalence. Historical constitution can coexist with present-state sufficiency. A withdrawn relation can remain historically significant. A test that merely reads an added label can manufacture discrimination without establishing an explanatory contribution. Dynamic relevance also leaves the evaluation of generative and reproductive effects open.

Causal attribution remains an additional task. A finite countermodel shows that two specifications can agree on the entire supplied observational law while disagreeing on a defined intervention. The result identifies a concrete information gap. It motivates explicit comparison designs, source criteria, and revision conditions rather than treating continuation inequality as a complete causal argument.

Section 2 establishes the relational criterion and action conventions. Sections 3 and 4 develop histories, exact recovery, and compatible answers. Section 5 examines conflicting constraints, source-preserving retraction, and time-aligned historical evidence. Section 6 examines future tests and availability. Section 7 coordinates those tests with uncertain comparison pairs, distinguishing pointwise efficacy, uniform witnesses, and coverage by fresh tests. Section 8 derives dynamically sufficient joint records, and Section 9 gives the finite illustration. Sections 10, 11, and 12 address evidence, philosophical interpretation, and model choices. The final discussion states objections and a bounded program for empirical development.

2 Relational Interpretation and the Abstract Base

This section establishes the language in which historical formation and potential effects will be compared. It first specifies the action model and the additional interpretive structure that a social application requires. It then develops the focal relational criterion, distinguishes several forms of dependence, and examines the explanatory work that order effects can perform. The objective is a usable analytical framework with visible assumptions. Its adequacy for a particular social inquiry depends on how the formal distinctions connect to participants, practices, records, and admissible comparisons.

2.1 Configurations, Mechanisms, and Distinctions

The base supplies a carrier of configurations, a composition rule for mechanisms, and a distinction regime. Table 1 introduces the main domains used throughout the paper. The separation of these domains matters because the same written symbol can otherwise drift between a possible configuration, an actual episode, and an investigator's description of that episode.

Assumption 2.1 (Action and distinction). Let S be nonempty and let the semigroup M act on S through $\alpha : M \times S \rightarrow S$. Write $\alpha(m, x) = mx$ and require $(nm)x = n(mx)$. Let $\sim$ be an equivalence on S preserved by every mechanism:

$$x \sim y \implies mx \sim my.$$

An operation label $b \in B$ specifies a map $T_b : S \rightarrow S$ arising from the declared mechanism interpretation.

The action law determines how a compound mechanism acts once its factors and their order are given. It provides a compositional account of effects on the represented variables. The equivalence determines which effects remain distinguishable at a selected level. Preservation ensures that equivalent configurations receive equivalent successors under a common mechanism, so the induced action on $S/\sim$ is well-defined. These are mathematical commitments whose content should be examined before social labels are attached.

For example, two configurations might agree on whether an activity has been completed while differing in the authorization under which it could be completed. If a permitted future operation reads the authorization, agreement on the completion flag alone may fail preservation. The investigator then faces an explicit choice: retain a finer distinction, restrict the operation repertoire, or accept a reduced description that requires additional memory. Calling the coarse observation a state would leave this choice unresolved unless its update adequacy were demonstrated.

The starting framework is the action-and-distinction architecture of Part I [1]. The present paper develops a narrower question inside that architecture: which records support claims about formation and prospective efficacy. Its variables and examples are additional constructions. The inherited architecture provides a common mathematical setting without deciding the identity, social meaning, or evidential status of every object subsequently represented.

Table 1: Domains and interpretive responsibilities in the historical-relation model

Symbol	Mathematical role	Interpretation to be supplied
S	Nonempty configuration carrier	Features represented in a possible situation
M, α	Semigroup and total left action	Mechanisms and their composition
$\sim$	Action-preserved equivalence on S	Distinctions retained at the selected level
B, T_b	Labels and maps $S \rightarrow S$	Individuation of operations and their effects
H, η	Histories and endpoint map	Candidate formation sequences and their scope
$\rho : H \rightarrow Z$	Retained record	Available representation of a history
$g : H \rightarrow V$	Historical query	The past-directed question being answered
$o : S \rightarrow Y$	Current observation	Output relevant to the predictive inquiry
$U_b : H \rightarrow H$	History update	Consequence of recording further activity
$\mathcal{C} \subseteq S^2$	Comparison domain	Contrasts licensed by the investigation

2.2 Labels, Effects, and Word Conventions

The distinction between an operation label and its effect prepares the historical argument. Two labels can carry different descriptions of how an arrangement was formed even when they induce the same transformation on the selected carrier. Conversely, the same label can be used across contexts whose effects differ if the context has been suppressed. Both possibilities place obligations on the coding scheme.

For $w = b_1 \cdots b_k$, words execute from left to right:

$$T_w = T_{b_k} \circ \cdots \circ T_{b_1}.$$

Thus $T_{uv} = T_v \circ T_u$ for a word u followed by a word v . The empty word ϵ has $T_\epsilon = \text{id}_S$. It represents zero steps in this word calculus. A semigroup identity, if the original semigroup has one, need not have been required to act as the identity under an unrestricted semigroup-action convention. Defining the empty word directly avoids assigning it to an old element whose action has only been proved idempotent.

The map from labels to transformations can be many-to-one. Suppose n and i both set an authorization variable to one. Their equality of effects on that variable says that the selected update model disregards their different formation descriptions. It leaves open whether another observation, participant, or subsequent operation would distinguish them. A model that later introduces a review procedure sensitive to formation must revisit the earlier predictive equivalence. The enlarged dynamics can use information that the narrower dynamics did not represent.

The distinction also works in the other direction. A label such as approval may stand for a procedure whose outcome depends on a committee, a governing rule, or a prior record. A deterministic map requires those dependencies to be fixed, represented in the configuration, or explicitly abstracted through a separate modeling convention. An ambiguous label cannot acquire a determinate effect through notation alone. For an empirical application, the interpretation of T_b should specify which conditions are included and which remain outside the model.

A word specifies a sequence of modeled operations. Whether a corresponding sequence occurred is an evidential question. A record of a planned sequence and a record of an executed sequence can contain the same labels while supporting different claims. The history domain will therefore require a declared status: possible histories, reconstructed histories, actual histories under an idealized description, or some combination whose components remain identifiable.

2.3 The Focal Relational Criterion

The working definition identifies a family of relations for which historical formation and potential efficacy are jointly relevant. Its purpose is to organize an inquiry, rather than to classify every use of the word relation. The definition should be read together with the operational and evidential obligations that follow it.

Definition 2.2 (Historically formed potentially efficacious relation). A relation within the present scope is a relatively reidentifiable dependence pattern with a declared account of historical formation, whose presence or alteration can affect a specified subsequent dynamic or feasible continuation under at least one admissible comparison.

The phrase relatively reidentifiable requires a criterion for treating appearances as continuations of a pattern. The criterion can depend on participants, offices, recognized obligations, a document lineage, or a stable functional arrangement. Each choice answers a different identity question. A membership change can preserve an institutional relation under an office-based criterion while ending a relation individuated strictly by its participants. The model should state which continuity claim it intends.

Historical formation concerns the sequence or process through which the pattern came to hold. A formation account can be coarse, such as a classification into two routes, or detailed, such as a dated record of proposals, revisions, acceptances, and implementation. Its resolution should match the historical query. A classification adequate for one question can lose distinctions required by another. The paper’s recovery criterion supplies a precise test for that dependence.

Potential efficacy concerns a difference under a declared continuation. The definition permits a relation to matter before its effect is realized. An authorization can alter the outcome of an available attempt even when the attempt has yet to occur. A dependency can become visible only when a supporting condition is removed. A latent obligation can affect a future review despite leaving the currently observed output unchanged. These are possible interpretations of the criterion, each requiring an appropriate operation and observation model.

The comparison clause is essential. A pair of configurations may differ in several respects, and a subsequent difference can depend on any of them. The mathematical witness shows a distinction between the represented configurations under a selected test. Attributing that distinction to the focal relation requires a defensible account of the contrast. This additional requirement is especially important when the relation is represented by a large pattern rather than a single independently manipulable variable.

The definition is deliberately revisable. Some historically significant relations may remain outside its efficacy condition because the chosen model supplies no defensible future test. They can still be retained as historical research objects. Conversely, a potential-effect witness can be mathematically well-defined while its proposed social meaning remains unsupported. The analytical criterion provides a focal scope, and the adequacy of that scope remains a philosophical and empirical question.

2.4 Incidence and Structured Dependence

An incidence relation identifies which participants stand in a specified relation. A dependence model specifies how the arrangement enters possible changes. This subsection separates those representational tasks and considers their coordination.

For a participant domain I , an incidence relation may take the form $R \subseteq I \times I$. Its ordered pairs distinguish roles: a relation from a participant to an office can have a different meaning from its reverse. A symmetric graph discards that orientation by design. A higher-arity relation can be needed when a dependence involves a participant, an institution, a task, and a governing condition together. Reducing such an arrangement to pairwise edges may be useful, provided the omitted coordination remains explicit.

Incidence alone supplies limited dynamic information. The presence of an edge says which instances satisfy the declared predicate. It leaves the transition consequences open until rules connect that predicate to operations. A model can therefore contain the same incidence structure and different dynamics, or different incidence structures with identical observed dynamics. The interpretation of an edge and its role in an update law should be recorded separately even when both are stored in one implementation.

A dependence pattern can also extend beyond a single edge. An activity may require a chain of recognitions, a jointly maintained procedure, or access through several intermediaries. Describing that arrangement by one relation can be appropriate at a chosen level, but the compression should be justified for its intended use. If removal of one intermediary changes feasibility, the reduced relation must retain enough information to represent that effect or declare the intervention outside its scope.

Reidentification compounds the issue. Suppose two appearances count as the same participant. A timeless incidence predicate on participant classes is well-defined only when replacing appearances by equivalent ones preserves incidence truth. A changing relation usually requires a context or time coordinate. Otherwise a participant's earlier and later appearances may be merged even though the relation changes between them. The resulting inconsistency arises from the coordination of identity and incidence, rather than from the mere use of relations.

A historically formed dependence pattern can accordingly be represented through several linked components: participant appearances, a continuity criterion, incidence at a context, formation records, and dynamic rules. This is an enriched model. The base action does useful work within it, but the social interpretation of those components remains an input whose warrant should be examined. The benefit of enrichment lies in exposing the assumptions that an informal discussion can leave implicit.

2.5 Formation, Maintenance, and Present Dependence

Formation and maintenance refer to different temporal roles of activity. The distinction clarifies several apparent contradictions about whether a relation depends on its history. This subsection develops the distinction without assuming that one role always has explanatory priority.

A formation process explains how a relation entered the represented history. A maintenance process explains how conditions supporting its continuation are renewed or preserved. The two can involve different participants and mechanisms. A procedure that creates an authorization may differ from the procedure that periodically renews it. An arrangement formed through one route can later depend on resources that played little role in its origin.

A current state can contain a sufficient record of the consequences of formation. In that case, prediction under a specified repertoire can proceed from the present record. Historical formation remains a fact about the represented trajectory. It need not remain an additional input to every predictive calculation. This distinction prevents a confusion between having a history and requiring the entire history for a selected forecast.

Conversely, a current snapshot can omit a formation feature that a later operation will read. A review procedure may treat otherwise similar records differently according to their origin. If the operation is included, the sufficient state must retain the relevant historical feature or equivalent information. The required memory is relative to the update system and the observation task. It can be small even when the underlying history is long.

Maintenance can also change what formation information matters. A renewal procedure might preserve the initial formation label, replace it with the latest renewal route, or combine both. These are distinct historical queries. A statement that a relation retains its origin requires a definition of origin persistence and an update rule for its record. Without that rule, the same label can shift between the first formation, the most recent formation, and the currently effective basis.

The paper's authorization illustration deliberately uses a most recent formation query. That choice permits a finite record and makes revision visible. It also leaves other questions open. Counting all episodes, distinguishing continuous from interrupted authorization, or recording the identity of every participant would require a richer history query. The example demonstrates a method of designing a record; it supplies a specific solution only for its declared question.

2.6 Order Effects and Their Interpretive Scope

Order sensitivity is a property available directly in the base language. Its formal clarity makes it useful for calculation, while its generality limits the social conclusions it can support. This subsection states the predicate and separates its legitimate use from stronger interpretive claims.

Define

$$R_{\text{ord}}(m, n, x) \iff (nm)x \not\sim (mn)x. \quad (1)$$

The predicate compares two sequences containing the same mechanisms in opposite order. It is definable from the action and the distinction regime. A witness establishes that order matters at the selected configuration and level of discrimination.

Order sensitivity can reveal a dependency in a specified operational model. For example, preparing a condition before an attempt can produce a different result from preparing it afterward. The interpretation depends on what preparation and attempt denote. Anonymous maps can satisfy exactly the same predicate. Their algebraic behavior leaves employment, friendship, recognition, and authority unspecified.

Commuting operations also can be relevant to a shared arrangement. Translations by one and by two on the integers commute while changing the same coordinate. In a social model, two independently sufficient inputs might commute under an observation that records only a completed outcome. The absence of an order witness would then concern this particular predicate, rather than establish the absence of every dependence relation.

The distinction regime introduces another sensitivity. Two operation orders can produce different raw configurations while remaining equivalent under $\sim$. A finer regime may expose an order effect that the coarser regime merges. The refinement changes what the investigator distinguishes in a fixed dynamic model. A claim that a social relation has thereby appeared requires a further argument about the target phenomenon.

A useful interpretive test holds the base fixed while varying the additional social assignment. If two permitted enrichments assign different formation meanings to the same mathematical data, every base-only expression has the same value in both. A successful base definition must therefore rely on restrictions that exclude one assignment or on additional symbols that encode the relevant distinction. The test identifies a missing premise; it leaves open which enriched formulation will be most useful.

2.7 Philosophical Commitments and Research Status

The framework treats historical formation and potential efficacy as analytical obligations that can be examined separately and then combined. This final subsection locates that position relative to the paper's ontological suspension and its practical research aims.

Ontological suspension permits the inquiry to investigate how a pattern is represented, tracked, and tested while leaving its ultimate mode of being unsettled. The formal existence of a set, function, or quotient establishes a mathematical construction. Its adequacy for describing a social relation requires interpretation and evidence. This division allows philosophical discussion to continue at the level of explanatory and representational commitments without turning a convenient carrier into an inventory of social reality.

The historical criterion also leaves explanatory depth open. An account can reconstruct a sequence without explaining why it occurred. It can identify a condition whose removal changes the modeled future without explaining how participants understand that condition. It can predict an observed output while omitting the reasons by which a procedure is justified. These achievements answer different questions. A comprehensive account should state which are pursued and how their evidence relates.

Potential efficacy carries a similar plurality. A dependence can enable, constrain, sustain, reproduce, or redirect an activity. The mathematical witness records a difference under a declared comparison. Its sign, desirability, legitimacy, and distribution among participants require further criteria. The present framework keeps generative and reproductive roles open to evaluation through those criteria, rather than assigning a fixed value to either role.

The practical objective is disciplined articulation. The framework should help a researcher specify a relation, explain the record that supports its formation account, identify a possible test of its prospective role, and locate the assumptions on which those claims depend. The subsequent sections develop the information and dynamic requirements of that objective. Their formal results remain conditional tools for an inquiry whose substantive interpretations are revisable.

3 Histories and Question-Relative Records

This section develops the retrospective component of the relational criterion. It distinguishes a represented history from a retained record, derives the condition for exact recovery of a historical query, and examines the assumptions under which recovery contributes to an empirical inquiry. The argument moves from word histories to successive compressions, then from a query's information requirements to the design of a sufficient record. This order makes the mathematical result intelligible without treating formal recoverability as evidence of actual occurrence.

3.1 Possible Histories and Actual Occurrence

The history domain determines the alternatives over which a recovery claim is evaluated. Choosing that domain is therefore a substantive part of the model. A theorem about all histories in a selected set can be mathematically correct while its application fails because the relevant actual history falls outside that set.

Fix an initial configuration x_* . In the simplest total deterministic model, let $H = B^*$ be the set of all finite words over the operation alphabet. A history $h = b_1 \cdots b_k$ determines a trajectory

$$x_0 = x_*, \quad x_j = T_{b_j}x_{j-1} \quad (1 \leq j \leq k).$$

The endpoint map is

$$\eta(h) = x_k = T_h x_*. \quad (2)$$

The equality follows by induction. At length zero, both sides are x_* . If it holds for h , appending b gives $\eta(hb) = T_b \eta(h) = T_b T_h x_* = T_{hb} x_*$.

This construction represents possible sequences under the specified maps. The membership of a word in H supplies a modeled possibility. A claim that the word occurred requires an actual-history assignment and evidence for that assignment. The distinction remains necessary even when every letter names a familiar practice. Familiar vocabulary can make a hypothetical path appear evidentially established unless its status is stated.

A domain can be restricted to histories consistent with a governing rule, an observed archive, or a time window. Such restrictions can improve relevance, but their basis should be visible. If a disputed formation route is excluded at the outset, a subsequent uniqueness result can reproduce that exclusion. The exclusion may be justified; the proof of recovery by itself supplies only its conditional consequence.

Initial conditions introduce another choice. A single x_* treats the starting configuration as fixed. When initial conditions are uncertain, histories can be pairs (x_*, h) with a candidate initial state. Two identical words may then produce different endpoints. Conversely, different initial states and different words can produce the same retained record. The recovery theorem below applies to this enlarged domain without alteration, but its fibres can become larger.

3.2 Temporal Resolution and History Individuation

A word history gives an order of operations at a chosen resolution. The relation between this representation and a social process requires decisions about episode boundaries, simultaneity, repetition, and unrecorded activity. This subsection makes those decisions explicit before considering compression.

An operation can denote a brief act, an extended procedure, or an aggregate episode. A negotiation represented by one letter may contain many offers and revisions. A more detailed alphabet can represent those steps individually. The two descriptions can share an endpoint while answering different historical questions. A coarse history is therefore best understood through a declared map from a more detailed domain, where such a domain is available, or through an explicit statement of the distinctions it intentionally retains.

Duration requires additional information. Word length counts labeled steps. It coincides with elapsed time only under a stated duration convention. If different operations take different amounts of time, a

duration assignment can be added to letters or occurrences. If duration depends on context, it belongs to the trajectory or occurrence record. A product of mechanisms alone generally loses the factorization needed to calculate such a quantity.

Occurrence identity presents a related issue. The words b and bb differ in the number of appearances of an operation label. When the effect of b is idempotent, both can have the same product and endpoint. The second occurrence can still matter to a query about repeated attempts, confirmation, expenditure, or testimony. To preserve that query, a record must retain occurrence information or another sufficient statistic whose update rule is proved.

Simultaneous or partially ordered activity can require a richer history domain. A total word order makes a specific sequencing commitment. When two activities are independent under the model, different linear orders may have equal effects. Equality of effects can justify endpoint compression for a specified task while leaving occurrence order historically unresolved. A partial-order representation would need its own definition of execution and observation; the present word model provides a controlled starting point for identifying that need.

The model also distinguishes an absent occurrence from an absent record. A documentary gap can correspond to inactivity, lost evidence, unrecorded action, or an unresolved coding decision. Treating every gap as an empty word would collapse these possibilities. The mathematical history domain should therefore state whether it represents activity itself or the sequence of records available to the investigator.

