The Ontological Status of Generative Order: An Inquiry into Emergent Precedence in Relational Systems

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Unreviewed research preprint (v2.4)
28 October 2026

Abstract

This exploratory inquiry examines when generative precedence can be derived from relational configurations and when a derived order warrants physical or ontological interpretation. A candidate generative landscape separates admissible transformations, conditioning regimes, equivalence relations, and observational interfaces. Established results on reachability and quotients supply the formal baseline. Conditional propositions and counterexamples examine joint-input compatibility, reconstruction under projection, resource-sensitive potential, transformations of landscapes, and stability of local ordering. Idealized chemical-reaction and stochastic aggregation benchmarks further distinguish conserved feasibility, kinetic prediction, memory, and approximate preservation across scales. Weak transitions may reorganize support-derived orders while finite-time predictions remain close. Comparative tests show that adequately specified graph, process, Petri-net, and Markov methods solve the paper's finite diagnostic problems. Accordingly, the proposal's distinctive role is a coordinated audit of the assumptions connecting generative rules, observable equivalence, effective dynamics, and ordering claims, rather than a new prediction method. Philosophical comparisons with structural realism, grounding, causal structure, and emergence delimit further interpretive commitments. The formal conclusions remain conditional; physical application requires independently justified laws and observables, and the practical usefulness of the audit awaits comparative evaluation.

Keywords: generative order; emergent precedence; relational systems; equivalence; generative landscapes; partial order; ontological interpretation

Discussion Paper Note

This manuscript initiates an exploratory generative-relational inquiry into the ontological status of order. It examines whether relations of precedence, stratification, and generative priority can be derived from conditions on a relational configuration space, and how such derivations vary with equivalence criteria, representations, admissible transformations, and scales of description. The inquiry begins with the conceptual and mathematical status of candidate generative landscapes before considering order as a possible derived structure.

The proposal draws selectively on mathematical order theory, symmetry and invariance, causal-set approaches to spacetime, renormalization, and philosophical work on emergence. These literatures offer distinct analytical resources with different physical and ontological commitments. The proposed framework preserves their differences while developing a conditional research programme. Terms such as *generative potential*, *level*, and *derived precedence* denote provisional constructs requiring further specification and evaluation.

Version 2.4 presents the ten-section inquiry and five technical appendices in a clean, publicly circulable preprint format. Its scholarly contribution is a coordinated audit of inferences from generative rules and observation to derived order; established order-theoretic, graph-theoretic, resource-sensitive, and stochastic results are credited as such. The central discussion now emphasizes the derivation of precedence, exact and approximate reconstruction, conditional potential, physical correspondence, and the differences among mathematical, explanatory, and ontological emergence. The appendices retain the finite chemical-kinetic models, exact and approximate aggregation calculations, stability tests, and controlled comparisons with standard graph, resource, and stochastic approaches. The mathematical results are supported by stated arguments and finite-model checks under their declared assumptions; the benchmarks use stipulated parameters and contain no original physical measurements. The preprint has not been externally peer reviewed. Independent proof checking, comprehensive full-text source assessment, and experimental tests remain outstanding. The status of any physically fundamental generative order remains open to further investigation.

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Notices

Research scope. Unreviewed research preprint, version 2.4, 28 October 2026. All ten sections are presented in full English prose; formal-consistency and physical-benchmark revisions have been consolidated into a ten-section main text and five technical appendices; source-level and substantive peer review remain outstanding. The paper offers preliminary conceptual arguments, elementary proved results, a controlled comparative-baseline assessment, and idealized chemical-reaction benchmarks, including a four-state microscopic aggregation with analytical time-correlation predictions. It reports no original measurements or independent experimental physical validation. The text has received no independent peer review.

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Suggested citation. Huang, Wanhong. "The Ontological Status of Generative Order: An Inquiry into Emergent Precedence in Relational Systems." Unreviewed research preprint, version 2.4, 28 October 2026.

1 Introduction

The present inquiry concerns the circumstances under which relations of generative precedence can be constructed, justified, and interpreted within evolving relational systems. Its purpose is to distinguish the availability of a mathematical ordering, the adequacy of that ordering as a scientific representation, and the further ontological claim that the represented order belongs to the constitution of the system. The discussion begins with candidate generative configurations and their admissible transformations, examines the roles of equivalence and observation in determining comparability, and then outlines a programme for investigating conditional strata, changing generative landscapes, and the interpretation of emergent order. The method is conceptual and mathematical, informed by selected results from physics and the philosophy of science. Physical examples serve as constraints on analogy and opportunities for future discrimination, while the paper's proposed constructions retain explicitly model-relative status.

Order appears across scientific explanations in markedly different forms. A causal relation distinguishes events connected by physically admissible influence; a computational dependency records which operations require the completion of others; a developmental process may exhibit dependence among conditions of formation; and a mathematical preorder organizes elements under a reflexive and transitive relation. Their shared vocabulary permits illuminating comparisons, although their identity cannot be presumed. The axioms and interpretation of an ordered set are established relative to a specified domain and relation, as made clear in mathematical order theory [13]. A relation that has predictive or explanatory force in one domain may encode a convention of description, an observational restriction, an actual enabling condition, or a substantive feature of a physical theory. The status of order thus depends upon the content of the relation as well as its formal properties. As a methodological benchmark, any claimed advantage over ordinary reachability, behavioral equivalence, or a domain-specific physical model should identify a concrete inferential distinction supplied by the additional generative-landscape description.

An especially significant distinction concerns *temporal order* and *generative precedence*. Temporal coordinates can facilitate the reconstruction of observed changes, whereas generative precedence concerns dependence among conditions and possible transformations within a declared model. In certain applications, the two relations may coincide. In others, multiple independent conditions may jointly permit a transformation; several configurations may be incomparable; or the direction of a proposed generative dependency may be underdetermined by the observations. Generative precedence is therefore introduced here as a question about structured possibility and constrained production, rather than as an immediate synonym for chronological succession. This formulation is compatible with the wider generative-relational preference for avoiding an ontologically privileged present or a fundamental division into past and future. Temporal descriptions remain legitimate tools whenever their physical assumptions and explanatory purpose are specified.

One motivation for separating these relations comes from the diversity of order structures already employed in physics. In causal-set theory, a locally finite partially ordered set is proposed as a candidate primitive structure of spacetime, with the order intended to represent causal relationships and with continuum geometry approached through suitable approximations [9]. The research programme developed here starts from a different conditional question. Given a class of relational configurations, criteria of effective equivalence, and an account of admissible generation, under which conditions might a partial order become an appropriate *derived* representation? The causal-set programme is therefore a relevant physical comparison, with its particular ontological commitments preserved as its own. Agreement at the level of an order-theoretic vocabulary would leave substantial dynamical, geometric, and empirical differences in place.

A second motivation comes from symmetry and invariance. Distinct descriptions may encode the same physical situation under an accepted equivalence, while a transformation between physically different situations may remain a genuine symmetry of a theory's laws. The treatment of symmetry, invariance, and symmetry breaking links mathematical representation with questions about objectivity and physical content [14]. For the present investigation, this suggests a disciplined priority: specify what counts as

a configuration, determine which descriptions are equivalent for the intended inquiry, and identify the transformations or observables that remain meaningful after this identification. Only then should a relation among the resulting objects be assessed as a candidate ordering. The procedure is intended to prevent artifacts of labeling, coordinates, or redundant descriptions from being mistaken for independent stages of generation.

The corresponding mathematical programme will proceed at several distinct levels. A *generative landscape* will provisionally denote a specified family of relational configurations together with admissibility conditions and potential transformations. An *equivalence criterion* will identify configurations whose differences are treated as immaterial for a particular question. A *generative dependence relation* will then be introduced under domain-specific conditions that make some configurations reachable, enabling, or constraining relative to others. A preorder may become available when the proposed relation is reflexive and transitive. Under a suitable quotient by mutual reachability, an antisymmetric order may be obtained. The conceptual significance of such constructions rests upon the interpretation and justification of their ingredients. In particular, the equivalence relation used to define the quotient requires its own warrant; mathematical quotienting alone provides neither physical identification nor an argument for ontological fundamentality.

This separation has two consequences for the concept of hierarchy. First, the existence of local comparisons permits an ordered structure without entailing a single global ranking. Mathematically incomparable configurations may remain so under the available evidence or may express genuinely independent pathways. Second, the language of *levels of generative potential* calls for an operationally explicit relation to feasible transformations. An increase in some defined capacity to generate further configurations need not coincide with increased complexity, greater thermodynamic entropy, enhanced fitness, improved welfare, or moral progress. These properties have different definitions and empirical domains. A scalar potential may be unavailable, and a vector-valued or partially ordered description may be more defensible. Even where a ranking is constructed, it may change with the chosen observation scale, resource constraints, or transformation rules. The proposed notion of level is consequently an object of investigation rather than an initial axiom.

A third motivation concerns scientific representations across scale. Renormalization-group methods illustrate how effective descriptions and their parameters can change with the scale of analysis, particularly in the study of critical phenomena [49]. Such work provides a rigorous body of examples for assessing the relationships among microscopic variables, equivalence of effective descriptions, and changes in macroscopic behaviour. The present inquiry does not infer a general generative hierarchy directly from renormalization. It asks how criteria of comparability and admissible generation can respond to changes of scale, and whether apparent order is robust under well-specified transformations of representation. A physical renormalization-group flow, where available, has specialized definitions and empirical significance. The broader formal vocabulary of generative landscapes must be assessed separately before any proposed correspondence can be accepted.

The interpretation of emergence requires comparable care. A feature may be novel and robust relative to a class of descriptions while admitting a deduction from an appropriate model under stated conditions; the role of a limiting procedure in such arguments deserves explicit scrutiny [12]. The present paper accordingly distinguishes three claims often brought together under the term *emergent order*. A *construction claim* states that an ordering relation follows from given mathematical definitions and assumptions. An *explanatory claim* states that this relation captures dependencies relevant to observed behaviour or explanatory comparison. An *ontological claim* maintains that the ordering reflects a feature of the world beyond the representational choices through which it was articulated. These claims differ in standards of support. A valid construction can coexist with a weak empirical fit, and a successful model can remain compatible with multiple ontological interpretations.

The role of representation is especially visible when histories or generative routes are only partially accessible. An observer may record a sequence of measured states, while the process of observation discards unmeasured variables, stochastic alternatives, intermediate conditions, and relations among distributed components. The same recorded sequence may then be consistent with several relational

mechanisms. Conversely, differences between two records may be representationally salient without establishing a difference in generative structure. Comparative analysis must specify the measurement map, relevant equivalences, model class, and evidential criterion for discriminating rival explanations. Where the data fail to identify a unique order, ambiguity is itself a substantive result. Such ambiguity may sometimes be reduced through new observations or interventions; in other cases it may persist because the chosen observational interface preserves only a restricted family of invariants.

A further difficulty arises when the landscape itself changes. Rules governing feasible transformation, admissible constituents, and available interaction types may depend upon an evolving configuration. A fixed graph of possible moves can therefore be a useful local representation while remaining inadequate as a universal model of continuing generation. Conditional precedence may emerge within one regime, weaken across a transition, or reorganize when interactions create new categories of configurations. A research programme concerned with generative change must keep the distinction between transitions *within* a declared landscape and transformations *of* the landscape's space of possibilities. Neither kind of transformation requires an unqualified assumption of universal increase, cumulative complexity, or a predetermined destination.

The paper uses the phrase *ontological status* in a precise and restrained sense. Its concern is the nature and evidential standing of proposed order structures within alternative accounts of what exists and how relations are generated. Several interpretations are left available: order might be primitive in a particular physical theory; it might be derived from a more basic relational dynamics; it might be dependent on observation and representation; or different aspects of order might occupy different explanatory positions across scales. The present work proposes criteria for distinguishing these possibilities, with appropriate attention to mathematical consistency, physical interpretation, empirical distinguishability, and the limits of analogy. It does not seek to settle the ultimate constitution of spacetime or generativity by the choice of vocabulary alone.

This stance bears on the broader generative-relational inquiry while preserving a distinct research object. Studies of political economy examine how heterogeneous participants produce and reproduce their enabling conditions, distribute control, and circulate value. Studies of norm formation examine how expectations and obligations become established through evolving interactions and institutions. Those questions may draw upon generative relations, yet their explanatory objects include historically specific actors, practices, and institutional arrangements. The present paper asks a logically prior modelling question: when does a formal relation of precedence emerge from stipulated conditions of generation, and how far can the resulting order be carried into physical or ontological interpretation? The separateness of these inquiries helps avoid turning a mathematical dependency into a moral hierarchy, an account of productive capability into a theory of fundamental physics, or a descriptive ordering into a justification of authority.

The intellectual contribution is methodological. Reachability preorders, quotient orders, and closure systems are established mathematical tools [13, 5], and structural realism and grounding already provide substantial philosophical accounts of relational priority [50, 32, 43]. G1 tests the warrants linking generative rules, equivalence, observation, and physical interpretation; it makes no priority claim for the constituent constructions. The unary reachability–potential duality in Section 6 is an explicit redundancy baseline, and the controlled comparisons in Appendix E show that specialized methods can solve every presented toy task with adequate inputs. Section 9.10 therefore locates the proposal in cross-method assumption auditing, whose practical value remains to be tested.

The paper proceeds through conditional formal and interpretive inquiries. It specifies candidate generative landscapes, their equivalences and compositional rules; examines reconstructed precedence and resource-dependent potential; then studies landscape change, critical-looking reorganizations, and possible physical interpretations. The associated proofs establish consequences of stipulated assumptions, while the counterexamples specify limits of inference. Cross-domain correspondences remain analogies until mappings, preserved properties, and empirical implications have been demonstrated. Subsequent research should assess the proposed diagnostics against ordinary graph reachability, observational selection, and established dynamical models, retaining the framework only where it yields demonstrable

additional discrimination.

The remainder of the planned paper follows this programme. Section 2 characterizes generative landscapes and their representational assumptions. Section 3 examines equivalence and invariance; Section 4 states candidate conditions for derived precedence; and Section 5 addresses the reconstruction of order through partial representations. Section 6 investigates levels of generative potential and local stratification. Sections 7 and 8 consider landscape transformation and critical transitions, respectively. Section 9 evaluates ontological interpretations and physical consistency, while Section 10 draws together the established results, remaining counterconditions, and proposed directions for research. The central methodological standard remains constant throughout: a relation earns its explanatory standing through the justification of its construction, and its ontological interpretation through additional physical and philosophical argument.

2 Generative Landscapes and Their Representations

The present section establishes the representational domain on which a relation of generative precedence might subsequently be defined. Its task is to specify candidate configurations, constraints on their admissibility, locally available generative transformations, observational interfaces, and comparisons between descriptions. The construction proceeds from a modest mathematical inventory toward conditions of representational adequacy, with examples showing why generative relations require more information than a set of states or a diagram of arrows. The argument is definitional and methodological: it supplies a candidate framework whose usefulness must be judged in particular applications, while leaving claims about physical ontology and the eventual existence of an order relation open.

2.1 Representational Domains and Candidate Configurations

A scientific representation begins with a choice of what is to count as distinguishable for a specified purpose. A field configuration, a graph of interactions, an algebraic structure, and a coarse statistical state provide different kinds of candidates. Their constituent objects, structural relations, and permitted attributes require declaration. The formal role of a configuration is to hold sufficient information for the questions to which a model is directed. It acquires a physical interpretation through an independently justified connection with measurement, intervention, or established theory.

Let q denote a research question together with a declared domain of application. The set D_q comprises available *descriptions*: coordinate records, labelled structures, simulated realizations, or other admissible data formats. A representation map $\rho_q : D_q \to C_q$ associates each description with a *candidate model configuration* in C_q. For the initial construction, C_q is the image of ρ_q, so that ρ_q is surjective. A modelling extension may introduce theoretical configurations without known descriptive realizations; such an extension must be recorded separately, since the representational guarantee provided by surjectivity would then cease to apply.

The distinction between D_q and C_q is functional. Descriptions may contain coordinates, labels, numerical precision, or encoding decisions that the model deliberately suppresses. A candidate configuration may conversely organize the retained information into additional structural relations. The map ρ_q specifies how a description enters the model; it gives no independent warrant for interpreting two descriptions with the same image as one and the same physical situation. The interpretive legitimacy of those identifications is a subsequent issue of equivalence and invariance, examined in Section 3 once that section is developed.

A useful example is a spatially distributed interaction network. A description might be an indexed edge list with measured attributes, whereas a candidate configuration is an attributed graph subject to declared modelling conventions. A permutation of vertex labels can leave the graph structure invariant. A measurement of orientation or location may nevertheless distinguish differently embedded realizations. Whether a relabelling is representational redundancy, a symmetry of a physical model, or a transformation

between different specimens depends on what has been included in q, D_q, and ρ_q. The language of relational configuration alone decides none of these matters.

The terms *relational* and *generative* also have distinct roles. A relational configuration encodes associations, incidences, interactions, constraints, or other specified relations. A generative account adds rules or conditions under which configurations can participate in the formation of further configurations. A relational dataset may be static and descriptive; a generative model carries an additional account of structured possibility. Conversely, a generative transformation can be described through algebraic operations whose relational content requires interpretation. The proposed framework can accommodate both while keeping their assumptions explicit.

2.2 Admissibility and Conditional Configuration Spaces

A set of candidate configurations is usually broader than the set licensed under an actual experimental regime, boundary condition, conservation law, institutional setting, or other effective constraint. Accordingly, the model needs a separate description of the conditions under which a candidate may be used. These conditions can be contingent, empirically measured, theoretically stipulated, or inherited from an established model. Their authority must be assessed independently.

Let K_q be a space of *conditioning regimes*, and let $A_q \subseteq C_q \times K_q$ be an admissibility relation. The admissible configuration set for a declared condition $\kappa \in K_q$ is defined in Equation 1.

$$C_{q,\kappa} = \{c \in C_q \mid (c, \kappa) \in A_q\}. \tag{1}$$

A conditioning regime can describe resource availability, admissible component types, geometric boundaries, operational restrictions, constitutive laws, or a collection of experimentally controlled parameters. In a field theory it might include boundary and regularity conditions; in a reaction network it might include the species admitted at an interface; in a developmental model it might specify environmental constraints. The notation treats these descriptions as model inputs while leaving their physical content to domain-specific justification.

This separation is essential when discussing what a system *could* generate. The collection of mathematically expressible configurations can greatly exceed the collection admitted by given resources or conservation conditions. Feasibility is thus conditional on the regime. A claim that some configuration possesses a "higher generative level" becomes ambiguous if the conditions that permit its construction are suppressed. The same configuration might be reachable under one resource regime and inaccessible under another. A global ranking that ignores the difference would import an unjustified stability assumption.

Conditions can also be relationally distributed. A coastal ecological model may involve water exchange, nutrient availability, population interactions, and human-managed infrastructures; these do not reduce naturally to a single resource count. A quantum or statistical physical model can involve a structured set of constraints rather than a finite list of scalar bounds. The set K_q is therefore an abstract indexing space and may itself carry topology, geometry, algebra, or empirical uncertainty when these additions are warranted. No linear order, total comparability, or privileged temporal parameter is assumed on K_q.

The admissibility relation should be distinguished from a rule of change. Membership in $C_{q,\kappa}$ states that a configuration is compatible with the stipulated regime. It does not imply the existence of a feasible operation producing that configuration from another. The difference between admissibility and constructibility will become central to the investigation of derived precedence in Section 4.

2.3 Local Generative Rules and Joint Conditions

Many generative descriptions employ single-input state transitions. Such rules are useful for deterministic dynamical systems and some computational processes, yet they can conceal the role of joint prerequisites. A chemical reaction may require several species, a structural assembly may require compatible

components, and a collective practice may depend upon concurrent institutional and material conditions. A preliminary landscape formalism should permit these relations to be recorded without first reducing them to a chain of unary state changes.

For each $\kappa \in K_q$ and each nonnegative integer m, let $G_{q,\kappa}^{(m)}$ be a collection of admitted m-input formation rules. The family is typed as in Equation 2.

$$G_{q,\kappa}^{(m)} \subseteq C_{q,\kappa}^{m} \times C_{q,\kappa}, \qquad m \in \mathbb{N}_0. \tag{2}$$

A tuple $((c_1, \ldots, c_m), c')$ in this relation records that the joint input family is admitted as sufficient for the production, assembly, or formal derivation of c' *under the declared rule semantics*. It does not by itself assert that the operation has been performed, that it occurs with positive physical probability, or that its result is unique. For $m = 1$, the rule can be drawn as a directed edge; for $m > 1$, a directed hyperedge or another multi-input representation retains the joint prerequisite. The case $m = 0$ denotes an explicitly licensed nullary introduction and requires a domain-specific account of its external or initial enabling conditions. Such introductions should never silently be interpreted as creation from nothing.

The displayed typing in Equation 2 presupposes that participating model configurations can be combined as inputs. In a theory where c_i represent entire, incompatible global states, arbitrary tuples will be meaningless. An application should therefore specify a compatibility predicate $J_{q,\kappa}^{(m)} \subseteq C_{q,\kappa}^{m}$ and require the input projection of $G_{q,\kappa}^{(m)}$ to lie in $J_{q,\kappa}^{(m)}$. This requirement can be implemented by gluing rules for local structures, composition at interfaces, tensor-product assumptions, or explicit compatibility checks. The appropriate form of composition belongs to the application, and the formalism does not identify all such composition laws.

A further distinction concerns *production* and *permission*. In one application, a rule represents a physically executable event; in another, it describes a logically valid derivation; in a third, it records a permissible intervention under stipulated institutions. These interpretations need separate labels and validation procedures. The common relation-theoretic notation enables comparison of formal constraints; it provides neither an equivalence of the processes nor their transfer across domains. Research on open reaction networks supplies a concrete example in which input/output interfaces, reaction rules, and dynamical realization can be formally related within a specific physical and mathematical framework [6]. Its results serve as a model of disciplined specification, rather than as a derivation of the present general landscape construction.

For later use, a *candidate generative landscape* is provisionally the structure given in Equation 3.

$$\mathfrak{L}_q = (D_q, C_q, \rho_q, K_q, A_q, \{G_{q,\kappa}^{(m)}\}_{\kappa,m}, \mathcal{O}_q). \tag{3}$$

Here $\mathcal{O}_q$ denotes a family of observational interfaces, defined below. Equation 3 is a modelling schema rather than a unique axiomatization. Equivalent bookkeeping arrangements may be preferable where the relevant objects are smooth manifolds, probability kernels, categories of typed processes, or field configurations. The analytical commitments are the separability of descriptions, configurations, conditions, generation rules, and observations, together with their explicitly declared relations.

Rule orientation is part of the specified model. For a reversible physical process, the reverse rule may also be admissible. For an inferential calculus, the direction may encode proof formation. A model may also leave a process direction unresolved. The orientation of a generating rule therefore cannot be used, without further argument, to postulate an ontologically fundamental temporal direction. Similarly, the composability of a particular pair of rules needs a compatibility and closure argument; a drawn path does not automatically assert that its endpoints stand in a justified global relation of precedence. The axioms governing such a relation are reserved for Section 4.

2.4 Observation Interfaces and Descriptive Adequacy

The generation rules and admissibility relation express a proposed structure of possibilities. Empirical inquiry encounters this structure through measurement procedures, recorded data, and theoretically interpreted observables. A landscape with no specified observational contact might remain a coherent formal

object, yet its claims about a physical system would be difficult to discriminate from alternative models. The interface between model configurations and obtainable records should therefore be explicit.

Let I_q index available observation procedures. For each $\alpha \in I_q$, an observational interface $O_{q,\alpha} : C_q \rightharpoonup Y_{q,\alpha}$ maps configurations in its stated domain of applicability to predicted observable outputs. The partial-map symbol emphasizes that a procedure may be inapplicable to some model configurations. When measurements are noisy, an observational interface may instead be represented by a probability kernel. A collection $\mathcal{O}_q = \{O_{q,\alpha}\}_{\alpha \in I_q}$ can encode multiple procedures with different resolution, calibration, or observational scope.

There is a fundamental distinction between an interface that produces a datum and an interface that accurately represents the structure that generated it. Two configurations can yield the same result under a chosen procedure while differing under another, or while supporting different subsequent operations. Conversely, a change in output can arise from calibration drift or a difference in observation conditions. Thus empirical indistinguishability relative to one instrument is a restricted relation, and its elevation to physical equivalence needs additional evidence.

The distinction between partial and complete observables in general-covariant mechanics demonstrates that questions about observability can depend strongly on the underlying theory's structure [41]. The present $O_{q,\alpha}$ is a generic modelling interface and is *not* intended as a restatement of Rovelli's technical notion of partial observable. That comparison is relevant because it cautions against attaching a single, unrestricted physical interpretation to a general-purpose map between mathematical spaces.

A simple factorization criterion makes one representational obligation precise. Suppose a directly defined descriptive measurement is $M_\alpha : D_q \to Y_{q,\alpha}$, and suppose that the representation map is surjective. A well-defined model-level map $O_{q,\alpha}$ satisfying the factorization in Equation 4 exists exactly when M_α is constant on each fibre of ρ_q.

$$M_\alpha = O_{q,\alpha} \circ \rho_q. \tag{4}$$

Indeed, constancy on fibres permits the assignment $O_{q,\alpha}(c) = M_\alpha(d)$ for any d with $\rho_q(d) = c$; surjectivity ensures existence of such a description, while the fibre condition makes the assignment independent of its choice. Conversely, factorization immediately gives equal measurements to descriptions with equal images. The constructed $O_{q,\alpha}$ is unique on C_q under the surjectivity assumption. This elementary result is a consistency test for an exact deterministic model. Real experimental settings may instead require calibrated uncertainty bounds, conditional kernels, or partial domains. Passing the consistency test is a necessary representational condition for the declared measurement, while empirical adequacy also requires that the measurements and state assignments reflect the intended phenomenon.

A model may project away variables that matter for generation while preserving the particular outputs an observer records. This possibility reveals why descriptive adequacy and generative adequacy differ. The factorization in Equation 4 concerns outputs. A stronger claim, that the rules themselves descend to a coarser representation without ambiguity, requires additional compatibility of the rule structure with the projection. Those requirements are examined later in the paper's treatment of equivariance and reconstruction.

2.5 Scale and Translation between Landscapes

A single research object often admits representations at several resolutions. Microscopic particle configurations, mesoscopic interaction networks, and effective continuum fields may support different kinds of inquiry. A comparison across these representations requires more than the observation that each can be called a configuration space. The admissible states, condition spaces, measurable outputs, and available transformations must be related by specified maps and, where appropriate, quantitative error criteria.

Consider two candidate landscapes $\mathfrak{L}_q^f$ and $\mathfrak{L}_q^c$, provisionally interpreted as finer and coarser descriptions of the same research object. A comparison may employ a state map $\sigma : C_q^f \rightharpoonup C_q^c$ and a condition map $\eta : K_q^f \rightharpoonup K_q^c$. These maps should be accompanied by a claim about which observable quantities are preserved. If O^f and O^c are corresponding observables and h translates fine-level outputs into

coarse-level outputs, one useful exact requirement is Equation 5.

$$O^c \circ \sigma = h \circ O^f \quad \text{on a declared common domain.} \tag{5}$$

In empirical applications, equality in Equation 5 may be replaced by a discrepancy bound appropriate to the observation type. The bound would require calibrated data, a declared metric or divergence, and uncertainty estimates. Exact commutation is a useful mathematical benchmark; approximate preservation is frequently the scientifically meaningful goal.

Preservation of observations does not ensure preservation of permitted transformations. For example, a coarser map may identify two fine configurations with different enabling conditions. Their shared image may then appear to possess operations available to only one of the fine configurations. Any assertion that fine-level generation is faithfully represented at the coarse level therefore needs a separate rule-compatibility test. In certain models an exact implication from fine rules to coarse rules is appropriate; in others a stochastic effective rule, a memory term, or a conditional restriction is required. The research task is to determine which case applies, rather than presuming that a projection automatically preserves generativity.