3.3 Successive Compressions of a History

A sequence can be reduced to a product, a transformation, or an endpoint. Each reduction answers a different computational question. This subsection identifies the information losses separately, so a successful endpoint calculation can be assessed without implying complete historical recovery.

For a nonempty history, let $e(h)$ be its semigroup product in execution order. The compression chain is

$$h \mapsto e(h) \mapsto \alpha(e(h), -) \mapsto \eta(h). \quad (3)$$

The first map uses equations in the semigroup. The second uses the action representation. The third evaluates a transformation at the selected initial configuration. Equality at a later stage can hold for inputs that remain distinct at an earlier stage.

An idempotent mechanism supplies a collision at the first stage. If $c^2 = c$, then the one-step word c and the two-step word cc have the same product. The product therefore cannot determine word length on a domain containing both histories. The conclusion follows from the exhibited pair alone. A more complicated reconstruction method cannot recover information that the retained input identifies.

A nonfaithful action supplies a collision at the second stage. Distinct semigroup elements can induce the same map on S . Their algebraic distinction might carry a meaningful label, or it might be redundant for the intended inquiry. The action map is sufficient for calculating effects on every represented configuration, yet it can omit a query whose answer depends on which element was applied. Such a query needs the element or another record that distinguishes its relevant classes.

Evaluation supplies a collision at the third stage. Two different maps can agree at x_* and differ elsewhere. An endpoint observation from one starting configuration then leaves their global action unresolved. An intervention that changes the starting condition might distinguish them, but that possibility requires an expanded test design. Equality of observed endpoints on one case establishes a narrower result than equality of transformations.

These losses are cumulative. If a query factors through an endpoint, it also factors through any richer record from which that endpoint can be computed. The reverse implication requires proof. A query recoverable from a word might fail at the product stage, a query recoverable from a product might fail at the map stage, and a query recoverable from a map might fail after evaluation.

The chain applies to nonempty words because a semigroup need not contain a zero-step element with identity action. The word calculus includes ϵ directly, with endpoint x_* . A formal unit can be adjoined

when a common algebraic domain is helpful. That addition should preserve the distinction between a new zero-step unit and an old algebraic identity whose action convention has yet to be checked.

3.4 Exact Recovery of a Historical Query

The recovery criterion characterizes what a retained record can determine on a declared history domain. It uses elementary set theory, but its force lies in making the information requirement explicit. The result also identifies a direct counterexample method.

Let $\rho : H \rightarrow Z$ be a retained record and let $g : H \rightarrow V$ be a target query. The query can concern formation mode, occurrence count, the presence of a particular operation, or a more structured historical property. Its codomain need not be numerical. The attained record set is $\rho(H) \subseteq Z$.

Proposition 3.1 (Recovery criterion). *A function $\bar{g} : \rho(H) \rightarrow V$ satisfying $g = \bar{g} \circ \rho$ exists if and only if*

$$\rho(h) = \rho(h') \implies g(h) = g(h') \quad (h, h' \in H). \quad (4)$$

Equivalently, $\ker \rho \subseteq \ker g$. When it exists, the function $\bar{g}$ is unique on the attained record set.

Proof. Suppose a function $\bar{g}$ satisfies the displayed factorization. If $\rho(h) = \rho(h')$, applying the same function to both sides gives

$$g(h) = \bar{g}(\rho(h)) = \bar{g}(\rho(h')) = g(h').$$

This proves necessity.

Conversely, suppose equality of retained records always entails equality of query values. For each $z \in \rho(H)$, the fibre $\rho^{-1}(z)$ is nonempty. Every member of that fibre has the same g value by the assumption. Define $\bar{g}(z)$ to be that common value. This definition is independent of the selected history. For any $h \in H$, the history belongs to the fibre at $\rho(h)$, so $\bar{g}(\rho(h)) = g(h)$. This proves sufficiency.

For uniqueness, let another function q satisfy $g = q \circ \rho$. For an attained z , choose h with $\rho(h) = z$. Then $q(z) = g(h) = \bar{g}(z)$. Every point in the function's domain is attained, so the two functions agree everywhere on that domain. $\square$

The restriction to attained records avoids an unnecessary extension problem. A record value outside $\rho(H)$ has no represented history from which to derive a query answer. A practical system can classify such a value as inconsistent, outside scope, or requiring a revised domain. Assigning an arbitrary answer to it would exceed the information supplied by the theorem.

A single collision disproves exact recovery. If $\rho(h) = \rho(h')$ and $g(h) \neq g(h')$, every procedure whose only input is the retained record receives identical input for two histories that require different answers. At least one answer must fail. The argument applies to any deterministic exact recovery procedure, independently of its computational sophistication.

The converse gives a design rule. A record can be sufficient without being a complete history, provided the query is constant on its fibres. A counter, a latest-label field, or a finite state of a query monitor can meet this condition. The useful question becomes which distinctions the target query requires and whether those distinctions can be retained under the updates the investigation anticipates.

3.5 Families of Queries and Record Design

A research project commonly asks several historical questions. Their joint information requirement can be stated without ranking them on a single scale. This subsection develops that requirement and examines its consequences for a shared record.

For queries $g_j : H \rightarrow V_j$ indexed by a set J , define the joint query $g(h) = (g_j(h))_{j \in J}$. Exact recovery of every component is equivalent to

$$\ker \rho \subseteq \bigcap_{j \in J} \ker g_j.$$

Indeed, constancy of the tuple on a record fibre is exactly constancy of each coordinate. A query family therefore specifies a partition of histories according to the answers the project intends to preserve.

This construction distinguishes disagreement about answers from disagreement about questions. Two investigators can agree on all recorded facts while seeking different classifications of the same history. One might ask for the first formation mode, another for the latest formation mode, and a third for whether a period of interruption occurred. A record designed for one question can be inadequate for the others without containing an internal contradiction.

Adding a query can require finer records. The old query family remains recoverable, while the new family distinguishes histories previously merged. The requirement is task-relative. It establishes the information needed for the enlarged inquiry and leaves the intrinsic importance of the new distinction to a separate argument.

A minimal current-answer record is the tuple of query values itself. It is sufficient for answering those questions at the current point. Its usefulness for future updates is a further matter. The update of a query value can depend on a feature of the history omitted by its current value. Section 8 derives the dynamic closure needed to resolve that issue.

This distinction matters for archival design. Retaining only present answers can be efficient when the query set is fixed and future updates are irrelevant. Retaining a richer history supports later reinterpretation and new questions, but it introduces costs of collection, storage, access, and interpretation. The formal recovery criterion clarifies the information tradeoff; it supplies no universal rule that every possible record should be retained.

3.6 Historical Recovery and Evidential Warrant

The theorem concerns a relationship among defined functions. Its application to an actual relation requires a justified mapping between the world, candidate histories, and observed records. This subsection states that bridge explicitly and identifies several ways it can fail.

Suppose an actual history h_{act} belongs to H and the observed record equals $\rho(h_{\text{act}})$. If the recovery criterion holds, applying $\bar{g}$ to the record yields $g(h_{\text{act}})$. Each premise performs distinct work. Membership places the case inside the model, record validity connects the evidence to the case, and factorization guarantees the information content needed for the query.

A model can fail the membership premise by excluding a relevant formation route. It can fail record validity through an erroneous document, an unreliable identification, or a mistaken alignment of records from different episodes. It can fail factorization because several admissible histories share the record. These failures require different repairs. Enlarging the history domain addresses the first, reassessing evidence addresses the second, and retaining additional information or reporting uncertainty addresses the third.

A named formation label also needs a coding criterion. If g is defined by whatever a source document calls the episode, the recovered answer concerns that document's classification. A claim about the episode's formation mode requires a further relation between the classification and the target concept. The two queries can be connected, but their identity should be argued rather than inferred from a shared word.

PROV-DM provides a useful comparison because it distinguishes records of entities, activities, use, generation, and derivation [3, Sections 2.1.2 and 5.2.1]. The present history model has its own domains and recovery theorem. The comparison supports careful specification of a record's relations; it leaves the truth of an individual formation account to evidence.

A recovered answer can therefore carry several statuses. It may be exact within a stipulated model, conditionally warranted given a source assessment, or unresolved because the candidate histories remain plural. A comprehensive research record should preserve these statuses. The ability to calculate a unique symbol is a mathematical result; the entitlement to assert the corresponding historical proposition is an additional achievement.

3.7 Predictive Sufficiency and Historical Significance

Historical and predictive uses can place different demands on the same record. This final subsection develops the distinction through counterfactual model changes and explains why neither use automatically supersedes the other.

Suppose two histories reach the same current configuration and every subsequent operation reads only that configuration. They then agree under all future observations generated by those operations. A query about their formation routes can nevertheless differ. A predictive quotient can merge the histories while a historical query separates them. Both results can be correct because they answer different questions.

A later operation that reads the formation record changes the predictive problem. Under the enlarged repertoire, the formerly merged histories may acquire a distinguishing continuation. The earlier prediction claim remains valid within its declared scope, while the expanded model requires a richer state. This is a traceable refinement of the model's responsibilities.

Historical significance can also remain relevant independently of a new predictive operation. An inquiry may seek to understand how a procedure developed, how a commitment was formed, or which alternatives were considered. Those questions can have their own evidential and explanatory aims. Predictive redundancy under one output measure does not settle their worth.

The reverse caution also applies. A formation distinction can be historically meaningful while contributing little to a particular forecast. A researcher should explain why the query belongs in a joint record instead of assuming that every remembered difference improves prediction. Section 8 permits both purposes to be retained explicitly and measures the additional distinctions they jointly require.

The record framework thus makes a limited but constructive contribution. It supplies a criterion for deciding whether a selected representation can answer a selected question. It also locates the points where a richer history, a revised query, an uncertainty report, or stronger evidence is needed. The next section develops the uncertainty report when exact recovery fails.

4 Uncertainty and Refinement in Historical Records

This section extends exact recovery to records compatible with several historical answers. It defines an answer set, proves its behavior under valid refinement, and examines conflicting evidence, incomplete domains, and changing questions. The construction provides a disciplined way to retain uncertainty. It also clarifies why a more detailed record can improve identification without guaranteeing a true or complete account of formation.

4.1 Compatible Historical Answers

A record can be informative even when it leaves more than one historical answer possible. The appropriate response is to represent the remaining alternatives and the conditions under which they were obtained. This subsection gives that representation on the history domain introduced in Section 3.

For an attained record $z \in \rho(H)$, define

$$\Gamma_\rho(z) = \{g(h) : h \in H, \rho(h) = z\}. \quad (5)$$

The set collects the answers supplied by histories compatible with the retained record. It is nonempty because the record is attained. Its members need not identify distinct histories: many histories can give the same query answer.

Proposition 4.1 (Singleton boundary of exact recovery). *The query g is exactly recoverable from ρ on H if and only if $\Gamma_\rho(z)$ has exactly one member for every $z \in \rho(H)$.*

Proof. If $g = \bar{g} \circ \rho$, every history in the fibre at z has value $\bar{g}(z)$, so the answer set is the singleton containing that value. Conversely, if every answer set is a singleton, define $\bar{g}(z)$ to be its unique member.

The definition is well-defined on attained records. For any history h , its value $g(h)$ belongs to $\Gamma_\rho(\rho(h))$ and therefore equals $\bar{g}(\rho(h))$. $\square$

The proposition separates uncertainty about a query from uncertainty about a complete history. A record can identify the formation mode while leaving the exact sequence of intermediate steps unresolved. Its compatible-history fibre can be large even when its answer set is a singleton. Conversely, two candidate histories can suffice to leave a crucial formation distinction unresolved.

Consider three represented histories: one classified as negotiated, one as imposed, and one as renewed after an interruption. Suppose the retained endpoint records only that authorization is currently active. The answer set contains all three labels if each history has that endpoint. A query asking only whether any formation occurred can have a singleton answer on the same record. The record's adequacy depends on the question.

The set carries a logical interpretation. Membership means compatibility with the declared model and record, given the chosen query. It supplies neither a frequency nor a degree of belief. Equal membership does not imply equal probability. A probabilistic assessment would require a distribution over histories or another justified uncertainty model. The distinction prevents a finite list of possibilities from silently becoming a uniform statistical model.

4.2 Valid Refinement of a Retained Record

A finer record can separate histories that a coarser record merges. The refinement theorem specifies the conditions under which the remaining answer set can only narrow. Its hypotheses exclude several changes that can appear to be refinement while actually altering the inquiry.

Let $\rho' : H \rightarrow Z'$ be a new record on the same history domain. Say that ρ' refines ρ when a map $f : Z' \rightarrow Z$ satisfies $\rho = f \circ \rho'$. This means that the old record can be recovered from the new one. The definition compares information about the same histories; it leaves the practical reliability of the new record to a separate assessment.

Proposition 4.2 (Answer-set refinement). *If $\rho = f \circ \rho'$, then for every attained $z' \in \rho'(H)$,*

$$\Gamma_{\rho'}(z') \subseteq \Gamma_\rho(f(z')). \quad (6)$$

Exact recovery from the coarse record implies exact recovery from the finer record.

Proof. Let $v \in \Gamma_{\rho'}(z')$. By definition, there is a history h such that $\rho'(h) = z'$ and $g(h) = v$. The refinement equation gives $\rho(h) = f(\rho'(h)) = f(z')$. Thus the same history witnesses $v \in \Gamma_\rho(f(z'))$, proving inclusion.

If the coarse answer set is a singleton, the finer answer set is a nonempty subset of it because z' is attained. It must therefore be the same singleton. Applying Proposition 4.1 proves the final statement. $\square$

The inclusion can be strict or an equality. Adding a formation document may distinguish negotiated from imposed routes. Adding an unrelated identifier may leave the query's answer set unchanged. More information in a record need not provide more information about every query.

Refinement also admits a useful compositional reading. If a third record refines the second, and the second refines the first, the composite of the forgetting maps recovers the first from the third. The corresponding answer sets form successive inclusions. This supports a traceable account of how an investigation narrows its historical alternatives.

A record can be more detailed in ordinary language without satisfying the mathematical refinement condition. A revised summary may add one field and discard another. Two source systems may use incompatible identifiers. A new coding scheme may redefine the event boundaries. These changes require an alignment argument before the theorem applies. The apparent increase in detail alone supplies insufficient grounds for an inclusion claim.

4.3 Candidate Histories under Multiple Pieces of Evidence

An inquiry often combines several pieces of evidence rather than receiving one complete record. Candidate sets provide a simple representation of that process. This subsection states the logical construction and the assumptions that connect it to an actual history.

Let each accepted evidence item determine a compatibility set $E_i \subseteq H$. The surviving candidates are

$$K = \bigcap_{i \in I} E_i, \quad G(K) = g(K) = \{g(h) : h \in K\}. \quad (7)$$

Adding another accepted item E replaces K by $K \cap E$. Since $K \cap E \subseteq K$, applying the fixed query gives $g(K \cap E) \subseteq g(K)$.

This inclusion follows directly from set membership. An answer in $g(K \cap E)$ has a witness history that belongs to both K and E . The same history belongs to K , so it also witnesses membership in $g(K)$. The derivation uses a common history domain and query, together with the interpretation of evidence through compatibility sets.

The logical update can be useful even when evidence items come from different media. A dated log can constrain operation order, a document can constrain a formation label, and a participant record can constrain which actors appeared. Combining them requires a common account of episode identity. If the records refer to different episodes, their intersection may incorrectly eliminate valid histories or produce an artificial contradiction.

An empty candidate set has a specific diagnostic meaning. The chosen domain and accepted evidence predicates have no common witness. Possible explanations include inconsistent evidence, an inadequate history domain, an incorrect alignment, or an overly strict coding rule. The empty set provides a reason to inspect these premises. It supplies no historical answer by itself.

A nonempty candidate set can still be misleading if the actual history has been excluded. Suppose the model admits only negotiated formation, while a real case followed a different route. Every candidate can agree on the label and produce a singleton answer. The agreement is conditional on the domain; it cannot repair the exclusion. Domain adequacy is therefore part of the evidential argument, even when the internal set calculation is exact.

4.4 Truth Conditions and False Precision

A mathematically unique answer can be warranted only under premises that connect the model to its target. This subsection states those premises and examines how false precision can arise from apparently successful refinement.

Assume an actual history $h_{\text{act}} \in H$ belongs to every accepted compatibility set. Then it belongs to their intersection K , and its query answer belongs to $g(K)$. If $g(K) = \{v\}$, it follows that $g(h_{\text{act}}) = v$. The argument is short, but each membership premise is substantive.

A false record can remove the actual history while leaving one alternative. The candidate set shrinks and the answer set becomes a singleton, yet the unique answer can be wrong. The refinement theorem remains true as a statement about the imposed predicates. What fails is the claim that those predicates preserve the actual case.

This possibility distinguishes precision from warrant. Precision concerns the size or specificity of the retained answer set. Warrant concerns the justified relationship between the evidence, the model, and the proposition asserted. A more precise answer can be less warranted if its narrowing depends on unreliable exclusions. The model should preserve the source and status of each exclusion so that the apparent gain can be reviewed.

A related problem arises when several records share a common source. Their agreement may reflect reproduction of the same assertion. A purely logical intersection can still be defined, but a probabilistic interpretation of corroboration requires an account of dependence among sources. The existence of several documents does not by itself supply several independent observations. The present set model therefore keeps corroboration strength outside its formal conclusion.

The query's coding can also introduce false precision. A binary classification can force a complex formation process into negotiated or imposed categories even when mixed or uncertain cases are relevant. The resulting singleton may reflect the codomain's exclusions. A richer codomain can include mixed, indeterminate, or unresolved values when the research question warrants them. That revision changes the query and should be recorded accordingly.

4.5 Updating Uncertain Historical Records

Historical inquiry can continue while the represented relation changes. A candidate set then requires both a dynamic update and a new evidence update. This subsection distinguishes those operations and derives their composition under explicit assumptions.

Let $U_b : H \rightarrow H$ be a known deterministic history update. If K contains the candidate histories before the update, the candidate successors are

$$U_b(K) = \{U_b h : h \in K\}.$$

If new evidence permits only histories in E , the revised set is

$$K' = U_b(K) \cap E. \quad (8)$$

The first step advances each candidate through the modeled activity. The second filters the resulting candidates by the newly accepted evidence.

The order matters. Evidence about the successor history should be applied to successor candidates. Applying it to the old histories can test a different proposition. An implementation needs the time or episode reference of each predicate, especially when a field such as current authorization changes through the update.

If the actual old history belongs to K , the actual update is U_b , and the actual successor satisfies the new evidence predicate, then the actual successor belongs to K' . This conditional preservation follows by direct substitution. Uncertainty about which operation occurred requires taking a union over the candidate updates before filtering, together with an explicit operation-evidence model.

Query answers alone may be insufficient to perform this update. Two candidate sets can have the same current set of answers while their histories evolve to different answer sets. The update therefore can require information beyond $g(K)$. This is the uncertainty counterpart of the dynamic-closure problem developed in Section 8.

The construction concerns the investigator's represented uncertainty. The actual relation follows one history under the assumed case description. Its state and the investigator's candidate set have different roles, even when both are represented mathematically. Keeping those roles distinct prevents a narrowing of uncertainty from being described as a change in the relation itself.

4.6 Changing Questions and Reopening History

A later inquiry can ask a question that an earlier record was never designed to answer. The resulting uncertainty can increase even when the available evidence has improved. This subsection explains why that outcome is compatible with the refinement theorem.

The theorem holds for a fixed query g . If the project replaces it with a finer query g' , the answer sets belong to a different codomain. A comparison requires a map relating the two questions. For example, a detailed sequence classification may imply a coarse formation label. Recovering the coarse label then remains possible even when the detailed sequence is unresolved.

A record can therefore be sufficient for the original project and insufficient for a later one. This is a limit of scope rather than an automatic defect in the original design. The practical question is whether foreseeable future uses justify retaining additional history. That decision involves research aims and record-governance considerations beyond the factorization theorem.

Reopening a history can also enlarge the candidate domain. A newly recognized formation route may fit the existing evidence. The compatible-answer set can expand because the old domain excluded that possibility. Such an expansion concerns model revision, whereas the monotonicity theorem concerns evidence refinement inside a fixed domain. Recording the distinction prevents a justified revision from appearing to violate the formal result.

Interpretive disagreements can be represented through several query maps applied to a shared history domain. One coding scheme might classify a procedure by its formal authorization, another by its participants' effective options. Their disagreement can persist even when the sequence of events is agreed. The mathematical framework can display that difference while leaving its philosophical resolution open.

The comprehensive use of a record therefore includes its genealogy: which questions motivated its fields, which domains were assumed, which evidence narrowed the alternatives, and which later revisions changed the inquiry. These matters belong in the accompanying research state even when the manuscript presents only the current formulation.

4.7 Interpretive Consequences for Historical Relations

The answer-set construction contributes a modest form of rigor to historical claims. It distinguishes an unresolved alternative from an omitted one, a singleton answer from a uniquely reconstructed history, and valid information refinement from an unjustified exclusion. This final subsection states the philosophical consequences within the paper's scope.

A historically formed relation can remain identifiable even when its formation account is uncertain. Current incidence, continuity, or operational evidence can support tracking the pattern while several formation routes remain compatible with the record. The uncertainty should be attached to the historical query, rather than automatically transferred to every claim about the relation.

Conversely, a well-supported formation account can coexist with uncertainty about present efficacy. A record may establish how an authorization arose while leaving its current enforcement or practical availability unresolved. Historical and prospective claims need their own evidence. Their conjunction becomes stronger only when both components are warranted.

The construction also gives a precise response to the temptation to treat a polished narrative as a uniquely determined history. Narrative coherence can organize a candidate account, while exact recovery requires constancy of the target answer over all admissible histories sharing the record. A competing compatible history with a different answer identifies a specific unresolved issue.

The resulting stance supports inquiry through revisable constraints. Evidence can narrow alternatives, formal analysis can expose missing information, and new questions can reopen the domain. The mathematical results discipline those movements without deciding in advance which historical interpretation will survive them.

5 Revision of Historical Inferences under Conflicting Evidence

This section develops the revision of historical inferences when active evidence conflicts or when new evidence concerns a later stage of the history. It separates preserved source records from interpreted constraints, derives consistency-restoring retractions, examines duplication and reliability, and establishes a time-aligned update rule. The analysis keeps logical consistency, historical truth, and evidential warrant distinct.

The compatible-answer construction presumes a candidate-history set supported by the retained evidence. That set can become empty. An investigator then needs to examine the constraints, their interpretation, and the background domain. Simply continuing to intersect evidence sets would preserve the conflict without identifying the assumptions responsible for it.

5.1 Source records and active constraints

Let H be a nonempty domain of candidate histories. A finite set A indexes interpreted evidential constraints $C_e \subseteq H$. For $B \subseteq A$, define

$$K_B = \bigcap_{e \in B} C_e, \quad K_\emptyset = H.$$

The set K_B contains histories satisfying every constraint currently selected in B .

The distinction between a source and its constraint is essential. A document can support several interpreted propositions. One proposition can also appear in several documents derived from a common source. The constraint labels therefore need an explicit relation to the records from which they were obtained.

An active constraint can be withdrawn when its interpretation is challenged, its temporal scope changes, or its evidential support proves inadequate. Such withdrawal changes the current inference. The original record, former interpretation, and reason for revision can remain preserved.

This separation permits serious revision without erasing the trajectory of inquiry. A historical account can change while its sources and prior claims remain available for scrutiny. The formal active set identifies which constraints currently support the candidate-history intersection.

5.2 Conflict sets and their minimal members

A conflict set is a subset $B \subseteq A$ for which $K_B = \emptyset$. Every superset of a conflict set is also a conflict set: adding constraints can only reduce the compatible histories.

A conflict is inclusion-minimal when each proper subset is consistent. Minimality identifies a set of constraints that cannot all remain active under the background model, with every member needed for that particular conflict.

The word *minimal* concerns inclusion. A minimal conflict can contain more constraints than another minimal conflict. The two can expose different sources of incompatibility. Their cardinalities alone supply a limited comparison.

Let $\mathcal{C}_{\min}(A)$ be the family of minimal conflicts. Since A is finite, every conflict contains a minimal one. The history domain itself can be infinite; the finite assumption concerns the active constraint set.

The background domain also matters. If H excludes the actual history, a conflict can arise even when the sources are accurately interpreted. The model should therefore permit revision of the candidate domain as a separate diagnostic option.

5.3 Retractions that restore consistency

For a proposed retraction $R \subseteq A$, the retained active set is $A \setminus R$. The following result identifies exactly which retractions restore a nonempty candidate set.

Proposition 5.1 (Consistency-restoring retractions). *Under the stated finite active-set assumptions,*

$$K_{A \setminus R} \neq \emptyset \iff R \cap B \neq \emptyset \text{ for every } B \in \mathcal{C}_{\min}(A).$$

The inclusion-minimal successful retractions are the minimal hitting sets of the minimal conflict family.

Proof. Suppose the retained constraints are consistent. If a minimal conflict B were disjoint from R , it would lie inside $A \setminus R$. Then $K_{A \setminus R} \subseteq K_B = \emptyset$, a contradiction. Thus R intersects every minimal conflict.

Conversely, suppose R intersects every minimal conflict but the retained constraints are inconsistent. Their finite set contains a minimal conflict B . Since $B \subseteq A \setminus R$, that conflict is disjoint from R , contradicting the assumption. The equivalence therefore holds for all retractions. Taking inclusion-minimal members gives the final statement. $\square$

Conflict and hitting-set duality has an established antecedent in Reiter's theory of diagnosis. Theorem 4.4 and Corollary 4.5 characterize diagnoses through conflicts and minimal conflicts [5]. The present proposition applies that structure to finite sets of historical-evidence constraints.

The interpretation differs from an assessment of a source's truthfulness. A retraction can concern a coding assumption, a date assignment, or the applicability of an assertion to the focal relation. The theorem identifies consistency-restoring options within the declared representation.

5.4 A conflict with several consistent repairs

Let

$$H = \{h_1, h_2, h_3\},$$

and define

$$C_a = H \setminus \{h_1\}, \quad C_b = H \setminus \{h_2\}, \quad C_c = H \setminus \{h_3\}.$$

The three constraints jointly exclude every candidate history. Each pair remains consistent.

Thus $\{a, b, c\}$ is the unique minimal conflict. The minimal retractions are $\{a\}$, $\{b\}$, and $\{c\}$. Each restores a different singleton history:

$$K_{\{b,c\}} = \{h_1\}, \quad K_{\{a,c\}} = \{h_2\}, \quad K_{\{a,b\}} = \{h_3\}.$$

The calculation offers three consistent accounts. It supplies no reason to select one account as true. Additional evidence or a justified assessment of the constraints is required for that selection.

The example also shows why a singleton candidate set can provide false confidence. After an unsupported retraction, the remaining intersection can be precise while excluding the actual history. Precision and validity of the evidential steps remain separate.

An inquiry should therefore record the status of a repaired account. It can be a candidate reconstruction, an account conditional on withdrawing one interpretation, or a supported historical claim after further checking. The same set-theoretic output can occupy these different epistemic positions.

5.5 Duplicate constraints and apparent evidential weight

Duplicate records can affect a count-based revision rule even when they leave the logical content unchanged. The following example isolates that representation dependence.

Let $H = \{h_0, h_1\}$, with

$$C_a = C_b = \{h_0\}, \quad C_c = \{h_1\}.$$

The minimal conflicts are $\{a, c\}$ and $\{b, c\}$. Their minimal hitting sets are

$$\{c\}, \quad \{a, b\}.$$

A minimum-cardinality retraction rule chooses $\{c\}$.

If the repeated constraint is represented only once, the conflict has two members and its singleton retractions tie in cardinality. Adding a duplicate label has changed the numerical preference.

The compatible-history intersection has remained unchanged because intersecting a set with itself leaves that set fixed. The change concerns the number of labels the retraction must remove. A count of labels therefore depends on how the evidence record is individuated.

For historical inquiry, a copied report can preserve useful information about circulation or public uptake. Treating the copy as independent corroboration requires further evidence about its source and production. The provenance relation should remain available when support is assessed.

The example leaves open the use of costs or priorities in a revision procedure. Such a procedure must state what its weights represent and why they are justified. Logical minimality and evidential reliability answer different questions.

5.6 Retraction, domain revision, and historical truth

An empty candidate set can result from several failures. A source can be inaccurate. Its interpretation can be mistaken. A date or participant can be misidentified. The operation model can be inadequate. The history domain can omit a relevant alternative.

The retraction theorem holds the background domain and constraint interpretations fixed while selecting active subsets. It therefore locates a specific family of revision options. A revision to H or to a constraint's meaning changes the problem and can change its conflict family.

A sound investigation should compare those options before assigning responsibility to a source. A conflict in the representation establishes that the active assumptions cannot jointly hold within that representation. It does not identify which assumption should be revised.

The original source record remains valuable after its constraint is withdrawn. It can establish that an assertion circulated, that an institution relied on it, or that an earlier investigation used a particular interpretation. Those are further historical queries with their own evidence requirements.

This distinction supports a durable research record. The current inference can be revised, while its genealogy remains traceable. A later correction can then explain both the revised account and the conditions under which the earlier account was formed.

5.7 Evidence about later states

Evidence must be aligned with the historical stage it concerns. Let

$$U : H_t \longrightarrow H_{t+1}$$

be a declared history update, let $K \subseteq H_t$ be the prior candidate set, and let $E \subseteq H_{t+1}$ encode evidence about the updated history.

Updating and then applying the evidence yields $U(K) \cap E$. To constrain the earlier histories first, the evidence must be pulled back through U . The relevant earlier set is

$$K \cap U^{-1}(E).$$

Proposition 5.2 (Time-aligned evidence filtering). *For any map $U : H_t \rightarrow H_{t+1}$,*

$$U(K \cap U^{-1}(E)) = U(K) \cap E.$$

Proof. A member of the left side has the form Ux , with $x \in K$ and $Ux \in E$. It therefore lies in $U(K) \cap E$. Conversely, if $y \in U(K) \cap E$, choose $x \in K$ with $Ux = y$. Since $y \in E$, this representative belongs to $U^{-1}(E)$. Thus y belongs to the left side. Injectivity of U is unnecessary. $\square$

Filtering K directly by E can be ill-typed because the sets concern different stages. Shared numerical codes can hide the mistake. On $S = \{0, 1\}$, let U flip the bit, let $K = \{0\}$, and let $E = \{1\}$. Updating then filtering gives $\{1\}$. Filtering the old state directly by $\{1\}$ gives the empty set.

The preimage gives the correct earlier constraint: $U^{-1}(\{1\}) = \{0\}$. The example shows why an observation's content must include the time or stage to which it applies, separately from the time at which the observation is received.

5.8 Retrospective queries and fixed historical events

Let $g : H_t \rightarrow Y$ be a query about the earlier history. After receiving the later-stage evidence, the compatible answers are

$$g(K \cap U^{-1}(E)).$$

The set can be smaller than $g(K)$, because later evidence rules out some earlier candidates.

This is a change in the inference supported by the evidence. The earlier actual history can remain fixed. The construction therefore represents retrospective learning without an operation that changes the past event.

The amount learned depends on the update and evidence. If many earlier histories lead to the same later record, observing that record can leave the historical query unresolved. A more discriminating later trace can narrow the alternatives.

A revised question can also change which earlier differences matter. An inquiry that initially asked whether authorization existed may later ask how it was formed or who participated. The historical domain and source records can be revisited for the new query, while the earlier question and answer retain their original scope.

The distinction between retrospective interpretation and historical alteration is central to the paper's account. Histories can have consequences for questions that emerge later. An expanded record makes those consequences available for investigation, subject to evidence and the limits of recovery.

5.9 Consequences for the potential-effect comparison

Evidence revision can alter the candidate comparisons used to assess potential effects. Withdrawing a constraint can enlarge the compatible-history set and introduce new possible pairs. Valid later evidence can narrow the set through a time-aligned update.

The witness analysis in Section 7 therefore needs the provenance of its comparison set. A common witness can appear after a justified narrowing, or disappear when a previously excluded alternative is restored.

Consistency of the comparison domain is a preliminary requirement. Historical identification, operation warrant, causal attribution, and practical testing remain further tasks. The formal revisions help locate which claim a change in the evidence actually supports.

The resulting framework permits a historical account to remain revisable while preserving its source trajectory. It gives conflict, retraction, later evidence, and prospective comparison separate mathematical locations within one inquiry.

6 Potential Effects under Declared Continuations

This section develops the prospective component of the relational criterion. It defines a potential-effect witness, distinguishes present differences from future differences, and examines the dependence of a test on the operation repertoire, observation, and comparison domain. It then extends the construction to unavailable operations and considers the philosophical interpretation of unrealized efficacy. The result is a conditional test framework whose empirical and causal use requires additional premises.

6.1 Observation Signatures and Comparison Domains

A prospective inquiry asks how represented configurations would respond to specified operations. Its mathematical input includes more than a current state description. This subsection identifies the maps and comparisons needed to make the question determinate.

Fix total deterministic maps $T_b : S \rightarrow S$ and an observation $o : S \rightarrow Y$. For $w \in B^*$, define

$$\sigma_x(w) = o(T_w x). \quad (9)$$

The signature is a function from continuation words to observable outputs. It includes the current observation at the empty word: $\sigma_x(\epsilon) = o(x)$.

The signature is relative to both the operations and the observation. A distinction can be invisible under one output map and visible under another. It can be unreachable through one repertoire and exposed through an enlarged repertoire. These changes should be described as changes in the test framework until an argument connects them to a change in the target relation itself.

Let $\mathcal{C} \subseteq S^2$ be a declared set of admissible comparison pairs. The set records which contrasts the investigation intends to interpret. It can encode a controlled contrast, an analytically stipulated comparison, or a class of empirically justified alternatives. Its status should be stated with the model.

The need for $\mathcal{C}$ follows from the difference between discrimination and attribution. Any two configurations with different responses supply a mathematical contrast. A relational inquiry needs a reason to associate that contrast with the focal pattern. If the configurations differ in authorization, resources, participants, and observation conditions, the resulting difference does not identify which feature carries the explanatory burden.

A comparison can be useful even when a real intervention is unavailable. It may support a conceptual counterexample or clarify a theoretical dependency. Its evidential status remains that of the stated comparison. The framework therefore permits several uses while requiring their premises to remain distinct.

6.2 Positive-Length Potential-Effect Witnesses

The focal definition concerns a possible subsequent difference. Restricting witnesses to positive-length words makes that temporal role explicit. This subsection states the definition and derives the distinction from current observational difference.

Definition 6.1 (Comparative potential-effect witness). For $(x, y) \in \mathcal{C}$, define

$$\mathcal{D}^+(x, y) = \{w \in B^+ : \sigma_x(w) \neq \sigma_y(w)\}. \quad (10)$$

The pair exhibits a potential future effect under the declared tests when $\mathcal{D}^+(x, y)$ is nonempty.

A witness contains three ingredients: a permitted comparison, a nonempty continuation, and different resulting observations. The definition is existential. It requires at least one continuation with the stated effect. It leaves open whether that continuation is selected, whether it is practically feasible under an enriched model, and whether its difference deserves a causal interpretation.

Present difference alone is insufficient. Let $S = \{0, 1\}$, let o be the identity, and let the sole operation r send both states to zero. The current observations of zero and one differ. Every positive word is a repetition of r , and each sends both states to zero. Therefore $\mathcal{D}^+(0, 1)$ is empty.

The converse distinction is equally important. Let two configurations share a current observation but differ in a variable read by a later operation. An authorization example has $o(r, e) = e$ and an activation operation that changes e only when $r = 1$. The states $(1, 0)$ and $(0, 0)$ currently agree, while activation separates them. Their potential difference is dormant under the current observation.

A universal efficacy requirement would be stronger: every selected continuation would need to distinguish the configurations. An erasing operation can defeat that requirement even when many other tests expose the dependence. The existential criterion matches the paper's focus on possible efficacy, while universal persistence is a separate property that a particular inquiry might also investigate.

The length restriction also distinguishes an operation that leaves the output unchanged from the absence of an operation. A positive identity-like test can carry a practical interpretation such as inspection or attempted confirmation, but its inclusion in the repertoire needs justification. The empty word is available mathematically; it should not silently become a performed test.

6.3 Future Equivalence and Current Observation

The witness definition induces an equivalence based on positive continuations. Comparing that equivalence with full continuation equivalence exposes a useful boundary of the concept. This subsection gives the complete relation and its proof.

Define $x E^+ y$ when every positive continuation gives equal observations. Let $x E_o y$ mean equality under all words in B^* , including the empty word. These definitions use the same fixed carrier, operations, and observation.

Proposition 6.2 (Current and positive-future agreement). *The relation E^+ is an equivalence preserved by every T_b . Moreover,*

$$E_o = \ker o \cap E^+, \quad E^+ = \bigcap_{b \in B} (T_b \times T_b)^{-1}(E_o). \quad (11)$$

If some nonempty word acts as id_S , then $E^+ = E_o$.

Proof. Reflexivity, symmetry, and transitivity follow from equality of the positive-word observation functions. For update preservation, suppose $x E^+ y$ and fix b . For any positive word w ,

$$o(T_w T_b x) = o(T_{bw} x) = o(T_{bw} y) = o(T_w T_b y).$$

The middle equality holds because bw is positive. Hence $T_b x E^+ T_b y$.

Agreement on all words consists of agreement at ϵ together with agreement on all positive words. This proves $E_o = \ker o \cap E^+$. Every positive word can also be written bw with $b \in B$ and $w \in B^*$. Equality under all such words is equivalent to $T_b x E_o T_b y$ for every b , proving the second identity.

Finally, suppose a positive word v acts identically. If $x E^+ y$, then $o(x) = o(T_v x) = o(T_v y) = o(y)$. Thus $E^+ \subseteq \ker o$, and the first identity gives $E^+ = E_o$. $\square$

The reset example shows why the identity-word assumption matters. Positive tests can erase a current distinction before observing it. A positive-future quotient then merges states whose present outputs differ. Such a quotient can be suitable for a narrowly prospective question, while a record intended to report the current output must also retain $\ker o$.

The theorem concerns observation equivalence, not entity identity. Configurations can agree under every declared future test and still represent different participants or historical episodes under a separate identity criterion. The equivalence states which distinctions the chosen test system preserves. An ontological identification would require an additional argument.

The result also sharpens a practical design question. If an inquiry intends current differences to count among prospective evidence, it can include a justified inspection operation whose action preserves the relevant state. That changes the test repertoire. Its social and physical feasibility should be examined, especially when observation itself alters the arrangement.