The renormalization group supplies a particularly important physical reference point. Under precise assumptions, changes in scale can be associated with flows of effective couplings and the emergence of robust macroscopic descriptions [49]. The maps introduced in this subsection have a more elementary role: they specify candidate comparisons between representations. Their existence alone yields neither a renormalization transformation nor a physical fixed point, universality class, or scale-invariant order. Such claims require the additional dynamics, observables, limiting procedures, and empirical or theoretical derivations characteristic of the particular physical application.

Scale also need not be linearly ordered. Several representations may preserve distinct and partially overlapping sets of observables. A molecular model and an ecological network model, for instance, may highlight different mechanisms without lying on a single hierarchy of resolution. An informative landscape programme can thus use a family of maps with partially defined comparisons and multiple domains. One of its objectives is to determine when a directed refinement or coarse-graining structure is justified and when the available descriptions remain only locally comparable.

2.6 Illustrative Landscapes and Representation Failures

Small examples clarify what the proposed formal inventory captures and what it leaves open. They are logical constructions rather than empirical models. In each case, the outcome depends upon the declared configurations, conditioning regimes, and generation rules.

Joint enabling conditions. Let $C = \{a, b, c\}$ and let a regime κ admit all three candidates. Suppose $G_\kappa^{(2)}$ contains the rule $((a, b), c)$, while the unary rule sets contain no rule from a to c or from b to c. The model states that a and b jointly suffice to form c under the declared interpretation. It supplies no justification for treating either input as individually sufficient. If one collapses the hyperedge into independent edges, a purported one-input accessibility relation appears that the original landscape never asserted. A later theory of precedence must distinguish joint prerequisite relations from reachability based on complete compatible states.

Reversible transformations. Let $C = \{u, v, w\}$ and suppose the unary rule family contains $u \to v$, $v \to u$, and $v \to w$, under a fixed condition. A model that interprets rules as admissible moves can express a reversible pair and a further option. It remains possible to study the mutual accessibility of u and v once a suitable rule-composition semantics is supplied. Any partial order formed from such relations would require careful treatment of cycles and suitable identifications. In particular, drawing the arrows does not itself establish antisymmetry on the original set.

Table 1: Constituent layers of a candidate generative landscape and their distinct evidential obligations.

Layer	Analytical function	Principal adequacy question
Description D_q and map ρ_q	Select and encode model configurations	Which distinctions are retained or suppressed?
Candidate configurations C_q	Specify structured states or arrangements	What constitutes a well-formed configuration?
Conditions K_q and A_q	Determine regime-relative admissibility	Which boundaries or resources delimit feasibility?
Rules $G^{(m)}_{q,\kappa}$	Specify compatible generative inputs and outputs	Which operations are justified and jointly enabled?
Observation family $\mathcal{O}_q$	Connect configurations with measurable predictions	What is observable, with what uncertainty?
Inter-landscape maps	Relate representations under declared changes of resolution	Which observables and rules survive translation?

Observationally merged configurations. Let $C = \{x, y, z\}$, and consider an observation map with $O(x) = O(y)$ and $O(z)$ distinct. Suppose a rule permits formation of z from x, while none permits the corresponding formation from y under the same regime. The observed value shared by x and y is insufficient to determine whether the relevant operation is available. Any coarse configuration class merging x and y would need to retain hidden context, provide a conditional or probabilistic rule, or acknowledge the resulting ambiguity. This example anticipates the reconstruction problem treated in Section 5.

Changing conditions. Consider a rule $a \to b$ admissible under κ_1 and inadmissible under κ_2. It is inappropriate to assign unconditional generative precedence from the rule alone. The landscape retains both the candidates and the conditions that delimit the rule's use. A more advanced model may describe how κ_1 is transformed into κ_2, including the production or depletion of resources. Such a model concerns the alteration of the landscape itself and is reserved for the investigation of changing generative conditions in Section 7.

Table 1 summarizes the representational layers established by the section. The table serves as a diagnostic: for any proposed physical or interdisciplinary application, each layer calls for independent specification and a corresponding check of adequacy.

These examples also show why an illustrative landscape should be accompanied by explicit counter-conditions. An apparent relation of order can be generated by an arbitrary choice of labels, an omitted conditioning parameter, an inappropriate reduction of joint prerequisites, or a many-to-one observational map. Each failure is potentially detectable at the representational level before a stronger ontological interpretation is attempted. Such failures should guide revision of the model's domain, interfaces, or rules, rather than be absorbed into an unsupported claim of universality.

2.7 Minimal Adequacy Conditions for Subsequent Order Analysis

The preceding construction yields a checklist for the inquiry into emergent precedence. First, a landscape should state its modelling target and the granularity of its configurations. Second, conditions of admissibility should be recorded separately from rules of possible production. Third, the arity, input compatibility, and interpretation of generating rules should be explicit. Fourth, observational interfaces and the limitations of their resolution should be identifiable. Fifth, changes of representation should declare the properties that they preserve and the failures they permit. These obligations concern logical coherence and the prospects of empirical discrimination; they do not assert a universal physical representation.

A further requirement is to retain the epistemic standing of each component. Candidate configurations may be directly observed, inferred through an inverse problem, generated by simulation, or stip-

ulated for conceptual exploration. An admissibility relation may be known from a conservation law or estimated from incomplete data. A rule family may represent an experimentally measured propensity, a theoretical selection rule, a logical derivation, or a proposal awaiting validation. These evidential differences matter when the model is used to claim that an order arises in the world. They should remain traceable through any later quotient, ranking, or reconstruction.

The resulting landscape is intentionally prior to a theory of equivalence and precedence. The map ρ_q already makes representational choices, but the identification of its fibres with physically or explanatorily equivalent configurations requires further criteria. Likewise, the generating rule family encodes allowed orientations and joint prerequisites, but a preorder or partial order requires specified closure, compatibility, and quotient conditions. Section 3 develops equivalence and invariance; Section 4 investigates the requirements for a derived precedence relation. The present section establishes the ingredients and their limits so that later mathematical conclusions can be traced to intelligible assumptions.

This sequence supports a constrained form of ontological inquiry. Under a successful physical interpretation, a landscape could articulate dependencies of real explanatory importance. Under an observationally underdetermined interpretation, it could instead characterize a family of candidate mechanisms compatible with the same evidence. Under a purely formal interpretation, it could clarify consequences of selected axioms. The distinction among these outcomes is the central safeguard against mistaking a useful representation for a settled ontology of generative order.

3 Equivalence and Invariance in Generative Configuration Spaces

Equivalence determines which differences among candidate configurations may be suppressed while retaining the features relevant to an inquiry. In a generative landscape, that determination has consequences for admissibility, observation, and the availability of transformations. The present section develops a taxonomy of equivalence relations, identifies the structures that can be preserved under isomorphism and symmetry, and specifies when generative rules admit an exact quotient representation. Its method combines elementary mathematical definitions with counterexamples and a bounded comparison to gauge theory. The resulting analysis prepares the conditions for derived precedence in Section 4, while preserving the distinction between representational identification and its physical interpretation. In particular, observational equivalence, mutual reachability, and equality of potential profiles are separately constructed relations; the restricted circumstances under which the latter two coincide are established in Proposition 6.3.

3.1 Domains and Criteria of Equivalence

Consider the candidate landscape $\mathfrak{L}_q$ introduced in Section 2, with description space D_q, configuration space C_q, conditions K_q, admissibility relation A_q, generating relations $G_{q,\kappa}^{(m)}$, and observation family $\mathcal{O}_q$. A claim that two objects are equivalent is incomplete until the objects' domain, the criterion of identification, and the properties intended to survive identification have been specified. Equivalence of data records, configurations, model realizations, and empirical systems can express different claims even where the same mathematical notation is used.

For a fixed condition $\kappa \in K_q$, let $C_{q,\kappa}$ denote the admissible configurations defined in Equation 1. An equivalence relation $E_{q,\kappa}$ on $C_{q,\kappa}$ is reflexive, symmetric, and transitive. Its associated quotient consists of classes $[c]_E$ of configurations related to c, and the canonical projection π_E maps an admissible configuration to its class. This set-level quotient exists once E is supplied. Whether the quotient carries observation maps, generative transformations, or useful physical content is a separate question.

Three immediate criteria merit separation. A *descriptive equivalence* groups records under an encoding convention, such as two files yielding the same image under ρ_q. A *structural equivalence* identifies configurations related by an isomorphism of specified relational structures. An *observational equivalence* identifies configurations that yield the same predictions for a declared observation family under

declared conditions. Each criterion refers to an interface and purpose; none entails the other without additional premises. In particular, a pair of records might have identical model images because the encoding discarded information that a subsequently designed experiment could recover.

It is useful to distinguish an exact equivalence relation from a similarity or tolerance criterion. A relation defined by distance at most ε is typically reflexive and symmetric, while transitivity can fail. For real-valued measurements, values 0, 0.7ε, and 1.4ε illustrate the problem: each adjacent pair is close, yet the first and last may exceed the tolerance. Forming equivalence classes by taking a transitive closure can aggregate configurations separated by many tolerance steps. Such aggregation is a further modelling decision, with consequences for interpretation. Approximate observational indistinguishability therefore calls for uncertainty models, metrics, or explicitly selected clustering conventions before it can serve as an exact quotient.

In a conditional landscape, the equivalence relation itself may depend upon κ. Two configurations can be indistinguishable under one measurement regime and distinguishable when an external field, boundary constraint, or additional sensor becomes available. An acceptable quotient construction should record this dependency. One may compare a family $\{E_{q,\kappa}\}$ across conditions, or construct a relation on conditioned pairs (c, κ); these alternatives answer different questions. The mathematical objects used below are conditional and local unless stronger cross-condition assumptions are explicitly supplied.

3.2 Structural Isomorphism and Relational Descriptions

A structural isomorphism concerns the preservation of relations declared significant in the model. Let a configuration c be presented as a structure with a carrier X_c, specified attributes, and a finite or otherwise declared signature of relations and operations. An isomorphism $f : X_c \to X_d$ is a bijection preserving and reflecting those relations and preserving the designated operations. The signature determines what qualifies as preservation. For attributed networks, an isomorphism may preserve edge incidence and types while allowing node labels to change. If geographic coordinates, distinguished boundary sites, or directed interactions are part of the signature, a proposed isomorphism must also respect those additions.

The familiar treatment of relabelling in graph theory demonstrates the explanatory usefulness of structural equivalence. Two differently indexed adjacency matrices may represent isomorphic graphs. Counting them as structurally distinct can inflate the apparent diversity of network types. Yet the graph isomorphism alone does not determine whether the corresponding empirical systems are interchangeable. Distinct experimental specimens, histories, locations, or available resources may persist beneath a shared graph structure. A network description that intentionally suppresses such properties can establish a valid combinatorial equivalence while leaving physical and historical identification open.

This distinction becomes especially important for heterogeneous generative systems. A biochemical interaction network, an engineered reaction network, and a diagram of institutional dependencies may have isomorphic incidence structures. Their material units, constraints, and semantics differ. An isomorphism of selected signatures can support the transfer of structural lemmas; it cannot, by itself, transfer conservation laws, empirical rate parameters, ethical categories, or warranted causal claims. The proposed inquiry should therefore make the *signature of comparison* available for inspection and record which additional properties remain external to it.

Isomorphism may also concern entire landscapes rather than individual configurations. Suppose landscapes $\mathfrak{L}_q$ and $\mathfrak{L}_{q'}$ have compatible signatures. An inter-landscape isomorphism would require appropriate bijections of configurations and conditions, preservation and reflection of admissibility, and coherent transport of each declared generating relation. A mapping between only the configurations is weaker. If two models have identical configuration sets but different resource conditions or joint-input rules, their generative structures can be inequivalent despite identity of the underlying sets.

Physical examples add another interpretative distinction. The permutation of labels for mathematically identical particles, the relocation of an experimentally prepared assembly, and the transformation of coordinates may each admit a mathematical group description. Their physical significance can differ radically, depending on the state space, observables, and theory of measurement. Symmetry and philo-

sophical discussions of identity have long emphasized the need to distinguish mathematical and physical concepts in these contexts [42]. The present formalism adopts this discipline without fixing one physical interpretation for all applications.

3.3 Symmetry Actions on Conditional Landscapes

An action of a group H on a configuration space provides a systematic way to generate candidate equivalence classes. The group axioms guarantee that the relation of lying on the same orbit is an equivalence relation. The content of a symmetry claim, however, depends upon which additional landscape structures are preserved. A permutation of C_q alone can be a mathematically valid action while altering admissibility, joint generation, or observations in ways material to the intended model.

For conditional systems, a useful starting point is a group H acting on both C_q and K_q. Call the action *landscape-covariant* when the action transports admissibility and generating relations across the corresponding conditions according to Equation 6.

$$\begin{aligned} (c,\kappa) \in A_q &\iff (hc, h\kappa) \in A_q, \\ ((c_1,\ldots,c_m),d) \in G^{(m)}_{q,\kappa} &\iff ((hc_1,\ldots,hc_m),hd) \in G^{(m)}_{q,h\kappa}. \end{aligned} \tag{6}$$

Here $h \in H$, and the condition applies for each arity m included in the landscape. Equation 6 defines covariance under a *simultaneous* transformation: all inputs, the output, and the relevant conditioning regime are transported together. It provides a precise sense in which a transformation preserves the landscape's rule structure across conditions.

At a fixed κ, the stabilizer $H_\kappa = \{h \in H \mid h\kappa = \kappa\}$ acts within $C_{q,\kappa}$. Its orbits then provide a conditional equivalence relation. A transformation with $h\kappa \neq \kappa$ relates two different conditioned domains. Treating it as an internal symmetry of the original fibre would suppress an actual change of experimental or operational regime. This distinction matters for systems with fixed boundaries, resource distributions, external fields, or marked reference points. The preserved mathematical structure may concern a family of models, even where a single experimental arrangement has fewer symmetries.

Covariance of rules also differs from invariance of a selected numerical observable. A scalar observable O is invariant under h when its evaluation is unchanged under the transformation, with due attention to conditions. A vector-valued or tensor-valued observable may instead transform by a prescribed representation of H; its components change while the law relating them is covariant. Conflating these two cases can lead to the premature removal of physically meaningful orientation, direction, or boundary information. The distinction between symmetry principles, their empirical interpretations, and their status as theoretical constraints is well established in philosophy of physics [14].

An effective group action need not be free. Some configurations possess nontrivial stabilizers, leading to orbits of different sizes or local geometries. The quotient can consequently have special strata or singular structures even when the original configuration space is regular. The present set-theoretic development requires no smooth manifold, measure, or orbit-space topology. Such structures may be introduced when a physical application warrants them. In particular, orbit structure alone does not produce a well-defined relation of generative precedence among the orbits.

3.4 Observational Invariants and the Resolution of Equivalence

Observational equivalence is defined relative to a declared family of measurable predictions. Suppose each $O \in \mathcal{O}_{q,\kappa}$ is defined on $C_{q,\kappa}$ and takes values in a specified observation space. Under exact deterministic predictions, one may set $c \sim_{\mathcal{O},\kappa} d$ when $O(c) = O(d)$ for each member of the family. For stochastic models, corresponding equality may concern full predicted probability laws under fixed interventions, rather than equality of individual measurement outcomes. The family of interventions, detector limits, and measurement uncertainty should accompany the equivalence claim.

For an arbitrary equivalence E on $C_{q,\kappa}$, an observable O descends to a function on the quotient precisely when it is constant on every E-class. This condition is expressed in Equation 7.

$$c\ E\ d \implies O(c) = O(d) \quad \iff \quad O = \overline{O} \circ \pi_E \text{ for a unique } \overline{O} : C_{q,\kappa}/E \to Y_O. \tag{7}$$

The implication in Equation 7 is required for every relevant pair; the right-hand factorization specifies how an invariant is evaluated on classes. The result is elementary, but its modelling consequence is substantial: a quotient that merges distinguishable predictions is unsuitable for an exact account of those predictions. It may remain useful as an explicitly approximate representation provided the loss and its error are characterized.

Invariance and completeness have different logical strengths. An invariant respects an already specified equivalence relation. A *separating* family of invariants distinguishes any two distinct equivalence classes, so equal values across the family imply equivalence. A physically accessible observation family may be invariant without separating classes. The existence of an abstract complete family of class indicators says little about practical measurement, because those indicators may lack operational realizations. A quotient can therefore be mathematically coherent and still be empirically underdetermined.

Conversely, equality of a restricted observation family can create equivalences too coarse for the available generative rules. Two configurations may share all monitored outputs while differing in the set of operations they enable. Observational indistinguishability may persist over a limited experiment and disappear under an intervention that exposes the latent difference. This gap connects symmetry analysis with inverse problems, experimental design, and the reconstruction task reserved for Section 5. It also cautions against inferring sameness of generative capacity from similarity of observed trajectories alone.

Observational equivalence should further be distinguished from physical identity. Operational indistinguishability can supply a powerful reason for treating two descriptions jointly in an empirical model. The step from that practice to a claim that the descriptions concern one physical state involves commitments about the relevant theory and the completeness of its empirical interfaces. The present paper keeps that interpretative step visible, particularly when comparing different scales and kinds of generative system.

3.5 Quotient Compatibility for Joint Generative Rules

The strongest requirement for an exact generative quotient concerns the preservation of rules themselves. A generating relation of arity m can depend upon the mutual arrangement of its inputs, so identifying configurations individually may erase information about which input tuples are compatible. The appropriate mathematical question is whether membership in the relation is determined entirely by the equivalence classes of its entries. This issue parallels the familiar requirement that algebraic operations respect congruence relations, while extending the discussion to possibly partial and relational generative rules [11].

Fix q, κ, and m, and write $C = C_{q,\kappa}$, $R = G^{(m)}_{q,\kappa} \cap (C^m \times C)$, and $\pi : C \to C/E$ for brevity. Define the product projection $\Pi = \pi^{m+1} : C^{m+1} \to (C/E)^{m+1}$. The relation R is *class-saturated* when equivalent replacement of each entry preserves its truth value. Equivalently, R contains every tuple that has the same projected classes as one of its members. The exact condition is stated in Equation 8.

$$R = \Pi^{-1}\big(\Pi[R]\big). \tag{8}$$

Proposition 3.1 (Exact descent of a relational generative rule)**.** *Let E be an equivalence relation on the admissible set C, and let $R \subseteq C^{m+1}$. There exists a relation $\overline{R} \subseteq (C/E)^{m+1}$ such that $R = \Pi^{-1}(\overline{R})$ if and only if R is class-saturated as in Equation 8. When such a relation exists, it is unique and equals $\Pi[R]$.*

Proof. Suppose $R = \Pi^{-1}(\overline{R})$. Since Π is surjective, $\Pi[R] = \overline{R}$, and hence $R = \Pi^{-1}(\Pi[R])$. Conversely, if Equation 8 holds, choose $\overline{R} = \Pi[R]$; then its inverse image is R. Surjectivity of Π also proves uniqueness. □

Proposition 3.1 is a set-theoretic criterion for faithfully expressing the original relational membership on independently specified configuration classes. It is deliberately stronger than requiring a symmetry group to carry complete generating tuples to other complete tuples. A group action applies one transformation simultaneously to the tuple. Product-class saturation permits replacing the representatives of its entries independently. Those are different invariance requirements. For ordinary single-valued total operations, the familiar congruence condition ensures that equivalent input tuples produce equivalent outputs; for the multi-input relations considered here, the specified preservation requirement depends on whether one seeks exact membership, a partial operation on classes, or a weaker existence claim.

It is always possible to form the *existential image* $\Pi[R]$ on quotient tuples. The mathematical relation thereby obtained asserts that *some* representatives support the rule. Without saturation, it gives no guarantee that a different compatible-looking selection of representatives supports the same rule. Consequently, existential and universal readings of an apparent class-level generative rule must remain separate. The danger is especially acute if a subsequent analysis composes class-level rules as though arbitrary representatives could be chosen independently at each step.

For a simple example, let $C = \{a_1, a_2, b_1, b_2, z_1, z_2\}$, and let a two-element group act by simultaneously exchanging subscripts 1 and 2 for each of the letters. Set $R = \{(a_1, b_1, z_1), (a_2, b_2, z_2)\}$. The relation is invariant under the simultaneous group action. The quotient has classes $[a]$, $[b]$, and $[z]$, and its existential image contains $([a], [b], [z])$. Yet (a_1, b_2, z_1) and (a_1, b_2, z_2) are absent from R. The inputs a_1 and b_2 are individually members of the displayed classes, while their mixed combination does not support the generative rule. A class-level relation interpreted as valid for all representatives would therefore fabricate a joint possibility.

One way to retain the missing information is to quotient the *space of complete tuples* by the diagonal group action, $(C^{m+1})/H_\kappa$, rather than quotienting each configuration independently. This records relative alignment among components, although it produces a different type of object and may require additional composition data. Other applications can retain hidden condition variables, store compatibility constraints, or use probabilistic effective relations with explicitly estimated uncertainty. The appropriate repair depends upon the intended semantics of generation. The formal result establishes a limitation of quotienting; it makes no universal recommendation that every landscape employ the strongest possible equivalence criterion.

3.6 Gauge Redundancy and Physical Symmetry

The term *gauge equivalence* carries domain-specific physical meaning. In classical and quantum field theories, certain transformations connect mathematical descriptions that are ordinarily interpreted as encoding the same physical situation, with important qualifications arising from constraints, boundaries, global structure, and the treatment of observables. The interpretation is justified within the relevant physical theory; the mere availability of an algebraic group action on a relational model does not confer gauge status.

A bounded comparison with general relativity helps locate the issue. The hole argument has been central to debates about diffeomorphism-related models, determinism, and the identification of physical possibilities [15]. It demonstrates how mathematical transformations can acquire interpretative significance through the structure of a physical theory and its account of represented states. It supplies neither an automatic identification rule for every generative landscape nor evidence that all relational systems possess the same gauge structure. Distinct applications must identify their own state spaces, empirical criteria, and interpretation of transformations.

It is useful to separate four statements. First, a transformation may act on the formal descriptions used by an analyst. Second, it may be an automorphism of the selected mathematical structures. Third, the transformation may preserve every operationally accessible prediction in a particular experiment. Fourth, a physical theory may interpret the transformed descriptions as the same physical state. These statements can be related by theoretical and empirical argument, but their equivalence is an additional achievement. In particular, invariance of a restricted observation family supports the third statement only

within that family's resolution.

Gauge considerations also reveal why boundary conditions deserve explicit modelling. Transformations conventionally classified as redundant in an interior domain can interact with boundary data or asymptotic charges, producing physically relevant distinctions under suitable theories. The present paper does not assume a general boundary-charge construction. It records the methodological consequence: a claim of physical redundancy must be accompanied by the domain and conditions under which the relevant transformations are regarded as redundancies. This concern aligns with the regime-sensitive action developed in Subsection 3.3.

A different but related distinction concerns spontaneous symmetry breaking or effective symmetry change. A symmetry of governing laws need not be a symmetry of a particular solution, equilibrium, or measured configuration. Similarly, the invariances of a high-resolution representation need not coincide with those of an effective lower-resolution theory. The transformation groups and their interpretations can vary with scale and state. Subsequent analysis of critical transitions in Section 8 will therefore treat the persistence or reorganization of equivalence criteria as a substantive question rather than assuming a universal quotient group.

3.7 Comparative Counterexamples and Diagnostic Cases

The distinctions above can be tested on compact examples. Each example isolates an inference that would require extra assumptions in a physical or empirical application.

Isomorphic configurations with different conditions. Suppose two interaction networks have identical graph incidence and identical node attributes in their chosen signatures. One is assembled in a resource-rich environment; the other operates under a restrictive boundary condition. The graphs are isomorphic, while the available generative rules can differ because their conditioning regimes are different. A quotient of graph types can preserve structural counts yet fail to preserve the rule family unless the conditions are included in the model or an appropriate cross-regime covariance has been established.

Invariant observations with different opportunities. Consider two admissible configurations x and y producing the same measured value $O(x) = O(y)$. Suppose a permitted transformation creates z from x under the declared condition, while the corresponding transformation is unavailable from y. The observation induces an equivalence that fails to preserve generation. Increasing the observation family might separate x and y; alternatively, the model could retain a latent compatibility flag. Neither response is entailed by the original measurement. The example illustrates why observed similarity of outcomes cannot serve as a complete test of latent generative capacities.

Compatible label changes and incompatible input mixing. The simultaneous subscript exchange in the example following Proposition 3.1 preserves the entire tuple relation, yet independently selecting representatives destroys a joint rule. The quotient of complete configurations carries less information than the quotient of compatible tuples. The point is structural rather than merely linguistic: multi-input generation may depend upon relative arrangement, provenance, or coupling that cannot be recovered from classes of isolated participants.

Conditional symmetry reduction. Let an unmarked interaction system admit a permutation action on two otherwise symmetric sites. A condition that designates one site as a fixed input port distinguishes the sites in the conditioned model. The former permutation still relates configurations across suitably transformed conditions, but it generally ceases to belong to the stabilizer of the fixed port. An analysis that quotients by the larger group within the conditioned system can erase the distinction between physically accessible and inaccessible operations.

Table 2: Equivalence criteria and preservation tests for generative landscapes.

Candidate identification	Preserved target	Required check or limitation
Descriptive encoding	Model image of a description	Determine which measured distinctions ρ_q suppresses
Structural isomorphism	Declared relational signature	Confirm admissibility and rule compatibility separately
Group-orbit equivalence	Simultaneous transformed structures	Restrict to a fixed-regime stabilizer or track condition transport
Observational equivalence	Predictions in $\mathcal{O}_{q,\kappa}$	Check factorization and the experiment's discriminating power
Generative class quotient	Membership in multi-input rules	Test product-class saturation or retain joint compatibility
Physical gauge equivalence	Theory-relative physical state identity	Supply field-theoretic and boundary-sensitive interpretation
Approximate similarity	Controlled loss under aggregation	Establish uncertainty and composition-error bounds

Approximate equivalence and cumulative loss. Suppose a sequence of configurations is linked by small observational changes, each below a chosen tolerance. Pairwise similarity along the sequence does not entail similarity of the endpoints. Treating the chain as one exact class can erase a large aggregate difference. A controlled coarse model might bound error along compositions or carry distributions of hidden configurations. The choice requires quantitative assumptions beyond the availability of a similarity threshold.

Table 2 gathers the diagnostic obligations exposed by these examples. It provides a practical sequence of checks prior to constructing any apparent hierarchy from equivalence classes.

A diagnostic framework need not demand that every system admit one optimal equivalence. Different questions may legitimately yield different classes, with some relations refined by others and some incomparable. The practical task is to state which properties a proposed grouping preserves, how those properties are tested, and which operations would expose distinctions suppressed by that grouping. An appropriately limited equivalence can still be scientifically valuable even when it falls short of physical identification or exact rule descent.

3.8 Equivalence as Preparation for Derived Precedence

A derived relation of precedence requires a clear account of its domain. If the relation is to hold between configurations, the identities of those configurations and the semantics of admissible steps must be specified. If it is to hold between equivalence classes, the relation must be shown to survive representative changes in the sense appropriate to its proposed interpretation. If a model identifies configurations through mutual reachability, the resulting equivalence depends upon the same rule-composition semantics that defines that reachability. These choices have different mathematical implications.

Section 4 will investigate the construction of preorder and partial-order structures from specified generative rules and their closures. The present section supplies three constraints on that investigation. First, an observational or structural quotient may erase rule distinctions needed to establish directed accessibility. Second, the existence of a symmetry action guarantees an orbit equivalence relation but leaves joint-input compatibility and order properties to further analysis. Third, even a mathematically exact quotient does not by itself establish that its order is a physical, causal, or ontologically primitive relation.

The proposed sequence also suggests a strategy for model comparison. One may begin with a declared landscape, specify candidate equivalences by explicit criteria, test preservation of admissibility and the relevant operations, and retain counterexamples when a quotient fails. Only then should a proposed precedence relation be derived and examined for closure, comparability, cycles, or antisymmetry.