6.4 Test Repertoires, Horizons, and Completeness

A theoretical signature ranges over a specified language of continuations. An investigation usually examines a smaller set. This subsection derives the monotonicity of detection and explains the limits of finite testing.

For a test language $K \subseteq B^+$, the detected witnesses are $\mathcal{D}^+(x, y) \cap K$. If $K_1 \subseteq K_2$, then

$$\mathcal{D}^+(x, y) \cap K_1 \subseteq \mathcal{D}^+(x, y) \cap K_2.$$

The inclusion follows because every word belonging to the first intersection also belongs to the second. Enlarging the test language preserves existing witnesses and may reveal additional ones.

A horizon restriction takes $K_h = \{w : 1 \leq |w| \leq h\}$. Agreement through horizon h leaves longer continuations open. A relation whose effect requires a preparatory sequence can remain undetected at a short horizon. The missing witness can arise from sequence length even when all individual operations have been tested.

The distinction between a complete finite model and a finite sample is consequential. For a finite deterministic carrier with an effectively available operation repertoire, iterative refinement can establish full continuation equivalence. Section 8 derives the stabilization criterion. A finite set of observed traces on an unrestricted or incompletely specified carrier lacks that completeness guarantee.

Actual observation can be narrower than even a planned test set. A test may be abandoned, blocked, or recorded ambiguously. A report of agreement should specify which continuations were actually executed, which were inferred through a validated model, and which remain hypothetical. The same mathematical signature can organize these distinctions without assigning them equal evidential status.

Operation enlargement can also invalidate a previously sufficient record. If a new operation reads a historical field omitted by the old model, the old continuation equivalence may split. This is a change in the inquiry's dynamic scope. A careful revision records the added operation and identifies the formerly merged histories it distinguishes.

6.5 Availability and Blocked Continuations

Some relations affect access to operations rather than the observed outcome of operations executable in both configurations. A comparison restricted to common executions can hide that role. This subsection introduces a convention that makes availability differences visible.

For partial operations, let $W(x) \subseteq B^*$ consist of words executable through every prefix from x . Introduce a fresh output $\perp$ and define

$$\hat{o}_x(w) = \begin{cases} o(T_w x), & w \in W(x), \\ \perp, & w \notin W(x). \end{cases} \quad (12)$$

The marker is distinct from every ordinary observation. It records the failure of the modeled execution criterion.

A word can now distinguish configurations in two ways. It can be executable in both and produce different outputs, or it can be executable in exactly one. The second case represents a difference in access under the chosen feasibility rules. Comparing only $W(x) \cap W(y)$ would systematically omit that distinction.

The blocked marker requires interpretation. It can stand for a prohibited attempt, an unavailable resource, an undefined operation, or a failure to execute. These meanings have different empirical implications. A richer observation can separate them if the inquiry needs that distinction. Collapsing them into one marker is a declared compression.

An absorbing blocked state provides one mathematical implementation. Every unavailable step enters that state, and subsequent steps remain there. This convention preserves a total word action on the enlarged carrier. It models the impossibility of recovering the original execution after a blocked prefix within that representation. A real process that can repair a failed attempt needs an explicit recovery transition.

The original semigroup equations also require care. Two words can have the same total product while differing in prefix feasibility. The word representation retains their execution structure until a factorization condition proves that availability descends to the product. The feasible-path companion paper develops that condition and its counterexamples.

6.6 Dormancy, Erasure, and Test-Induced Change

Potential efficacy can coexist with an observed trace that never reveals it. A test can also change the condition being tested. This subsection examines those possibilities and connects them to a limited philosophical comparison.

A dormant difference appears when current observations agree and some positive continuation separates them. An erasing prefix has the opposite effect: it maps the compared configurations to states that agree under every later observation. The authorization example contains both. Activation exposes the authorization difference, while withdrawal removes it before activation can do so.

A realized trace beginning with withdrawal therefore supplies limited evidence about the earlier potential. It can show what happened after the condition was removed. The earlier activation witness

remains a conditional consequence of the model and requires its own warrant. Repeated observation of erasing or bypassing continuations leaves that potential untested.

Lewis’s discussion of finkish dispositions provides a narrow comparison: the verified abstract identifies cases in which testing changes a disposition and complicates a simple conditional analysis [2]. The present construction uses explicit transition maps and a declared observation. It leaves the broader metaphysical analysis of dispositions open.

The comparison suggests a modeling question that can be pursued without adopting a theory of powers. Does the proposed test preserve the condition whose efficacy is being assessed, activate it, replace it, or remove it? Those alternatives lead to different transition sequences. A counterfactual statement about the original condition should distinguish them.

The social interpretation can be especially sensitive when participants respond to being tested. A declared assessment may change cooperation, recognition, or available information. Such reactions belong in the model if they affect the target comparison. Otherwise the claimed test may concern an idealized operation whose relation to practice remains unresolved.

6.7 Causal Attribution and the Status of Potential

A distinguishing continuation supplies a comparative result within the model. Causal attribution and empirical prediction require additional premises. This final subsection states the boundary and the constructive role that remains.

A pair (x, y) can differ in more than the focal relation. The observation contrast then establishes that the total represented difference matters under the test. A causal interpretation requires a defensible account of how the compared configurations vary and which background conditions are held fixed. The explicit comparison domain makes that responsibility visible.

The transition maps also encode assumptions about responses. A potential witness derived from them inherits their status. In a stipulated example, it is a model consequence. In an empirical application, it can become a prediction or causal claim only through evidence supporting the relevant operations and contrast. Section 10 gives a finite countermodel showing how observational agreement can coexist with intervention disagreement.

Potential can remain meaningful when unrealized. The definition refers to a member of a declared continuation space, so its truth within the model does not require the continuation to occur. Empirical warrant then concerns whether that continuation is admissible and whether the model’s response is justified. The absence of actualization leaves those questions open rather than settling them automatically.

The construction also leaves evaluation distinct from efficacy. A relation can enable an unwanted outcome, preserve an arrangement valued by some participants, or obstruct an activity valued by others. The witness records a difference under a selected output map. Its social appraisal requires criteria concerning the outcomes, participants, and distribution of consequences.

The resulting account supports a precise research claim: a historically formed pattern can be tracked through a declared record and tested for a possible subsequent role under stated comparisons. The historical and prospective components have separate proofs and evidential requirements. Their coordination, developed next, supplies a record suitable for both kinds of inquiry.

7 Potential Effects under Uncertain Comparison Designs

This section coordinates uncertainty about the compared configurations with the prospective witness construction. It distinguishes possible efficacy, efficacy across every compatible comparison, and the existence of one common test. It then derives the effect of valid evidence refinement and examines test families, sequential erasure, and joint compatibility. The construction adds explicit quantifiers to comparisons that can otherwise sound interchangeable in prose.

A historical record may leave several current configurations or formation histories compatible with the available evidence. The investigator then faces a family of potential-effect comparisons. A witness

for one member of that family can have a different significance from a witness shared by every member. The distinction concerns what the evidence and model jointly support.

7.1 Compatible Pairs and the Test Language

Let $\mathcal{C} \subseteq S^2$ be the declared domain of admissible comparisons. Let $K \subseteq \mathcal{C}$ be a nonempty set of pairs compatible with the retained evidence and comparison design. Each $k = (x, y) \in K$ represents one candidate contrast. The set is a joint record of pairings; its interpretation can include restrictions on background conditions, formation modes, or participants.

Fix a test language $L \subseteq B^+$. For $k = (x, y)$, write

$$D(k) = \{w \in L : o(T_w x) \neq o(T_w y)\}.$$

The set contains the tests that distinguish that pair under the declared model. A partial-operation version can use the blocked-output signature defined in Section 6. The quantifier analysis remains the same once the meaning of a witness is fixed.

Nonemptiness of K is an evidential requirement for the following interpretation. An empty compatible set can indicate conflicting records, an inappropriate candidate domain, or a failed model assumption. It requires diagnosis. The mathematical convention for an intersection over an empty family should remain separate from a claim that the evidence establishes every test.

The test language also needs an interpretation. It can contain a finite catalog of planned comparisons, every positive word below a horizon, or all positive words of the declared repertoire. Results about that language inherit its restrictions. Practical feasibility and the reliability of the transition model remain further inputs.

7.2 Three Forms of Potential-Effect Support

The first question is whether the retained alternatives permit some efficacy interpretation. The second asks whether efficacy holds for every retained alternative, allowing its witness to vary. The third asks whether a single test works across the alternatives.

Define the possible and uniform witness sets by

$$D_{\text{poss}}(K) = \bigcup_{k \in K} D(k), \quad D_{\text{unif}}(K) = \bigcap_{k \in K} D(k). \quad (13)$$

The union collects a test whenever some compatible pair supports it. The intersection retains a test only when every compatible pair supports it.

Definition 7.1 (Witness scope under comparison uncertainty). For nonempty K , the model supports:

$$\begin{aligned} \text{possible efficacy:} \quad & \exists k \in K \exists w \in L : w \in D(k), \\ \text{pointwise efficacy:} \quad & \forall k \in K \exists w \in L : w \in D(k), \\ \text{a uniform witness:} \quad & \exists w \in L \forall k \in K : w \in D(k). \end{aligned}$$

The term *pointwise* means that the witness may depend on the candidate comparison. It carries a stronger evidential statement about the alternatives than possible efficacy, yet it can leave the test-selection problem unresolved. A uniform witness supplies one word whose distinguishing role survives every alternative retained in K .

Proposition 7.2 (Implications among witness scopes). *A uniform witness implies pointwise efficacy. Pointwise efficacy implies possible efficacy. Each converse can fail.*

Proof. A uniform witness can be chosen for every member of K , which proves the first implication. For the second, select one member of the nonempty set K and use its pointwise witness. Possible efficacy can hold with two witness sets $\{a\}$ and $\emptyset$, while pointwise efficacy fails. Pointwise efficacy can hold with two nonempty disjoint witness sets, while their intersection is empty. The next subsection gives a dynamic realization of this second case. $\square$

Table 2: Primitive tests for two compatible comparison pairs. Both output states are absorbing under both tests.

Input state	Successor under a	Successor under b
x_A	z_0	z_0
y_A	z_1	z_0
x_B	z_0	z_0
y_B	z_0	z_1

These statements can guide the wording of a substantive claim. Evidence compatible with both an effective and an ineffective comparison supports possibility at the model level. Evidence under which every comparison has some effect supports a stronger conclusion. A test that can be selected without resolving the remaining comparison uncertainty requires the uniform condition or another explicit decision procedure.

7.3 A Dynamic Example with Incompatible Witnesses

The following model shows that the quantifier distinction can persist through arbitrarily long finite continuations. The failure is produced by the dynamics of testing, rather than solely by a short test catalog.

Let

$$S = \{x_A, y_A, x_B, y_B, z_0, z_1\}.$$

Set $o(z_1) = 1$, and let the observation of every other state equal zero. Both primitives a and b fix z_0 and z_1 . Their remaining values are given in Table 2.

Take

$$K = \{(x_A, y_A), (x_B, y_B)\}, \quad L = \{a, b\}^+.$$

The current observation agrees within both pairs. Test a distinguishes pair A , while test b distinguishes pair B .

Every first step enters one of the absorbing states. Consequently a word beginning with a has the same final observation as its first a -step. It distinguishes pair A and collapses pair B . A word beginning with b distinguishes pair B and collapses pair A . Every positive word has exactly one of these two first letters.

Thus each compatible comparison has a potential-effect witness, yet the witness sets are disjoint. Extending the horizon leaves their intersection empty. The absence of a uniform witness follows from irreversible loss of a comparison distinction at the first step.

The example is a stipulated test model. A social application would need evidence that the test changes the relevant condition in the represented way. Its analytic role is to show why a claim about efficacy across alternatives can coexist with difficulty in selecting a single informative test.

A sequence that includes both labels also fails to repair the problem. In ab , the first test has already collapsed pair B ; the later b -step preserves the collapsed output. In ba , the first test has already collapsed pair A . The execution history matters even though the labels of both candidate tests appear in each sequence.

7.4 Evidence Refinement and Opposite Changes in Witness Sets

Evidence refinement can reduce the compatible comparison set. Its effect on possible and uniform witnesses moves in opposite directions. This subsection derives both inclusions and explains their evidential meaning.

Proposition 7.3 (Witness sets under valid refinement). *If $\emptyset \neq K' \subseteq K$, then*

$$D_{\text{poss}}(K') \subseteq D_{\text{poss}}(K), \quad D_{\text{unif}}(K) \subseteq D_{\text{unif}}(K').$$

Pointwise efficacy on K implies pointwise efficacy on K' .

Proof. A member of the union over K' belongs to $D(k)$ for some $k \in K'$. Since that pair also belongs to K , the test belongs to the union over K . This proves the first inclusion. A member of the intersection over K belongs to every $D(k)$ with $k \in K$, hence to every such set indexed by K' . This proves the second inclusion. Finally, pointwise witnesses available for every member of K remain available for each member of its subset K' . $\square$

A narrower evidence set can remove a merely possible witness by eliminating the only comparison that supported it. The same refinement can establish a uniform witness by eliminating a comparison that defeated that test. Greater evidential precision therefore need not increase every measure of potential efficacy.

In the dynamic example, retaining only pair A makes every word beginning with a a uniform witness. It also removes words beginning with b from the possible-witness set. These changes are consistent: the evidence has narrowed what might be the case and strengthened what follows across the remaining possibilities.

The validity of the refinement is essential. Removing a compatible pair without evidential or design justification can manufacture a uniform conclusion. A study should record which new evidence or revised assumption excluded the pair. It should also examine whether the intended actual comparison remains in the candidate domain.

An empty refinement deserves separate treatment. If every candidate has been excluded, the formal set operations still have conventional values, yet the evidential interpretation has failed. The investigator must revisit the evidence, candidate space, or comparison assumptions before interpreting witness claims.

7.5 Coverage by a Family of Fresh Tests

A family of tests can cover the compatible comparisons even when a single uniform witness is unavailable. The meaning of coverage depends on how the tests are executed.

For each $w \in L$, define

$$V_w = \{k \in K : w \in D(k)\}.$$

A selected family $J \subseteq L$ covers K when

$$K = \bigcup_{w \in J} V_w. \quad (14)$$

This means that each compatible pair is distinguished by at least one selected test, evaluated from the pair's declared initial configurations.

Proposition 7.4 (Finite coverage from pointwise efficacy). *If K is finite and nonempty, pointwise efficacy guarantees a covering family with at most $|K|$ tests.*

Proof. Choose one witness for each member of K . The resulting collection covers every member by construction. Removing duplicate choices gives a test family containing at most $|K|$ distinct words. $\square$

The finite assumption is substantive. An abstract witness incidence can have $K = \mathbb{N}$ and one test w_n covering only the comparison indexed by n . Every comparison then has a witness, while every finite test family leaves infinitely many comparisons uncovered. This incidence example establishes the set-theoretic limit; a dynamic realization of it would be an additional construction.

Coverage also differs from concatenation. In the six-state example, $\{a, b\}$ covers the two comparison pairs when each test is evaluated from its declared initial pair. Executing ab or ba on one evolving pair has the erasure problem already derived. The coverage equation therefore needs fresh instances, justified restoration, or another execution interpretation.

For a proposed empirical design, access to comparable fresh instances can be a demanding assumption. Restoration may also alter the relation or fail to restore its history. A protocol using several tests must state how its instances are related and what supports treating their initial conditions as comparable.

7.6 Joint Compatibility and Comparison Constraints

A set of admissible pairs contains information about which configurations can appropriately be compared. Replacing it by separate marginal candidate sets can lose that information.

Let $S = \{0, 1\}$, let the observation be the identity, and let the sole test a act identically. Take

$$K = \{(0, 1), (1, 0)\}.$$

The test distinguishes every pair in K . Both marginal candidate sets are $\{0, 1\}$, but their Cartesian product also contains $(0, 0)$ and $(1, 1)$. Each diagonal pair has an empty witness set.

Consequently replacing K with its marginal product changes a uniform-witness conclusion into a failure of pointwise efficacy. The change comes from admitting new comparisons. It leaves the operation and observation maps unchanged.

In a relational inquiry, joint constraints might require the same background institution, a common resource condition, or a specified contrast between formation procedures. Such constraints are part of the comparison design. Their empirical justification remains open to challenge, while their formal role should remain visible.

The example also warns against interpreting a larger candidate set as automatically more cautious. Enlarging the set can be useful when the original evidence was overconfident. Enlarging it through incompatible pairings changes the question. A sound uncertainty model retains both uncertainty and the constraints supported by the comparison design.

7.7 Historical Evidence and Implementable Testing

The preceding results connect two parts of the paper's argument. Historical evidence can constrain which comparison pairs remain possible. Prospective operations determine the witnesses available for each pair. Their combination supports several different claims, whose quantifiers and practical requirements must remain explicit.

A historical record can establish pointwise efficacy without identifying a test that an investigator can choose from the retained information. Conversely, a common test can be available even while the exact formation history remains uncertain. The appropriate record depends on whether the inquiry seeks a historical answer, a potential-effect claim, or an executable test protocol.

An adaptive protocol adds further structure. It can choose a later test from earlier observations, provided those observations preserve enough information and the operations remain available. Its analysis requires an observation rule, a policy, and the dynamics of test-induced change. The set coverage result alone leaves that problem open.

The social interpretation should also retain the difference between testing a model and intervening in an arrangement. A conceptual comparison can use a stipulated fresh instance. An actual test can alter participants' expectations, records, or future willingness to cooperate. Those effects belong to the model when they influence the intended inference.

The construction supplies a precise extension of the potential-effect criterion. It allows a study to state what follows for some compatible comparison, what follows for every compatible comparison, and what a single selected continuation can demonstrate. Historical uncertainty, test choice, and practical execution remain coordinated through explicit assumptions.

8 Joint Records for Prediction and Historical Inquiry

This section constructs a dynamically sufficient record for a predictive observation and a historical query together. It first specifies history updates, then derives the greatest compatible equivalence preserving both tasks. It examines a failure of current-value storage, gives a finite realization, and proves a finite refinement procedure and a minimality property. The construction supplies a positive response to the historical compression problem while keeping its dependence on the chosen tasks explicit.

8.1 History Updates and Joint Observations

A historical query can change when further activity is recorded. To retain it dynamically, a model must represent both the current answer and the effect of subsequent updates on that answer. This subsection specifies the required domains.

Assumption 8.1 (Updated history system). Let H be nonempty and closed under total maps $U_b : H \rightarrow H$ for $b \in B$. Let $o_H : H \rightarrow Y$ be a predictive observation and $g : H \rightarrow V$ a historical query. Define the joint current observation $j : H \rightarrow Y \times V$ by $j(h) = (o_H(h), g(h))$. Use the left-to-right word convention for compositions of updates.

For word histories, $U_b h = hb$ and $o_H = o \circ \eta$. A formation query might record the latest formation label or the number of formation episodes. Other history domains can include documents, dates, or occurrence identities, provided their updates are specified. The theorem concerns the declared system and applies independently of which interpretation is ultimately adequate.

Closure ensures that every update word used in a signature has a value in the history domain. A domain containing only actually completed records may fail this condition because many hypothetical continuations have yet to occur. The mathematical construction therefore uses a represented continuation domain. Its relation to actual histories remains an evidential question.

Partial updates require additional treatment. One can restrict the word language by prefix feasibility or introduce a blocked state with a declared readout. The total theorem below applies directly only after such a representation has been justified. A partial operation cannot be treated as total merely by suppressing its undefined cases.

The joint observation places historical and predictive uses beside one another. It leaves their philosophical importance unordered. A record can be required to answer both even when one coordinate adds nothing to the other. The resulting size measures the information needed for the selected combination, rather than the intrinsic worth of either task.

8.2 Continuation Signatures and Separate Equivalences

A record that supports future updates must retain distinctions that can affect later answers. Continuation signatures represent all such answers at once. This subsection defines the separate and joint signatures before deriving their relation.

For each history, define

$$\sigma_h^o(w) = o_H(U_w h), \quad \sigma_h^g(w) = g(U_w h), \quad w \in B^*. \quad (15)$$

Let E_o be equality of the first signature and E_g equality of the second. Each is an equivalence because equality of functions is reflexive, symmetric, and transitive.

Define the joint signature by

$$J_h(w) = (o_H(U_w h), g(U_w h)). \quad (16)$$

It records the pair of answers after every common continuation. The empty word retains both current values. Longer words retain the distinctions needed by later updates.

The construction has a standard mathematical antecedent in deterministic behavior semantics, where observations are indexed by finite input words [6, Example 10.2(5)]. The present application selects a historical query and a predictive observation as its joint output. The social interpretation of those coordinates is supplied by the research problem.

A signature can be infinite even when the query has a finite codomain. Its mathematical existence establishes an exact representation. Whether a finite record implements the same distinctions is a separate issue addressed below. The distinction between existence and practical representation should remain visible when the history domain contains arbitrarily long words.

8.3 The Compatible Joint Quotient

The main theorem identifies the coarsest equivalence that retains both current observations and remains compatible with all updates. This subsection gives the full proof in stages, including the quotient maps and their minimality under inclusion.

Proposition 8.2 (Compatible joint record). *Under Assumption 8.1, equality of joint signatures is $E_j = E_o \cap E_g$. The relation E_j is the greatest equivalence preserved by every U_b and contained in $\ker o_H \cap \ker g$. The quotient H/E_j supports both observations and the induced deterministic updates.*

Proof. First, equality of $J_h(w)$ and $J_{h'}(w)$ for every word is equivalent to equality of both coordinates for every word. Hence $E_j = E_o \cap E_g$. The intersection of equivalences is an equivalence. Taking $w = \epsilon$ also gives $E_j \subseteq \ker o_H \cap \ker g$.

Second, suppose hE_jh' and fix an update b . For every word w ,

$$J_{U_b h}(w) = J_h(bw) = J_{h'}(bw) = J_{U_b h'}(w).$$

The equality uses $U_w U_b = U_{bw}$ under the execution convention. Thus $U_b h E_j U_b h'$, proving update preservation.

Third, let E be any equivalence preserved by all updates and contained in both current-observation kernels. If hEh' , induction on word length gives $U_w h E U_w h'$ for every word. The base case is the assumed pair. The induction step uses preservation by the next update. Since E retains both current observations, their values at these updated histories agree. Consequently $J_h = J_{h'}$, so $E \subseteq E_j$.

Finally, define

$$\bar{U}_b([h]) = [U_b h], \quad \bar{o}([h]) = o_H(h), \quad \bar{g}([h]) = g(h).$$

Update preservation makes the first map representative-independent. Containment in the two observation kernels makes the remaining maps representative-independent. They give the claimed quotient system. $\square$

Greatest under inclusion means coarsest as a partition. The theorem merges every pair that can be merged while retaining the stated update and observation requirements. A finer equivalence can preserve additional information. A coarser one must fail at least one requirement.

The proof also shows the exact sense in which future closure is necessary. Current agreement alone permits histories whose answers later diverge. Update preservation rules out that possibility through induction. The full signature collects the distinctions that this induction can expose.

A preexisting preserved equivalence can be included through intersection when the project requires it. Intersecting two preserved equivalences retains preservation because each update respects both relations. The resulting record can be finer than the joint quotient. Its additional distinctions need their own interpretive purpose.

The theorem supplies a conditional construction, not a universal social state definition. Different observations, history queries, or operations can produce different quotients. The dependence is a strength when it makes research commitments explicit and a limitation when those commitments remain inadequately justified.

8.4 Failure of Current-Answer Storage

Storing the current query value can appear sufficient until a later update requires information that value has discarded. A small counterexample isolates the issue. This subsection derives the failure and constructs a finite repair.

Let $H = \{a\}^*$, let the predictive observation be constant, and define

$$g(h) = \begin{cases} 1, & |h| = 2, \\ 0, & |h| \neq 2. \end{cases}$$

The histories ϵ and a both currently have query value zero. After appending a , their values become zero and one. Therefore the kernel of the current query is not preserved by the update.

The failure concerns a specific missing distinction. A zero answer can mean that length two lies ahead, has just not been reached, or has already been passed. These histories share the present answer while requiring different future answers. A current Boolean field cannot update itself deterministically without additional memory.

Define a record

$$r(h) = \min(|h|, 3) \in \{0, 1, 2, 3\}.$$

Its update is $F(r) = \min(r + 1, 3)$, and its query readout is one exactly at record value two. Direct substitution gives $r(ha) = F(r(h))$. Thus the record supports both exact current recovery and deterministic updating for every history.

The value three represents all lengths at least three. These histories can be merged because every future extension keeps the query value zero. The record therefore discards irrelevant detail while preserving the distinctions required by the task. It illustrates a constructive alternative to retaining the complete history.

All four record values are necessary for this query under the declared update. For two values $a < b$ in $\{0, 1, 2, 3\}$, the smaller value satisfies $a \leq 2$. Append $2 - a$ letters. The smaller value reaches two, while the larger reaches a value greater than two or remains at three. The query outputs differ. Hence all four values have distinct continuation signatures.

The example also clarifies the meaning of a finite sufficient record. It need not resemble the original query's vocabulary. A research question about a particular episode count can require a progress record whose categories include before, at, and beyond the target. The usefulness of the record lies in its demonstrated behavior, rather than the apparent simplicity of its labels.

8.5 Finite-Horizon Refinement

A finite deterministic system permits an explicit procedure for constructing continuation classes. This subsection derives the recursion, stabilization condition, and bound on strict refinements. The proof supplies the completeness criterion used in the worked example.

Let X be a finite nonempty domain, with total updates U_b and joint observation $j : X \rightarrow J$. Define

$$E_k = \{(x, y) : j(U_w x) = j(U_w y) \text{ for every } |w| \leq k\}. \quad (17)$$

At horizon zero, $E_0 = \ker j$. For the next horizon,

$$E_{k+1} = E_0 \cap \bigcap_{b \in B} (U_b \times U_b)^{-1}(E_k). \quad (18)$$

To derive the recursion, split the tested words into the empty word and words bw whose remaining suffix has length at most k . Agreement at the empty word requires E_0 . Agreement after every possible first letter requires the successor pair to belong to E_k . These conditions are jointly equivalent to agreement through horizon $k + 1$.

Proposition 8.3 (Finite stabilization). *The sequence $E_0 \supseteq E_1 \supseteq \dots$ is decreasing. If $E_{k+1} = E_k$, then E_k equals full continuation equivalence. If $N = |X|$ and c_0 is the number of E_0 classes, at most $N - c_0$ strict refinements occur.*

Proof. Agreement under a larger set of words entails agreement under its subsets, proving the decreasing property. Suppose $E_{k+1} = E_k$. The recursion implies that every E_k -related pair has E_k -related successors under every update. The relation also retains the current observation because it is contained in E_0 . Induction along arbitrary finite words therefore gives agreement of every future observation. Thus E_k is contained in full continuation equivalence. The converse inclusion follows from the definition of E_k .

Every strict refinement of an equivalence on a finite set splits at least one class and increases the number of classes by at least one. Starting from c_0 classes, the number can increase at most to N . Hence there are at most $N - c_0$ strict refinements. $\square$

The result separates a stabilization test from an arbitrary stopping horizon. Equality at two successive refinement stages proves closure because it entails preservation by every update. Merely reaching a chosen horizon supplies a bounded result unless the equality condition has been checked.

The unary query example yields class counts 2, 3, 4, 4. At horizon zero, only the target length is separated. One further step distinguishes the immediately preceding length, and the next distinguishes the initial length from the permanently post-target class. The final repeated count is accompanied by equality of partitions, which establishes stabilization.

Finite states alone do not guarantee an executable algorithm if the operation repertoire is infinite and lacks an effective specification. The proof quantifies over every update. A practical computation needs a finite alphabet or a justified procedure that covers the relevant update effects. This obligation should be stated with any claim of exhaustive checking.

8.6 Minimality among Deterministic Records

The quotient theorem can be expressed through a concrete record-size comparison. This subsection proves that every exact deterministic record for the same tasks must retain at least the continuation distinctions.

Let $\rho : H \rightarrow Z$ be a record with update maps $F_b : \rho(H) \rightarrow \rho(H)$ satisfying $\rho U_b = F_b \rho$. Suppose the joint current observation factors through ρ . These conditions say that the record has a deterministic internal update and a sufficient readout.

Proposition 8.4 (Record factorization through continuation classes). *Under these assumptions, $\ker \rho \subseteq E_j$. The map*

$$q : \rho(H) \rightarrow H/E_j, \quad q(\rho(h)) = [h]_{E_j},$$

is a well-defined surjection. If the attained record set is finite, it has at least as many elements as H/E_j .

Proof. Suppose $\rho(h) = \rho(h')$. Applying a common update gives equal records because both are transformed by the same F_b . Induction extends this equality to every update word. The joint observation factors through the record, so the corresponding updated observations agree. Hence $h E_j h'$.

This implication makes the definition of q independent of the chosen representative of a record value. Every continuation class contains some history, whose record maps to that class, so q is surjective. The finite cardinality conclusion follows from surjectivity. $\square$

The minimum is relative to a fixed domain, update system, and query family. Changing any of those inputs changes the comparison. A smaller record obtained by abandoning a historical question solves a different task. A smaller record obtained by restricting histories relies on that restriction. The proposition makes these tradeoffs inspectable.

Record size also differs from practical cost. Two encodings with the same number of values can have different storage, update, or interpretation costs. An infinite but simply specified record may be easier to describe than a large finite table. The cardinal result identifies necessary distinctions; an implementation study requires further criteria.

8.7 Task Coordination and Philosophical Interpretation

A joint record can preserve a historical distinction even when it is irrelevant to one predictive output. This final subsection explains that coordination and the limits of the quotient's philosophical significance.

The intersection $E_o \cap E_g$ makes the two tasks explicit. A pair of histories remains merged only if both tasks agree under every continuation. A historical query can split a predictive class, while a predictive operation can split a historical class. The quotient records the combined information requirement.

This construction avoids reducing historical significance to predictive usefulness. It also avoids assuming that every historical distinction must affect the selected output. The project can retain a query for explanatory, interpretive, or evidential reasons while stating its predictive role separately.

A conflict can arise between the desire for compression and the desire to support later questions. The theorem provides an exact minimum for the declared tasks. It leaves unforeseeable future tasks outside that guarantee. A research archive can therefore preserve more than a working model uses, with a clear distinction between archival retention and the model's sufficient state.

The quotient also should remain distinct from the identity of a relation. It identifies histories according to selected future answers. Two episodes can share a continuation class while being different episodes. A single continuing relation can pass through several classes as its relevant properties change. Tracking the relation requires its own continuity criterion.

The positive result is a method of coordination. Historical formation and prospective efficacy can be represented in one updateable record when their information requirements are made explicit. The next section applies this method to a finite authorization model and derives the resulting classes in full.

9 An Authorization-Formation Illustration

This section applies the historical recovery and joint-record constructions to a finite authorization model. It specifies the configuration variables and every update, derives the loss of formation information, enumerates the reachable enriched records, and proves the predictive and joint quotients. It then changes one modeling assumption to show how a historical distinction can become predictively relevant. The example is stipulated for analysis; its social vocabulary names proposed interpretations whose empirical adequacy remains open.

9.1 Configuration Variables and Operational Scope

The example uses a small carrier so that every relevant consequence can be inspected. Its simplicity also makes the omitted features visible. This subsection defines the variables and explains the scope of each.

Let $S = \{0, 1\}^2$, with $x = (r, e)$. The variable r represents whether a focal authorization condition currently holds. The variable e records whether a focal activity has occurred at least once. Begin at $x_* = (0, 0)$ and use full equality as the base equivalence. The predictive observation is $o(r, e) = e$.

The two variables have different temporal roles. Authorization can be established and removed. The occurrence flag is persistent: once the represented activity has occurred, later updates retain that fact. It therefore records an accumulated historical property inside the current configuration. This already illustrates how a present state can contain a sufficient summary of part of its history.

The occurrence flag intentionally omits the number of activities, their duration, their participants, and their consequences beyond the chosen completion fact. A model of repeated activity would need a counter or a richer occurrence record. A model of an activity that can be reversed would need a different state variable. The once-completed interpretation gives the current Boolean update its meaning.

The authorization variable also compresses a complex possible target. It records one condition under a declared criterion. It leaves the source of authority, its recognition by participants, its legitimacy, its enforcement, and its practical accessibility open. Those features can be added when a research question requires them. Their absence is a scope condition of the illustration.

The starting state fixes both variables. If the initial case were uncertain, the analysis would use a set of candidate initial states or an enriched history domain. The present choice allows a complete reachable-record enumeration and isolates the formation-query issue.

9.2 Formation, Activation, and Withdrawal Updates

The operation rules encode the example's substantive assumptions. This subsection gives each map and derives its immediate interpretation. Equal effects of differently named formation operations are a deliberate feature of the construction.

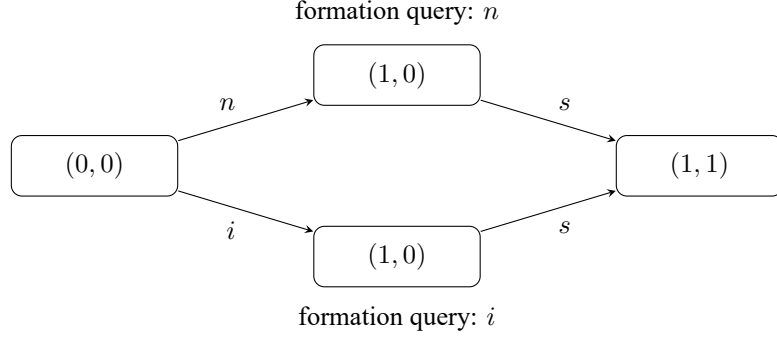


Figure 1: Distinct represented formation histories with the same current configuration and activation effect. Formation labels survive only in an additional record. The arrows represent possible operations.

Use four labels:

$$\begin{aligned}
 n(r, e) &= (1, e), & i(r, e) &= (1, e), \\
 s(r, e) &= (r, e \vee r), & d(r, e) &= (0, e).
 \end{aligned} \tag{19}$$

The labels n and i stand for hypothetical negotiated and imposed formation routes. The operation s attempts activation, and d withdraws the current authorization condition.

The formation maps are equal on the two-coordinate carrier. Both establish $r = 1$ and preserve the occurrence flag. Their distinct labels retain a difference in the represented formation history. The selected operational variables disregard that difference. The free semigroup of nonempty labeled words therefore acts nonfaithfully on S .

Activation reads the current authorization. If $r = 1$, it sets $e = 1$. If $r = 0$, it preserves the state. The operation is total, so an unauthorized attempt remains a modeled operation with an unchanged endpoint. A system that prohibits the attempt would require partial availability, while a system that records a refusal would require another observation or variable.

Withdrawal sets $r = 0$ while preserving e . It removes the current enabling condition and retains the fact of earlier activity. This rule distinguishes the persistence of a past occurrence from the persistence of its enabling relation. A different application might require withdrawal to reverse or invalidate an outcome, but that would be a different transition model.

All four maps are idempotent on this carrier. Repeating formation, activation, or withdrawal can leave the same endpoint as a single application. The word history nevertheless retains the repeated occurrences. This provides a concrete instance of the distinction between effect compression and historical counting.

9.3 Formation Collisions and Historical Queries

Two one-step histories reach the same operational configuration while carrying different formation labels. This subsection derives the resulting recovery failure and states precisely which historical question is being retained.

Let $g(h)$ be the most recent formation label appearing in the word history, with value none before any formation operation. Thus $g(n) = n$ and $g(i) = i$. Figure 1 shows their common endpoint and common activation outcome. Table 3 adds withdrawal and activation continuations.

The endpoint map satisfies $\eta(n) = \eta(i) = (1, 0)$, while $g(n) \neq g(i)$. Proposition 3.1 therefore proves that the endpoint cannot recover this query on a history domain containing both words. The witness pair is sufficient to establish the failure.

The equality extends beyond the current endpoint. For every continuation w , determinism gives $T_w\eta(n) = T_w\eta(i)$. Hence all future e observations agree. Under this operation repertoire, the formation distinction is predictively redundant for the selected output while remaining historically queryable from the word record.

Table 3: Selected word histories and the most recent formation query

Word from $(0, 0)$	Endpoint (r, e)	Observation e	Query g
n	$(1, 0)$	0	n
i	$(1, 0)$	0	i
nd	$(0, 0)$	0	n
id	$(0, 0)$	0	i
ns	$(1, 1)$	1	n
is	$(1, 1)$	1	i

Withdrawal provides another collision. Both nd and id reach $(0, 0)$, and their formation queries remain different. The initial empty history has the same endpoint but query none. The endpoint therefore merges a never-formed case with two differently formed and subsequently withdrawn cases. A present absence of authorization leaves its formation history unresolved.

The choice of most recent formation is material. A query about the first formation would update differently after a second formation operation. A query about the complete sequence would require more information. A query about the currently effective basis might clear its value upon withdrawal. These alternatives should remain separately specified; a single field called origin can otherwise shift among them.

9.4 Dormant Efficacy and Erasing Continuations

The example separates historical formation from present potential efficacy. This subsection derives an activation witness and explains how withdrawal removes it without settling the earlier historical question.

Compare $x = (1, 0)$ and $y = (0, 0)$. Their current observations agree: $o(x) = o(y) = 0$. Activation gives $sx = (1, 1)$ and $sy = (0, 0)$, so $o(sx) = 1 \neq 0 = o(sy)$. If the pair belongs to the admissible comparison domain, $s \in \mathcal{D}^+(x, y)$.

The witness identifies an unrealized prospective role. Authorization changes the modeled result of an activation attempt even before the activity occurs. The definition requires the available test and its stipulated response, rather than an actual recorded activation. Empirical use would require evidence for the condition, the operation, and the admissibility of the contrast.

Withdrawal erases the operational difference: $dx = dy = (0, 0)$. Every later common continuation then has the same effect because it starts from the same state. The witness set for the post-withdrawal pair is empty. This calculation concerns the altered pair; it leaves the pre-withdrawal activation witness intact as a conditional statement about the earlier configurations.

The distinction also clarifies an observed trace. If an actual sequence begins with withdrawal, its later agreement can establish what happened after authorization was removed. It leaves the earlier counterfactual response to activation dependent on the model. The trace supplies neither an actual activation witness nor a refutation of the stipulated earlier potential.