The approach makes the warrant for each inference inspectable and revisable. It allows equivalence to be a substantive part of generative theory rather than an incidental notation placed before an unexplained hierarchy.

As a matter of ontological interpretation, the relation between identity and order remains open. In some physical models, the equivalence classes may have a direct connection with invariant quantities or constrained state spaces. In other applications, they may express useful levels of description with unavoidable unresolved distinctions. The aim of the present formalism is to clarify the passage from chosen identifications to derived mathematical claims and thereby expose the additional evidence required for stronger physical conclusions. That clarification is the prerequisite for any disciplined inquiry into the ontological status of generative precedence.

4 Conditions for Derived Precedence

A claim of generative precedence requires an account of the transformations through which one configuration, collection of prerequisites, or admissible state can lead to another. The present section derives several order structures from explicitly declared generative rules and examines the assumptions that sustain those derivations. It begins with reachability for unary transformations, develops a closure semantics for joint generation, and then tests the compatibility of ordering with equivalence, conditioning regimes, and physical interpretation. The method is constructive and comparative: formal results establish what follows from specified relations, while counterexamples delimit conclusions that would require additional information. Throughout, precedence refers initially to an orientation within a mathematical representation; the status of that orientation as causal, temporal, or ontological is an independent question.

4.1 Domains and Orientation of the Precedence Relation

Section 2 introduced a landscape with conditioned admissible configurations $C_{q,\kappa}$ and multi-input rules $G^{(m)}_{q,\kappa}$. Section 3 distinguished several equivalence relations whose preservation properties depend upon the intended operations. A relation of precedence requires a further choice: its terms may be individual configurations, globally admissible system states, compatible families of resources, derivations, or appropriately defined equivalence classes. Each domain supports a different question, even when one notation is useful across several domains.

For the constructions below, fix the research question q and regime κ, and abbreviate $C_{q,\kappa}$ by C. A rule $((c_1, \ldots, c_m), d) \in G^{(m)}_{q,\kappa}$ specifies a directed formation possibility under the declared semantics. Its orientation may express a permitted inference, a feasible assembly, a physical transition, or a choice of explanatory decomposition. Thus the word *precedence* will mean an induced relation of accessibility or derivability until supplementary assumptions supply a more specific interpretation. A local rule must be distinguished from its compositional closure, and the closure must be distinguished from the relation obtained after identifying configurations.

Three interpretive readings can be distinguished without conflating them. *Derivational precedence* concerns whether a construction, proof, or derivation can be obtained from specified premises. *Operational precedence* concerns whether an admissible sequence of interventions or processes can be executed under declared resources and constraints. *Physical causal precedence* requires a theory-specific criterion connecting the orientation to causal structure and its empirical interpretation. A formally valid derivational order can be useful even when no corresponding operational experiment exists. Conversely, a physical model may have time-reversal or gauge symmetries whose correct treatment leaves the choice of a globally preferred orientation unsettled.

Order theory studies reflexive and transitive relations and the extra antisymmetry condition that characterizes partial orders [13]. Abstract reduction systems provide a complementary setting in which elementary steps, their closures, and issues of termination or confluence are defined before their interpretation in an application [5]. The proposal here uses these established mathematical resources to specify

a candidate generative-order construction. The distinctively generative question concerns the adequacy of the chosen rule family and the compatibility of its construction with heterogeneous prerequisites and changing regimes.

4.2 Unary Reachability and the Induced Preorder

Suppose first that the intended units are complete admissible states and that the application provides a composable binary transition relation $T_\kappa \subseteq C \times C$. The use of complete states is important: a transition records a permitted step of the whole represented system, rather than an isolated component's change while its remaining prerequisites are ignored. The binary relation can arise from a suitable $G^{(1)}_{q,\kappa}$, but a physical implementation may require the construction of an expanded state space containing conserved quantities, environmental states, or other context.

A finite directed path is a sequence $c_0, \ldots, c_n$ with $(c_{j-1}, c_j) \in T_\kappa$ for each $1 \leq j \leq n$. We include the empty path $n = 0$, representing reflexivity as a convention of reachability. The induced relation is defined in Equation 9.

$$c \preccurlyeq^T_\kappa d \quad \Longleftrightarrow \quad (c,d) \in T^*_\kappa := \bigcup_{n=0}^{\infty} T^n_\kappa, \qquad T^0_\kappa = \Delta_C. \tag{9}$$

Here T^n_κ denotes n-fold relational composition and Δ_C is the identity relation. The union expresses finite-path reachability; a system requiring infinite-limit constructions or continuous trajectories needs its own closure notion. In particular, a state accessible only as an asymptotic limit is generally distinct from one reachable by finitely many licensed steps.

Proposition 4.1 (Preorder from composable transitions)**.** *For any set C and binary transition relation T_κ, the finite-path relation $\preccurlyeq^T_\kappa$ defined by Equation 9 is reflexive and transitive. Define $c \simeq^T_\kappa d$ exactly when $c \preccurlyeq^T_\kappa d$ and $d \preccurlyeq^T_\kappa c$. The induced relation on $C/\simeq^T_\kappa$, given by $[c] \leq^T_\kappa [d]$ when $c \preccurlyeq^T_\kappa d$, is a well-defined partial order.*

Proof. Every $c \in C$ reaches itself by a path of length zero. Concatenation of a path from c to d with a path from d to e yields a path from c to e, establishing transitivity. Reflexivity and transitivity imply that mutual reachability $\simeq^T_\kappa$ is an equivalence relation. If $c \simeq^T_\kappa c'$ and $d \simeq^T_\kappa d'$, transitivity carries $c \preccurlyeq^T_\kappa d$ to $c' \preccurlyeq^T_\kappa d'$, proving independence of representatives. Reflexivity and transitivity descend to classes, and mutual comparability of two classes entails their equality by construction. Hence the quotient relation is antisymmetric. □

Proposition 4.1 is a general fact about abstract reachability. Its value here is diagnostic: it shows precisely which part of the proposed generative order follows mathematically once T_κ and compositional path semantics have been accepted. It does not prove that every element of C corresponds to a physical state, that the transitions are dynamically possible, or that the quotient classes represent ontological identity. In a finite transition graph, $\simeq^T_\kappa$ partitions vertices into strongly connected components. The graph of components has no directed cycles between distinct components; its reachability yields the same partial order. The resulting structure may be highly incomplete, with many incomparable pairs.

The orientation carries additional information. If $c \preccurlyeq^T_\kappa d$ denotes access from c to d, the reverse relation represents access in the opposite direction, a generally different property. Reversing every stipulated rule reverses the reachability order on the quotient, where the corresponding reversal is meaningful. No universal criterion in elementary order theory singles out which direction should be called generatively prior. That choice comes from the rule semantics and, in physical applications, the theoretical and empirical account of the chosen transformations.

4.3 Joint Generation and the Closure of Available Prerequisites

A unary path semantics is insufficient when an output requires several jointly available inputs. Consider rules $((a,b),d)$ and $((d,e),f)$. Possession of a alone need not permit d; a path diagram $a \to d$ would

suppress the role of b. Likewise, a record of $d \to f$ would omit the role of e. Such projections can turn a requirement of joint compatibility into a misleading succession between individual components. The multi-input formalism from Equation 2 preserves these prerequisites at the rule level.

One useful semantics treats a collection $S \subseteq C$ as the pool of configurations currently available as reusable premises. Fix a declared background pool $B \subseteq C$ that represents shared enabling resources or conditions explicitly admitted to the derivation. Nullary rules may be represented through this background only when their enabling conditions have been documented; external inputs do not silently enter at intermediate stages. Define the one-step expansion operator in Equation 10.

$$\Phi_{\kappa,B}(S) = S \cup B \cup \left\{ d \in C \;\middle|\; \begin{array}{l} \exists m \geq 1,\ (c_1, \ldots, c_m, d) \in G^{(m)}_{q,\kappa}, \\ c_1, \ldots, c_m \in S \cup B \end{array} \right\}. \tag{10}$$

The operator is monotone under inclusion: enlarging the available pool cannot invalidate the existence of a licensed derivation in this particular semantics. The monotonicity assumption is substantive. A process in which using a consumes a, or combining ingredients excludes another operation, may require explicit inventory states and resource accounting. Equation 10 is therefore appropriate for persistent knowledge, reusable capabilities, or an idealized availability semantics. It is an abstraction for consumptive systems unless those dynamics have already been encoded in the candidate configurations and rules.

For finite derivations, define $S_0 = S \cup B$, $S_{n+1} = \Phi_{\kappa,B}(S_n)$, and the resulting closure in Equation 11.

$$\mathrm{Cl}_{\kappa,B}(S) = \bigcup_{n \in \mathbb{N}_0} S_n, \qquad S_0 = S \cup B, \quad S_{n+1} = \Phi_{\kappa,B}(S_n). \tag{11}$$

Every rule has finite arity by the standing definition. Consequently, if all prerequisites of a rule are contained in the union on the right of Equation 11, they appear together at some finite stage, and the output appears at the next stage. The closure thus contains the results of every finitely justified, iterated rule application. The same observation establishes its stability under $\Phi_{\kappa,B}$, even when the domain C is infinite.

Proposition 4.2 (Closure properties of reusable joint generation)**.** *Under the finite-arity, reusable-premise semantics of Equations 10 and 11,* $\mathrm{Cl}_{\kappa,B}$ *is extensive, monotone, and idempotent:* $S \subseteq \mathrm{Cl}_{\kappa,B}(S)$*;* $S \subseteq T$ *implies* $\mathrm{Cl}_{\kappa,B}(S) \subseteq \mathrm{Cl}_{\kappa,B}(T)$*; and* $\mathrm{Cl}_{\kappa,B}(\mathrm{Cl}_{\kappa,B}(S)) = \mathrm{Cl}_{\kappa,B}(S)$.

Proof. Extensivity follows from $S \subseteq S_0$. Monotonicity follows by induction on the stages, because the existence of all premises in a pool is preserved under enlargement. Every finitely derived configuration and all background elements belong to $\mathrm{Cl}_{\kappa,B}(S)$. Finite arity then ensures that any rule whose premises belong to that closure also has its output in the closure, making it stable under $\Phi_{\kappa,B}$. A second iteration therefore adds nothing. □

Proposition 4.2 does not require that every derivation have a uniquely determined output. It allows multiple distinct outputs from the same premises and leaves alternative construction paths unresolved. In particular, confluence and determinism are separate questions. The closure identifies what is derivable under the rule family; it gives no unique genealogy, probability of realization, or account of how a physical system chooses among competing possibilities. This distinction is central to the later treatment of generative potential in Section 4 and the planned Section 6.

4.4 Precedence among Resource Pools and Generative Capacities

The closure semantics provides two relations that can look similar while expressing different comparisons. The first asks whether a target pool T can be assembled using a given initial pool S and the fixed background B. The second asks whether every configuration obtainable from S is also obtainable from T. These correspond, respectively, to an oriented relation of finite derivability and an inclusion comparison of generative capacities. Their orientations are opposite in an important case.

For $S, T \subseteq C$, define the pool-derivability relation in Equation 12.

$$S \preccurlyeq^{\mathrm{gen}}_{\kappa,B} T \quad \Longleftrightarrow \quad T \subseteq \mathrm{Cl}_{\kappa,B}(S). \tag{12}$$

Its reflexivity follows from extensivity. If S derives T and T derives U, monotonicity and idempotence imply that U lies in $\mathrm{Cl}_{\kappa,B}(S)$; thus Equation 12 is transitive. Mutual pool derivability is equivalent to equality of the resulting closures. Accordingly, the quotient of pools by equality of their closures again supports a partial order. Unlike the preceding binary transition construction, this derivation faithfully records joint inputs under the assumed reusable-premise semantics.

A distinct measure-free capacity order is given in Equation 13.

$$S \sqsubseteq^{\mathrm{cap}}_{\kappa,B} T \quad \Longleftrightarrow \quad \mathrm{Cl}_{\kappa,B}(S) \subseteq \mathrm{Cl}_{\kappa,B}(T). \tag{13}$$

This is a preorder on pools and becomes a partial order on distinct closed sets under inclusion. Whenever $S \preccurlyeq^{\mathrm{gen}}_{\kappa,B} T$, one obtains $\mathrm{Cl}_{\kappa,B}(T) \subseteq \mathrm{Cl}_{\kappa,B}(S)$. Consequently, $T \sqsubseteq^{\mathrm{cap}}_{\kappa,B} S$. A pool from which another pool can be derived has at least as broad a set of derivable configurations as that particular derived pool under the fixed background. This reversal is a mathematical consequence of the definition, and it prevents a misleading inference that later derivation necessarily means greater generative capacity.

The capacities under discussion are sets of possibilities, not scalar magnitudes of value, complexity, welfare, or physical energy. Inclusion need not compare two closures: each may contain possibilities unavailable from the other. A numerical summary, such as the cardinality of a finite closure, can obscure these differences. It can also depend strongly on the granularity of C, the treatment of equivalent descriptions, and the declared admissibility regime. A genuine claim about distinct *levels* of generative potential therefore requires an additional stratification criterion, whose assumptions and empirical meaning will be developed in Section 6.

The role of B warrants caution. If a configuration can be generated from the background alone, that configuration appears in $\mathrm{Cl}_{\kappa,B}(S)$ for every initial pool S. It would be unreasonable to infer that every particular S independently produced the configuration or was materially necessary for its availability. To evaluate dependence on a specific participant, one must compare counterfactual backgrounds and rule sets, or trace derivations that actually use the participant's contribution. Similarly, the ability of a resource pool to derive an outcome expresses sufficiency within a model; identifying necessary or causally efficacious inputs calls for additional intervention or provenance criteria.

4.5 Quotients, Cycles, and the Risk of Fabricated Paths

Several quotients are available, and their differences matter. The equivalence relation $\simeq^{T}_{\kappa}$ in Proposition 4.1 is constructed from the very reachability relation that it subsequently orders. By contrast, the descriptive, structural, observational, or symmetry-based equivalences of Section 3 are specified independently. Quotienting a transition system by one of those relations can create paths that were absent in the original model if different representatives of one class are incorrectly spliced together.

Consider the finite set $C = \{u, a_1, a_2, v\}$, with only two transitions, $u \to a_1$ and $a_2 \to v$. Suppose an observational criterion identifies a_1 and a_2, producing the class $[a]$, while leaving u and v distinct. The existential quotient graph has $[u] \to [a]$ and $[a] \to [v]$. It therefore exhibits a two-step path $[u] \to [v]$. In the original graph, there is no transition out of a_1 toward v and no path from u to v. The quotient has fabricated a composite path by silently switching representatives at the intermediate node. This is a failure of faithful path reconstruction, even though each projected edge corresponds to some original edge. More strongly, the projected image of concrete reachability need not itself be transitive; the precise counterexample is established in Proposition 5.2.

An exact relational saturation condition, such as Proposition 3.1, suffices to avoid that particular ambiguity for a binary rule: any representatives of the relevant equivalence classes support the projected edge. Other sufficient structures include explicit lifting properties for paths, maintained compatibility witnesses, or a quotient representation of whole compositional processes. A weaker existential quotient can remain analytically useful if its claims are labelled as possible at the class level and the risks of unliftable compositions are tracked. There is no need to demand universal saturation of an empirical approximation when a controlled abstract interpretation is its intended role; the inferential strength must simply match the preservation guarantee.

In the multi-input setting, the splicing problem acquires a further dimension. Independently quotienting inputs can conceal whether they form a compatible tuple at all. The simultaneous symmetry of a full tuple may preserve the original generative relation while permitting no arbitrary mixing of separately chosen class representatives. Section 3.5 gave an explicit example of this distinction. An order derived from an existentially quotiented rule family may therefore reflect a broader set of class-level possibilities than any single configuration permits. The remedy is to retain sufficient coupling, witness, provenance, or condition information before interpreting quotient reachability operationally.

Cyclic reachability raises a different issue. If $a \to b \to c \to a$, all three are mutually reachable and lie in one reachability class. Quotienting them yields a partial order on classes, but does not assert that the original states are physically identical, simultaneously available, or observationally indistinguishable. A reversible physical process may exhibit rich distinctions within a strongly connected component. The quotient suppresses those distinctions for the particular purpose of ordering mutually accessible regions. Its adequacy must therefore be judged against the research question; a quotient suitable for a reachability diagram may be unsuitable for thermodynamic observables, intervention costs, or detailed dynamics.

4.6 Conditional Orders and the Persistence of Comparisons

Every order constructed above is indexed by κ, by the rule family, and, where relevant, by the background pool and semantics of composition. A different regime can admit new rules, remove former rules, alter admissible configurations, or change compatibility of joint inputs. Thus two order relations may differ even if they are described on sets with similar names. A scientific comparison between them requires explicit correspondence of the underlying domains and of what a transition is taken to mean.

When two regimes share a common admissible domain C and possess unary transition relations satisfying $T_\kappa \subseteq T_\lambda$, their path closures satisfy $T_\kappa^* \subseteq T_\lambda^*$. This elementary monotonicity assertion records inclusion of opportunities for reachability under expanded rules. It does not entail monotone growth of the number of distinct classes in the quotient partial order. Adding a reverse edge can merge two previously distinct strongly connected components; adding forward edges can create new comparisons while leaving other components separate. Changes in the available rule set can therefore reorganize the quotient as well as expand reachability.

For joint generation, an analogous comparison holds under fixed C and B if every rule available in κ remains available in λ: then $\mathrm{Cl}_{\kappa,B}(S) \subseteq \mathrm{Cl}_{\lambda,B}(S)$ for each pool S. The statement requires actual inclusion of properly typed rule relations, with the same input and output interpretations. When the admissible sets differ, a cross-regime translation or embedding must instead be supplied. A newly available configuration can represent an expanded physical opportunity, a more detailed representation, or a change in measurement conventions; the formal inclusion by itself does not discriminate among them.

A further source of ambiguity is the mixing of paths across regimes. Suppose the step $a \to b$ is permitted only in regime κ and $b \to c$ only in regime λ. Their union contains a path from a to c, but a physical or institutional system may have no feasible process that executes both steps in sequence. A cross-regime reachability relation therefore requires licensed transitions between regimes, or a common enlarged state (c, κ) on which regime changes are explicitly represented. Such a state space can incorporate switching constraints, costs, or boundary conditions. The framework has no need to assume a single privileged present or a universal chronological parameter: the available relations among regime-labelled configurations are specified by the application.

This distinction matters especially for historically changing systems. A rule may be recorded as available in one institutional setting and unavailable in another; technological capabilities may alter what is composable; effective physical descriptions may reorganize under changes of scale. A single diagram assembled from observations across different conditions can consequently resemble an unconditional hierarchy while lacking a licensed common composition law. Sections 7 and 8 will study transformations of the rule family itself and critical reorganizations of derived order; the present section gives the fixed-regime baseline against which those changes can be identified.

4.7 Counterexamples and Formal Adequacy Tests

The construction supports several compact tests of candidate claims about generative hierarchy. These examples are formal probes, with their interpretation deliberately limited to the assumptions declared in each case.

Reversibility and mutual reachability. In a model with $a \to b$ and $b \to a$, reflexive-transitive reachability relates both configurations in both directions. The quotient places them in one class. If a and b carry distinct physical observables, this quotient remains a statement of mutual accessibility alone. It supplies neither a physical identification of the states nor a strict precedence relation between them.

Complementary prerequisite pools. Let the only generative rule be $((a, b), d)$ and let $B = \varnothing$. Then $d \in \mathrm{Cl}(\{a, b\})$, while $d \notin \mathrm{Cl}(\{a\})$ and $d \notin \mathrm{Cl}(\{b\})$. Any binary account claiming that a alone or b alone generates d is a change of semantics. The example shows why hyperedge structure and explicit input compatibility must accompany an interpretation of precedence.

Independent pathways and incomparability. Suppose the rules admit $a \to x$ and $b \to y$ with no cross-paths or common resources. The model yields no comparison between a and b by reachability. Declaring one a higher generative level would require a criterion beyond the given transitions. Incomparability is an informative outcome of the formal construction rather than a failure to force a total order.

Rule augmentation and class merging. Begin with $a \to b$ and no reverse transition. The two states occupy distinct quotient classes, ordered by reachability. Adding $b \to a$ makes them mutually reachable, merging their classes. A new feasible transformation has increased directed accessibility while decreasing the number of distinguishable reachability classes. This limits any simple association between expansion of possibilities and proliferation of hierarchical levels.

Infinite approximation and finite reachability. Let $c_n \to c_{n+1}$ for nonnegative integers n, and suppose an additional candidate c_∞ is declared as a limit under a topology. Finite-path closure from c_0 contains each c_n, yet the limit belongs only if the model separately licenses a rule or another closure operation. An inference from topological convergence to generative reachability therefore requires a bridge principle.

Table 3 summarizes the checks that a domain-specific application should pass before a derived precedence structure is assigned explanatory weight.

These tests also discipline the comparison with causal-set research. Causal-set approaches study locally finite partially ordered sets with causal interpretation under their own physical postulates [9]. The present construction proceeds from a selected relation of generative accessibility and obtains a partial order by quotienting mutual reachability, or from inclusion comparisons of rule closures. Its mathematical resemblance to an order-theoretic aspect of causal-set models does not establish local finiteness, spacetime reconstruction, Lorentzian causal structure, or physical equivalence. Those properties require additional assumptions and independent evidence. Causal-set theory supplies an instructive comparison precisely because its physical commitments are explicit and therefore distinguishable from a general mathematical reachability construction.

4.8 Interpretive Conditions for Generative Precedence

A disciplined claim of derived generative precedence can now be given a layered interpretation. At the first layer, the analyst specifies admissible configurations and generative transformations. At the second, a composition semantics defines which finite sequences or multi-input derivations are licensed. At the third, a preorder is derived and any identification required for antisymmetry is made explicit. At the fourth, the resulting order is compared with observations, interventions, or a domain-specific

Table 3: Formal and interpretive tests for candidate generative precedence relations.

Target of inference	Necessary modelling statement	Characteristic failure or open question
Composable state transition	Typed relation on complete states and licensed concatenation	Suppressed conservation, boundary, or inventory conditions
Finite reachability	Declared finite-path closure and direction	An asymptotic limit presented as a finite result
Joint-prerequisite generation	Explicit compatible input tuples and resources	A hyperedge projected into spurious unary arrows
Pool-derived precedence	Fixed background and reusable-premise closure	Material consumption or spontaneous background contributions
Equivalence-class order	Mutual reachability or a separate preservation/lifting criterion	Fabricated paths through unrelated representatives
Regime comparison	Common domain or verified cross-regime translation	Compositions that require incompatible conditions
Physical causal interpretation	Physical theory and evidence supporting the proposed orientation	Accessibility, correlation, and causation conflated

physical theory. Each layer provides a testable or inspectable obligation; acceptance at one layer leaves the stronger claims at subsequent layers open.

The formal results are modest but precise. Proposition 4.1 proves that composable unary reachability induces a preorder and that its mutual-reachability quotient is partially ordered. Proposition 4.2 proves that reusable, finite-arity joint generation gives an extensive, monotone, idempotent closure. Equations 12 and 13 show why the ability to derive a target and the breadth of a generative repertoire need separate directions of comparison. None of these results produces a unique order for an unspecified physical system. Their premises include the domain, allowed rule composition, resource semantics, and selected background conditions.

The principal interpretive difficulty concerns the word *prior*. A partial order on mutual-reachability classes captures directed accessibility without itself defining a temporal coordinate, an irreversible physical process, or a relation of causal necessity. With complete-state unary rules, singleton inputs, and an unrestricted outcome domain, reachability and the reverse inclusion of potential profiles are mathematically equivalent (Proposition 6.3). This special-case identity helps locate the additional content, if any, contributed by joint prerequisites, restricted observables, and resource-conditioned initial pools. A closure order on resource pools expresses a conditional inclusion or derivation relation without establishing the intrinsic superiority of one pool. A hierarchy of explanatory levels may be defined through a coarse representation while changing under refinement. These distinctions invite precise comparisons among models whose relevant properties survive changes of encoding, observational interface, and conditioning regime.

In some applications, evidence may support a physical causal interpretation of particular order relations. In others, the warranted outcome will be an operational, inferential, or explanatory structure whose physical realization is uncertain. Retaining this range of interpretations is consistent with the paper's ontological suspension. A derived order may be an informative feature of a relational system's representation without being declared an ultimate constituent of the world.

Section 5 will therefore investigate the reconstruction problem: what an observer can infer about precedence when measurement and coarse-graining reveal only projections of the generating landscape. The false-path example in Subsection 4.5 supplies an immediate warning that a clear order in observed classes can conceal noncomposable underlying transitions. The research task is to determine which claims remain invariant under faithful representations, which require stronger intervention evidence, and which remain underdetermined. Only such discrimination can support a credible transition from derived mathematical precedence toward an ontological account of generative order.

5 Representational Mediation and Order Reconstruction

A derived order is formulated within a representation, whereas evidence for that order is obtained through particular measurements, observation schedules, interventions, and changes of descriptive scale. The present section examines the inferential passage from observed records to claims about generative precedence. Its purpose is to distinguish exact preservation of a proposed order, partial reconstruction under limited access, and observational patterns that admit several generative explanations. The analysis first identifies what an observational interface makes available, establishes results about the projection of transition systems, and then considers stochastic aggregation, uncertainty over candidate models, inverse inference, and discriminating research designs. These constructions provide a methodological bridge from the algebraic order relations in Section 4 to the more conditional account of generative potential developed in Section 6. The argument remains model-relative: the reliability of a reconstructed order depends on the adequacy of its configurations, rule semantics, and evidence-generating procedures.

5.1 Observation Interfaces as Conditions of Reconstruction

The landscape introduced in Equation 3 distinguishes candidate configurations C_q, conditioning regimes K_q, and observational interfaces $\mathcal{O}_q$. Fix a question q and regime κ, and abbreviate $C_{q,\kappa}$ to C. A research design normally specifies an observation programme E: a collection of instruments, admissible interventions, recording intervals or sampling windows, and rules for matching records to represented configurations. The programme determines which information can count as evidence concerning a transition relation $T \subseteq C \times C$ or, in a joint-generation model, the family $G^{(m)}_{q,\kappa}$.

An exact deterministic observational map $\pi : C \to Y$ will serve as an initial simplification. The elements of Y are observation classes, and π need only be surjective onto its image. Two configurations with the same observation may differ in internal composition, latent resources, dynamical accessibility, or available joint prerequisites. Accordingly, the equality $\pi(c) = \pi(c')$ means indistinguishability under the declared interface. Its promotion to a structural or physical equivalence requires the separate criteria discussed in Section 3. In empirical work, the interface may be a partial function, a collection of condition-indexed maps, or a probability kernel $P(y \mid c, E)$. The deterministic setting is used first because it isolates the logical consequences of representational aggregation.

Observation programmes also vary by what they record. A measurement that observes only initial and final configurations supplies less information than a fully resolved transition record. An endpoint record may be compatible with a direct one-step transformation, a sequence with unobserved intermediates, or a transformation that depends on unrecorded external conditions. Conversely, recording two successive output categories need not establish that a single concrete system moved from a representative of the first category to a representative of the second. Such claims require a justified procedure for matching observations to trajectories and identifying the relevant unit of change. This requirement is particularly important where different physical components enter and leave the measurement domain, where a dataset is assembled from multiple subjects or specimens, or where the description changes during observation.

Representational reconstruction is consequently an inverse problem. A model proposes configurations and rules, supplies a forward account of obtainable observations, and is assessed against records. Inverse-problem theory distinguishes available measurements, prior information, and the relationships connecting model parameters to data [47]. The present proposal adds a specific target of inference: the orientation and comparability of a derived precedence relation. Its target is narrower than reconstructing an entire physical ontology, while its evidential requirements can still be substantial.