The admissible comparison remains an additional assumption. In the example, it is analytically stipulated that the two configurations differ in the authorization variable while the other represented variable is fixed. A social case may involve unrepresented changes accompanying authorization. The causal interpretation then needs a richer account than this two-coordinate contrast.

9.5 Construction of the Enriched Record

A finite record can retain the selected formation query without storing every complete word. This subsection defines that record, derives its updates, and proves that it is sufficient for the declared historical and predictive tasks.

Use

$$z = (r, e, g), \quad g \in \{\text{none}, n, i\}.$$

The first two coordinates update by Equation (19). Operation n sets $g = n$, operation i sets $g = i$, and operations s, d preserve g . The initial record is $(0, 0, \text{none})$.

The update rule for g follows from the query definition. Appending a formation label makes that label the most recent one. Appending activation or withdrawal introduces no new formation label, so the old answer persists. Induction on word length proves that the record's g coordinate equals the query on every represented history.

The first two coordinates similarly equal the endpoint by their update definition. The record therefore supplies both e and g as direct readouts and updates deterministically. It meets the conditions of Proposition 8.4, although it may retain distinctions beyond the smallest joint quotient.

Exactly nine enriched records are reachable. Before any formation, only $(0, 0, \text{none})$ is reachable because activation and withdrawal preserve it. Once a formation has occurred, the most recent label is n or i . For each label $\ell \in \{n, i\}$, all four pairs (r, e) are reachable through

$$\ell, \quad \ell d, \quad \ell s, \quad \ell sd,$$

with endpoints $(1, 0)$, $(0, 0)$, $(1, 1)$, and $(0, 1)$ respectively. These records are distinct, and the preceding argument excludes every other value.

The enumeration proves both reachability and completeness. Listing nine successful examples alone would prove only a lower bound. The invariant that $g = \text{none}$ implies $(r, e) = (0, 0)$ supplies the upper bound, while the two formed labels each allow at most four pairs.

9.6 Predictive Classes and Joint Classes

The enriched record can be compressed differently for prediction alone and for prediction together with the formation query. This subsection derives both quotients and the stabilization counts.

Under the observation e , the nine reachable records have three continuation classes. The first contains all records with $(r, e) = (0, 0)$, including the initial record. The second contains those with $(r, e) = (1, 0)$. The third contains every record with $e = 1$.

The first two classes share the current output zero and separate under s . Every record in the third class continues to output one because all updates preserve the completed flag. Within each of the first two classes, the first two coordinates agree and future maps ignore g , so all future outputs agree. These arguments establish both separation between classes and equivalence within them.

Under the joint observation (e, g) , there are seven continuation classes. The initial record forms one. For each formed label, the two records with $e = 0$ remain separate according to r , while the two records with $e = 1$ merge. Thus the count is $1 + 3 + 3 = 7$.

Different formation labels separate immediately through g . Within a fixed label, authorization separates the two incomplete records under activation. The completed pair remains equivalent because activation and withdrawal preserve the label and completed output, while later formation operations replace the label identically in both records. This verifies closure under every update.

The horizon refinement counts are 2, 3, 3 for e and 5, 7, 7 for (e, g) when the repeated stable stage is displayed. At horizon zero, prediction retains two outputs, while the joint task retains five attained output pairs. One-step tests split the incomplete authorization conditions. The next stage leaves both partitions unchanged, so Proposition 8.3 establishes full continuation equivalence.

The difference between nine, seven, and three has a precise interpretation. Nine counts the reachable values of the convenient enriched record. Seven counts the distinctions necessary for the joint task. Three counts those necessary for the predictive output alone. These numbers measure different representational responsibilities.

9.7 A Formation-Sensitive Review Variant

A changed operation can make a previously redundant historical distinction affect prediction. This subsection gives an explicit variant and shows why the old two-coordinate state then becomes insufficient.

Add a review operation v on the enriched record. If $g = i$, it sets $r = 0$ and preserves e, g . For the other formation values, it leaves the record unchanged. This is a stipulated rule for a hypothetical review. Its selection of a label carries no general judgment about actual formation practices.

After history n , the record is $(1, 0, n)$. After i , it is $(1, 0, i)$. Applying v preserves authorization in the first and withdraws it in the second. Following review with activation therefore yields $e = 1$ in the first and $e = 0$ in the second. The continuation vs distinguishes the formerly predictively equivalent histories.

The old two-coordinate record cannot implement v deterministically. Its common input $(1, 0)$ would need to map both to $(1, 0)$ and to $(0, 0)$, depending on the omitted formation label. This is the exact information gap identified by the record-update criterion.

The variant preserves the validity of the old equivalence within its original repertoire. It changes the model by adding an operation that reads historical information. The resulting refinement is a traceable response to an expanded task. A claim that the histories have identical complete futures would need to specify which repertoire the word complete covers.

The example also illustrates a research possibility. A formation record can have little effect under routine operation and become consequential under review, conflict, or revision. An empirical application would need evidence that such a procedure exists and responds to the proposed historical distinction. The formal variant identifies what that evidence would need to support.

9.8 Interpretive Limits and Empirical Development

The illustration establishes exact results for a declared finite model. This final subsection identifies the substantive questions that remain before its social labels could support an empirical explanation.

Negotiated and imposed are hypothetical coding categories. A real formation episode can involve mixed procedures, unequal options, disputed recognition, or changes over time. The present binary distinction supplies a controlled collision example. An empirical inquiry would need definitions, source criteria, and a procedure for ambiguous cases.

Authorization also can differ from effective access. A formally permitted activity may require resources or cooperation omitted from the model. An unauthorized activity may still occur through other means. Adding those possibilities changes the activation rule and can change the continuation classes. The formal method remains available, while the particular nine-record enumeration belongs to the stated assumptions.

The model deliberately separates descriptive classification from normative evaluation. Formation labels can matter to an inquiry about legitimacy, responsibility, or procedural quality, but a label alone supplies no complete judgment. The relevant standards, affected participants, and evidence need their own treatment.

The example's strongest contribution is methodological. It demonstrates endpoint information loss, dormant efficacy, erasure, finite historical memory, task-dependent compression, and refinement under a new history-sensitive operation within one inspectable system. Each conclusion has an explicit witness or proof. The next section turns to the evidential and causal requirements that a substantive application would add.

10 Evidence and Causal Interpretation of Relational Effects

This section examines the bridge from exact model results to claims about an actual historically formed relation. It separates evidential coding, predictive discrimination, and causal attribution; derives a finite countermodel of observational underdetermination; and develops a research design for testing the proposed interpretation. The aim is to identify the premises that a substantive application must support, while preserving the conditional value of the earlier formal constructions.

10.1 Claims and Their Evidential Targets

A formation account contains several kinds of claim whose evidence can overlap without being interchangeable. This subsection identifies their targets so that an empirical study can avoid treating one successful observation as support for the entire account.

An occurrence claim concerns whether a represented episode happened. An ordering claim concerns the temporal or procedural relation among episodes. A formation classification concerns how an episode meets the adopted definition of a formation mode. A continuity claim concerns whether later appearances belong to the same relation under the tracking criterion. A potential-effect claim concerns a possible response under a specified continuation. A causal claim concerns the role of the focal relation in that response.

These claims can depend on different records. An activity log may support an ordering claim while leaving participants' interpretations unresolved. A formation document may support a declared procedure while leaving its enactment open. An observed outcome may support a predictive comparison while leaving the formation route unidentified. The model should record which source supports which proposition.

The distinction also applies to negative findings. Failure to find a formation document concerns the search and available archive. It can support a claim about the record's incompleteness without establishing that formation never occurred. Failure to observe a distinguishing continuation concerns the tested repertoire. It can leave potential efficacy unresolved when relevant tests remain unperformed.

A useful evidence ledger therefore links propositions to sources, coding rules, and unresolved alternatives. The ledger is an aid to inquiry rather than an authority that promotes an interpretation automatically. Its strongest benefit is traceability: a later challenge can identify which premise of a formal or historical claim needs revision.

10.2 From Source Material to Query Values

The map $g : H \rightarrow V$ treats a historical query as well-defined on the represented domain. An empirical coding procedure must justify that treatment. This subsection examines the decisions that connect source material to a query value.

A formation category needs criteria that can be applied to the available record. If negotiated and imposed are used, the inquiry should specify which features distinguish them and how mixed or disputed cases are represented. A source's own label can be retained as one observation while the investigator's classification remains separate.

The distinction between source language and analytical language is especially important when a term carries evaluation. A document can describe a procedure as voluntary, cooperative, or legitimate. Recovering that description establishes what the document says, given a reliable transcription. Endorsing the corresponding classification requires an additional argument relating the source's usage to the research criterion.

Temporal alignment also matters. A record created after a dispute may describe an earlier episode through a revised vocabulary. That record can be valuable evidence, but its date of production and its target episode perform different roles. Treating them as the same time coordinate can distort the formation sequence.

A coding rule should allow the evidence to remain insufficient when the relevant distinction cannot be resolved. Forcing every case into a binary classification can hide uncertainty in the query definition. A structured value can distinguish the proposed answer, its evidential status, and the alternatives still compatible with the record.

The formal recovery theorem then applies to the declared query. It does not validate the coding rule by itself. A careful application reports both the information result and the interpretive argument that makes the query meaningful.

10.3 Predictive Discrimination and Causal Attribution

A continuation can distinguish two configurations without identifying the causal contribution of one selected feature. This subsection states that difference and explains why a causal comparison needs additional assumptions.

The witness condition $o(T_w x) \neq o(T_w y)$ compares the complete represented configurations. If x and y differ only in a focal coordinate under a justified intervention model, the contrast can contribute to a causal argument. If several relevant conditions differ, the same inequality leaves attribution open.

Even a single displayed coordinate can conceal several changes. An authorization variable may summarize recognition, access, information, and enforcement together. Changing its value in the model can stand for a bundle of interventions. A claim about one component requires either a finer representation or an argument that the bundle is the intended causal object.

The transition maps themselves encode causal or predictive assumptions according to their use. A map fitted to observed transitions can summarize association. A map interpreted under intervention requires a justification that it remains valid when the focal condition is changed in the specified way. The same functional notation does not erase that difference.

Pearl's account of causal inference explicitly distinguishes observational information from the additional assumptions needed for causal questions [4]. The present framework uses that distinction to delimit its own continuation tests. Its finite countermodel below is a stipulated illustration of the information gap.

An admissible comparison domain $\mathcal{C}$ should therefore state more than which pairs are mathematically available. It should identify the intended contrast, the background conditions held fixed, the operation interpretation, and the evidence supporting transfer from the represented pair to the case. These are components of the argument, rather than details to be inferred from an inequality.

10.4 A Binary Countermodel of Observational Underdetermination

A small construction shows why complete knowledge of a specified observational law can leave an intervention question unresolved. This subsection defines two models, derives their observational agreement, and calculates their different intervention answers.

Let $U \in \{0, 1\}$ have a fixed distribution with $\Pr(U = 1) = p$. Consider the two structural specifications

$$\mathcal{A}: \quad X = U, \quad Y = X, \quad \mathcal{B}: \quad X = U, \quad Y = U. \quad (20)$$

These equations define a stipulated example. The variables carry no empirical identification with a particular social case.

In $\mathcal{A}$, substitution of $X = U$ into $Y = X$ gives $Y = U$. In $\mathcal{B}$, that equation is already explicit. Hence both models produce the observed pair $(X, Y) = (U, U)$. Their observational law assigns probability $1 - p$ to $(0, 0)$ and p to $(1, 1)$.

Define an intervention $X := x$ to replace only the equation determining X , while retaining the other equation and the distribution of U . Under this expressly stated intervention semantics, $\mathcal{A}$ gives $Y = x$. Model $\mathcal{B}$ still gives $Y = U$.

Proposition 10.1 (Shared observations and different intervention contrasts). *The two models in Equation (20) have identical observational laws for every $p \in [0, 1]$. Their mean-outcome contrasts between interventions $X := 1$ and $X := 0$ equal one in $\mathcal{A}$ and zero in $\mathcal{B}$.*

Proof. The substitution argument above proves pointwise equality of the observed pair for each value of U , hence equality of the induced law for every distribution. Under intervention, the mean outcome in $\mathcal{A}$ is x , so the contrast is $1 - 0 = 1$. In $\mathcal{B}$, the mean is p under either intervention, so the contrast is $p - p = 0$. $\square$

Any procedure whose input is only the shared observational law receives identical input from the two models. It must return the same answer for both. Since their correct intervention contrasts differ, the procedure cannot identify the contrast throughout this two-model class without an additional discriminating premise or source of evidence.

The conclusion is deliberately local. It concerns the specified models and intervention semantics. It establishes a logical limitation of the supplied observations. It leaves open which additional assumptions or experiments would be justified in a substantive application.

10.5 Implications for Relational Comparisons

The countermodel identifies an information gap that can recur when a relation is represented through observed association. This subsection translates the lesson into the paper's terms without assigning the example an empirical interpretation.

Suppose X is proposed as a relational condition and Y as an activity output. Observed agreement between them can fit a model in which changing X changes Y , or a model in which both depend on another condition. The observed pattern alone leaves these alternatives open within the illustrated class.

Historical records can help distinguish such alternatives, but their contribution needs an explicit argument. A sequence showing that the relational condition preceded the activity supplies temporal information. Precedence can constrain a proposed causal account while leaving other common causes or mechanisms possible. The record should be linked to the exact alternatives it excludes.

Mechanism evidence can also matter. A documented procedure may specify that an activity reads a particular authorization field. Observations of its operation can support that specification. The inference remains conditional on the procedure's actual use, the validity of the record, and the treatment of exceptions. The formal transition map summarizes those commitments once warranted.

A formation history can be causally relevant through several routes. It can leave a current condition that affects activity, preserve a record later consulted by a review, or alter the available operation repertoire. These routes require different state variables and tests. A single generic statement that history matters can conceal their differences.

The model can therefore guide evidence collection by identifying discriminating observations. If two candidate accounts disagree only under a review operation, routine activity records may leave the issue unresolved. If they disagree about formation labels but share all selected future outputs, a historical source inquiry may be more relevant than another predictive test.

10.6 Intervention Semantics and Feasible Contrasts

An intervention is more than a symbolic replacement of a variable in an empirical application. This subsection examines the conditions under which a formal contrast can represent a feasible and meaningful change.

A relational variable can depend on several participants or institutional conditions. Changing it while holding everything else fixed may be a useful analytical idealization, yet a real procedure may necessarily alter several features together. The investigator should state whether the target is an idealized contrast or a feasible intervention.

Different procedures can produce the same displayed variable value while carrying different side effects. Establishing authorization through two routes can leave different records, expectations, or practical dependencies. A binary variable may conceal those differences. An intervention description should therefore specify the procedure when its history or side effects matter.

The comparison can also change the observation process. Participants may disclose different information after a procedure, or a record system may begin tracking a previously invisible activity. An observed output difference can then combine a change in activity with a change in its measurement. A richer observation model is needed to separate those possibilities.

Feasibility imposes another constraint. A modeled operation can require resources, authority, or cooperation absent from a particular configuration. The blocked-continuation convention represents

this issue abstractly. A substantive account should identify which prerequisites make the comparison available and which remain hypothetical.

These considerations leave a constructive role for formal interventions. They make the intended contrast precise and expose the assumptions needed for transfer. The requirement is to retain the intervention's interpretation alongside its equation, so the mathematical consequence can be assessed at the appropriate level.

10.7 A Research Design for Formation and Potential Effects

The preceding distinctions support a concrete sequence of research operations. This subsection organizes them around the focal relation and identifies what would count against the proposed account.

Begin by specifying the target pattern and its continuity criterion. Identify the participant appearances, incidence structure, and context needed to track it. Record competing identity criteria when they produce different trajectories. This step establishes the object whose formation and efficacy will be investigated.

Next define the historical query and candidate history domain. Preserve the source material separately from the analytical coding. Construct the retained record and test whether the query factors through it. If recovery fails, report compatible answers and identify the missing distinction through a witness pair.

Then specify the predictive observation and operation repertoire. State which operations are total, which have availability conditions, and which responses remain conjectural. Define the admissible comparison pairs and search for positive continuation witnesses. A witness should be accompanied by its status: stipulated, model-derived under supported assumptions, or empirically tested.

The historical and prospective records can then be combined. Apply the joint-closure test to determine whether the chosen summary updates deterministically. If a current field is insufficient, construct a richer record or restrict the intended task with an explicit rationale. The finite authorization example illustrates this operation.

Finally assess causal and interpretive alternatives. Seek a competing account that preserves the available observations while changing the claimed formation or intervention answer. Such a countermodel identifies an unresolved dependency. Evidence that discriminates between the accounts can strengthen the claim; a surviving alternative should remain in the research state.

10.8 Failure Conditions and Revisable Conclusions

A useful formal interpretation should specify conditions under which it would be revised. This final subsection identifies those conditions for the present research program.

The formation account requires revision if a newly supported history fits the retained record and changes the target answer. The record can remain accurate while its claimed sufficiency fails on the enlarged domain. The appropriate response is to update the answer set and reconsider the query or evidence.

The predictive account requires revision if a justified operation distinguishes histories that the proposed state merges. This exposes a failure of update sufficiency. A new state variable or revised operation scope can repair the model, provided the change is recorded and its consequences are rechecked.

The causal interpretation requires revision if a defensible alternative model fits the evidence while assigning the difference to another condition. The existence of a mathematical witness then remains valid within the original model, while its attribution loses uniqueness. Additional evidence or a narrower claim is needed.

The social interpretation requires revision if the coding categories fail to track the practices or distinctions the inquiry intends to study. A formally consistent classification can still be a poor conceptual representation. Philosophical analysis and source engagement therefore remain part of the method.

The strongest conclusion available from the combined framework is conditional and explicit. A relation can have a recoverable formation account and a modeled potential effect when the record, dy-

namics, comparisons, and evidence meet their respective requirements. The value of the framework lies in making those requirements inspectable and in preserving the alternatives that remain unresolved.

11 Philosophical Interpretation of Historical Dependence

This section develops the philosophical commitments and tensions exposed by the formal results. It considers relational identity under change, the significance of formation when prediction can use a present record, the interpretation of unrealized efficacy, and the role of absent or extinguished relations. It also examines a circularity risk in the selection of tests and separates dynamic relevance from evaluation. The discussion advances proposed interpretations and objections within the paper's ontological suspension.

11.1 Relational Individuation and Continuity

Tracking a relation through a history requires a criterion for deciding which appearances belong to the same continuing pattern. This subsection examines the relation between that criterion and the operational state used for prediction.

A relation can be individuated through its participants, its institutional positions, its function, its recognized history, or a combination of these features. Each criterion can survive some changes and fail under others. A relation between two named persons can end when one participant changes, while a relation between offices can persist through succession. A functional arrangement can continue despite replacement of its documents, while an archival query may distinguish every version.

These alternatives should be treated as different analytical proposals until their relation has been argued. A model that uses one criterion for ordinary tracking and another for a counterexample can create an apparent paradox through equivocation. The criterion should therefore travel with the history domain and the comparison design.

The predictive quotient supplies a separate identification. It merges histories that agree under the selected future observations. That agreement can hold for different relations or different episodes. Conversely, one continuing relation can pass through several predictive classes as its relevant conditions change. The quotient's equivalence and the relation's continuity serve distinct purposes.