5.2 Distinct Generative Accounts with Matching Observations

Observational agreement is compatible with substantial differences in generative structure. Consider a set of observed categories $Y = \{A, B, D\}$ and a latent configuration space $C = \{a, b_1, b_2, d\}$ with $\pi(a) = A$, $\pi(b_1) = \pi(b_2) = B$, and $\pi(d) = D$. In one model, the transitions $a \to b_1$ and $b_1 \to d$ are

admitted. In a second model, $a \to b_1$ and $b_2 \to d$ are admitted, while a transition connecting b_1 to b_2 is absent. If an observation programme records only the existence of transitions between output categories, the models can produce the same two arrows $A \to B$ and $B \to D$. Their concrete reachability relations differ: the first admits a path from a to d, and the second does not.

This example indicates two forms of ambiguity. The first concerns *state identity*: repeated measurements classified as B may concern different underlying states. The second concerns *rule composition*: two transitions that appear consecutive under a classification may fail to be executable consecutively within a single admissible system. The distinction survives even when the detector assigns the observed categories without error. Better measurement precision within an unchanged classification scheme does not automatically recover the state distinctions suppressed by the scheme itself.

A separate ambiguity arises from joint generation. Suppose an output d requires the compatible pair (a, b), as represented by a rule in $G^{(2)}_{q,\kappa}$. A record displaying a followed by d under an unobserved background availability of b may resemble a unary rule $a \to d$. An alternative model that represents b as an explicit prerequisite will make different predictions when b is unavailable or its compatibility with a changes. A meaningful reconstruction therefore includes the scope of controlled backgrounds and the resources that may be silently treated as constant. This is an inferential obligation distinct from the mathematical transitivity proved for complete-state reachability in Proposition 4.1.

There are further sources of observational agreement: differing transition rates can generate similar aggregate frequencies over selected sampling windows; different rules can coincide on a restricted set of initial configurations; and different condition-indexed landscapes can agree on one regime while diverging under another. These possibilities support a graded vocabulary of reconstruction. An order may be *identified* within a declared model family and design, *compatible* with the available observations, or *underdetermined* among several admissible candidates. An identified order under one design may remain underdetermined when the research question expands to unmeasured conditions or alternative representational domains.

5.3 Projected Transitions and Apparent Paths

The ambiguity concerning state identity has a concise mathematical expression. Let $T \subseteq C \times C$ be a binary transition relation on complete configurations, and let $\pi : C \to Y$ be a surjective observation map. Its *projected one-step relation* $\overline{T} \subseteq Y \times Y$ contains the pair (y, y') when there exist $c, d \in C$ with $\pi(c) = y$, $\pi(d) = y'$, and cTd. The projected relation records that some representatives admit a transition. It does not state that every representative of y can execute a transition into y'.

There are two candidate representations of observable reachability. One first computes concrete finite-path reachability and then projects the endpoints. The other first projects the elementary transition relation and then computes paths among the observed classes. Their relationship is described by Equation 14.

$$(\pi \times \pi)(T^*) \;\subseteq\; \big((\pi \times \pi)(T)\big)^*, \qquad T^* := \bigcup_{n \geq 0} T^n. \tag{14}$$

Here $(\pi \times \pi)(R)$ is the image of a binary relation R, and the star denotes reflexive–transitive closure on the appropriate domain. Equation 14 follows because projecting each step of a concrete path produces a path in the projected relation. The inclusion can be strict, as the previous example demonstrates: A reaches D in $\overline{T}^*$ even though a cannot reach d through the admitted concrete transitions. Projecting a relation and closing it under finite-path composition are therefore operations that need not commute.

Proposition 5.1 (Projected-path inclusion and compositional ambiguity)**.** *Let $\pi : C \to Y$ be surjective and $T \subseteq C \times C$. Then Equation 14 holds. There exist finite C, Y, T, and π for which the inclusion is strict.*

Proof. If cT^*d, there is a finite T-path from c to d. Applying π to each vertex produces a finite path in $\overline{T}$, so $(\pi(c), \pi(d)) \in \overline{T}^*$. For strictness, take $C = \{a, b_1, b_2, d\}$ and $T = \{(a, b_1), (b_2, d)\}$ with $\pi(b_1) = \pi(b_2) = B$ and distinct images A, D of a, d. Then $(A, D) \in \overline{T}^*$ through B, whereas $(A, D) \notin$

$(\pi \times \pi)(T^*)$: the only representative of A is a, and the only representative of D is d, with no concrete connecting path. □

The same example exposes a further order-theoretic qualification. Even though T^* is reflexive and transitive, its image $(\pi \times \pi)(T^*)$ need not be transitive. Transitivity can be recovered by taking the closure of the projected relation, yet that operation may introduce precisely the unliftable paths that a reconstruction was meant to avoid.

Proposition 5.2 (Failure of preorder descent under observational projection)**.** *There exist a finite configuration set C, a surjection $\pi : C \to Y$, and a transition relation T such that T^* is a preorder on C, while $(\pi \times \pi)(T^*)$ fails transitivity on Y. Consequently, the image of concrete reachability under an arbitrary observation map need not be a preorder and cannot in general be used directly to form a quotient partial order.*

Proof. Use the configuration set and transition relation in Proposition 5.1. The image of T^* contains (A, B) and (B, D), because aTb_1 and b_2Td, respectively. It excludes (A, D), since no T-path connects a to d. Thus the image relation is not transitive, whereas T^* is reflexive and transitive by its construction. □

Proposition 5.2 distinguishes three mathematically different operations: the concrete preorder T^*; the projected concrete-reachability relation $(\pi \times \pi)(T^*)$, which records existential witnesses for endpoint pairs; and the abstract path closure $((\pi \times \pi)(T))^*$, which is always a preorder but may concatenate transitions with incompatible representatives. A justified class-level ordering requires a preservation or lifting argument, or an explicitly weaker semantics for the abstract relation. This distinction remains relevant when a projected relation happens to be transitive in a particular dataset: transitivity alone would not establish faithful correspondence of every abstract path to a concrete one.

Proposition 5.1 is an elementary fact about relational images; its diagnostic significance is the potential overstatement introduced by an observational quotient. The apparent A-to-D path is *abstractly composable* but does not correspond to a concrete path in the supplied model. This is a logical failure of reconstruction, independent of noise or statistical significance. For an application requiring material resources, it may correspond to splicing observations from incompatible inventories, interventions, or local environments. The remedy is to preserve sufficient internal state information or to establish a condition under which all projected paths can be lifted to admissible concrete paths.

5.4 Representative-Stable Successors and Path Lifting

A sufficient condition for accurate reconstruction can be stated in terms of the possible successor classes of each concrete configuration. Write $\mathrm{Succ}_\pi(c) = \{\pi(d) \mid cTd\}$. The proposed *representative-stability condition* requires configurations with the same observed image to have the same sets of observable successor classes. Equation 15 states the condition.

$$\pi(c) = \pi(c') \quad \Longrightarrow \quad \mathrm{Succ}_\pi(c) = \mathrm{Succ}_\pi(c') \qquad (c, c' \in C). \tag{15}$$

The condition is stronger than simple equality of present observations. It asserts preservation of a relevant part of the transition potential after observational aggregation. It can be investigated for a fixed regime and choice of complete states; extending it across regimes, interfaces, or scales calls for additional checks.

Proposition 5.3 (Representative stability and uniform path lifting)**.** *For a surjection $\pi : C \to Y$ and a binary relation $T \subseteq C \times C$, the representative-stability condition of Equation 15 holds if and only if every finite $\overline{T}$-path beginning at y_0 can be lifted to a T-path from* each *prescribed representative $c_0 \in \pi^{-1}(y_0)$. Under either condition, equality holds in Equation 14.*

Proof. In the forward direction, fix c_0 over y_0. An abstract edge $y_0\overline{T}y_1$ has a witness from some representative $\widetilde{c}_0$ of y_0. Representative stability gives a successor c_1 of c_0 over y_1; induction constructs a concrete lift of any finite abstract path. Conversely, suppose every abstract path admits a lift from each representative. If $\pi(c) = \pi(c') = y$ and $y' \in \mathrm{Succ}_\pi(c)$, the one-edge abstract path $y\overline{T}y'$ can be lifted from c',

so $y' \in \mathrm{Succ}_\pi(c')$. Exchanging c, c' gives the reverse inclusion of successor-class sets. The equality of projected concrete reachability and abstract reachability follows by lifting from a representative of each initial observation class and applying Proposition 5.1. □

For the unlabelled transition systems considered here, representative stability is precisely the condition that the kernel equivalence of π match successor behavior in both directions. In the terminology of process semantics, this corresponds to a *strong bisimulation equivalence* on an unlabelled transition graph, with the labels and observation classes suppressed. Bisimulation is an established concept in concurrency theory [35]; partition-refinement algorithms provide standard computational baselines for checking related behavioral partitions [37]. Proposition 5.3 is a direct, elementary reformulation for G1's observable path-lifting question, rather than a new theory of bisimulation. The equivalence between uniform lifting and representative stability is stronger than mere equality of the two existential reachability relations.

To see the distinction, let $C = \{a, b, z\}$, $\pi(a) = \pi(b) = A$, $\pi(z) = Z$, and let the only nontrivial transition be aTz. The projected concrete reachability and the abstract path closure both consist of the identity pairs and (A, Z). Yet the path $A\overline{T}Z$ has no lift from b, so representative stability fails. Agreement of existential endpoint relations therefore does not certify interchangeable starting representatives, even in a graph with only two observable classes. Under the uniform-lifting condition, all three constructions discussed after Proposition 5.2 coincide as relations on observation classes; their common relation is a preorder.

The path-lifting property supplies a disciplined bridge between observed and concrete reachability. It does not automatically preserve the potential profiles of Section 6: an observation map may merge configurations with different resource pools or suppress outcome distinctions in Q. Agreement of abstract paths and existential concrete reachability thus leaves a separate profile-preservation question whenever opportunity sets, rather than endpoint accessibility alone, are the object of inference. It is a sufficient condition for the exact equality of the *existentially projected* reachability relations. It does not imply that the observation map is injective, that microscopic details are dynamically irrelevant for every research purpose, or that the reconstructed relation identifies physical causation. Moreover, representative stability concerns one-step observable successors and is stronger than the equality of reachability images alone. It may fail even when the particular pair of configurations under study has an accurately reconstructed path.

The requirement also differs from the exact relation-saturation criterion for quotients developed in Section 3. Saturation states that the membership of each concrete input–output tuple is uniform over its quotient fibre. Representative-stable successors allow differing concrete edges among the representatives while requiring a matching set of *output classes*. A quotient can consequently be adequate for observable reachability even if the original transition relation does not descend to an exact concrete-membership relation on equivalence classes. The appropriate criterion depends upon whether the research question concerns literal rule preservation, reachability of observation classes, resource-sensitive derivations, or physical identity.

5.5 Stochastic Aggregation and Scale-Dependent Dynamics

A deterministic projected transition graph loses information about the frequency, probability, and conditions of transitions. When the represented system is stochastic, coarse-graining may also affect whether the aggregated record has the same dynamical form as the fine-scale model. Consider a finite time-homogeneous Markov chain with state space C and transition matrix P, together with a surjection $\pi : C \to Y$. A standard sufficient condition for the aggregated process to be Markov with a well-defined transition matrix independent of its initial distribution is equality of the total transition probabilities from states in one observation class to every destination class. This strong lumpability condition is expressed in Equation 16.

$$\sum_{z \in \pi^{-1}(y')} P_{xz} = \sum_{z \in \pi^{-1}(y')} P_{x'z}, \quad \text{whenever } \pi(x) = \pi(x'), \quad y' \in Y. \tag{16}$$

Under Equation 16, the common sum defines the aggregated transition probability from $\pi(x)$ to y'. The condition belongs to established Markov-chain aggregation theory [31]. It is included here as an instructive comparison with the path-lifting criterion, not as a general dynamical axiom for generative landscapes. The two criteria address different objects: the first is equivalent to uniform lifting of finite paths in a binary transition graph; the second preserves a Markov transition description under probabilistic aggregation assumptions. For a finite chain with support relation xTy whenever $P_{xy} > 0$, strong lumpability implies representative stability of support successor classes, while the converse generally fails. Thus a quotient can be exact for possible paths while carrying inadequate information for quantitative dynamics.

For an empirical coarse-graining, apparent changes in order may arise from finite sampling, a choice of time resolution, conditioning on a selected subpopulation, or real changes of the generative regime. A coarse observable that appears to move from A to D may suppress intermediate states because a detector samples them too slowly. Another observable may merge several microstates that have different outgoing transitions, producing a process with history dependence at the coarse level. Whether that history dependence should be represented by a memory kernel, additional hidden states, or a refined observation scheme is a model-selection question. The coarse process may still admit an effective description for its intended predictive task, but its adequacy for claims about generative precedence should be assessed separately.

A useful distinction is between *observational scale* and *generative scale*. The former describes a measurement programme's ability to resolve differences; the latter concerns the configuration and rule structure relevant to formation possibilities. A change in observational resolution may create or remove visible comparabilities without changing the modeled underlying transitions. A physical renormalization analysis, by contrast, requires an independently justified effective dynamics and scale transformation, as in the established renormalization framework [49]. Accordingly, an observed hierarchy across data resolutions does not alone warrant interpretation as a renormalization flow or a physical stratification of generative potential.

5.6 Partial Identification and Discriminating Interventions

An empirical reconstruction usually begins with a model family rather than a single known transition system. Let $\mathcal{T}$ be a declared family of candidate transition relations on a common configuration domain C and under a fixed conditioning regime. Let $\mathcal{M}_E(T)$ denote the records predicted by candidate T through a specified observation programme E, including any measurement-error model and allowed experimental interventions. For observed data D and a declared adequacy relation $\sim_E$, the data-compatible family is defined in Equation 17.

$$\mathcal{T}_E(D) = \{T \in \mathcal{T} \mid \mathcal{M}_E(T) \sim_E D\}. \tag{17}$$

The meaning of compatibility must be determined by the research design. It may denote exact matching in a logical example, a confidence or credible set under a statistical model, or a bounded predictive discrepancy under calibration constraints. The family is not assumed to contain the true physical dynamics: a misspecified configuration space or excluded mechanism can make apparently narrow inference misleading. If $\mathcal{T}_E(D)$ is empty, the proposed model family and its observation assumptions are jointly inconsistent with the chosen adequacy criterion.

Where $\mathcal{T}_E(D)$ is nonempty, one can distinguish a relation that holds for at least one compatible candidate from a relation shared by all compatible candidates. Under the common-state-space restriction, the latter is given in Equation 18.

$$R_E^{\text{shared}}(D) = \bigcap_{T \in \mathcal{T}_E(D)} T^*. \tag{18}$$

Each T^* is a preorder, so the intersection in Equation 18 is also a preorder on C. The relation contains precedence claims robust *within the specified candidate family and evidence criterion*. It does not automatically descend to an observation quotient; Proposition 5.1 explains why computing reachability after aggregation can introduce unsupported pairs. Nor does a shared path assertion establish the cause of

the transitions. The use of a common C in Equation 18 matters: comparing candidates with different state spaces requires an explicitly justified correspondence among configurations, which can itself be uncertain.

The inverse problem becomes more informative when proposed models generate distinct results under new measurements or interventions. An intervention that selectively removes an enabling resource may discriminate between a unary account and a joint-prerequisite account. Recording a previously hidden state variable may differentiate two pathways that appear identical under a coarse classification. Varying conditioning regimes can reveal whether a rule is stable or tied to a particular enabling environment. The inference also requires the distinction between a manipulated condition and a variable merely associated with the outcome. Established causal-inference research analyzes the assumptions linking observational associations, interventions, and causal structure [38, 45]. The generative-order programme uses this literature as methodological guidance, while allowing applications in which formal derivability or physical accessibility, rather than causal effect estimation, is the target.

The degree of identification is conditional. A model may be identified relative to a restricted measurement schedule, initial-state distribution, and background regime, yet remain distinguishable under another schedule or intervention. Such conditionality is scientifically useful: it indicates which further observations could revise an ordering hypothesis. It also prevents the intersection of the presently entertained models from being mistaken for evidence that the research programme has exhausted all plausible generative explanations.

5.7 Model Uncertainty and the Composition of Reachability

An observationally compatible family of transition models can support several distinct meanings of a "possible" path. The shared-precedence relation in Equation 18 retains endpoint reachability warranted by every candidate. A more permissive assertion states that at least one candidate admits the path. A third construction combines all candidate edges and computes reachability in the resulting graph. The latter operation allows an edge supplied by one model to be followed by an edge supplied only by a different model. Where the candidates represent competing global hypotheses, this composition changes the inferential question: it permits switching the governing hypothesis between steps without a modeled regime transition.

Fix a common set C and a nonempty family $\mathscr{T}$ of binary transition relations on C. Denote the intersection of the candidate edge sets by $T_\forall$ and their union by $T_\exists$. Define $R_{\text{possible}} := \bigcup_{T\in\mathscr{T}} T^*$ and $R_{\text{shared}} := \bigcap_{T\in\mathscr{T}} T^*$. The relations satisfy the bounds stated in Proposition 5.4.

Proposition 5.4 (Reachability bounds for a fixed uncertain model family)**.** *For any nonempty family $\mathscr{T}$ of relations on the same set C, the following inclusions hold.*

$$(T_\forall)^* \;\subseteq\; R_{\text{shared}} \;\subseteq\; R_{\text{possible}} \;\subseteq\; (T_\exists)^*. \tag{19}$$

The shared relation is a preorder; the possible relation is reflexive and need not be transitive. The first and last inclusions can be strict.

Proof. For each $T \in \mathscr{T}$, inclusion $T_\forall \subseteq T \subseteq T_\exists$ and monotonicity of finite-path closure give $(T_\forall)^* \subseteq T^* \subseteq (T_\exists)^*$. Taking the appropriate intersection and union establishes Equation 19; nonemptiness of $\mathscr{T}$ supplies the middle inclusion. An intersection of reflexive, transitive relations is reflexive and transitive. A union of reflexive relations is reflexive, while transitivity of a union is unconstrained. Strictness is exhibited by the examples immediately below. □

For strictness on the left, take two candidate graphs on $\{a, b, c, d\}$. The first contains $a \to b \to c$; the second contains $a \to d \to c$. Each candidate admits $a \to^* c$, yet the candidates share no nonidentity edge. Thus $aR_{\text{shared}}c$ while $(a, c) \notin (T_\forall)^*$. For strictness on the right, take $T_1 = \{(a, b)\}$ and $T_2 = \{(b, c)\}$ on $\{a, b, c\}$. The union graph contains a path $a \to b \to c$ while neither candidate contains any $a \to^* c$ path. Here R_{possible} is also nontransitive, because it contains (a, b) and (b, c) but omits (a, c). Both examples concern mathematical candidate sets, with no statistical frequencies attached.

The distinction matters when uncertainty represents alternative explanations of a single regime. The shared relation reports claims surviving the candidate set; the possible relation reports claims supported by at least one complete candidate; and closure of the edge union reports paths permitted after combining the candidate rules. The third object becomes appropriate only with an additional interpretation, such as an explicitly defined switching system whose configurations record which regimes can change and under what conditions. Switching without such a rule risks treating epistemic uncertainty as a physical capacity to alternate between incompatible dynamics. This is structurally related to the incompatible-representative problem of Proposition 5.1, although the suppressed information now concerns model identity rather than concrete configuration identity.

Probabilistic uncertainty introduces a further distinction. A transition can be physically stochastic within a specified kernel P, while the investigator remains uncertain which kernel P applies. For a family of kernels, a thresholded support graph depends on a declared probability threshold, the time scale of the kernel, and the conditioning information. A union of candidate high-probability edges need not describe a high-probability path under any single kernel. Moreover, an assertion that a path has positive probability is weaker than an assertion about its likelihood, cost, or reliable attainability. Partially observable Markov decision models offer established methods for representing state uncertainty and action-dependent observations [30]; the present reachability bounds isolate a logical issue that persists independently of the choice of statistical estimator.

A comparative evaluation can therefore report, for each proposed precedence pair, whether it holds throughout a credible or otherwise declared model family, in at least one candidate, or only after unconstrained rule pooling. The family itself should be accompanied by its construction method, measurement programme, and adequacy threshold. Shrinking the family through new evidence may enlarge shared reachability and contract possible reachability, although newly admitted models or altered state descriptions can change these comparisons. The partial order, if one is derived after quotienting mutual reachability, must likewise be stated relative to the chosen inferential relation. The bounds in Proposition 5.4 are elementary and establish no probabilistic coverage guarantee for the data-compatible family.

5.8 Finite Benchmarks against Established Transition Models

The formal scope of G1 can be assessed against simpler models that already answer specific reconstruction questions. The following finite examples compare a projected directed graph, a behavioral quotient, a Markov aggregation, and a resource-sensitive generating system. They isolate what an additional generative-relational description clarifies and what it leaves to established methods. Table 4 summarizes the comparisons. A reproducible enumeration and the exact small examples are supplied with the editable source as `benchmark_checks.py`.

Graph-theoretic path composition. Consider the four-state example of Proposition 5.1. Concrete reachability, projected reachability, and closure after projection differ for the observable pair (A, D). A baseline graph algorithm that computes strongly connected components and reachability on the projected graph reports the apparent path $A \to B \to D$ correctly *for that graph*; the same assertion is unlicensed as an existential concrete path. This is a representation error rather than a failure of the graph algorithm. An exact benchmark compares both constructions and reports the number of apparent observable pairs lacking concrete witnesses. The supplied finite check exhausts all 2^{16} directed graphs on four labelled vertices under a fixed three-class surjection, a deliberately small domain chosen for complete enumeration rather than physical realism.

Behavioral partition refinement. A second graph baseline asks whether the observation classes are stable under all available one-step transitions. Proposition 5.3 identifies this test with uniform lifting, a standard bisimulation-style property for the unlabelled graph. The distinction matters because existential equality of coarse and concrete reachability can hold even when individual representatives have different successors. A graph quotient judged adequate for one endpoint query may therefore fail as a behavioral

Table 4: Formal benchmark comparisons for observed generative order. The examples are fully specified finite models; they are neither empirical measurements nor evidence of physical emergence.

Baseline model	Valid target under stated conditions	Information omitted by a weaker representation
Projected directed graph	Reachability within the projected graph	Concrete representative compatibility
Unlabelled behavioral quotient	Uniform observable path lifting	Transition probabilities and joint inventories
Aggregated Markov chain	Initial-distribution-independent class transitions under lumpability	Hidden internal rates when lumpability fails
Unary output graph	Reachability with unary reusable rules	Joint prerequisites and resource withdrawal
Joint-rule closure	Derivability from an admitted input pool	Consumption, probabilities, and physical evidence unless modeled separately

substitute for an unobserved starting configuration. G1's contribution at this stage is chiefly to specify which reconstruction claim is intended, not to supersede partition-refinement methods.

Stochastic behavior with shared graph support. Let the states be x_1, x_2, z with $\pi(x_1) = \pi(x_2) = X$, $\pi(z) = Z$. Consider two representatives with self-transition probabilities 0.9 and 0.1, respectively, and probabilities of reaching the absorbing z of 0.1 and 0.9. Both representatives have observable successor classes $\{X, Z\}$, so the graph support satisfies representative stability and every observable path lifts. Their probabilities of observing Z in one step are 0.1 and 0.9, however; the lumpability equality in Equation 16 fails. The same observable reachability preorder therefore admits markedly different transition probabilities under the same support graph. A credible physical or empirical reconstruction may require rates, observation schedules, and initial distributions in addition to reachability.

Joint prerequisites hidden by unary rules. Let a denote a productive configuration, r an enabling resource, and z the designated output. A unary account containing $a \to z$ predicts that the output is derivable from the singleton initial pool $\{a\}$. A joint-input account containing only $(a, r) \to z$ predicts z from $\{a, r\}$ and excludes it from the closure of $\{a\}$ with empty background. If observations are confined to trials in which r is supplied, both accounts agree on whether z is available. Withholding r separates the models. The distinction relies on the semantics of multi-input formation and the experimental ability to intervene on the prerequisite. A finite reachability graph of full inventory states can represent it, but a unary graph over product labels alone generally cannot; state completion is a valid competing remedy.

These comparisons constrain the role of a proposed landscape framework. In the unary, fully observed case, ordinary reachability already supplies the relevant precedence order. Where observational classes preserve behavior, established quotient techniques provide a mathematically complete baseline for that target. Where probabilistic rates or consumptive inventories matter, the landscape must acquire corresponding state variables and rules before its ordering claims can be empirically assessed. The added value of G1 lies in declaring the boundaries between these tasks, tracking how equivalence and formation conditions change their answers, and identifying observations that could discriminate rival reconstructions. The benchmark does not establish the framework's empirical superiority, and the finite examples have no statistical generalization to physical domains.

5.9 Comparative Applications and a Reconstruction Protocol

The following cases are schematic illustrations of research design, rather than empirical results. They show why the choice of observational interface changes the content of an order claim. Table 5 summarizes the corresponding inferential obligations.

Table 5: Inferential obligations when reconstructing generative precedence from mediated observations. The entries are methodological diagnostics rather than measured findings.

Evidence or construction	Licensed assessment	Principal unresolved issue
Observed endpoint pairs	Compatibility with selected access paths	Hidden intermediates and causal attribution
Projected transition graph	Possible transitions between observation classes	Representative compatibility and spurious composition
Stable successor classes	Exact lifting of finite observed paths under stated assumptions	Physical adequacy and joint-resource semantics
Aggregated Markov kernel	Effective transition probabilities under a lumpability criterion	Hidden memory and initial-regime dependence
Interventions and model comparison	Exclusion of candidates with divergent predictions	Confounding, model misspecification and external validity

Distributed assembly and hidden prerequisites. A fabrication process may require two components and an environmental condition to produce an assembled unit. A dataset of completed assemblies records outcomes at the end of each production cycle, while a process log records inputs and resource constraints. The first representation may suggest a simple succession of product categories. The second can test whether each alleged transition is possible from a complete admissible state and whether multi-input formation was silently reduced to unary reachability. A test that varies a single prerequisite while holding other enabling conditions within a declared range can discriminate among compatible models. Such a test licenses claims only for the particular mechanism, resource regime, and measurement design examined.

Ecological interaction and observational windows. A coarse survey can classify distinct ecological configurations by a shared indicator such as population abundance. Two configurations with similar abundance may differ in nutrient exchange, species interactions, or habitat constraints, leading to different subsequent possibilities. A reconstructed ordering among observed categories may therefore reflect a partial projection of an interaction system. Repeated measurements with complementary indicators, together with appropriately designed intervention or natural-experiment evidence where ethically and practically feasible, can constrain the candidate model family. Ecological interdependence further raises an ethical constraint: observational precision alone does not justify manipulating an ecosystem to settle a theoretical ambiguity.

Physical observables and model redundancy. In a gauge-theoretic or relativistic model, coordinate descriptions may vary while suitably defined physical observables remain invariant. A reconstruction procedure should distinguish different coordinate descriptions of one theoretical situation from different physical situations that happen to agree under a restricted collection of observables. The philosophical literature on symmetry and the hole argument helps locate the interpretive stakes [15, 42]. The elementary observation quotient developed here makes no claim to recover the correct gauge quotient of a particular theory. A physically meaningful reconstruction requires that theory's constraints, observable algebra, and treatment of redundancies.

A disciplined reconstruction programme may proceed through an ordered sequence of checks. The first check specifies the object of inference: single-step transitions, finite reachability, joint-prerequisite derivations, or a relation of physical causation. The second identifies the observation map or kernel, the sampling protocol, and the relevant conditioning regimes. The third inventories latent distinctions and records whether the observational equivalence classes are preserved by the proposed operations. The fourth compares the closure of the projected relation with the projection of concrete closure, applying a path-lifting or other appropriate preservation test where possible. The fifth constructs a family of data-compatible candidate models, records remaining ambiguities, and identifies accessible observations likely to discriminate them. The final check states the intended scope of the derived order: a math-

ematically valid relation, an operationally supported reconstruction, an empirically adequate effective dynamics, or a stronger physical interpretation justified by the particular theory.

The sequence need not be followed as a rigid algorithm. Inference can require iterations between measurement design, model revision, and conceptual clarification. Its principal advantage is traceability: a reader can identify the stage at which a structural assumption enters, a latent distinction is suppressed, an empirical comparison is performed, or an interpretive commitment is added. The results in this section establish exact elementary constraints on some reconstruction operations, while the proposed research protocol states methodological requirements for more complex empirical settings. Neither converts limited observations into a universal ordering of generative systems.

5.10 Reconstruction Limits and the Transition to Generative Potential

The analysis yields a differentiated account of mediation. An observational quotient may faithfully preserve specified invariants while altering the set of apparent reachability paths. The failure can occur because equivalent observed states have different successors, because incompatible representatives are spliced across transitions, or because joint prerequisites are omitted. Representative stability is equivalent to uniform lifting of observable paths from every starting representative; probabilistic lumpability supplies a stronger, quantitative criterion for preserving an effective Markov description. Data-compatible model families permit the degree of identification to be stated without assuming uniqueness where the evidence does not support it.

These distinctions also clarify the philosophical position of generative order. A mathematical preorder derived from a declared transition system is secure within its assumptions. A reconstructed preorder inferred from observations carries the additional uncertainties associated with measurement, model choice, and inverse inference. A physically interpreted precedence relation requires further theoretical commitments and, where relevant, discriminating empirical support. The distinctions are related, yet each carries an independent warrant. The word *emergent* is therefore most informative when the level of representation, the preserved structure, the generating conditions, and the sense of novelty or stability have been specified.

Section 6 turns from the reconstruction of reachable relations to the comparison of *generative potential*. Reachability establishes whether a destination lies along an admissible path in a declared model. Generative potential may instead concern the range, diversity, robustness, or condition-dependent structure of further feasible transformations. Their relation depends on generative semantics and on the target outcomes. Under full-domain unary reachability the unrestricted profiles are exact reachability successor sets, as proved in Proposition 6.3; with differentiated initial resources, joint formation rules, or outcome restrictions, this identification may fail. A reconstructed order can consequently be adequate for an endpoint-reachability task while providing little information about resource-conditioned opportunities. The next section will investigate which stratifications of potential can be defined locally, how their comparisons depend on chosen conditions, and what evidence would justify using such strata as analytical levels.

6 Levels of Generative Potential and Local Stratification

The analysis of derived precedence establishes conditions under which admissible transformations can support a preorder, while the reconstruction analysis shows how measurement and coarse-graining may alter its apparent structure. A further issue concerns the *generative potential* attributed to a configuration or an available collection of prerequisites. Potential concerns the possibilities that a specified system makes available under declared conditions; it therefore requires a definition separate from the realized course of a system and from the mathematical orientation of a precedence relation. This section develops conditional potential profiles, investigates inclusion-based comparison and local stratification, identifies admissible scalar summaries, and examines how resources, viability requirements, observation, and changing regimes delimit those constructions. The argument is formal and methodological.

Its proposed levels remain model-relative analytical objects whose physical or ontological significance requires additional evidence.

6.1 Potential as a Conditioned Space of Possibilities

A capacity comparison requires an explicit account of what is available, what might be produced, and how much of the generative process is permitted to occur. In Section 4, reusable joint generation was expressed through $\mathrm{Cl}_{\kappa,B}(S)$, with a fixed admissible domain $C_{q,\kappa}$, a background pool B, and finitely justified applications of the rules $G^{(m)}_{q,\kappa}$. The resulting closure answers a derivability question. It does not specify an intrinsic magnitude attached to each configuration. A description of potential must identify an initial-resource convention, a comparison domain, and an operational or formal horizon.

Fix an inquiry q, a regime κ, an admissible configuration space $C = C_{q,\kappa}$, and background resources $B \subseteq C$. Let $\mathcal{I}_{\kappa,B} : U \to \mathcal{P}(C)$ assign to each comparison object $c \in U$ an admitted initial pool, where U may be a subset of configurations or a separately typed set of resource-bearing situations. The assignment is part of the model: $\mathcal{I}_{\kappa,B}(c)$ can contain c itself, accompanying instruments, usable relations, and other resources whose co-availability is explicitly licensed. We use the formation operator $\Phi_{\kappa,B}$ of Equation 10. To make a finite-horizon comparison, define an iterative family by $S_0(c) = \mathcal{I}_{\kappa,B}(c) \cup B$ and $S_{n+1}(c) = \Phi_{\kappa,B}(S_n(c))$. Each stage counts one additional simultaneous round of permissible derivations; this is a bookkeeping depth, independent of any presupposed universal physical time.

Let $Q \subseteq C$ designate a common set of outcome configurations relevant to the research question. The finite-stage potential profile is given in Equation 20.

$$\mathcal{P}^{[h]}_{\kappa,B,Q}(c) := Q \cap S_h(c), \qquad S_0(c) = \mathcal{I}_{\kappa,B}(c) \cup B, \quad S_{n+1}(c) = \Phi_{\kappa,B}(S_n(c)). \tag{20}$$

The set in Equation 20 is a *profile* of individually derivable outcomes from an admitted pool. It records an existential possibility for each outcome and does not assert that all outcomes can be obtained together within the same consumptive run. Its interpretation must respect the semantics of the generating rules: an output may be a formally derivable formula, a reusable design, an assembled component, or a whole-system state, according to the modeled domain. The choice of Q prevents irrelevant differences among initial descriptions from dominating a comparison, although a selective Q also introduces an identifiable dependence on the researcher's objective.

The limiting finite-derivation profile is $\mathcal{P}^{[\infty]}_{\kappa,B,Q}(c) = Q \cap \mathrm{Cl}_{\kappa,B}(\mathcal{I}_{\kappa,B}(c))$. The superscript ∞ denotes the union of all finite derivation depths, rather than an assumption about realizations at an infinite physical time. The definition leaves out trajectories requiring genuinely infinite premises, limiting processes, or non-finitary formation rules. Applications admitting such operations should specify them separately. Where a rule requires the joint presence of several inputs, membership of an output in the profile is justified by a derivation tree whose leaves are supplied by the initial pool and permitted background resources.

A further distinction concerns the object to which potential is ascribed. The possibility set of a *configuration* depends on how the configuration is related to the available resources and enabling interfaces. The possibility set of a *resource pool* depends directly on that pool. Consequently, comparing c with d is generally insufficient without stating why $\mathcal{I}_{\kappa,B}(c)$ and $\mathcal{I}_{\kappa,B}(d)$ are appropriate counterparts. A richer physical description may reveal distinct inventories or interactions within configurations that looked identical under a coarse observational scheme. In such cases the profile itself becomes an object of empirical reconstruction in the sense of Section 5.

6.2 Inclusion Comparisons and the Role of Background Resources

Set inclusion is a parsimonious initial comparison of potential. With Q, B, κ, and h fixed, write $c \trianglelefteq^{[h]}_{\kappa,B,Q} d$ if $\mathcal{P}^{[h]}_{\kappa,B,Q}(c) \subseteq \mathcal{P}^{[h]}_{\kappa,B,Q}(d)$. The relation states that the observed target opportunities available from c under the specified construction are also available from d. It makes no claim about the relative desirability, energetic cost, probability, or ethical use of the attainable outcomes. In particular, it is possible for

two configurations to have equally broad cardinalities while their corresponding opportunities remain disjoint.

The first formal result makes explicit the monotonicities already implicit in reusable-premise closure.

Proposition 6.1 (Conditional expansion of potential profiles)**.** *Under the finite-arity, reusable-premise semantics of Section 4, fix κ, B, and Q. For any $c \in U$ and integers $h \leq k$, $\mathcal{P}^{[h]}_{\kappa,B,Q}(c) \subseteq \mathcal{P}^{[k]}_{\kappa,B,Q}(c)$. If $\mathcal{I}_{\kappa,B}(c) \subseteq \mathcal{I}_{\kappa,B}(d)$, then $\mathcal{P}^{[h]}_{\kappa,B,Q}(c) \subseteq \mathcal{P}^{[h]}_{\kappa,B,Q}(d)$ for each finite h, and the same inclusion holds for finite-derivation limits.*

Proof. The operator $\Phi_{\kappa,B}$ is extensive and monotone under the declared semantics. Extensivity gives $S_h(c) \subseteq S_{h+1}(c)$; induction yields inclusion at any later finite stage. For nested initial pools, the inclusion holds at stage zero. Monotonicity of $\Phi_{\kappa,B}$ preserves it by induction. Intersection with a common set Q preserves both inclusions. Taking unions over finite stages proves the limiting assertion. □

Proposition 6.1 supplies a useful consistency check. If a proposed implementation of reusable joint generation produces fewer attainable outcomes after merely adding admitted resources and retaining every formerly available rule, its definitions or implementation require inspection. The result has a limited domain: a new resource can impose safety constraints, trigger exclusion rules, consume a scarce input, or transform the governing regime. Such mechanisms require their own representation and can change the monotonicity conclusion. A physical system may also impose compatibility restrictions on entire resource pools, in which case the assumption that a larger pool contains every earlier feasible combination needs justification.

A generative possibility set can therefore grow with derivation depth while becoming less accessible under a resource budget, changing external conditions, or additional regulatory requirements. The mathematical statement about h is a theorem of the fixed reusable-premise model; a statement about development, evolution, or cumulative historical progress would require a considerably richer theory. It is especially important to distinguish an increased set of formally derivable candidates from increased freedom to realize them in a finite-resource system.

6.3 Potential Equivalence and Local Partial Orders

Inclusion of potential profiles defines a preorder on the comparison domain U: it is reflexive and transitive by elementary set inclusion. Define $c \approx^{[h]}_{\kappa,B,Q} d$ when the two profiles are equal. Quotienting by this equivalence yields a partial order under profile inclusion. This construction supplies a precise sense in which local levels of potential may be discussed without presuming that every pair of candidate configurations is comparable.

Proposition 6.2 (Partial order of potential profiles)**.** *For fixed κ, B, Q, and h, the relation $\trianglelefteq^{[h]}_{\kappa,B,Q}$ is a preorder on U. Equality of profiles defines an equivalence relation $\approx^{[h]}_{\kappa,B,Q}$, and the induced inclusion relation on $U/\approx^{[h]}_{\kappa,B,Q}$ is a partial order.*

Proof. Inclusion of subsets of Q is reflexive and transitive. Equality of the assigned subsets is an equivalence relation. Replacing a representative by one with an equal profile leaves the inclusion judgment unchanged, so inclusion descends to equivalence classes. Mutual inclusion of two profiles implies their equality and therefore equality of their classes, giving antisymmetry. □

Proposition 6.2 is an order-theoretic construction of the kind discussed in standard lattice and order theory [13]. The quotient is distinct from the mutual-reachability quotient of Proposition 4.1 and from the equivalence structures of Section 3. Equal potential profiles do not entail structural isomorphism, observational identity under an enlarged measurement family, or physical gauge equivalence. They mean that the declared comparison set and generative rules assign the same presently considered outcome possibilities to the configurations.

A partial order permits incomparable profiles. Take $Q = \{u, v, w\}$ and three modeled configurations with potential profiles $\{u\}$, $\{v\}$, and $\{u, v\}$. The first and second profiles are incomparable; each is

included in the third. A local comparison can therefore identify an upper profile relative to two other profiles while declining to rank the first pair. The terminology of *level* gains precision when it refers to such explicitly constructed relative relations, rather than to an asserted universal ladder. A partition into tiers based on additional conventions can be useful, but the convention should be disclosed.

Reachability and potential inclusion admit an exact correspondence in one restricted case. Suppose C is the complete-state domain, all rules are the unary transitions of a fixed $T \subseteq C \times C$, the background is empty, each initial pool is the singleton $\{c\}$, the outcome domain is $Q = C$, and all finite derivation depths are admitted. The potential profile then becomes $\mathcal{P}_T(c) := \{z \in C : cT^*z\}$. Equation 21 states the resulting correspondence.

$$cT^*d \quad \Longleftrightarrow \quad \mathcal{P}_T(d) \subseteq \mathcal{P}_T(c). \tag{21}$$

Proposition 6.3 (Unary reachability and unrestricted potential profiles)**.** *Under the stated unary, singleton-input, full-outcome, and finite-path assumptions, Equation 21 holds. Moreover, $\mathcal{P}_T(c) = \mathcal{P}_T(d)$ exactly when c and d are mutually reachable. Hence the potential-profile quotient is order-dual to the mutual-reachability quotient, with the order of profiles taken by set inclusion.*

Proof. If cT^*d, transitivity gives $dT^*z \Rightarrow cT^*z$, hence $\mathcal{P}_T(d) \subseteq \mathcal{P}_T(c)$. Conversely, reflexivity gives $d \in \mathcal{P}_T(d)$, so inclusion implies $d \in \mathcal{P}_T(c)$ and therefore cT^*d. Equality of profiles implies both reachability directions by the same argument, and mutual reachability implies equality. The quotient statement follows. □

Proposition 6.3 is an elementary consequence of reflexive–transitive closure. It identifies a precise setting in which two apparently different order constructions contain the same information with opposite orientations. For instance, with $a \to b$ and $a \to c$ and terminal b, c, the full reachable-outcome profile of a contains both terminal states; the profiles of b and c lie below it by inclusion. The special-case duality should be tested before claiming that a generative-potential ordering adds information beyond ordinary graph reachability.

The correspondence changes when initial resources or target observables are restricted. As a first counterexample, take unary rules $a \to b$ and $b \to a$, together with the joint rule $((a, r), z)$, and fix $B = \varnothing$, $Q = \{z\}$, $\mathcal{I}(a) = \{a, r\}$, and $\mathcal{I}(b) = \{b\}$. Although a and b are mutually reachable by the unary rules, z belongs to the potential profile of a and is absent from that of b: the latter pool never acquires r. As a second counterexample, take distinct configurations a, b, z with unary rules $a \to z$ and $b \to z$, let both initial pools be singletons, and set $Q = \{z\}$ at one derivation round. The profiles of a and b coincide even though neither configuration reaches the other. Thus profile equality and mutual reachability can separate under explicitly changed resource or outcome conventions. Their comparative meaning must be stated together with the initial-pool assignment, target domain, and regime.

The exact unary correspondence also has a stability qualification. In a stochastic physical interpretation, the positive-support graph can change under arbitrarily small rate perturbations, whereas a finite-horizon, probability-thresholded outcome profile may be preserved under a strict error margin. The construction and its proof are given in Proposition D.3. Consequently, comparisons of generative potential across measurements should declare whether "accessible" means positive-probability access, access above a calibrated probability level, or guaranteed access across an uncertainty family. These distinctions refine the notion of a local level without changing the elementary order-theoretic facts proved above.

6.4 Stratification Procedures and Scalar Summaries

A finite profile-order poset admits several stratification conventions. One convention places minimal classes in a first layer and repeatedly assigns higher indices to classes whose predecessors have already been assigned. Equivalently, a rank can be defined by the maximum length of a strict chain terminating at each class. For the finite quotient poset P, Equation 22 states this rank under the convention that minimal elements have rank zero.

$$r_P(x) = \max\big\{n \in \mathbb{N}_0 : \exists x_0 \lneq x_1 \lneq \cdots \lneq x_n = x \text{ in } P\big\}. \tag{22}$$

Equation 22 yields a monotone integer summary of strict order in a finite poset: a strict predecessor has lower rank. Classes assigned an equal rank may still differ substantially in their profiles, and the rank depends upon the chosen finite comparison domain. Removing an intermediate class may change rank while leaving every surviving pairwise inclusion unchanged. The strata therefore express the organization of a selected comparison collection. Infinite or non-well-founded posets need a separately justified stratification method and may resist finite integer ranks.

Another summary associates a nonnegative measure μ with a measurable outcome domain Q and defines $V_\mu(c) = \mu(\mathcal{P}^{[h]}_{\kappa,B,Q}(c))$, whenever these profiles are measurable. The resulting scalar is monotone under potential inclusion. In a finite comparison domain it may be implemented as a weighted count. Yet monotonicity is only one adequacy condition. Equal measures can conceal entirely different sets of opportunities, while a large number of trivial outcomes can outweigh a small collection of difficult or consequential transformations under uniform weights. Unspecified outcome types, incomparable resource costs, and the choice of μ prevent a bare scalar from serving as a universal magnitude of generativity.

A scalar can be chosen to count reachable distinct forms, the measure of feasible parameter regions, the number of robust strategies, or the maximum depth attainable under a resource constraint. These quantities should be introduced as separate observables, with explicit target classes and operational protocols. Moreover, a greater profile by inclusion may contain possibilities of harmful interference alongside beneficial possibilities. The mathematical comparison thereby remains descriptive until a domain-specific account of value, welfare, ecological integrity, or other evaluative standards is independently supplied. Sen's distinction between capabilities and realized achievements offers a relevant philosophical comparison concerning opportunity and actuality [44]; the present construction concerns model configurations generally and imports none of the capability approach's human-welfare commitments into its formal definitions.

The difference between scalar and structural comparison matters for scientific explanation. Two generative configurations may have the same number of accessible outputs while differing in how those outputs are related, in the resources that each production requires, and in whether several outcomes can coexist. Counting outputs also ignores the spatial, physical, or social organization of their dependencies. A vector of measurements can retain more information than one score, although vector comparison requires a declared componentwise order and may still leave pairs incomparable. The defensible meaning of a *higher level* is thus tied to a specified partial order, profile measure, and interpretation. A physically meaningful scalar potential in thermodynamics or mechanics has additional structure and governing laws; the present use of the word *potential* does not claim those properties.

6.5 Consumptive Resources and Jointly Attainable Outcomes

The reusable-premise semantics of Equation 20 is suitable for logical derivation, reusable designs, and other settings in which applying a rule leaves its premises available for further applications. Material formation can instead transfer or consume the resources that enable a transformation. A condition-sensitive profile then needs an account of inventories and of the feasibility of a *joint* outcome within a single run. This subsection introduces a minimal resource-aware alternative and identifies the additional comparisons it supports. The construction is a familiar kind of discrete-event model, closely related to Petri-net marking and firing semantics [36]; its use here is comparative rather than an assertion of a novel semantics.

Let the resource types form a finite index set $\{1, \dots, d\}$ and let a marking $m \in \mathbb{N}^d$ record their nonnegative integer amounts. A rule r specifies an input vector $\alpha_r \in \mathbb{N}^d$ and an output vector $\beta_r \in \mathbb{N}^d$. It may fire from m if $m \geq \alpha_r$ componentwise. The consumptive transition is defined in Equation 23.

$$m \xrightarrow{r} m - \alpha_r + \beta_r, \qquad m \geq \alpha_r. \tag{23}$$

The source and target in Equation 23 are whole inventories, while the rule specifies the required co-availability of its inputs. A marking can also carry location, compatibility, or safety distinctions if these

are essential to the inquiry; the simple vector model assumes each resource type is internally interchangeable. Let $\text{Reach}_{\leq h}(m)$ be the set of markings reachable by at most h rule firings, including the initial marking. For a set $Q \subseteq \mathbb{N}^d$ of *demand vectors*, define the coverability-based profile in Equation 24.

$$\mathcal{V}_Q^{[h]}(m) := \{q \in Q \mid \exists m' \in \text{Reach}_{\leq h}(m),\ m' \geq q\}. \tag{24}$$

Membership in $\mathcal{V}_Q^{[h]}(m)$ means that one reachable inventory covers a specified demand. It is a statement about an individual query q and its attainable marking. The coexistence of two individually coverable demands is tested by a *third* query requiring both in the same marking, when such addition accurately represents the practical requirement. The interpretation is conditional on the chosen granularity of resource types, allowable firings, and the horizon.

For example, suppose the initial marking contains one token of resource A and no tokens of products B and C. The only rules consume one A to produce either one B or one C. The demand for one B is coverable, as is the demand for one C, while the demand vector containing both one B and one C is unreachable. The reusable-premise closure of the symbolic productions $A \Rightarrow B$ and $A \Rightarrow C$ contains both symbols, so treating that closure as a concurrently realizable inventory would overstate the available joint production. Conversely, if outputs are reusable design descriptions rather than consumptive inventories, the closure model may be the appropriate one. The relevant difference resides in the semantics of the generating rules, and it can be tested directly for a declared finite model.

The resource-aware construction yields several implications for generative comparison. Adding a new rule can enlarge the set of existentially reachable inventories while altering which individual runs can maintain critical supplies. Counting individually attainable products can miss rivalry among outputs that require the same scarce precursor. A safety requirement or replenishment mechanism changes the relevant target demands and allowable trajectories. Where conservation laws apply, admissible transformations must respect the corresponding invariants. In the one-token example, the total token count is conserved, so a demand requiring two tokens is excluded by a directly checkable invariant. Such restrictions need not have an analogue in an abstract closure of reusable propositions.

The profiles of Equation 20 and Equation 24 therefore answer different questions. The first tests formal derivability under reusable-premise rules. The second tests the coverability of demands under consumptive, inventory-level transitions. Neither, by itself, measures welfare, value, physical energy, or a subject's moral standing. In a physically grounded application, resource quantities must be associated with actual conserved or constrained magnitudes, rather than being treated as arbitrary tokens merely because a Petri-net representation is convenient. In addition, an interpretation of generated products as simultaneously available must be supported by a reachable whole-system state and by the relevant observation programme. These distinctions make the assessment of generative potential more discriminating without assuming that a single notion of potential should govern every domain.

6.6 Constrained Continuation and Viable Possibilities

Reusable-premise closure is well suited to a model of cumulative availability, yet many questions about generative potential involve finite stocks, exclusive commitments, conservation constraints, and the ability to remain within a permissible region. In such settings an attainable outcome and a continuing capacity to operate can diverge. The existence of a path to an output does not ensure that the path preserves sufficient resources for any subsequent step. A stronger account should keep track of the evolving inventory, environmental constraints, and admissibility of intermediate configurations.

Consider a state-complete space X whose elements encode the variables necessary to decide whether a transition can occur, including relevant resource levels or mutually exclusive commitments. Let $T \subseteq X \times X$ denote a fixed, composable transition relation, and let $K \subseteq X$ be a declared set of admissible continuing states. Define the finite-step viability domain in Equation 25.

$$\mathcal{V}_K^{[h]}(T) = \big\{x_0 \in K : \exists x_1, \ldots, x_h \in K \text{ with } (x_{j-1}, x_j) \in T \text{ for } 1 \leq j \leq h\big\}. \tag{25}$$

The expression in Equation 25 defines the initial states from which at least one trajectory of *exactly* h licensed transitions remains in K; $\mathcal{V}_K^{[0]}(T) = K$. For this precise convention, longer-horizon viability is more demanding: $\mathcal{V}_K^{[h+1]}(T) \subseteq \mathcal{V}_K^{[h]}(T)$, because an admitted length-$(h+1)$ trajectory has an admitted prefix of length h. Thus two horizon-dependent sets naturally move in opposite directions under fixed assumptions. Reachable-outcome profiles expand as more finite derivations are admitted, while the set of initial states supporting a longer admissible continuation can contract.

This elementary contrast does not depend on a privileged metaphysical present. It compares finite strings of permitted transitions from a designated boundary or initial condition. Such a boundary is an explanatory choice in the model. A physical account with time-reversal symmetry, stationary constraints, or globally imposed boundary conditions can use different orientations or two-sided trajectory requirements. A system with continuous dynamics may instead require differential inclusions and a specified state constraint. Viability theory provides rigorous mathematical resources for this kind of constrained-dynamics inquiry [4]. The present finite-step definition is an elementary discrete analogue, and the manuscript does not claim to derive the established viability theorems.

Two configurations may have identical sets of individually reachable outcomes yet differ in the number of safe continuations or in the resources surviving those continuations. Moreover, a state can reach a high-valued output through a destructive pathway that compromises its later ability to sustain any admissible operation. In a technical system, this distinction may concern available energy, memory, or heat dissipation. In ecological systems it may concern habitat integrity and regenerative cycles. In institutional systems it may concern the maintenance of cooperative structures. These are schematic translations requiring domain-specific dynamics and evidence, rather than universal instances of one literal physical potential.

There is a further reason to distinguish continuation capacity from simple outcome abundance. A model with choices can have a large set of terminal outputs but only a few trajectories that remain robust to interruptions. A small outcome set may be sustained by diverse feasible pathways. Counting endpoints, paths, and robust policies measures different features; the definitions of admissibility, uncertainty, and strategy determine which account is fit for a particular inquiry. Local stratification may therefore require more than one ordered profile rather than a single apparent hierarchy.

6.7 Generative Potential and Realized Trajectories

The interpretation of potential turns on the difference between a permitted continuation and a continuation selected or realized within the model. Suppose a record contains an observed sequence $x_0, x_1, \ldots, x_n$ that is compatible with the declared transition relation. The record provides evidence for those particular transitions, conditioned on the adequacy of the observations and the associated physical interpretation. It normally gives only partial information about alternative paths, including those not selected, interrupted, or inaccessible to the observational apparatus. Inferring complete generative potential from one record requires supplementary assumptions or deliberate tests.

Consider two transition models on $X = \{a, b, c\}$. In the first, $a \to b$ is the sole nontrivial transition; in the second, $a \to b$ and $a \to c$ are licensed. An observation of the trajectory $a \to b$ is compatible with both. The two models nevertheless assign different reachability profiles to a. The difference can be investigated by sampling alternative boundary conditions, studying the enabling conditions of the additional edge, or intervening on a relevant resource where intervention is scientifically justified and ethically permissible. The inferential problem is an instance of the reconstruction issues discussed in Section 5, now formulated for unrealized opportunities rather than for a uniquely selected reconstructed path.

Even if a model fixes its permissible paths, the relative frequency of particular realizations requires an additional probabilistic or selection mechanism. One might add a transition kernel, control policy, decision rule, or statistical ensemble. None follows from reachability alone. The probability of an outcome and its membership in the possibility profile are distinct statements: positive probability may imply formal accessibility under a properly specified stochastic transition system, while formal accessibility

supplies no numerical probability of actualization. In physically constrained settings, a path could be technically licensed but practically inaccessible within the energy, budget, or duration relevant to an experiment.

The distinction has wider methodological implications. Data on realized products, observed biological forms, or institutional decisions commonly reveal only selected regions of the feasible domain. Repeated success can support a hypothesis of reliable realization, but may leave unexplored generative branches. Conversely, a failure to observe an outcome can result from low probability, weak measurement, missing prerequisites, or the absence of the alleged generative relation. Inference should specify which explanation is being tested. A single scalar called "generative potential" would obscure these differences unless its operational meaning were constrained and the accompanying uncertainty reported.

6.8 Regime Dependence and Robust Comparability

A local hierarchy formed under one regime may change when the rule set, admissible domain, resource assumptions, or observation target varies. This dependence matters when the declared objective is a comparison that survives more than one experimental condition. A particularly transparent construction takes an explicitly selected family $\mathcal{K}$ of regimes and compares profiles defined on a shared outcome domain Q with a common interpretation of the comparison objects U. Appropriate cross-regime translations must be supplied if configurations or targets have different types.

For each $\kappa \in \mathcal{K}$, suppose the profiles $\mathcal{P}_\kappa^{[h]}(c)$ have been defined using the corresponding fixed resources, rule families, and observation conventions. Define the *regime-robust inclusion* relation in Equation 26.

$$c \trianglelefteq_{\mathcal{K}}^{[h]} d \quad \Longleftrightarrow \quad \forall \kappa \in \mathcal{K},\ \mathcal{P}_\kappa^{[h]}(c) \subseteq \mathcal{P}_\kappa^{[h]}(d). \tag{26}$$

Equation 26 is a preorder, because it is an intersection of inclusion preorders on the same comparison domain. Equating objects whose profiles agree for every considered regime produces a quotient partial order by the same argument as Proposition 6.2. This robustness is relative to the declared family $\mathcal{K}$. It should be distinguished from robustness against probabilistic perturbations, worst-case control guarantees, or stability under unconstrained future transformations. A relation can be robust across a narrow selection of regimes while failing in a regime that was omitted from consideration.