Relational individuation can also be distributed. A dependence pattern may persist through the coordinated behavior of several participants without having a single explicit record that carries its identity. A model can represent such persistence through an appearance structure and continuity rule. Its adequacy then depends on evidence about the pattern, rather than the mere availability of a name or document.

This approach leaves ontological priority open. The model can begin with participants and derive a pattern, or begin with a structured configuration and identify relatively stable participant roles within it. The choice determines an explanatory architecture. The formal success of one architecture in a scoped task does not by itself establish the ultimate priority of its primitives.

11.2 Counterfactual Change and Identity Criteria

Potential-effect comparisons can alter features used to identify the relation. This creates a philosophical difficulty that the comparison domain must handle. This subsection examines the difficulty and several possible responses.

A statement about removing a relation while holding everything else fixed can be coherent when the relation is represented by an independent condition. It becomes more delicate when the relation partly constitutes the identities or capacities of its participants. Removing an office relation, for example, may also change which role an actor occupies. The counterfactual comparison then needs to specify what remains comparable.

The mathematical pair (x, y) avoids an internal contradiction by placing both configurations in a carrier. Its social interpretation still requires an alignment between the cases. The investigator might track the same persons while allowing their roles to change, compare corresponding positions in two

structures, or examine two alternative histories from a common earlier situation. These are different comparisons with different explanatory targets.

A relationally constituted object can therefore be studied through counterfactuals without assuming that every feature survives removal of the relation. The task is to state which features define the comparison and which change with the intervention. An unrestricted claim that everything else remains the same can conceal an impossible combination of commitments.

The comparison domain $\mathcal{C}$ provides a formal location for these choices, but its inclusion rule requires argument. Declaring a pair admissible records a modeling decision. It does not prove that the cases are socially comparable. A philosophical audit should test whether the proposed contrast respects the adopted identity and dependence criteria.

The issue also affects empirical design. Two observed cases can share a surface classification while differing in the background relations that make the classification meaningful. Matching their displayed attributes may leave the relevant relational structure unaligned. The framework encourages a richer description of the contrast when those differences matter.

11.3 Historical Constitution and Present-State Sufficiency

A relation can be historically constituted while its selected future behavior is determined by a sufficient present record. This subsection examines why the two claims can coexist and what explanatory questions remain after state compression.

Historical constitution concerns the process through which a relation acquired its current form, standing, or role. Present-state sufficiency concerns whether a chosen record supports prediction under a declared update system. The first is a claim about a history and its significance; the second is an information and dynamics claim. Their domains overlap without coinciding.

The authorization illustration makes the distinction concrete. A formation operation establishes the condition, while the current authorization flag suffices for predicting activation in the narrow model. The flag can summarize a consequence of the history. Its predictive sufficiency leaves the formation route unresolved because different histories produce the same value.

An explanation of the present can seek more than a predictive initial condition. It can ask why this condition arose, which alternatives were available, or how the current practice became recognizable. Those questions require historical information even when the subsequent transition calculation uses a small state. A compressed predictor and a historical explanation can therefore serve complementary purposes.

The reverse caution is that a detailed narrative can contain more history than a particular explanation requires. A formation account should identify which parts of the sequence carry the proposed explanatory role. The recovery framework can test whether a smaller record preserves the relevant query, while philosophical analysis evaluates whether the query captures the explanatory interest.

The phrase history matters should consequently be unpacked. It can mean that an earlier process produced a present condition, that a present record contains information about that process, that future operations consult the record, or that the process is relevant to interpretation or evaluation. These claims can support one another, but each needs its own premises.

11.4 Potentiality, Dormancy, and Test Conditions

The potential-effect definition represents unrealized differences through possible continuations. This subsection examines the philosophical advantages and limitations of that representation.

The definition permits an arrangement to be efficacious without a presently observed difference. A condition can remain dormant until an operation encounters it. The formal witness identifies a way the difference could become observable, relative to the declared dynamics. This gives potentiality a precise comparative role.

The witness remains relative to test conditions. Some continuations may preserve the relation, some may activate it, and some may erase it before its effect can appear. A universal statement about efficacy

across all tests can therefore be too strong for the intended question. An existential witness is weaker, but it also requires a reason to regard the witnessing test as relevant and admissible.

The distinction between ordinary and exceptional conditions can matter. A relation may affect routine activity, appear only under disruption, or matter only under an artificial test. The same existential definition can include all three. A substantive inquiry should distinguish them through the test language and its interpretation rather than allowing the quantifier to flatten their practical differences.

Potentiality also can concern access rather than outcome. An arrangement may determine which operations can be attempted while leaving the results of common operations unchanged. The blocked-signature construction represents that role. Its philosophical significance depends on the difference between a modeled option, a practically accessible action, and an action that a participant can recognize and choose.

The formalism therefore gives a disciplined vocabulary for potentiality without completing its theory. It states what counts as a witness inside a model. Evidence, feasibility, agency, and interpretation determine what the witness supports in an actual inquiry.

11.5 Historical Significance after Operational Extinction

A relation can cease to have any future effect under the selected model while retaining historical significance. This subsection examines the boundary of the focal criterion and prevents that criterion from erasing its own historical objects.

Suppose two histories differ in formation, but all represented conditions carrying that difference have been removed. Under the selected operations and observation, every future continuation agrees. Their positive witness set is empty. The result concerns the present test system.

The earlier relation can still belong to a historical account. It may explain how a practice developed or why a later configuration became possible. A query about its formation or duration can remain meaningful even when the current dynamic model contains no remaining effect. Such a query is retrospective and can be retained through the history record.

The working definition is therefore a scope criterion for relations whose historical formation and prospective role are jointly investigated. It leaves room for historical objects that satisfy only the retrospective part at the time of inquiry. Exclusion from the focal class should be stated in those terms.

A broader theory might investigate the persistence of effects through records, habits, institutions, or material arrangements. That investigation would need to add and justify the relevant variables. The mere possibility of some unmodeled trace should not be used to declare every past relation presently efficacious. Such a declaration would make the criterion difficult to challenge.

The boundary gives a useful research obligation: identify the represented carrier of a claimed continuing effect. A formation document, an available procedure, or a maintained dependency can supply a candidate. Its connection to a future output requires a specified operation and evidence. Where that connection remains unresolved, the claim should retain its conjectural status.

11.6 Absence, Withdrawal, and Missing Records

Absence can appear in several roles within a relational inquiry. This subsection distinguishes an absent condition, a historical withdrawal, and a missing record, then examines their different dynamic consequences.

An absent condition is a value in the current configuration under a declared coding rule. Withdrawal is an operation or episode that removes a condition. A missing record is a limitation of the investigator's evidence. These distinctions can coincide in a case, but the coincidence requires support.

The authorization example shows why the distinction matters. The initial history and a formed-then-withdrawn history can share $r = 0$. Their current condition agrees while their histories differ. An endpoint record merges them, whereas the formation query distinguishes them.

An absent condition can also affect feasibility. If activation requires authorization, absence blocks or changes the result of the attempt. The model need not reify absence as a separate entity to represent

that dependence. It can specify how a variable value enters an update or availability rule.

A missing record supports a different kind of uncertainty. It can leave the actual condition unresolved or permit several formation histories. Treating it as proof of absence would convert an epistemic gap into a state claim. The candidate-history construction keeps that inference visible.

Withdrawal adds temporal and agentive questions. The model can calculate its effect while leaving open who performed it, why it occurred, and how it was understood. Those questions can become additional historical queries or interpretive studies. Their importance should determine the enrichment, rather than the convenience of one existing state variable.

11.7 Circularity in the Selection of Efficacy Tests

A sufficiently permissive test framework can manufacture a witness for any recorded difference. This subsection examines that risk and proposes a methodological safeguard.

Suppose a historical label is added to a record and a new operation simply reports that label. Two differently labeled histories then have a distinguishing continuation by construction. The mathematical result is valid, but its explanatory interest depends on why the reporting operation and output matter to the inquiry.

If the sole purpose of adding the operation is to guarantee that the focal relation has an effect, the argument risks circularity. The relation's efficacy is encoded in the test definition and then announced as a discovered consequence. The derivation adds little beyond the chosen readout.

A historical query can still be important in its own right. The joint-record theorem permits it to be retained explicitly. The safeguard is to distinguish historical-query preservation from a substantive claim about prospective consequences. The latter should identify an independently motivated target activity or observation.

Independence here is methodological, rather than a requirement that every variable be causally unrelated. The research question should supply a reason to care about the output before the witness is used to establish the relation's role. A review procedure, for example, can be a meaningful target if its actual function and consequences are supported.

The safeguard also improves falsifiability. A useful test specification permits the possibility that the focal distinction will have no effect under the selected target. A framework that guarantees a positive answer through every coding choice cannot discriminate between competing relational accounts in the intended way.

11.8 Generative Roles, Reproductive Roles, and Evaluation

Dynamic effects can create, maintain, constrain, or reproduce arrangements. This subsection separates those descriptive roles from the criteria by which their consequences are evaluated.

A generative role can refer to the production of a configuration, capacity, or available path under a declared model. A reproductive role can refer to the maintenance or recurrence of an arrangement. The same mechanism can perform both roles at different levels or times.

Neither role has a fixed evaluative sign. Maintaining an arrangement can preserve valued cooperation or sustain an objectionable dependency. Producing a new option can expand a participant's agency or enable an unwanted consequence. The formal role leaves those judgments open.

The observation map already reflects a selection of relevant outputs. A study that records completion alone can miss its distribution, cost, or meaning. Enlarging the observation can reveal consequences previously merged. The refinement is justified by an expanded evaluative or explanatory task.

Formation history can matter to evaluation even when current outputs agree. Different procedures can be assessed through standards concerning participation, recognition, or responsibility. Such standards require explicit argument. A formation label alone supplies only the coded historical distinction.

The formal framework can assist evaluation by clarifying which facts a judgment depends on and which records recover them. It does not derive the standard of evaluation from the existence of a quotient,

witness, or finite state. Keeping that boundary visible preserves the philosophical question instead of concealing it inside the notation.

11.9 The Scope of the Combined Account

The paper combines a historical record and a prospective test without claiming that their combination exhausts a social relation. This final subsection states the resulting research position and its open tensions.

The combined account is useful when an inquiry needs to track a dependence pattern, reconstruct its formation, and evaluate its possible role under specified changes. The record theorems determine information requirements. The continuation tests determine modeled differences. The evidential analysis states what supports their application.

Several questions remain open. A relation's identity can depend on contested continuity criteria. A formation category can remain disputed despite agreement on events. A potential witness can be inaccessible to direct testing. A current state can be sufficient for one purpose and inadequate for another.

These tensions provide directions for research. They should remain visible in the project state rather than be resolved through a more confident description of the same equations. A comprehensive formal treatment earns its scope by making the unresolved premises precise.

The resulting position is an analytical proposal: historical formation and potential efficacy can be coordinated through explicit records and update rules. Their social interpretation remains open to philosophical argument and empirical revision.

12 Model Choices and Limits of Historical Memory

This section compares alternative representations of historical memory. It examines exact finite records, queries requiring unbounded memory, partial updates, finite-horizon approximations, noisy observations, and stochastic extensions. The purpose is to identify which earlier results transfer directly and which require new assumptions. The comparison keeps mathematical possibility, computational feasibility, and empirical adequacy separate.

12.1 Complete Histories and Sufficient Summaries

A complete word history retains every labeled step at the chosen resolution. A sufficient summary retains only the distinctions needed for specified queries and updates. This subsection compares their roles without assuming that either is universally preferable.

Complete histories provide a straightforward exact record for any query defined on the word domain. Their information content can support later questions whose relevance was previously unknown. Their costs include potentially unbounded length and the need to interpret a growing sequence. They also remain incomplete relative to aspects of a process omitted by the chosen alphabet.

A sufficient summary can be much smaller. The latest formation label in the authorization example has three possible values despite histories of arbitrary length. Combined with the operational variables, it updates exactly because the query depends on a simple rule: the latest formation replaces the previous label, and other operations preserve it.

The reduction is justified by the query and update proof. A summary named origin or history cannot be presumed sufficient without that proof. The field's semantic meaning and update rule determine which histories it merges. A new query can require a finer summary even when the old record remains adequate for its original purpose.

A research archive and a working model can therefore have different retention policies. The archive can preserve source records and alternative interpretations, while the working model uses a compact state for a bounded task. Links between them allow the summary's genealogy to remain inspectable without requiring the predictive calculation to carry every source detail.

12.2 Families of Historical Questions and Exact Retention

A record is often expected to answer several questions at once. This subsection derives the exact retention condition for a family of historical queries and examines the stronger promise of answering questions whose relevance has yet to be decided. The argument extends the single-query factorization criterion while preserving the difference between a working summary and an archive.

Let H be the history domain and let $(g_j : H \rightarrow Y_j)_{j \in J}$ be a declared query family. The index j identifies a question, and $g_j(h)$ is its answer in the stipulated model. Define

$$E_J = \bigcap_{j \in J} \ker(g_j). \quad (21)$$

Two histories are related by E_J exactly when every selected question gives the same answer on them. This is an equivalence, because equality of each function's values is an equivalence and intersections preserve reflexivity, symmetry, and transitivity.

Proposition 12.1 (Exact retention for a query family). *A record $\rho : H \rightarrow Z$ permits exact recovery of every g_j if and only if $\ker \rho \subseteq E_J$. The attained joint record*

$$h \mapsto (g_j(h))_{j \in J}$$

is the coarsest exact record for these present queries, up to relabeling its attained values.

Proof. For each j , the single-query factorization theorem requires g_j to be constant on every fibre of ρ . Requiring this for every index says precisely that any pair in $\ker \rho$ belongs to every $\ker(g_j)$, which is the stated inclusion.

The joint record has kernel E_J . Each query can be recovered by projecting to its coordinate. Every other exact record has a kernel contained in E_J , so it distinguishes at least the pairs distinguished by the joint record. This proves coarseness among records for the selected present queries. A bijective change of the attained labels preserves its partition and its recovery properties. $\square$

The qualification concerning present queries is essential. Updating a record introduces a second responsibility: equal retained records must yield equal successor records whenever the same update is applied. The minimal joint record for today's selected answers can fail this responsibility. The continuation construction supplies the additional refinement needed for repeated exact updates. The two results therefore answer successive questions: which distinctions current readouts require, and which further distinctions their future evolution requires.

Adding a query shrinks E_J or leaves it unchanged. The resulting partition becomes finer when the new question distinguishes histories previously merged. If its answer can already be recovered from the existing joint record, the partition remains unchanged. This gives a concrete test for whether a proposed historical field contributes new information to the declared inquiry.

Deleting a query can enlarge the equivalence and permit a smaller working record. Such a change is a revision of the task. It supplies no argument that the omitted question has lost historical importance. A research project should retain the reason for the revision so that an efficient current model does not silently become an assessment of which earlier distinctions deserved preservation.

12.3 Exact Answers to Unrestricted Future Questions

An archive can have a broader purpose than a fixed working model. This subsection derives the implication of requiring a record to answer every Boolean question on the selected history domain. The statement is deliberately strong, making its retention consequence explicit.

Proposition 12.2 (Unrestricted exact-query retention). *A record $\rho : H \rightarrow Z$ permits exact recovery of every function $g : H \rightarrow \{0, 1\}$ if and only if ρ is injective.*

Proof. Suppose the record answers every Boolean query. For distinct histories h, h' , consider the indicator of the singleton $\{h\}$. It gives value 1 on h and 0 on h' . If $\rho(h) = \rho(h')$, a readout from that common record would have to give both values. Thus distinct histories must have distinct records, proving injectivity.

Conversely, an injective record has a well-defined inverse on its attained image. Any query on H can be evaluated by recovering the unique history associated with the record and applying the query. This gives exact recovery. $\square$

The proposition concerns the information carried by the record, independently of how convenient the inverse is to compute. A lossless encoding can use different symbols or exploit regularity while retaining injectivity. The claim does not require that every history be displayed in its original surface format during every calculation.

It also concerns a selected domain. A complete word retains the labels and order that the model chose to encode. It can leave out tone, intention, undocumented participants, informal practices, or other features absent from the alphabet. Injective retention on H therefore preserves every distinction in H ; it leaves the adequacy of H for the social episode as a separate question.

The strong query family helps articulate a preservation tradeoff. A project that guarantees only a fixed list of readouts may compress histories whenever those readouts agree. A project that wants later researchers to ask unanticipated questions on the retained domain has a reason to preserve more. The theorem identifies the formal limit: an unrestricted promise of exact answers requires lossless retention on that domain.

This does not rank every possible query equally. A Boolean question can distinguish histories without serving the project's explanatory purpose. The argument uses such questions to test the breadth of the promise being made. The choice of a smaller query family can be reasonable, provided its limits remain part of the description of the record.

A practical architecture can coordinate both responsibilities. A working state carries the distinctions needed for current updates and predictions. A source archive retains richer histories and the evidence for their interpretation. A provenance link permits the working state to be reconstructed, challenged, or refined. Its value lies in preserving a route between the compact calculation and the richer source, with each record's responsibility stated explicitly.

12.4 Record Cardinality and Formation Histories

The retention distinctions can be quantified in a simple formation model. This subsection derives cardinal lower bounds and compares four historical or operational questions on the same word family. The bounds concern exact coding, leaving physical storage and substantive importance to their own analyses.

For finite H , Proposition 12.1 implies

$$|\rho(H)| \geq |H/E_J| \quad (22)$$

whenever ρ answers the selected query family exactly. Choose one history from each E_J class. Histories from different classes disagree on at least one selected query, so an exact record must assign them distinct values. This proves the inequality directly.

A fixed-width binary code of length b has 2^b possible values. For a nonempty query quotient, (22) therefore yields

$$b \geq \lceil \log_2 |H/E_J| \rceil. \quad (23)$$

The coding convention matters. The bound concerns a fixed maximum field length with exactly b binary positions. Variable-length codes, storage overhead, and the compression of typical histories require additional specifications.

Take length- ℓ words over two labels, negotiation n and imposition i , where $\ell \geq 1$:

$$H_\ell = \{n, i\}^\ell, \quad \ell \geq 1.$$

Both labels set the same operational authorization bit to one. This stipulated action deliberately omits their formation distinction from the operational variable. Every nonempty word therefore has the same endpoint and the same transformation of that bit.

The complete sequence query has 2^ℓ attained answers: each position permits either label independently. Exact sequence recovery consequently requires at least ℓ fixed-width bits. In contrast, the number of negotiation labels can take the values $0, 1, \dots, \ell$, giving $\ell + 1$ answer classes. Its lower bound is $\lceil \log_2(\ell + 1) \rceil$ bits.

The latest formation label has two attained values and requires one fixed-width bit on this nonempty domain. The operational endpoint has one value and requires zero additional bits once the model and initial seed are fixed. These are different exact records for different questions, despite their common underlying history family.

At length three, the complete sequence distinguishes eight histories. The count query distinguishes four groups: zero, one, two, or three negotiation labels. The latest-label query distinguishes two groups. For example, nni and ini agree on their latest label and differ on their counts; nni and nin agree on their counts and differ on their latest labels. Neither of these two historical summaries determines the other.

Their joint record is therefore more informative than either alone, although it still merges some sequences. This is a concrete instance of (21): the joint partition is the intersection of the two kernels. The combined record should be sized by its attained pairs, which can be fewer than the Cartesian product of all separately possible values. Counting unattained combinations would overstate its exact information requirement.