For a simple reversal, let $Q = \{u, v\}$ and consider two configurations a, b. In regime κ, their profiles are $\mathcal{P}_\kappa(a) = \{u\}$ and $\mathcal{P}_\kappa(b) = \{u, v\}$; in regime λ, they are $\mathcal{P}_\lambda(a) = \{u, v\}$ and $\mathcal{P}_\lambda(b) = \{v\}$. Each regime admits a strict comparison, but the comparison points in opposite directions. Over the two-regime family, the pair is incomparable under Equation 26. The example is a formal countercondition to a universal ordering inferred from a single context. It also shows why an experimentally observed difference in generative capacity may require replication under varied enabling conditions before a broad comparative claim is justified.

A different research objective could assign a probability distribution or a policy-relevant weight to the selected regimes and compare expected outcome values. Such a weighting would introduce additional assumptions concerning the distribution of regimes, the evaluative scale, and acceptable variation. It could reveal useful operational priorities while losing the strict inclusion guarantees of Equation 26. For physical applications, the regime family may be parameterized by control settings or boundary conditions; for historical or social applications, the family may be delimited by institutional or material settings. Across all cases, the comparability claim should state how the allegedly corresponding configurations and targets have been identified.

6.9 Comparative Cases and Criteria for Evidential Assessment

The constructions can be illustrated across several domains, provided their differing generative semantics remain visible. These cases are proposed modeling exercises rather than evidence for a single universal theory. Table 6 states the principal objects of measurement and the interpretive risks of each comparison.

Modular computation and proof generation. Let configurations encode formal expressions, verified modules, or executable compositions. In a declared logical calculus or typed programming environment, the rule family specifies which joint inputs admit an output. The relevant potential profile may consist of derivable proofs or executable interfaces within a bounded depth. Some derivations can be verified mechanically, making rule membership relatively transparent. Resource-limited execution, however, requires additional bounds on memory, computation, and type compatibility. Counting proofs or outputs is also sensitive to syntactic equivalence: many distinct descriptions may realize the same result. The quotient criteria of Section 3 and the reconstruction conditions of Section 5 therefore remain necessary.

A constrained technological process. A process model may include raw components, energy supplies, environmental constraints, and successive machine configurations. Closure under reusable premises approximates the list of conceivable products if each prerequisite remains available. A state-complete consumptive model instead tracks inventory and permits a viability test for continued operation. Two production configurations can therefore occupy different relative positions according to the comparison of products, resource burden, recoverability, or safe continuation. Which comparison matters depends on the research question; the model should not convert a larger product set into an unsupported assertion of greater economic or normative value.

Ecological and evolutionary possibilities. Ecological models may assign possible continuations to states characterized by species interactions, resource availability, and environmental conditions. A potential profile can describe outcomes permitted by the selected model, while observations trace only particular historical realizations. Variation in habitat and environmental regimes may reverse local inclusion comparisons. The mathematical model cannot by itself establish an organism's welfare, evolutionary destiny, or the ecological legitimacy of intervention. Empirical validation would require independently defensible ecological mechanisms, state measurements, uncertainty bounds, and appropriate safeguards for research activity.

Physical effective descriptions. At one scale, an effective model may permit a class of transformations that become differently represented at another scale. A profile comparison is meaningful only after specifying the translation between domains and the invariants preserved by that translation. Renormalization supplies a physically substantial but theory-dependent example of how effective descriptions may transform with scale [49]. The present section proposes neither a renormalization-group flow for generative potential nor a claim that higher values of its scalar summaries correspond to increasing physical complexity. Such results would require a particular action, dynamics, coarse-graining procedure, and tests of the resulting observables.

The comparative programme can be evaluated through a sequence of research decisions. First, specify whether the target is a set of derivable configurations, a space of admissible trajectories, a viable continuation domain, or a distribution over realizations. Second, state the input-resource convention and identify the joint prerequisites suppressed by any simplified representation. Third, define comparable outcome types, distinguish the selected observational quotient from physical equivalence, and record the resource or horizon bounds. Fourth, test whether an inclusion relation survives perturbations of the generating conditions, measurement procedures, and choice of outcome domain. Fifth, investigate cases where the proposed rank disagrees with a profile comparison, where realized trajectories conceal branching opportunities, or where a larger option set conflicts with safe continuation. The latter cases are potential falsifiers of claims that a particular scalar captures the intended explanatory object.

The section yields a bounded conception of generative levels. Conditional profiles support preorders and quotient partial orders; finite rank and scalar summaries may describe selected aspects of that order. Constrained viability, realized trajectories, and cross-regime robustness provide additional, only partially aligned comparisons. These mathematical distinctions do not establish a single hierarchy of entities, a privileged orientation of ontological succession, or a necessary historical tendency toward larger generative repertoires. They enable clearer hypotheses about which features of a generative landscape are

Table 6: Analytical comparisons of local generative potential and their principal inferential limits. Each row identifies a model-relative construction, rather than an empirical finding.

Comparison	Licensed inference	Additional requirement
Outcome-profile inclusion	Relative accessibility of declared targets	Common domain, rules, conditions and horizon
Equality of potential profiles	Equivalence for the selected outcomes	Preservation tests for richer operations
Finite poset rank	Layer within a specified collection	Disclosure of domain and ranking convention
Scalar profile measure	Monotone summary under inclusion	Outcome measure and evaluative interpretation
Finite-step viability	Admissible continuation within constraints	State completeness and transition semantics
Cross-regime inclusion	Stability across specified regimes	Valid translations and regime coverage
Realized path	Evidence of an observed permissible course	Testing of unobserved alternatives

preserved across conditions and which depend on the choice of representation. Section 7 will extend the fixed-regime analysis by allowing the landscape and its rules of formation to transform, so that changes in potential and changes in the structure that defines potential can be studied together.

7 Transformations of Generative Landscapes

The preceding construction compared reachability, equivalence, and generative potential within a declared landscape and a specified collection of effective conditions. A more demanding inquiry concerns circumstances in which the domain of admissible configurations, the generation rules, or the descriptive interfaces are themselves transformed. This section develops a formal vocabulary for such changes and examines their consequences for derived precedence. It first separates operations within a landscape from transformations between landscapes, then gives conditions for comparing admissible paths across distinct representations. It subsequently analyzes rule revisions, changing potential profiles, the reorganization of equivalence classes, and historical dependence. The argument is a structural investigation supported by illustrative models; the proposed constructions require independent empirical or physical interpretation in each application.

7.1 Landscape Transformations as Distinct Analytical Relations

Recall the candidate landscape $\mathfrak{L}_q$ of Section 2, with descriptions D_q, configurations C_q, representation ρ_q, conditioning regimes K_q, admissibility A_q, generating relations $G^{(m)}_{q,\kappa}$, and observation interfaces $\mathcal{O}_q$. These components have different formal functions. A transition between admissible configurations holding their types and rules fixed is a relation internal to one landscape. A transformation of the landscape may change the very objects by which those transitions are defined. The distinction becomes especially consequential when a scientific model admits changing constraints, acquired operations, modified interfaces, or the creation of new component types.

Let I index a declared family of candidate landscapes, $\{\mathfrak{L}_i\}_{i\in I}$, where each $\mathfrak{L}_i$ has the same formal signature but may contain different configuration domains, conditions, or rule families. The index labels an episode, context, experimental regime, or representational proposal. Its mathematical structure is left open. In particular, using an index does not supply a universal clock, an intrinsic order of creation, or an ontologically preferred present. An application may give I a physical time parameter, a partial order of events, a graph of experimental interventions, or an unordered collection of alternative models; those choices require separate justification.

The transformations of interest are specified by a declared relation $M \subseteq I \times I$ together with cross-landscape correspondences. For each selected $(i, j) \in M$, a relation $W_{ij} \subseteq C_i \times C_j$ associates configurations when a comparison is warranted. An admissibility correspondence $H_{ij} \subseteq K_i \times K_j$ may similarly relate conditions. This minimal representation has two merits. It accommodates comparisons where the configuration spaces differ in size, dimension, or mathematical type, and it avoids requiring a unique counterpart for each original configuration. Where a genuine structure-preserving map is available, W_{ij} can be chosen as its graph. Where several translations are plausible, a relation represents the ambiguity explicitly.

A useful discipline is to distinguish three readings of M. It may represent a *modeled transformation* actually licensed by a process; an *intervention relation* recording what an investigator can change; or a *comparison relation* between two theoretical descriptions. Only the first reading, under a physically supported dynamics, directly speaks to the occurrence of a change in the target system. An arrow in a comparison diagram does not establish a causal mechanism. The rest of the section uses the neutral expression *landscape transformation* and states the required reading locally.

For path-based applications, one can construct a disjoint union of regime-tagged states $\widetilde{C} = \bigsqcup_{i \in I} \{i\} \times C_i$. A within-landscape edge preserves the tag i and follows a chosen unary transition T_i. A cross-landscape edge changes the tag according to M and follows a separately licensed correspondence. The latter edge may encode a physical alteration of the rules or an intervention; a merely descriptive alignment should remain outside the realized transition relation. Maintaining the tags is essential: suppressing them can make transitions performed under mutually incompatible rules appear directly composable.

7.2 Domains, Rules, and Interfaces under Transformation

A landscape can vary through several analytically separable dimensions. A change in admissibility can enlarge or contract $C_{i,\kappa}$ while leaving the formal rule schema approximately stable. A rule transformation can add, disable, or condition a generating operation. A change of representational domain can introduce descriptions for phenomena previously omitted or identify descriptions formerly distinguished. An observational transformation can improve a sensor, modify its resolution, or alter the observable variables without changing the underlying modeled transformations. Finally, a change of component types or of compatibility predicates can modify the arity and composability of the generating operations themselves. These possibilities frequently interact, but treating them separately helps discriminate physical change from epistemic refinement.

Table 7 records the main distinctions together with the question that would have to be answered before interpreting the change as evidence for a reorganization of generative order. The categories are diagnostic rather than a taxonomy of physical causes.

These distinctions also identify different interpretations of nonstationarity. A changing empirical distribution may arise under fixed rules whose inputs vary. The same empirical record could be modeled through changing transition probabilities, a drifting observation mechanism, or revisions of the permitted configuration space. Temporal-network research offers a relevant example of why the activation of relations and their ordering can matter for inferred network dynamics [27]. This comparison motivates explicit records of contact or edge activity; its results are not taken to establish the landscape-update formalism proposed here.

The question of a *new* generative rule deserves special care. An investigator may discover an operation already present in the target domain, an engineering intervention may introduce a new capability, or an evolving system may alter the conditions for subsequent operations. Those cases share a representational pattern—a changed set $G_{i,\kappa}^{(m)}$—while differing in causal and ontological status. An evidential account should specify whether the change occurred in the studied system, in the experimental setup, in the model's epistemic coverage, or in several of these at once.

Table 7: Types of landscape transformation and the associated requirements for order comparison. A change in a modeled object alone does not establish a corresponding physical change.

Object transformed	Possible effect in the model	Requirement for interpretation
Admissibility and background	Altered feasible configurations or premises	Verified condition and resource correspondence
Generation rules	Added or withdrawn derivation paths	Stable rule typing and operational semantics
Input compatibility	Changed combinations of jointly usable configurations	Evidence for interfaces, gluing, or coexistence
Descriptions and configuration types	Different state distinctions and expressible entities	Typed translation and provenance of refinement
Observational interface	Revised inferred order with unchanged proposed mechanisms	Measurement calibration and reconstruction tests
Rule-update mechanism	Altered opportunities for subsequent rule changes	An explicit higher-level update account

7.3 Path Preservation between Different Landscapes

A comparison of order requires a relation between configurations and a criterion for preserving admissible processes. Consider two binary transition systems (C_i, T_i) and (C_j, T_j) obtained under explicitly paired conditions. Let $W_{ij} \subseteq C_i \times C_j$ be the declared cross-landscape correspondence. A forward simulation condition requires each T_i-step from a represented source to be matched by a finite T_j-path from its chosen counterpart, ending at another related pair. Equation 27 states this condition in a form that exposes its existential commitment.

$$\begin{aligned} & (c, c') \in W_{ij}, \quad c\, T_i\, d \\ \Longrightarrow \quad & \exists d' \in C_j : \; c'\, T_j^*\, d' \text{ and } (d, d') \in W_{ij}. \end{aligned} \tag{27}$$

The use of T_j^* allows one modeled step to be realized by several steps, or by no steps when the related endpoints already coincide. This is appropriate for many comparisons of granularity, but an application demanding positive operational work can replace T_j^* by its non-reflexive or bounded-length counterpart. The condition also presupposes that W_{ij} and the paired regimes have been justified independently; an arbitrarily chosen total correspondence can make the conclusion vacuous.

Proposition 7.1 (Preservation of reachable counterparts)**.** *Suppose W_{ij} satisfies Equation 27. For any $(c, c') \in W_{ij}$ and every $d \in C_i$ with $c\, T_i^*\, d$, there exists $d' \in C_j$ such that $c'\, T_j^*\, d'$ and $(d, d') \in W_{ij}$.*

Proof. Choose a finite T_i-path from c to d. For a path of length zero, take $d' = c'$. For a positive-length path, apply Equation 27 to the first step and to the given related pair. It supplies a related successor and a finite T_j-path. Repeating the argument along the remaining steps and concatenating the finite matching paths yields the claimed counterpart. The construction is a finite induction. □

Proposition 7.1 is a conditional statement about simulation, not a declaration of physical continuity or ontological identity. It guarantees that each reachable endpoint has some related reachable endpoint. Different source paths can be matched by different target paths; the correspondence need not be injective, unique, or reversible. The reverse simulation condition, if independently satisfied, gives a stronger reciprocal comparison, although two-way simulation still leaves open whether the systems have the same observables, causal explanations, or material resources. For multi-input rules, a unary simulation is also insufficient to preserve joint prerequisites. There the adequate cross-landscape condition must relate compatible *tuples*, rule applications, and outcomes, much as the quotient-saturation analysis of Section 3 required preservation of a whole generative relation.

This distinction prevents an easy conflation between similarity of trajectories and persistence of generative structure. Two landscapes may produce observationally similar paths even though one relies upon a resource available only in the first regime. Conversely, a rule correspondence may be faithful while

the observed paths differ because initial conditions or interventions select different branches. A defensible comparison should state which level—path, rule, joint compatibility, potential, or observation—is preserved, and whether the correspondence is sufficient for the intended inference.

7.4 Changing Rules and Conditional Potential

Section 6 defined finite-depth potential profiles under fixed rules, background pools, and target domains. When the rules vary, potential becomes a family of quantities rather than an attribute with one stable value. Even configurations that retain the same name across regimes may acquire or lose possibilities. A particularly transparent case holds the relevant configuration domain, compatibility convention, initial pool, target set, and finite-depth counting convention fixed, while changing only the set of admissible rules.

Suppose two rule families G_i and G_j are defined on a common set C and every rule application of G_i is also admitted by G_j, with the same input–output meaning and any background B unchanged. Write $S_h^i(S)$ and $S_h^j(S)$ for the stage-h derivable pools obtained using the reusable-premise expansion procedure of Section 4. Equation 28 records the resulting comparison.

$$G_i^{(m)} \subseteq G_j^{(m)} \text{ for every } m \quad \Longrightarrow \quad Q \cap S_h^i(S) \subseteq Q \cap S_h^j(S), \;\; h \in \mathbb{N}_0. \tag{28}$$

Proposition 7.2 (Potential inclusion under compatible rule addition)**.** *Under the common-domain, compatible-typing, reusable-premise assumptions stated before Equation 28, its conclusion holds for every initial pool S and finite depth h, and for the unions of all finite derivation stages.*

Proof. The two stage-zero pools coincide. If the pool constructed with G_i is included in the one constructed with G_j at stage h, every joint input tuple available for an i-rule at the next stage remains available in the j-pool. Rule inclusion ensures that the corresponding output is also added by the j-expansion. Induction proves the stagewise inclusion. Intersection with Q and union over finite stages preserve it.□

The inclusion in Proposition 7.2 is *pointwise across regimes*: it compares the potential of the same initial pool under an expanded rule family. It does not guarantee preservation of comparisons *between* distinct initial pools. To see this, let the initial pools be $\{a\}$ and $\{b\}$, the common outcome domain $Q = \{x, y, z\}$, and the initial unary rules be $a \to x$, $b \to x$, and $b \to y$. At depth one, the profile from a is $\{x\}$ and the profile from b is $\{x, y\}$. Add the rules $a \to y$ and $a \to z$, retaining all earlier rules. The new profiles are $\{x, y, z\}$ and $\{x, y\}$, respectively. Both have expanded or stayed fixed, yet their comparative inclusion is reversed. Thus regime-wise profile expansion and stability of the local stratification are logically different properties. The latter requires its own cross-regime test.

The proposition states a controlled comparison, and its qualifications carry most of its interpretive weight. In a consumptive process, adding an available operation may introduce competition for scarce resources; the set of individually derivable outputs can then diverge from a set of jointly executable plans. A new rule may also alter stability, deactivate an earlier rule, change the relevant configuration type, or impose constraints that were absent in the earlier regime. If any of these conditions changes, a claim of monotone expansion requires renewed analysis. An increase in formal potential does not establish increased viability, welfare, or physical complexity. The section's central concern is how the space of allowed generation changes under declared transformations, including cases of contraction and incomparability.

The common-domain assumption can be relaxed only by providing a translation. A relation W_{ij} may identify some configurations or output types across the two landscapes, but inclusion of their profiles is meaningful only after the targets and resource assumptions have been aligned. A comparison based solely on the cardinality of outputs may miss a change in the kinds of possibilities: five derivable outputs in one landscape and six in another say little about their respective compatibility, continuation conditions, or scientific significance. The appropriate object may therefore be a diagram of translated profile sets or a family of partial comparisons rather than a difference of scalar "generativity."

A transformation can also create conditions for subsequent changes of the rules themselves. To model such a capacity, one must specify a higher-level update relation whose objects include the relevant

rule descriptions. For example, a state may encode both an ordinary configuration and an editable rule set, with licensed operations on the pair. This is a valid extended-state model if the allowed update operations and their resource dependencies are independently specified. Calling such a transformation *self-generative* adds a philosophical interpretation; it does not follow merely from writing a rule-update arrow. When the update mechanisms themselves vary, the same modelling question recurs at another descriptive level. A finite research model may choose a useful boundary for this recursion without thereby locating an ultimate ontological foundation.

7.5 Reconfiguration of Equivalence and Derived Order

If a landscape's generating operations change, its reachability preorder and associated quotient poset may change even when the underlying configurations remain fixed. Consider $C = \{a, b\}$ with $T_i = \{(a, b)\}$. The induced preorder contains a strict reachable direction from a to b. If a transformed regime admits $T_j = \{(a, b), (b, a)\}$, the two configurations become mutually reachable and belong to one quotient class. Under a different transformation admitting only (b, a), the strict direction reverses. In each example the state labels remain unchanged, while the ordering construction depends on the rule family. The examples exhibit a change of model-derived structure without presuming an ontological reversal of time or a universal scale of complexity.

Changes of observational resolution can reorganize the quotient for a separate reason. A new sensor may distinguish two formerly indistinguishable configurations; a coarse effective description may merge them. Both can change the number of classes and the relations apparently connecting them. Section 5 showed how an observational projection can introduce apparent paths when incompatible underlying representatives are spliced. A cross-regime comparison adds another potential source of splicing: paths observed in two different regimes can be combined after erasing their regime tags, even if the actual change of regime never occurs in the model. Keeping transitions internal to each tagged landscape and specifying licensed cross-landscape transitions guards against that inference.

An analogous issue arises for the equivalence relations used in potential stratification. In one regime, configurations can share the same target profile; under changed rules, their profiles may diverge. Two configurations once placed in the same stratum may therefore separate. Conversely, the addition of common possibilities may equalize their profiles. A comparison of "levels" across landscapes needs a correspondence of the underlying configurations and of the relations defining the levels. A coincidence of integer ranks alone is insufficient: ranks may remain constant after a substantive change in the potential sets, while a harmless extension of a finite comparison collection may change ranks without changing any previously existing pairwise inclusion.

The resulting methodological principle is specific. An observed change in derived order should be decomposed into changes in generating rules, admissible resources, observational distinctions, and the selected comparison domain. These factors can interact and cannot always be separately identified from an observational record. The formal models make their possible roles explicit; actual attribution requires independently discriminating evidence.

7.6 History Dependence, Memory, and Relational Orientation

In many systems, present observable descriptions do not suffice to determine the available subsequent transformations. The relevant constraint can be the history through which the system arrived at its current configuration: prior resource expenditure, acquired compatibility, material wear, stabilized institutions, or an inherited repertoire of operations. A history-dependent model therefore assigns the admissible rules or conditions to a structured history H, with a law of the schematic form $G(\mathsf{H})$ or $\kappa(\mathsf{H})$. This formal dependence can be operational even when the chosen observable configuration appears unchanged. It also makes explicit which historical distinctions are retained and which are discarded.

Consider two histories H_1 and H_2 that lead to the same observable configuration y. Under H_1, a compatible auxiliary resource is present and a transformation to z is admitted; under H_2, the resource has been consumed, making the transformation inadmissible. A model in which y is the entire state

would either incorrectly license the transition in both cases or incorrectly exclude it in both. An extended representation retaining the resource condition or a relevant history summary resolves the ambiguity. The extension should be chosen in relation to the intended predictions or causal explanation; recording every detail of a trajectory is rarely a useful or observationally feasible default.

Path dependence has established applications beyond the present abstract construction. Arthur's increasing-returns model of competing technologies demonstrates, within its economic assumptions, how historical contingencies can influence technological lock-in [2]. Its relevance here is structural: rules of development and accumulation can make different histories support different reachable futures. The model provides neither an empirical demonstration of universal lock-in nor a general physical law of generative precedence. In other domains, history can affect accessible possibilities through different mechanisms, including hysteresis, plasticity, material fatigue, ecological succession, or information retained in an adaptive controller. Each claimed mechanism calls for its own theory and evidence.

The orientation of a historical relation must also be distinguished from a metaphysical division into past, present, and future. An empirical record may support an order of interventions, reactions, or observed events, and a mathematical model can use that order to define its allowed continuations. The GR inquiry leaves open whether this orientation is fundamental, emergent from effective dynamics, imposed by boundary conditions, or a feature of the observational protocol. Even when a transition graph is directed, the direction is fixed by its stated semantics. Reversible physical processes, bidirectional interactions, and histories with only partial event comparability require their own adequate representations.

The possibility of reconstructing an evolving landscape from observations is therefore conditional on observability of the relevant rule regimes and history variables. Distinct hidden rule histories may generate the same finite observations but different potential profiles. A measurement programme may distinguish them through interventions, resource tracking, repeated experiments under controlled changes, or independent access to intermediate constraints. In the absence of such evidence, the appropriate outcome is a set of admissible reconstructions, rather than an attribution of one uniquely determined generative history.

7.7 Hidden-State Memory and Effective Generative Dynamics

History dependence admits a sharper operational test when the model distinguishes a sufficiently specified internal state from its recorded observation. Consider a finite hidden-state transition system with deterministic observation map $\pi : X \to Y$. Even if its law on X is Markovian, observations in Y need not be Markovian, because the distribution of hidden configurations within one observation class can depend on the preceding observations. The resulting dependence is a property of the chosen coarse description under the specified probability law, rather than an independent claim that the underlying system has acquired a new ontological direction.

A finite deterministic illustration makes the issue precise. Take hidden states p, q, x_1, x_2, b, c, with paths $p \to x_1 \to b$ and $q \to x_2 \to c$. Let the observational images be P, Q, A, A, B, C, respectively. Assume the initial distribution assigns positive weight to both p and q, and that transitions along the displayed paths occur with probability one. The observed history (P, A) is followed by B, while (Q, A) is followed by C. Both histories have the same current observation A, but give different conditional distributions for the next observation. Thus a single time-homogeneous one-step transition probability on Y cannot reproduce both conditional predictions. The hidden-state system itself remains a first-order Markov model. The example is a logical diagnosis of effective memory, with no empirical frequencies or claims about physical irreversibility.

Under a specified hidden Markov kernel $P(x, x')$, observation map π, and initial distribution, a conditional belief $\mu_t(x) = \Pr(X_t = x \mid Y_0, \ldots, Y_t)$ can summarize the information from observed histories. Whenever the new observation has positive conditional probability, the deterministic-observation filtering update is defined in Equation 29.

$$\mu_{t+1}(x') = \frac{\mathbf{1}\{\pi(x') = y_{t+1}\} \sum_{x \in X} \mu_t(x) P(x, x')}{\sum_{z \in X} \mathbf{1}\{\pi(z) = y_{t+1}\} \sum_{x \in X} \mu_t(x) P(x, z)}. \tag{29}$$

Equation 29 describes an epistemic state updated under declared dynamics and observations. It requires a known or explicitly estimated transition kernel and a correctly specified observation mechanism. Belief-state modeling in partially observable decision processes is well established [30]. The present example uses the idea to identify when a proposed generative order over the observed categories has silently discarded information relevant to continuation.

One response is to refine the state description. In the toy model, retaining the immediately preceding observed category alongside the current category distinguishes the two histories, although such a finite-memory enlargement is not sufficient for every partially observable process. Another response is to retain a distribution over hidden states or a set of candidate histories; the resulting effective representation may be infinite-dimensional or computationally demanding. A third is to narrow the inference to statements that are robust over the hidden configurations compatible with the record. These approaches carry distinct commitments about observability, computation, and the empirical adequacy of the hidden-state law.

The example also separates *memory of a process* from *uncertainty about the process*. A known hidden-state kernel can yield non-Markovian observations because different histories induce different hidden-state beliefs. A family of unknown kernels introduces a further epistemic uncertainty of the type examined in Section 5.7. The latter cannot be resolved merely by including one additional observed lag, and composing candidate-model edges may still produce impossible trajectories. For dynamically changing generative rules, the regime variable may need to be represented as part of an extended state, with explicitly allowed transitions between regimes.

These distinctions make the link between history and derived order conditional. Reachability over fully specified states, reachability over observed categories, and predictions conditioned on an observed history can disagree while each remains mathematically well defined. In physically instantiated models, evidence for a hidden memory variable must come from mechanisms or tests adequate to the domain: material inventories, physiological adaptation, environmental persistence, or controlled experiments may offer different routes. Ontological suspension remains appropriate because a successful history-sensitive representation establishes predictive or explanatory adequacy under its stated conditions rather than a fundamental metaphysics of temporal becoming.

7.8 Comparative Applications and Transition Criteria

The formal constructions can guide several kinds of inquiry while leaving substantial domain-specific work to be done. In a rewriting calculus, a rule extension can be defined exactly and the potential-inclusion result can be mechanically checked for finite derivations. If the language of terms or types changes, a translation preserving well-formedness becomes necessary. A formal demonstration in this setting establishes a property of a chosen calculus; extrapolation to cognition, biology, or physical evolution requires an additional argument connecting the formal rules to the target domain.

For a material production system, the landscape may include configurations of machines, inventories, energy supplies, and permissible assemblies. A newly available fabrication operation expands a set of designs under a reusable-premise abstraction. It may simultaneously require scarce materials, maintenance, or safety constraints that contract feasible long-run operation. The observation that a capability has appeared becomes scientifically meaningful only with a description of its provenance, enabling conditions, and operational limits. An engineering model with physical conservation and consumptive resources may be more appropriate than pure closure for assessing actual production.

An ecological system can exhibit changed compatibility relations and available transformations as species interactions, boundary conditions, or environmental regimes vary. Comparisons must account for distinct explanatory timescales, evidence uncertainty, and the difference between representationally recognized possibilities and observed ecological effects. A network representation with time-dependent active edges may capture part of this variation, while a richer process model is needed when the dynamics modify the nodes, species types, or formation rules themselves. The source of any proposed historical direction must come from the relevant ecological and evolutionary mechanisms.

Physical effective descriptions provide a further, sharply bounded comparison. Changing scale can

change the couplings and observables retained in a model, and renormalization relates such descriptions within specific theories [49]. A generic relation W_{ij} between landscapes is insufficient to establish that it represents a renormalization-group transformation. Such an identification would require a defined physical model, a coarse-graining map or flow, consistency with its symmetries and dynamics, and an account of relevant observables. Section 5 already cautioned against inferring concrete paths from abstract coarse-grained paths. Section 7 adds that a physical change of generating constraints and a change of effective description are separate hypotheses, even when both alter the mathematical ordering of the represented configurations.

A practical research programme can evaluate landscape transformations by identifying the intended unit of change; recording which aspects of the configuration space, rule family, resource condition, and observational interface vary; and declaring the correspondences through which comparisons are made. Path simulations and rule-compatibility checks can test whether proposed precedence relations survive the transformation. Potential profiles can be compared only after target domains and background conventions are aligned. Finally, repeated observations and designed interventions can discriminate an actual rule change from a change in the model or measurement. Failure of the proposed correspondence, an unstable ranking under reasonable translations, or a history dependence omitted from the state description should narrow the associated interpretive claims.

The hidden-state diagnostic additionally establishes that a first-order Markov model on fully specified states can yield history-dependent predictions under coarse observation; any reconstructed effective order must therefore declare its state and memory conventions. The section's result is a conditional framework for evolving generative landscapes. It provides a method for distinguishing internal transitions, transformations of the possibilities of transformation, and changes in their representation. It also establishes limited path-preservation and rule-inclusion results, clarifies the reorganization of equivalence and order, and identifies historical dependence as a possible source of differing potentials under apparently identical observations. These findings are mathematical and methodological. They do not imply a general tendency toward greater generativity, a privileged direction of time, or the unique recovery of a physical ontology from observed changes. Section 8 will examine a narrower possibility: transformations that produce critical reorganizations of derived order and whose relation to symmetry breaking, phase transitions, and scale changes can be tested in specified physical models.

8 Critical Transitions and the Reorganization of Order

The structural analysis of generative landscapes distinguishes transformations of admissibility, rules, representation, and observation. A more specific issue concerns transitions at which such changes reorganize the derived order qualitatively. This section studies that issue through parameterized transition systems, elementary stability results, and limited comparisons with equilibrium and dynamical critical phenomena. It first defines a criterion for structural change in a derived preorder, then examines transition motifs, symmetry, thermodynamic criticality, and renormalization. It closes with a diagnostic procedure for deciding which kind of transition a proposed example actually supports. The mathematical results concern explicitly specified formal systems; physical applications require independent models of observables, dynamics, and limiting behavior.

8.1 Parameterized Landscapes and Structural Transition Criteria

Let Λ be a parameter space and consider a family of candidate landscapes $\{\mathfrak{L}_\lambda\}_{\lambda\in\Lambda}$ equipped with paired conditioning regimes. The parameter may encode a coupling, a resource threshold, a control intervention, a boundary condition, or a choice of effective description. Its interpretation is an additional empirical commitment. For clarity, the first construction holds the finite configuration domain C fixed and assigns each λ a unary transition relation $T_\lambda \subseteq C \times C$. The reachability preorder T_λ^* and mutual-reachability equivalence E_λ then determine the quotient partial order $\mathscr{O}_\lambda = (C/E_\lambda, \preccurlyeq_\lambda)$ as established in Section 4.

If the relevant rules are multi-input, the construction can instead use resource-pool closure and its inclusion preorder; a unary reduction then requires an expressly defined operational semantics.

For two parameter values, structural similarity requires more than equal numbers of configurations. An *order isomorphism* between $\mathscr{O}_\lambda$ and $\mathscr{O}_\mu$ is a bijection between their quotient classes that preserves and reflects the respective orders. In an application involving named configurations, measured observables, or physically interpreted sectors, a suitable comparison may require the bijection to respect further labels and a declared cross-landscape correspondence. A purely unlabeled order isomorphism can conceal a substantial redistribution of physical roles; conversely, a newly introduced description may divide a previously coarse observational class without an underlying change in the target process.

For a real parameter interval, call $\lambda_\star$ a *structural transition point* for the declared family of order structures when arbitrarily close parameter values on the considered domain yield different order-isomorphism types. A weaker, correspondence-sensitive criterion can register changes in specified precedence comparisons even when the abstract unlabeled poset remains isomorphic. These definitions provide precise meanings for a qualitative reorganization of modeled order. They leave open whether $\lambda_\star$ is a critical point in statistical mechanics, a bifurcation of a dynamical system, an intervention threshold, or an artifact of model selection. A physical identification needs the additional signatures appropriate to its discipline.

The parameter itself carries no necessary temporal orientation. The family Λ may consist of experimental alternatives, spatially varying conditions, proposed cosmological regimes, or different mathematical models. Traversing a parameter interval is a comparison operation until a theory supplies an actual process that changes the parameter. Likewise, a directed edge in T_λ expresses the stipulated meaning of the modeled transformation; it does not by itself install a fundamental arrow of physical time.

8.2 Local Stability of Derived Precedence

A sharp alteration of an order relation can follow from the switching of an admissible transition. Whether such switching is inevitable near a given parameter value depends upon the regularity of the rule that determines admissibility. Consider a finite candidate-edge set $E_0 \subseteq C \times C$ and a collection of continuous real-valued functions $f_e : \Lambda \to \mathbb{R}$, one for each candidate edge. For $\Lambda \subseteq \mathbb{R}$, define the active relation through the strict threshold rule of Equation 30.

$$T_\lambda = \{e \in E_0 : f_e(\lambda) > 0\}. \tag{30}$$

The threshold values can represent a formal decision boundary for licensing a transformation. They need not coincide with a physically measurable energy barrier, rate, or actual event probability. For example, representing a stochastic transition as present whenever its estimated probability exceeds a fixed cutoff adds a measurement convention that can create threshold changes even when the probability varies smoothly.

Proposition 8.1 (Local constancy away from activation boundaries)**.** *Suppose C and E_0 are finite, Λ contains an open neighborhood of λ_0, and each f_e in Equation 30 is continuous at λ_0. If $f_e(\lambda_0) \neq 0$ for every $e \in E_0$, there exists a neighborhood U of λ_0 on which T_λ, its reachability preorder, and its mutual-reachability quotient order are constant.*

Proof. For each candidate edge e, continuity and $f_e(\lambda_0) \neq 0$ give a neighborhood U_e on which f_e retains its sign. The finite intersection $U = \bigcap_{e \in E_0} U_e$ is a neighborhood of λ_0. The membership of each candidate edge in T_λ is therefore the same throughout U. Consequently the active relation, its reflexive transitive closure, and the equivalence and order derived from that closure are unchanged. □

Proposition 8.1 gives a useful diagnostic: a continuously parameterized finite threshold model can change its transition structure only at an activation boundary, provided the candidate domain and decision rule remain fixed. The result is a fact about the chosen combinatorial representation. Different decisions about edge thresholds, additional latent configurations, or a change of generative regime can move the apparent transition boundary. When the edge set is infinite, sign stability for each separate edge need

not yield one shared neighborhood; accumulation of switching thresholds can generate more intricate behavior. When configurations themselves change, common-domain comparisons and typed translations become necessary.

The proposition also illustrates a methodological tension between statistical smoothness and categorical comparison. A weighted edge may vary continuously while the binary relation formed by thresholding it changes discontinuously. One should therefore retain the continuous weights and their uncertainty when they carry empirical meaning, while reporting the chosen threshold and sensitivity of the inferred order. Calling the resulting discontinuity an ontological rupture would exceed what the formal construction warrants.

8.3 Cycles, Quotients, and Order-Reorganization Motifs

A small transition network shows how a single new edge can reorganize an entire quotient order. Let $C = \{a, b, c\}$, with $a \to b$ and $b \to c$ active for every parameter value. Introduce $c \to a$ precisely for $\lambda > 0$. For $\lambda \leq 0$, the mutual-reachability classes are the three singletons, ordered as a three-element chain. For $\lambda > 0$, the directed cycle makes all three configurations mutually reachable; their quotient is a one-element partial order. The cardinality of the underlying configuration space remains fixed, while the order-isomorphism type changes at the edge-activation boundary. In this toy example, the formal relation is sufficient to establish the reorganization, yet it contains neither a thermodynamic limit nor a measured singularity.

The contrast becomes stronger if one attaches heterogeneous roles to the configurations. Suppose a represents an input arrangement, b an intermediate assembly, and c a completed device that can be disassembled into the original arrangement once a new recycling operation is permitted. Mutual reachability then expresses the availability of paths under a stipulated reusable-premise semantics; the physical feasibility of continual recycling depends on resources, dissipation, material losses, and admissibility restrictions. If the new operation consumes an essential reagent, the one-element quotient may be an artifact of omitting resource state. A more faithful expanded configuration space could preserve distinctions the simplified graph has eliminated.

Order changes can also be expressed through growth of the transition relation. For fixed C, if $T_\alpha \subseteq T_\beta$, then $T_\alpha^* \subseteq T_\beta^*$. Consequently, two configurations already mutually reachable under T_α remain mutually reachable under T_β. Distinct strongly connected classes may merge as edges are added, whereas a split requires that some transitions be removed or the domain or comparison semantics change. This elementary monotonicity refers to the equivalence partition induced by reachability. The quotient orders at different parameters need not be related by a simple inclusion, since their underlying equivalence classes can differ.

Qualitative reorganization admits several structural patterns: the creation of a new comparable pair; loss of a comparison under rule withdrawal; merger or splitting of mutual-reachability classes; reorientation of a comparison under changed rules; and changes in the incomparability pattern of local potential profiles. Each pattern has distinct implications for an interpretation of generative levels. The profile order in Section 6 may remain stable while the reachability quotient changes, or it may change under an altered target domain while the transition graph stays fixed. Maintaining these distinctions prevents the phrase *level transition* from grouping dissimilar mathematical events under a single explanatory mechanism.

8.4 Symmetry, Stabilizers, and the Status of Equivalence

Symmetry provides another route through which the organization of relational descriptions may change. Let a group G act on a configuration space C, and suppose a physical or formal rule family is equivariant under that action in a specified regime. An individual configuration c has a stabilizer $G_c = \{g \in G : g \cdot c = c\}$. A change in the stabilizer type of selected solutions, equilibrium states, or dynamically preferred configurations can organize a comparison of phases or branches, provided the group action and the meaning of the selection are defined. In a physical example, a globally symmetric law can admit

ordered states whose individual stabilizers have less symmetry than the law. Earman's philosophical account emphasizes the importance of the physical and interpretive setting in connecting symmetry and objectivity [14].

Three distinguishable relations matter here. First, symmetry of the *law* or rule schema concerns transformations preserving its form. Second, symmetry of a *selected state* concerns the subgroup leaving that state unchanged. Third, an *equivalence relation* used to quotient descriptions depends on which transformations the particular theory regards as representational redundancy. The reduction of a state stabilizer need not change the symmetry group of the law. Nor does a change in stabilizer automatically determine a new reachability order: the latter additionally depends on the allowed transitions among configurations. A physical order parameter may discriminate phases while remaining conceptually separate from the generative-precedence relation of this paper.

The global-versus-gauge distinction is especially consequential. Local gauge transformations are ordinarily treated as redundancies of a physical description; gauge-related representatives should be identified at the appropriate stage of constructing physical states or observables. Elitzur's lattice-gauge result establishes, within its stated setting, a restriction on spontaneous breaking of unfixed local gauge symmetry [16]. It is therefore inappropriate to use an arbitrary change of gauge representative as evidence that distinct physical configurations or generative levels have appeared. Gauge-fixed descriptions, gauge-invariant order parameters, boundary effects, and other precise formulations require their own physical analysis. The present framework makes no general inference from symmetry breaking to an ontologically fundamental precedence relation.

Suppose an observational map records only an invariant under a group action. Then different configurations may be observationally indistinguishable while retaining distinct transition neighborhoods or compatibility relations. Section 3 established the need for congruence-like preservation conditions before multi-input generating rules descend exactly to a quotient. A symmetry change can therefore interact with order reconstruction in at least two ways: by altering physically admitted transformations, and by changing what the observational interface resolves. Discriminating these mechanisms is essential when a seemingly new hierarchy emerges from an experimental record after symmetry-sensitive measurements are introduced.

8.5 Thermodynamic Criticality and Dynamical Transition Mechanisms

In equilibrium statistical mechanics, criticality has a more specific content than the appearance or disappearance of an edge in a graph. A physical model normally specifies microscopic degrees of freedom, interactions, an ensemble and its control parameters, and observables whose behavior is studied under a suitable thermodynamic limit. Near continuous phase transitions, singular behavior of thermodynamic functions and associated scaling relations are central topics of critical-phenomena theory [21]. The meaning of critical exponents, correlation length, and universality depends on the model and the quantity being measured. A finite system can show pronounced crossover behavior without the ideal singularities associated with an infinite-system limit.

A generative-landscape interpretation of such a model would have to decide what the configurations are and which transformations count as generation. Microstates, mesoscopic blocks, thermodynamic phases, fluctuations, and trajectories are related but different domains. A spin flip allowed by a dynamical updating rule is not automatically a causal production relation between phases; a transition between thermodynamic phases is not automatically a generative precedence ordering. In equilibrium systems satisfying detailed balance, configurations can communicate in both directions within a stochastic dynamics even when their equilibrium probabilities differ greatly. Using mutual reachability alone might collapse nearly all accessible microstates into one class, leaving little of the phase structure to be explained. More discriminating constructions may involve metastability, coarse-grained sectors, transition timescales, large-deviation rates, or observables with appropriate limiting behavior.

Dynamical critical phenomena likewise require distinctions among equilibrium statics, relaxation, conserved quantities, and the kinetic laws governing fluctuations. Hohenberg and Halperin classify

dynamic critical behavior in specified classes and relate critical scaling to hydrodynamic and kinetic structure [26]. Their account supports the methodological requirement to include actual dynamics when discussing critical slowing down or the response of a system to perturbations. A generic order relation contains no relaxation times, noise kernels, conservation constraints, or experimentally estimated critical exponents. Those objects must be supplied through domain-specific modeling before the framework can make quantitative predictions.

Some systems exhibit abrupt changes in the set of attractors, basins, or viable trajectories as parameters vary. Such transitions can motivate changes in a generative potential profile, especially when the profile is defined over a finite horizon and under explicit feasibility restrictions. However, a change of stability, a topological rearrangement of basins, and a nonanalytic equilibrium phase transition are different mathematical events. They can coincide in a carefully studied model and can separate in another. A useful claim about critical reorganization should specify the observable signatures and the theory linking them, rather than inferring a physical mechanism from a resemblance in diagrams.

8.6 Renormalization and Scale-Relative Generative Structure

Renormalization offers a disciplined comparison because it relates effective descriptions through carefully defined changes of scale. Kadanoff's block-spin perspective uses collective variables on cells larger than microscopic scales and smaller than the correlation length in the relevant setting [29]. The Wilson–Kogut review develops renormalization-group methods for critical behavior and scaling [49]. These approaches provide an established framework for understanding how descriptions at different scales can retain selected physical information while changing their couplings and degrees of freedom. The present relational formalism is at most a candidate language in which particular scale comparisons could be represented.

A proposed scale map $\pi_{\ell \to L}$ between microscopic and effective descriptions should specify the observable variables retained, the ensemble or dynamics represented, and the rules for obtaining effective interactions. The exact form of a renormalization flow depends on the physical model and coarse-graining procedure. In a generative landscape, the temptation is to treat a coarse-graining projection as an order-preserving morphism automatically. The projected-path counterexample in Section 5 shows why that inference fails for arbitrary transition graphs: concatenating coarse transitions can join different representatives of an intermediate coarse class. An effective physical dynamics may possess a valid autonomous description under additional assumptions, but such autonomy is a result to establish, rather than a property inherited by every quotient.

Fixed points of a renormalization-group flow and scaling operators describe scale behavior within a specified theoretical construction. Their eigenvalue classifications, including relevant and irrelevant directions, refer to how perturbations transform under repeated coarse-graining. They should not be conflated with generative-precedence comparability or with a hierarchy of ontological importance. A variable that is irrelevant for the critical fixed point may still matter for boundary conditions, finite-size behavior, or other questions. Likewise, an effective level that is useful for prediction may have neither a unique microscopic reconstruction nor a privileged metaphysical status. Goldenfeld's pedagogical account underscores the role of model assumptions and scale-dependent reasoning in the theory of phase transitions [24].

One way to examine a proposed connection is to require a commutation or controlled approximation between admissible microscopic evolution and effective evolution. If a candidate diagram fails to commute, one should determine whether the mismatch arises from eliminated memory, changed noise statistics, hidden interactions, or an invalid assumption about composable paths. The investigation can then decide whether the coarse description supports an exact transition relation, an approximate stochastic kernel, a memory-bearing effective model, or only an observational summary. A claim that a generative order *emerges under renormalization* is scientifically substantive only when the microscopic and effective objects, their translations, and the evidence for the claimed order are jointly specified.

Table 8: Levels of claims about reorganized generative order and their principal tests. The rows distinguish formal, observational, physical, and ontological interpretation.

Claim type	Evidence or demonstration required	Principal alternative to exclude
Formal structural transition	Explicit rules and a change in precedence or quotient-order type	Threshold or domain convention
Observed order change	Repeated measurement and stable reconstruction under justified interfaces	Changed resolution or missing state variables
Symmetry reorganization	Defined group action, invariant dynamics, and state-sensitive observables	Confusion of gauge and physical equivalence
Physical critical phenomenon	Model-specific scaling, singular or finite-size signatures, and limiting assumptions	Crossover or noncritical bifurcation
Effective scale reorganization	Valid coarse-graining map and tested effective dynamics	Fabricated paths or omitted memory
Ontological interpretation	Explanatory discrimination among competing physical accounts	Reification of an effective representation

8.7 Operational Tests for Proposed Order Transitions

The distinctions above suggest a comparative diagnostic rather than a universal criterion of criticality. Table 8 identifies the minimal evidential commitments appropriate to several transition claims. A single example can legitimately satisfy more than one category; that conjunction should be established by separately checking the relevant criteria.

For a finite formal model, the initial test is often algorithmic. A researcher can enumerate parameter-dependent transitions, compute strongly connected components and the quotient order, and locate switching boundaries. The exercise can be accompanied by perturbations of the thresholds and by alternative state granularities. For stochastic models, estimated transition probabilities and uncertainties should be retained in addition to thresholded graphs. A sensitivity analysis can show whether an alleged transition survives alternative credible operationalizations. For continuously varying parameters, a finite-size analysis or a study of scaling with system size may be required before a thermodynamic interpretation becomes warranted.

For a physical experiment, the hypotheses should be formulated before the indicators are selected. One candidate explanation may posit a new dynamical channel; another may posit a change in measurement resolution; a third may posit an effective coarse description whose memory dependence was omitted. Such hypotheses can sometimes be discriminated by targeted perturbations, repeated observations across scales, and independent measurement of enabling conditions. The same reconstructed order under different observation models may support a limited invariance claim; disagreement can motivate refinement rather than immediate rejection of the research programme. The identifiability limits identified in Section 5 remain active throughout this procedure.

The formalism also supplies explicit revision conditions. An alleged structural transition should be redescribed if it vanishes under a better grounded admissibility rule. A claimed phase transition should be narrowed to a finite-system crossover if the required scaling or limiting evidence is absent. A proposed symmetry change should be reinterpreted if it concerns different gauge representatives of the same physical state. A derived hierarchy should remain model-relative if several empirical models with incompatible order structures fit the observations equally well. These outcomes are productive constraints on explanation: they identify the information needed to make claims about order more discriminating.

8.8 Interpretive Boundaries and the Ontological Question

The present section identifies a precise and limited sense in which generative order can reorganize discontinuously. A family of admissible transformations can change its reachability classes and resulting

partial order at a structural threshold. In a finite model, local constancy follows under explicit continuity and nonvanishing-margin assumptions, while abrupt quotient changes can result from the activation of a single edge. These statements concern derived mathematical structure. Their proofs are independent of claims about material phase transitions, conscious experience, or a fundamental architecture of spacetime.

Physical criticality becomes relevant when a particular target theory supplies a dynamical or statistical-mechanical interpretation that survives empirical and theoretical scrutiny. Symmetry, critical exponents, relevant perturbations, and effective levels have established uses in physics, yet they describe different mathematical and operational objects. The relational programme may help compare their structural roles while preserving the distinctions that make each theory informative. Ontological interpretation remains an additional step: one must assess whether the order expresses a physically stable relation, an effective explanatory organization, a contingent measurement convention, or another suitably justified possibility.

The next section, *Ontological Interpretation and Physical Consistency*, will examine how these conditional constructions bear on philosophical accounts of emergence and on the compatibility of generative precedence with established physical constraints. That inquiry will treat causal, representational, and ontological order as distinct candidates for interpretation and will preserve the possibility of multiple adequate descriptions. The main conclusion of Section 8 is methodological: a critical-looking reorganization of a derived order becomes informative when its generating rules, equivalence criteria, observables, limiting assumptions, and alternative explanations are made explicit.

9 Ontological Interpretation and Physical Consistency

The mathematical derivation of a preorder from admissible transformations establishes a relation within a specified generative landscape. Assigning physical or ontological significance to that relation introduces further questions about the interpretation of configurations, the status of equivalence, the laws governing transformations, and the information recoverable from observations. This section evaluates those questions through comparisons with causal structure in relativistic spacetime, gauge equivalence, reversible dynamics, quantum and effective descriptions, and philosophical accounts of emergence. The method is a constrained comparison of formal requirements and physical commitments. Its objective is to identify which components of a derived-order model might receive a physically defensible interpretation, and which remain features of the chosen representation.

9.1 Levels of Interpretation and the Burden of Physical Identification

A derived relation can be interpreted at several levels with different conditions of success. At the *mathematical level*, a relation is fixed by an explicitly specified domain, generative rules, closure operation, and equivalence criterion. The proofs in Sections 4–8 concern properties of such constructions: reflexive transitive closure induces a preorder; mutual reachability yields a partial order on equivalence classes; and changes in admissible transformations can reorganize the resulting quotient. These results have validity within their assumptions irrespective of whether any corresponding physical system exists.

At the *representational level*, configurations and transformations are candidates for describing features of some target. The analyst must specify what a configuration denotes, what it means for an operation to be admissible, which resources and boundary conditions are held fixed, and how observations can distinguish relevant differences. The distinction is consequential because two descriptions can generate the same quotient poset while encoding substantially different mechanisms. The reconstruction analysis in Section 5 shows that coarse observation can also introduce apparent paths absent from the underlying transition system. A mathematical relation may therefore be unambiguous within a model while its mapping onto the target remains partly identified.

At the *explanatory level*, a derived order earns support by identifying constraints, enabling conditions, or differences in possible transformations that matter for independently accessible phenomena.

Such support can include predictions about intervention-dependent feasibility, preservation of transition structure across changes of representation, or successful reconstruction of trajectories from partial observations. The adequacy of an explanation depends upon the problem being explained. A chemical reaction network, a computational rewrite system, and a physical causal relation may each possess order-like structures while assigning different meanings to dependence and reachability. Structural resemblance alone supports a comparative hypothesis, with further interpretation depending on the laws and evidential practices of the relevant discipline.

At the *ontological level*, an additional commitment is made concerning the constituents or organization of reality. One might propose that the order is primitive, that it emerges from more basic physical relations, or that it is a scale-dependent but objective feature of a physical regime. Another account may regard the order as a useful representational abstraction without assigning independent reality to its classes. These positions cannot be settled by a proof of antisymmetry or by the availability of a quotient. Their relative merits depend on an account of physical identity, explanatory scope, and empirical consequences. The present investigation preserves these alternatives as live interpretive possibilities, in keeping with its ontological suspension.

A useful way to expose the additional commitment is to specify a physical interpretation map. Let $\mathscr{O}_\kappa = (C/E_\kappa, \preccurlyeq_\kappa)$ be the quotient order derived under a declared regime, and let $\mathcal{P}_{\text{phys}}$ represent physically individuated states, events, processes, or structures within an independently specified theory. An interpretation $\iota : C/E_\kappa \to \mathcal{P}_{\text{phys}}$ requires criteria for physical identity and an independently justified target relation $\trianglelefteq_{\text{phys}}$. A particularly demanding correspondence would satisfy Equation 31.

$$[c]_{E_\kappa} \preccurlyeq_\kappa [d]_{E_\kappa} \quad \Longleftrightarrow \quad \iota([c]_{E_\kappa}) \trianglelefteq_{\text{phys}} \iota([d]_{E_\kappa}). \tag{31}$$

Equation 31 is a proposed standard for faithful interpretation, rather than a property supplied by the generative-landscape axioms. Depending on the target theory, one might require only preservation in one direction, observational agreement within tolerance, or correspondence among selected domains. Injectivity, surjectivity, and the physical meaning of the relation would then need separate specification. In particular, a relation introduced by defining the right-hand side to mirror the left would offer little independent physical warrant. The evidential value arises when the target relation has support external to the formal construction.

9.2 Structural Realism and the Interpretation of Relational Invariants

The interpretation of a generative landscape intersects with structural realism, yet the intersection has several distinct philosophical forms. Worrall's proposal about structural continuity across scientific theory change concerns the possible persistence of mathematically articulated relations when descriptions of theoretical entities undergo revision [50]. Ladyman's analysis distinguishes epistemic forms of structural realism from more ambitious ontic versions [32]. In an ontic structural-realist proposal, relational structure may receive an explicitly fundamental status, rather than serving primarily as the part of a theory to which a realist's epistemic commitment is restricted. French develops a metaphysical defence of this latter position in terms of physical laws, symmetry, and modality [22]. These are substantive positions in the philosophy of science with different commitments concerning objects and relational structure. G1's formal vocabulary is compatible with their discussion while leaving their central realist commitments undecided.

For a generative landscape, three kinds of structure can be compared: the mathematical relations specified at the descriptive level, the invariants preserved under the declared maps between descriptions, and the physical relationships supported by an independently motivated theory. They need not coincide. Two distinct transition systems can have isomorphic mutual-reachability quotient posets while differing in the number of concrete configurations, the organization of joint prerequisites, or the trajectories available under an intervention. The poset is consequently an invariant of a selected construction; its isomorphism type is a comparatively coarse invariant of the full generative landscape. The preservation of this invariant across an empirical theory change could be explanatorily significant, but the invariance alone does not establish which underlying constituents, mechanisms, or modalities are real.

The distinction has implications for the use of relational language. The existence of a relation on a mathematical set entails neither the metaphysical priority of relations over relata nor the physical redundancy of the relata. Such claims need further arguments about the individuation of physical states, permissible transformations, and explanatory dependence. Likewise, an observational equivalence relation need not reproduce physical gauge equivalence: its classes may reflect limited measurement rather than a physical identification. In G1, ontological suspension keeps open an epistemic reading of preserved structure, an effective realist interpretation under independent physical warrant, and stronger ontic proposals. These interpretations can be compared by specifying what must remain invariant, the observation or intervention by which the preservation is assessed, and the additional physical commitments required by each position.

9.3 Ontological Dependence and Operational Generative Precedence

A second philosophical comparison concerns metaphysical grounding, relative fundamentality, and relations of ontological building. Schaffer presents grounding as a central question about what is fundamental within an ontology [43]. Fine's account distinguishes explanatory varieties of ground, whose interpretation depends on the kind of dependence under consideration [20]. Bennett examines a family of building relations, including grounding, composition, and realization, and characterizes their connection with questions of fundamentality [8]. This literature provides an established philosophical treatment of dependence and priority that should be distinguished from the operational reachability of G1.

The difference can be stated precisely. Let a configuration be reachable from another under a stipulated set of admissible operations. This is a claim about the existence of a licensed derivation within a model. Reflexivity follows from allowing an empty derivation; transitivity follows from the modeled composition of paths. Both claims are formal consequences of reachability conventions. The corresponding metaphysical claim that one fact grounds another concerns an explanatory and ontological relationship whose meaning has to be supplied independently. A change of stipulated transition rules may change reachability while leaving the truth of a proposed grounding claim unaddressed. Conversely, the grounding of a fact may be proposed without a corresponding executable operation in the G1 landscape.