Across arbitrary history lengths, the distinctions change. The count of negotiation labels can grow without bound, so exact counts require infinitely many possible record values. The latest label remains a two-valued field on nonempty histories, with a third marker when the empty history is included. It updates by overwriting the previous label whenever a new formation operation is recorded. Complete sequence retention updates by appending the new label.

The operational representation remains constant after the first formation operation. This makes the central philosophical point mathematically explicit: uniformity of current operational effect can coexist with large differences in historical information. Whether a particular difference matters depends on the selected question. Its omission from the operational variable supplies no inference that the difference failed to occur.

12.5 Retention Decisions and Historical Interpretation

The bounds clarify the price of exact distinctions, while interpretation determines why they deserve attention. This subsection connects the formal comparison to the research practice of preserving insights and revising explanations.

A latest-label summary can be adequate when the institution's current rule depends only on the latest recognized formation. A count can be relevant when repeated attempts, renewals, or interventions are part of the explanatory question. A full sequence can be needed when order distinguishes gradual negotiation from later imposition, or when an apparent repetition has a different meaning after an intervening episode. Each use requires a substantive interpretation of the labels and their evidence.

The mathematical record should not conceal disagreement over those interpretations. Two investigators can agree on a word's stored symbols and disagree about whether one episode warrants the label negotiation. A faithful source record, the coding decision, and the formal history are then related artifacts with different epistemic roles. Lossless retention of the coded word preserves the coding; it does not by itself verify it.

A small current summary can also encourage retrospective overconfidence. If the retained field reports that the latest formation was negotiated, a reader may infer that the relationship was always negotiated or that earlier coercive episodes ceased to matter. The field's update rule warrants only its latest-label answer. Recovering the stronger historical claim requires another query and enough retained information to answer it.

Conversely, keeping a long sequence does not automatically produce an explanation. A list of episodes can preserve many distinctions while leaving their relevance unarticulated. The explanatory task connects selected differences to the question, the comparison design, and the modeled continuations. Preservation provides material for that task and permits its later revision.

The distinction between archive and working state thus serves both rigor and openness. The working state has a declared responsibility and a proof or argument for its sufficiency. The archive retains the historical material from which another question or model might be developed. Their coordination supports compact reasoning without requiring that the present inquiry exhaust the meaning of the histories it studies.

12.6 Finite Queries with Unbounded Memory Requirements

A query can have only two possible answers and still require an unbounded exact deterministic record. This subsection proves that boundary through a unary example. The example is mathematical and supplies a limit on a general inference from small outputs to small states.

Let $H = \{a\}^*$ and define $g(h) = 1$ exactly when the length of h is prime. Suppose there were a finite record $r : H \rightarrow Z$ with a deterministic update F and a readout $q : Z \rightarrow \{0, 1\}$ satisfying

$$r(ha) = F(r(h)), \quad g(h) = q(r(h)).$$

The record sequence along repeated a updates takes values in the finite attained set.

Proposition 12.3 (A binary query without a finite exact record). *The prime-length query has no finite exact deterministic record under unary history extension.*

Proof. Finiteness gives indices $i < j$ with $r(a^i) = r(a^j)$. Let $d = j - i > 0$. Applying the same number of further updates to these equal records gives $r(a^n) = r(a^{n+d})$ for every $n \geq i$. Applying the readout gives $g(a^n) = g(a^{n+d})$ throughout that range.

Choose a prime $p \geq i$. Primes are unbounded: if only finitely many existed, a prime divisor of one plus their product would lie outside that list. Periodicity then implies that the output at $p + pd = p(1 + d)$ equals the output at p . The latter equals one, while $p(1 + d)$ is composite because both factors exceed one. This contradiction excludes the assumed finite record. $\square$

The example demonstrates a structural possibility, rather than recommending prime episode counts as a social variable. Its role is to show that a finite answer set gives no general guarantee of finite update memory. The relevant complexity lies in how future answers depend on the history.

The earlier query asking whether length equals two admits four exact record values. The contrast between the two examples isolates the role of the query. Both have a Boolean output and the same unary history domain. Their continuation requirements differ.

A bounded horizon can restore finite representation for a bounded task. If only histories up to a fixed length are considered, storing that length is finite. The resulting record answers a restricted question. Its sufficiency should be reported with the bound.

12.7 Partial History Updates

Some histories permit an update while others do not. A reduced record must retain that definedness distinction as well as the successor information. This subsection derives the exact partial-update criterion.

Let $U : D \rightarrow H$ be a partial update with domain $D \subseteq H$, and let $\rho : H \rightarrow Z$ be a record. A reduced partial update is intended to act on the attained record set $\rho(H)$.

Proposition 12.4 (Partial-update descent). *An exact reduced partial update exists if and only if two conditions hold: membership in D is constant on every ρ fibre, and*

$$\rho(h) = \rho(h'), \quad h, h' \in D \implies \rho(Uh) = \rho(Uh').$$

The reduced domain is then $\rho(D)$, and the reduced update sends $\rho(h)$ to $\rho(Uh)$.

Proof. If a reduced partial update exists, one record value has one definedness status. Histories with that record must therefore either all belong to D or all lie outside it. Within the domain, the reduced update is a function, so equal input records give equal successor records. This proves necessity.

Conversely, define the reduced domain as $\rho(D)$. Fibrewise constant definedness ensures that a history lies in D exactly when its record belongs to this domain. For a record in the domain, choose a history with that record and send it to $\rho(Uh)$. The second condition makes the value independent of the choice. The construction is exact by definition. $\square$

The first condition is often the neglected one. A record can predict the outcome of an operation whenever it is executed while failing to identify whether execution is available. A model of authorization or access needs both responsibilities when its state is used for planning or testing.

A blocked-state totalization can incorporate definedness into an ordinary observation. Its adequacy depends on the interpretation of blocking and recovery. The partial criterion shows what information that totalization must preserve before its total-map proofs apply.

12.8 Finite Horizons and Approximate Responsibilities

A finite horizon can provide a useful bounded record when full continuation equivalence is too large. This subsection examines what that record guarantees and how its limits should be stated.

The relation E_k preserves answers under words of length at most k . It is a legitimate task-specific equivalence on the fixed system. Its classes can be used to summarize bounded future observations.

The horizon relation need not be preserved by every update at the same horizon. A first update consumes part of the budget: agreement through $k + 1$ steps guarantees agreement of successors through k steps. Using E_k as an unqualified permanent state can therefore lose required future distinctions.

A representation can include the remaining horizon as part of its state or retain a family of summaries indexed by the budget. Such a design must specify how the budget changes under execution. Resetting the horizon after every step asks a different question from maintaining one total budget.

In a finite exact model, stabilization of the refinement sequence provides a route from bounded agreement to full agreement. Without stabilization, the bounded result should remain bounded. The distinction protects the inference from an arbitrary stopping choice.

An empirical horizon can also be temporal rather than a count of operations. Relating the two requires durations or a timing model. A short word can represent a long procedure, and a long word can represent many rapid updates. The chosen budget should match the research question.

12.9 Noise, Tolerance, and Equivalence

Observed records often contain uncertainty or measurement error. Replacing exact equality with numerical closeness can appear natural, but it changes the mathematical structure. This subsection identifies the first obstacle.

For real-valued observations, suppose two values are declared equivalent when their difference is at most 0.7. Then 0 and 0.6 are close, and 0.6 and 1.2 are close, while 0 and 1.2 are separated. The relation is not transitive.

The quotient proofs in this paper require an equivalence. A tolerance relation therefore cannot replace equality without a new construction. Taking its transitive closure produces an equivalence, but chains of small differences can merge values far beyond the original tolerance.

Another option is to retain sets or intervals of possible observations. Their update requires an uncertainty model and can overapproximate the actual possibilities. The candidate-set construction provides one exact logical starting point when evidence is represented by compatibility predicates.

A statistical treatment would require distributions, error assumptions, and a criterion for comparing predictions. Its record sufficiency question differs from exact factorization. The present results identify the exact boundary from which such an extension must proceed.

Noise also can affect historical coding rather than a numerical measurement. Uncertain episode boundaries or formation labels can be represented through alternative histories. That representation preserves disagreement without forcing a numerical distance on categories whose meaning is qualitative.

12.10 Stochastic and Nondeterministic Extensions

A social application can require several possible successors for one operation or a probability distribution over successors. This subsection states the additional responsibilities such extensions would introduce.

A nondeterministic update assigns a set of possible successors. A record sufficient for deterministic updates can become insufficient if different underlying histories yield different successor sets. The relevant reduced update must preserve the selected notion of possibility.

A probabilistic update assigns weights as well as possibilities. Two histories can share the same possible outputs while differing in their probabilities. A quotient based only on support would then answer an availability question and lose quantitative prediction.

Sequential choice introduces another layer. A policy can depend on the information available after each observation. Comparing open-loop words alone can miss adaptive strategies. The record must then support the information and decision structure of the policy.

These extensions are research obligations in the present paper. Their definitions of equivalence and minimality require separate proofs. The deterministic results remain useful as baseline cases whose assumptions are explicit.

The practical choice should follow the question and evidence. A stochastic model can be appropriate when response variation has a justified distributional interpretation. A set-valued model can be more appropriate when possibilities are known but probabilities remain unresolved. Exact determinism can still serve a bounded analytic example.

12.11 Computational and Interpretive Costs

A mathematically sufficient record can remain difficult to compute or interpret. This final subsection distinguishes those costs from the existence results.

The full signature requires answers for every continuation. A finite quotient can be found through refinement when the carrier and updates are effectively available, but the number of states or operations can make enumeration expensive. A proof of finiteness alone leaves practical tractability open.

Constructing the state space can be harder than reducing it. An empirical model requires choices about variables, operations, and possible histories before the algorithm runs. The formal refinement procedure can check the consequences of those choices; it cannot supply their substantive warrant.

Interpretability also matters. A quotient class can combine histories whose common meaning is difficult to state. A slightly richer record may be easier for researchers to inspect and explain. The minimum cardinality result therefore supports a comparison, rather than a universal implementation rule.

The model-choice lesson is to state the responsibility of each record. A complete history supports broad retrospective use at its chosen resolution. A sufficient summary supports specified queries and updates. An approximation supports a declared bound or error criterion. Their value depends on the research task they can actually perform.

13 Objections and Research Obligations

This section tests the proposed account against objections concerning formal triviality, interpretive circularity, inaccessible comparisons, record costs, and the breadth of the relational criterion. Each objection is paired with a limited response and an unresolved obligation. The objective is to identify what the framework contributes and the conditions under which its current formulation should be revised or abandoned.

13.1 Elementary Mathematics and Explanatory Contribution

The recovery criterion follows from constancy on fibres, and the joint quotient uses an established form of continuation equivalence. An objection is that elementary mathematics cannot support the philosophical ambition associated with historically formed relations. This subsection distinguishes the level of the formal result from the contribution of its application.

The mathematical results are elementary or use established constructions. Their value here lies in exposing commitments that an informal account can merge. A composite adequate for endpoint calculation can lose formation information. A current answer can fail to update itself. A historical label can remain relevant to one question while being redundant for another. Each distinction has a precise condition and a counterexample.

The account would become trivial if it merely renamed those conditions without improving an inquiry. Its usefulness therefore requires a substantive application in which the distinctions change the record design, the interpretation of a result, or the evidence sought. The authorization example demonstrates the logical possibility of such a use, while empirical development remains open.

A stronger contribution claim would require comparison with existing methods for the same task and evidence that the present coordination clarifies or resolves a problem those methods leave unclear. The manuscript's source comparisons identify antecedents and differences; a broad originality claim remains outside the established result.

The response to the objection is therefore limited. Elementary mathematics can discipline a philosophical formulation, but its simplicity places an obligation on the author to show the conceptual work it performs. The paper should be judged by that work and its declared scope.

13.2 Historical Labels and Circular Explanation

The model supplies formation labels and then proves that a richer record can retain them. An objection is that this only reproduces information introduced by definition. This subsection clarifies the legitimate result and the point at which an explanatory claim would become circular.

A label's retention is a representational result. It establishes that a chosen summary preserves a query under specified updates. The proof does not discover the formation label or establish its empirical truth. Those tasks belong to the coding and evidential argument.

The result nevertheless can diagnose a real information problem. If an existing endpoint record merges histories that a warranted query distinguishes, the recovery criterion proves that the record is inadequate for that use. A proposed enriched record can then be tested for sufficiency. This is more specific than the general recommendation to retain history.

Circularity arises when the label is treated as its own explanation. Calling a route negotiated and then explaining its significance solely by the presence of that label leaves the category's meaning and relevance unsupported. The same risk appears when a test is added only to report the label and its output is presented as independent evidence of efficacy.

The repair requires an independently articulated query, coding criterion, and explanatory target. The relation between them should be argued. A historical classification can support an interpretive judgment or enter a real review procedure, but the transition from classification to consequence needs its own premises.

An unresolved obligation remains for empirical use: establish that the selected formation categories track distinctions worth making in the target inquiry. Mathematical well-definedness is one requirement in that argument, not its completion.

13.3 Counterfactual Accessibility and Empirical Testing

Potential-effect witnesses range over modeled continuations, some of which may be difficult or impossible to observe directly. An objection is that the criterion can become empirically unconstrained. This subsection states the conditions that preserve its testability.

A witness can have several statuses. It can be a stipulated analytic example, a counterfactual consequence of independently supported rules, or an observed response under an admissible comparison. The manuscript should identify which status applies. A hypothetical witness gains no empirical support merely from a formal derivation.

A model can still be constrained when the focal continuation is unobserved. Evidence about its constituent operations, historical analogues within a justified scope, or competing predictions can bear on the transition rules. Such support requires a substantive argument about transfer. The present framework identifies the relevant dependency without supplying that argument in advance.

A useful interpretation should also permit failure. If every possible observation is accommodated by changing the operation map after the fact, the model loses a discriminating role. The research state should record which transitions were specified before a test and which were revised afterward.

The comparison domain requires similar discipline. A witness whose pair becomes inadmissible whenever the expected difference fails provides weak support. The criteria for admissibility should be stated independently of the desired outcome where the inquiry permits.

The remaining obligation is to design a bounded study with explicit observation opportunities and revision conditions. A complete theory of every possible relation would be too broad to test through the present example. A specified authorization arrangement provides a more assessable starting point.

13.4 Record Burdens and the Limits of Retention

The argument for preserving historical distinctions can be mistaken for a general demand to retain every trace. An objection is that such retention can impose costs and produce new misinterpretations. This subsection states the scope of the record recommendation.

The recovery theorem is question-relative. It identifies the information needed for a chosen query. The joint-record theorem adds the information needed for its updates. Neither requires a complete archive when a smaller record meets those responsibilities.

A richer record can support future questions, but it can also contain errors, disputed labels, or details whose later use exceeds their original evidential scope. Preservation of provenance should include the status and limits of the record, rather than turning stored text into an authoritative fact.

Retention choices also involve the people whose histories are recorded. A research design should explain why a field is needed and which uses the evidence supports. Those considerations can justify a bounded summary or restricted access without altering the mathematical result.

The framework's contribution is to make the tradeoff explicit. If a field is omitted, the resulting query limitation can be stated. If it is retained, the project can identify the responsibility that motivates it. The formalism supplies an information analysis, while the retention decision needs further reasons.

13.5 Scope of the Relational Criterion

The focal definition combines historical formation, reidentification, and potential efficacy. An objection is that relations outside this combination can remain important to social inquiry. This subsection treats that objection as a boundary of the proposal.

An extinct relation can remain historically significant even when the selected future tests reveal no effect. A mathematical incidence relation can be useful without a formation account. A temporary interaction can matter even when long-term reidentification is irrelevant. The definition does not need to absorb every such case.

Its scope is justified when the research question requires the three components together. The framework then helps specify their distinct evidence and information requirements. A different question can use a different relational criterion.

The breadth of the continuation repertoire also affects the scope. A narrow repertoire can leave a relation operationally silent. An unrestricted repertoire can make efficacy too easy to manufacture. The selection should follow the substantive target.

The obligation is to state a reason for the chosen scope and preserve counterexamples that challenge it. A revised definition should retain the earlier formulation and explain which problem the revision solves.

13.6 A Bounded Next Application

A first empirical application can follow one authorization arrangement through formation, possible use, and revision. This subsection states the design at a level that makes the formal claims reviewable while leaving case selection open.

The study would identify the relation and its tracking criterion, then construct competing formation histories from the available evidence. A current-status record would be compared with an enriched event and process record. The historical queries would be specified before testing their recovery.

The operational model would identify a small set of possible uses, withdrawals, and review procedures. Each map would carry its evidential status. The study would distinguish an unused authorization from one whose availability or practical effect is unresolved.

The joint-record analysis would identify which historical fields are needed to predict the specified outputs. A field retained for interpretation or evaluation would remain explicitly separate from a field required by the predictive update.

The proposed account would be revised if credible histories outside the domain fit the record, if a newly justified operation splits a merged state, or if the causal contrast depends on an omitted condition. These failure conditions connect the formal model to a substantive research process.

The next application should produce both a positive construction and a documented limitation. That combination would show whether the framework improves the inquiry beyond its finite illustration.

14 Conclusion

This paper has developed a coordinated representation of historical formation and potential future effects within a declared generative relational model. Its central distinction separates the represented history, the record retained from that history, the current operational configuration, and the continuations used to test prospective relevance. Each component supports a different inference and requires its own interpretation.

The recovery theorem characterizes the information needed for a historical query: the query must be constant on every retained record fibre. Compatible-answer sets extend the construction when exact recovery fails. Valid refinement can narrow those sets, while evidential warrant additionally requires an adequate history domain and a reliable connection between records and the actual case.

The joint continuation construction addresses dynamic use. A record that retains current answers can still omit information needed to update them. Equality of all future predictive and historical answers supplies the greatest compatible equivalence preserving both tasks. The finite refinement proof and minimality result explain how that construction can be realized and assessed in a finite deterministic model. The boundary examples show why finite output sets, partial updates, and noisy observations require additional care.

The authorization illustration connects these results within one complete example. Differently labeled formation histories share an endpoint and all selected operational futures. A finite enrichment retains their historical distinction. The same model displays dormant efficacy, its erasure through withdrawal, and a refinement when a new operation consults the formation record. These are conditional consequences of stipulated rules, with empirical adequacy remaining open.

The philosophical interpretation preserves several distinctions that the equations alone cannot settle. Behavioral equivalence differs from relational identity. Predictive sufficiency leaves room for independent historical significance. A positive witness requires a relevant test and an admissible comparison. Causal attribution, normative evaluation, and the adequacy of formation categories need further argument and evidence.

The uncertainty extension sharpens the prospective claim. Every compatible comparison can have a witness while a common witness remains unavailable. Valid evidence refinement can reduce the possible-

witness set and enlarge the uniform-witness set. A covering family of fresh tests can also differ from a sequential experiment because an early operation can erase a later distinction. These results make the relation between evidence, quantifier scope, and executable testing explicit.

The proposed research method makes these requirements inspectable. It invites an inquiry to specify a relation, reconstruct its candidate histories, test the sufficiency of its records, identify possible effects, and preserve countermodels that challenge the interpretation. A bounded application can now examine whether the distinctions clarify an actual formation-and-use history. The framework's further development depends on that confrontation between formal requirements, philosophical interpretation, and evidence.

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