The quotient order of mutual reachability does not erase the difference. Its antisymmetry is obtained by identifying configurations that can reach one another, while many accounts of metaphysical building posit an asymmetric explanatory relation between appropriately individuated relata. This quotient construction may contain minimal elements without supplying a unique metaphysical foundation; an infinite quotient order need not be well-founded. Even in finite examples, a minimal equivalence class expresses a property of the specified transition relation, not an unconditional claim about what exists most fundamentally. A modelling decision to label such a class *foundational* would therefore carry an additional interpretive commitment.

This comparison also helps distinguish productive causation from ontological dependence. The joint-premise closure in Section 4 concerns derivability with available inputs and declared resource conventions. Such derivability might be used to model production, computation, intervention, or a component of causal dynamics. Grounding, constitution, causal influence, and generative feasibility may exhibit overlapping dependency patterns while obeying different laws of composition and identity. A useful connection would require a domain-specific bridge principle identifying which modeled generative steps warrant which kind of dependence. The present paper provides a language in which the bridge can be stated, and counterexamples that make its validity open to assessment.

9.4 Relativistic Causality and the Distinctive Commitments of Causal Sets

Relativistic causal structure provides an especially demanding comparison because it assigns physical content to a precedence relation. In suitable relativistic spacetimes, the light-cone structure constrains possible causal influence and is connected with conformal geometry. Results concerning causal or chronological structure establish strong reconstruction relationships under specified spacetime assumptions; for example, Malament's work is an important part of the literature on the relation between time-

like structure and spacetime topology, with qualifications involving causality conditions and the type of structure preserved [33]. Such results show that an order-like relation can have substantial geometric information content. They also demonstrate why the identity of the relation and the hypotheses of its reconstruction theorems matter.

Causal-set theory makes a distinctive further proposal. Its basic structure is a locally finite partially ordered set intended to represent a proto-causal relation, with order and cardinality connected respectively to causal structure and spacetime volume in suitable continuum approximations. The programme develops questions of manifoldlikeness, dynamics, Lorentz-invariant discretization, and the reconstruction of continuum geometry [9, 46]. Within this approach, the partial order and local finiteness belong to the physical proposal rather than being obtained simply by forming a quotient of an arbitrary transition network. Some causal sets admit no straightforward continuum interpretation, and the existence of a partial order alone supplies neither a spacetime dimension nor a physically adequate continuum limit.

By contrast, G1 begins from a specified collection of candidate configurations and relations of admissible transformation or generation. The quotient by mutual reachability is an order-theoretic operation whose inputs can be provided by many models. It typically encodes a relationship between descriptions of whole configurations or resource pools, whereas causal-set order is intended to relate elements interpreted as elementary events. The distinction between a configuration and an event is therefore a substantive translation problem. A state transition may involve many events, a causal relation may hold without corresponding to a directly executable state transformation, and multiple resources may be jointly necessary to license a generative step. In a relativistic theory, a putative influence relation also needs compatibility with its light-cone constraints. Defining an arrow between two mathematical configurations is insufficient to establish those properties.

A physically informative connection would require a mapping between types of objects as well as preservation of relevant physical relations. For a causal-set comparison, one would need to explain what counts as an event, what constitutes the physical causal relation, how local finiteness or an appropriate substitute is established, and which dynamical or observational predictions discriminate the proposal. If generative rules are claimed to produce a causal set, their composition would additionally need to yield a relation satisfying the required axioms with appropriate physical significance. If causal order is assumed as an ingredient of those rules, the investigation would instead concern consequences of an inherited structure. These two programmes have different explanatory ambitions and should be reported separately.

The causal-set comparison also illustrates an important asymmetry in explanatory direction. A derived-order construction can help organize dependencies among candidate models of spacetime; causal-set theory can use causal order as a primitive ingredient in constructing physical models. Both employ order-theoretic mathematics, yet the derivational roles differ. The contribution of G1 lies in making those roles explicit and in specifying conditions under which an apparently fundamental order might admit a conditional derivation. It supplies, at this stage, a formal methodology rather than a competing derivation of relativistic geometry.

9.5 Gauge Redundancy, Diffeomorphisms, and Relational Observables

Physical identity under symmetry introduces a second test of the proposed equivalence relations. The distinction established in Section 3 separates isomorphism of descriptions, equivalence under a selected group action, observational indistinguishability, and physical gauge equivalence. These relations may coincide for a particular physical theory, although their coincidence requires justification. In gauge theories, some changes of mathematical variables represent redundant descriptions of a physical situation. Identifying the corresponding physically equivalent states depends on the theory's constraints, observable content, and treatment of boundaries. Gauge-related descriptions therefore carry a meaning stronger than the existence of a combinatorial symmetry of a generative network [14, 16].

The hole argument in general relativity makes the philosophical stakes especially clear. Metric and matter-field descriptions connected by diffeomorphisms can raise questions about the identity of space-

time points, determinism, and the physical content of a model. The associated literature examines when differing mathematical assignments represent distinct physical possibilities and when their difference should be understood as descriptive redundancy [15]. Choosing to identify such descriptions is an interpretive position supported by particular features of the physical theory; it is not a generic consequence of the existence of a smooth transformation or a group action. Furthermore, a choice of coordinates and an active transformation of fields play different roles, even though their mathematical formulations can be related.

A generative landscape may assign a group action to candidate configurations. The action can produce orbits, and orbit classes can support an invariant representation. Their empirical standing depends on whether the transformations preserve the target's physically meaningful data and whether boundary or interface observables distinguish cases that the chosen quotient would merge. A transformation that is invisible to an observational instrument need not be a physical gauge transformation. Likewise, two descriptions may agree on selected invariants while differing in another observable or in the compatibility of their joint generative operations. Section 3 addresses the latter issue with an exact saturation requirement for quotienting multi-input relations.

Relational observables offer one route toward physically meaningful representations. In constrained physical systems, observable quantities may be characterized through relations among measurable quantities rather than by absolute labels; Rovelli's treatment of partial and complete observables provides a domain-specific comparison [41]. Background-independent approaches to quantum gravity likewise attend to the physical content of geometric variables and the role of gauge and diffeomorphism constraints [3]. These programmes supply substantive physical machinery. A general observation map $\mathcal{O}_q$ in G1 is an interface specified by the modeler and should not be identified with a Dirac observable, a relational observable, or a complete observable unless the corresponding conditions have actually been demonstrated.

One constructive consequence follows. If an order relation changes when merely descriptive gauge choices are varied, the proposed order lacks the invariance ordinarily expected of a physical quantity under those choices. Conversely, agreement across admissible gauge descriptions is a useful consistency test, while remaining weaker than a proof that the order is physically realized. In theories with boundaries or nontrivial gauge charges, even the determination of which transformations are redundant can be delicate. An adequate application should therefore identify its physical equivalence classes independently and examine whether the generative quotient descends to those classes without losing empirically relevant distinctions.

9.6 Reversible Dynamics and the Interpretation of Direction

Derived precedence has an apparent orientation: one configuration can be reachable from another under the declared transformations. A physical temporal arrow, however, requires additional structure. In many applications, the laws of motion permit reversible evolution at the relevant theoretical level. If the model contains both directions of each physically admissible transition, its directed reachability graph can place many configurations in the same mutual-reachability class. An impoverished quotient order may then be mathematically accurate for the model while being unhelpful for explaining irreversible macroscopic behavior. The choice of represented configurations, resources, histories, and boundary conditions can decisively affect which relation is obtained.

It is therefore helpful to distinguish the orientation of a transition rule, the direction of a selected solution trajectory, the ordering of events within a causal theory, and the statistical asymmetry found in thermodynamic phenomena. Reversing a mathematical transition edge is a change of formal relation; implementing a time-reversal operation on a physical theory may additionally transform momenta, fields, or other variables. A reversible law can exhibit trajectories whose observed macroscopic behavior has a preferred thermodynamic direction under appropriate conditions. Philosophical investigations of the temporal arrow emphasize the role of asymmetric boundary conditions and the danger of presupposing in an explanation the temporal orientation it seeks to account for [40]. The present paper accordingly

treats orientation as something to be justified locally.

For example, consider a system that can reversibly assemble and disassemble a structure under controlled operations. At the level of a simplified assembly graph, the two configurations may be mutually reachable. In a larger physical description, the assembly and disassembly may involve energy exchanges, environmental changes, catalysts, or apparatus states that the simple graph suppresses. A model recording the full combined system can distinguish the conditions under which each operation occurs. The answer to whether there is a preferred generative direction may therefore change with the boundary selected for the system and with the conditioning regime. Neither an assumed universal arrow nor an assumed universal reversibility follows from the availability of a graph model.

History-dependent models introduce another complication. Two configurations that are observationally equivalent at a chosen resolution can have different enabling resources or different possible continuations because of their distinct histories. Sections 6 and 7 describe such cases through conditional potential profiles and changing landscapes. If a physical application requires memory, the configuration space must include enough of the relevant history or internal state to support the claimed dynamics. Otherwise a Markovian graph can produce apparent pathways obtained by conflating histories that cannot be jointly realized. The appropriate state description is therefore part of the explanation, not an incidental technical detail.

These distinctions also preserve the non-privileging of temporal location adopted by G1. A generative relation may encode conditional dependence among possibilities without asserting an absolute present from which a unique future unfolds. When a physical theory supplies a time orientation or directed boundary-value problem, that orientation may legitimately enter the local model. When a comparison concerns alternative parameter regimes or mutually compatible constraints, a temporal narrative may add assumptions the analysis does not require. The relevant explanatory question is which physical or operational commitments license the direction of the relation.

9.7 Quantum Descriptions and the Limits of Reachability Semantics

Quantum theory presents a further boundary to classical generative reachability. A deterministic relation between configurations records which transformations a model licenses. Quantum descriptions can instead involve amplitudes, incompatible measurement contexts, interference, and correlations whose meaning depends on a physical state space and dynamical operators. A graph that includes an edge whenever a transition has nonzero amplitude may be useful for certain combinatorial questions, but its reflexive transitive closure generally discards the quantitative and phase information required for quantum predictions. In particular, concatenating possible edges as though they were independently executable classical steps can misrepresent quantum processes when measurement, conditioning, or interference affects the relevant amplitudes.

The issue concerns the interpretation of the elements of C as well as the relation. Candidate configurations might denote basis states, measurement outcomes, density operators, experimental arrangements, or effective macroscopic sectors. These objects have different composition rules and different physical claims of identity. A quotient by a chosen observational equivalence may capture the information relevant to one experiment while concealing distinctions important for another. An effective order defined on decohered or coarse-grained sectors would require the physical approximations that make those sectors meaningful. Its validity need not extend to a fundamental ontology of quantum states.

Similarly, background-independent programmes cannot be assimilated to a single generic order formalism. The role of geometrical variables, constraints, dynamical amplitudes, and physically meaningful observables differs among approaches to quantum gravity [3, 46]. A proposal connecting G1 to such theories would need to identify the relevant quantum objects, specify a rule of composition or dynamics appropriate to the theory, and show what operational content is retained by its proposed preorder. The abstract possibility of representing quantum models by relational structures is a starting point for investigation. It leaves the principal mathematical and physical work to be performed.

This limitation also suggests a fruitful separation of research questions. A classical reachability rela-

tion can organize the support of transitions in a candidate model. A probabilistic model can attach transition probabilities, conditional on specified regimes and measurements. A quantum proposal may need amplitudes and composition operations sensitive to phase and contextual structure. These descriptions may exhibit related order-like features under particular approximations, but their connections should be established through explicit mappings and proofs. The preliminary G1 formalism offers a vocabulary for identifying the missing pieces of such a mapping rather than a substitute for them.

9.8 Emergence, Renormalization, and Cross-Scale Physical Robustness

The physical interpretation of order can also depend on scale. Sections 5 and 8 show that projection and parameter changes can alter a derived order, sometimes generating apparent pathways or merging mutual-reachability classes. In physical science, effective descriptions can nevertheless be robust and explanatorily successful even when they omit microscopic detail. The renormalization-group literature provides a sophisticated account of scale transformations and the behavior of effective theories under specified conditions [49, 29, 24]. These established constructions should be distinguished from a general map that merely forgets features of a generative landscape.

Cross-scale consistency calls for a declared relation between the variables, transformations, and observables at the scales under comparison. The simple existence of a projection $\pi : C \to \bar{C}$ leaves open whether it preserves admissible generation or creates spurious composite paths. Under the path-lifting conditions of Section 5, observed pathways can be connected back to concrete pathways. In other circumstances the relevant correspondence may be statistical, approximate, or restricted to particular observables. A proposed scale-relative generative order should specify which of these standards it intends to satisfy, alongside a domain of applicability and a sensitivity analysis for neglected variables.

Philosophical accounts of emergence offer complementary interpretive distinctions. Novel, robust behavior can be related to lower-level models through deductions, including deductions involving limiting operations; the role of mathematical limits and their relation to physically finite systems must be assessed separately [12]. A scale-dependent order may therefore be emergent in the sense of exhibiting robust explanatory patterns without requiring the inference that it has become a new fundamental substance or primitive relation.

A further comparison arises from discussions of emergent macroscopic patterns in quantum theory. Wallace's account of emergent structure in the Everett interpretation explicitly relies on additional physical ingredients, including decoherence, to motivate the relevance of higher-level patterns [48]. The philosophical lesson relevant to G1 is limited and useful: an effective pattern can be a serious candidate for physical interpretation when the dynamics and empirical role of its coarse variables are specified. The analogous claim for a generative quotient requires its own account of which observables, transformations, and intervention-sensitive predictions survive. An isomorphism class of posets, in isolation, supplies a weaker premise than the physically grounded constructions in such examples.

A critical point in statistical mechanics, where supported by the appropriate theory and evidence, introduces stronger physical content than a threshold that merely alters the connectivity of a graph.

The G1 proposal should also allow a physically meaningful order to remain only partially reconstructible. Several microscopic descriptions may reproduce the same observed effective constraints, and an effective model can be successful over a restricted range without identifying a unique fundamental dynamics. In such cases, a modest realist position could attribute objective significance to stable relations among observables while declining to settle the ontology of the configurations used in one particular derivation. Alternatively, further interventions or cross-scale measurements may help discriminate among the candidate microphysical explanations. What matters for this paper is that emergence, explanatory autonomy, and ontological primitiveness are different propositions requiring different arguments.

9.9 Physical Benchmarks and the Scope of Effective Precedence

Physical application requires a declared correspondence between modeled states and the entities or observables of a target theory. Four detailed benchmark families in Appendices A–D examine the infor-

Table 9: Distinct physical claims supported by the benchmark families. The tests are analytical and conditional on the stated models.

Construction	Preserved or discriminated structure	Additional requirement for physical interpretation
Closed reaction inventory	Conserved moieties and feasible joint outcomes	Experimentally warranted reaction channels and stoichiometry
Kinetic models with common support	Distinct occupancies and relaxation times	Validated rate laws and time-resolved observations
Microscopic count projection	Exact closure versus hidden kinetic distinctions	Independent identification of observable count classes
Approximate aggregation	Finite-time predictive error under model mismatch	Relevant horizon, observation noise, and calibration
Threshold-defined order	Conditions for stable support or probability profiles	Justified thresholds, uncertainty bounds, and robustness margins

mation that a derived order preserves, the information it discards, and the kinds of additional structure needed for predictive adequacy. Their role is methodological. The chemical constructions use conventional conservation laws and Markov kinetics, while their interpretations as tests of generative precedence remain conditional on the chosen landscape and observation interface.

The closed-reaction benchmark in Appendix A separates stoichiometric admissibility from transition rates: positive-rate kinetic models may induce identical support-derived orders while having different stationary occupancies and relaxation times. Appendix B moves to a four-state microscopic conversion model with a three-class count observation. Exact count-level Markov closure holds under a specified symmetry of site rates; different microscopic rates can nevertheless agree on equilibrium counts while producing distinguishable time correlations. These comparisons make explanatory discrimination depend on conservation-aware states, rate-sensitive measurements, and the question being asked rather than on poset isomorphism alone.

The analysis in Appendix C evaluates approximate aggregation with a finite-horizon total-variation bound, exact examples of weakly unequal rates, and a declared noisy observation channel. A smaller observational discrepancy does not imply better dynamical closure when measurement itself suppresses distinguishability. Appendix D then distinguishes support stability from probabilistic proximity. Adding a transition with arbitrarily small positive rate may change the mutual-reachability quotient abruptly, even while finite-time distributional changes remain small. Threshold margins, interval graph envelopes, and horizon-limited hitting profiles supply different conditional criteria for stable order claims.

Table 9 compares the evidential content of these benchmarks. Their proofs and executable checks establish facts about specified finite models. General physical emergence, empirical adequacy, and a primitive ontological order require further inquiry into observable quantities, governing equations, experimental regimes, and competing physical representations.

9.10 Explanatory Contribution under Established Baselines

The formal apparatus of G1 builds upon established mathematics. Finite-path reachability and its quotient by mutual accessibility belong to graph and order theory; reusable-premise closure has familiar counterparts in algebra and rewriting; exact descent of relations requires standard quotient-compatibility conditions [13, 11, 5]. The nontransitivity of projected reachability is likewise an elementary counterexample in a specified representational setting. These constructions warrant precise proofs and explicit assumptions, while the manuscript claims priority for their coordinated application to a philosophical problem rather than for their general mathematical discovery.

The adequacy of G1 must therefore be tested against fully informed specialized methods. Full-state transition graphs solve concrete reachability questions; behavioral-equivalence and partition-refinement methods evaluate compatibility of observable transitions; marked Petri nets represent consumptive and

joint prerequisites; continuous-time Markov-chain methods test aggregation and kinetic predictions [37, 36, 10]. Appendix E gives four matched-information diagnostics. In each, the established baseline reaches the correct result. These examples demonstrate constraints on an under-specified generative-order description; they establish neither a new prediction algorithm nor superiority to competent use of the existing methods.

The proposed methodological contribution is a coordinated audit of four separately revisable decisions: (i) which transformations are licensed by generative rules and available resources, (ii) which configurations are identified by an equivalence relation, (iii) which distinctions survive the observational interface, and (iv) which independent physical relations warrant an interpretation of the resulting order. The audit can disclose incompatible composite paths, unlicensed joint outcomes, condition-dependent rankings, and physically underdetermined hierarchies. A practical advantage remains an empirical question. Matched-input studies would need to evaluate the correctness of warranted claims, detection of hidden assumptions, analyst effort, and transfer across domains, alongside the strongest combination of standard methods.

9.11 Comparative Criteria for Physical and Ontological Claims

The foregoing comparisons permit a practical assessment of proposed applications. A researcher claiming that a generative preorder expresses an objective physical order would first need a physically grounded account of its configurations and transformations. The next task is to defend the equivalence relation under which the order is formed: representational differences should be separated from distinct physical possibilities using the symmetries, constraints, and observables of the target theory. A third task is to establish how the proposed order interacts with independent physical relations, including causal propagation, conservation laws, reversible dynamics, or the relevant quantum operations. Finally, the account should report which observations or interventions could distinguish it from other models that produce similar order structures.

The warrant is cumulative: a formally correct quotient may still identify physically different configurations; valid dynamical predictions may leave primitive and effective ontologies observationally indistinguishable. The research programme in Table 10 links these kinds of warrant to concrete counter-conditions. That table serves as the manuscript's unified assessment guide, while the distinct physical benchmark results remain in Table 9. This division retains the difference between assessing an existing physical model and proposing a new ontological interpretation.

A useful comparative design would specify at least two candidate landscapes capable of reproducing the same observational record, and then identify an additional observation or intervention for which their predictions diverge. Section 5 provides the formal motivation through its family of observationally compatible transition models. If the competitors imply different reachability structures while remaining observationally indistinguishable, the order is only partially identified relative to that evidence. The experiment should target an actual difference in predicted physical outcomes, rather than merely compare their graph diagrams. In a theoretical-physics context, the required additional observable might concern correlation functions, transition rates, causal constraints, or scaling behavior, depending on the chosen model.

The possibility of failure is epistemically productive. A claimed order may turn out to be sensitive to a threshold that has little physical meaning, to a hidden resource state, to an unrecognized gauge choice, or to a mistaken assumption that individually permitted transitions can be composed. Each diagnosis specifies a possible correction: retain continuous transition weights, enlarge the state description, reformulate the quotient, or replace reachability with a composition law suitable for the physical theory. When no supported physical mapping survives, the construction can remain mathematically useful while its physical interpretation is withdrawn.

9.12 Interpretive Alternatives and the Scope of Ontological Suspension

Several philosophical positions remain consistent with the formal development. An instrumental interpretation can regard derived order as a tool for organizing and predicting specified observations. An effective structural interpretation can assign objective significance to stable constraints among physically accessible processes at a declared scale, while acknowledging representational underdetermination. A relational interpretation can regard the significance of configurations and order as arising through structured physical relations, provided the theory independently supports that reading. A more ambitious fundamental interpretation can propose that some order relation is constitutive of the physical domain, with the associated demand for precise dynamics and empirically discriminating predictions. These alternatives disagree about the referent and status of order while potentially employing similar mathematics.

Ontological suspension here means keeping the passage from formal model to physical claim open to scrutiny. It permits the investigation of each alternative through its characteristic commitments. The mathematical derivation of a partial order, the explanatory robustness of an effective model, and evidence for a particular physical theory can all contribute to an ontological argument, but their contributions are distinguishable. The same restraint applies when discussing the generative-relational programme more broadly: a useful interface with dynamical systems, physics, biology, or institutional inquiry establishes a possible research connection, with each field retaining its specific standards of evidence.

The preceding analysis suggests a bounded conclusion. Generative order can be derived from sufficiently specified rules and equivalence conditions, and the resulting structures can be transformed, compared, and tested for robustness. Their interpretation as causal, geometric, dynamical, or ontologically fundamental relations remains dependent on additional physical theory and evidence. Accordingly, a future application of G1 to spacetime or quantum processes would begin by choosing a physically meaningful representation and demonstrating the consistency of its generative transformations with the target theory. The final section consolidates the formal propositions, the counterconditions revealed by reconstruction and physical comparison, and the limited research agenda warranted by the present exploratory analysis.

10 Conclusion

The ontological status of generative order requires two inquiries to be kept connected and distinct. One concerns the mathematical conditions under which precedence can be derived from specified configurations and admissible operations. The other concerns the evidence and interpretation required to treat that derived relation as a feature of a physical target. The present study has developed a conditional framework for the first inquiry and a set of methodological tests for the second. Its contribution is an integrated programme of comparisons across generative rules, equivalence relations, observational interfaces, and physical models; the elementary order-theoretic results employed in that programme remain established mathematical tools.

10.1 Formal Derivation and Compositional Conditions

A candidate generative landscape separates its descriptive domain, admissible configurations, conditioning regimes, generation rules, and observational interfaces. This separation is important because a transformation becomes meaningful only relative to the configurations and background conditions that authorize it. In particular, the same notation for a directed connection can encode a rewrite step, an observed transition, an intervention-dependent possibility, or a proposed physical process. A useful comparison must preserve the interpretation assigned to the connection.

The formal results establish distinct, conditional warrants. Proposition 4.1 recovers the standard reachability preorder and its partial-order quotient. Proposition 3.1 specifies exact descent of a joint generative relation and identifies the representative-compatibility requirement. Proposition 4.2 gives reusable-premise closure properties; consumptive operations instead require inventory-aware feasibil-

ity. The stated assumptions determine which comparisons are mathematically licensed, while physical interpretation requires independent justification.

Generative potential provides a related, though condition-dependent, perspective. The unrestricted unary case admits the exact reachability–potential duality of Proposition 6.3; in that setting, an unrestricted profile adds no independent ordering information. Resource-conditioned initial pools, multi-input rules, chosen outcome sets, and continuation constraints can introduce distinctions absent from the unary baseline. The resulting levels are local to the declared regime. Potential breadth, actual realized trajectories, jointly feasible products, and sustained viability consequently support different kinds of comparison. A total numerical ranking is justified only when its valuation and aggregation rules have been independently specified.

10.2 Representation and the Stability of Inferred Order

The most consequential reconstruction difficulty occurs when a representation merges underlying configurations while retaining apparent paths between the resulting classes. Proposition 5.1 distinguishes the image of concrete reachability from the reachability relation formed after projecting transitions; Proposition 5.2 demonstrates that the image of a concrete preorder can itself lose transitivity. Proposition 5.3 characterizes a representative-stability condition that allows observable paths to be lifted uniformly to concrete paths. In this context, observation is an operation with mathematical consequences for what can be inferred, rather than a neutral view of a fixed ordering.

Further qualifications arise under uncertain models, consumptive prerequisites, and hidden-state dynamics. Shared reachability across admissible models differs from paths obtained by pooling their incompatible edges. Individually attainable products may fail to be simultaneously feasible under one inventory. An observable state sequence may exhibit memory even when its hidden microscopic dynamics are Markovian. These counterexamples identify distinct reasons for enriching a proposed generative landscape. They also prevent an abstract path relation from being interpreted as an unqualified capacity for production, prediction, or intervention.

Stability is therefore a family of questions rather than a single attribute. A quotient may be invariant under one mathematical presentation, unstable under a plausible change of observational resolution, and robust under a specified parameter regime. An arbitrarily weak positive transition can alter support-derived order while leaving finite-time predicted distributions close, as the benchmark in Appendix D shows. Threshold separation margins and finite-horizon probability margins supply conditional certificates for other notions of order. Their usefulness depends on documented measurement resolution, parameter uncertainty, relevant horizons, and the consequences of an incorrect comparison.

10.3 Physical Explanation and Ontological Commitments

An exact quotient order has a different evidential standing from a physically interpreted relation of precedence. A physical account requires independently warranted laws, admissible states, meaningful observables, and an interpretation map connecting the mathematical construction to a target theory. The chemical-reaction benchmarks in Appendices A–C make this distinction concrete. Conserved molecular inventories constrain physically admissible states; positive reaction rates determine time-resolved occupation probabilities; microscopic-to-effective count projections may preserve equilibrium distributions while losing exact Markov closure or temporal correlations. Approximate aggregation bounds provide additional domain- and horizon-dependent standards for acceptable predictions. None of these physical distinctions can generally be recovered from the isomorphism type of the unweighted reachability quotient alone.

Relativistic causality, causal sets, gauge invariance, reversible dynamics, and renormalization impose distinct interpretation requirements beyond the present operational model. The same quotient order can support instrumental, effective-structural, or more ambitious ontological readings. Structural realism and grounding supply additional philosophical premises concerning identity, dependence, and fundamen-

tality. None follows from the quotient construction alone, which also leaves global temporal orientation undetermined.

G1 accordingly uses *emergence* in a restricted derivational sense where the relevant relation is obtained from declared generative rules rather than posited as a primitive relation. A claim of explanatory physical emergence adds evidence that the derived relation captures an invariant or effective constraint with predictive significance. A stronger ontological claim would require further analysis of fundamental structure and its alternatives. These senses can remain open to different answers in different physical domains.

10.4 Evidential Warrants and Critical Alternatives

The inquiry's conclusions have different evidential warrants. Set-theoretic statements about quotient relations, finite paths, and profile inclusion follow from the declared definitions and hypotheses. Their proofs establish mathematical implications, with familiar counterparts in order theory, relational semantics, and stochastic-process theory. Finite enumerations and executable checks provide additional diagnostics for the implementations and specified small examples; they supplement the written arguments and are sensitive to the test domains chosen. The chemically motivated models introduce conservation constraints and explicit stochastic dynamics, but their quantitative predictions concern stipulated rates and idealized microscopic configurations. The philosophical interpretations require a further account of physical meaning and remain open to competing readings. Keeping these levels distinct is essential to assessing the thesis as an exploratory inquiry rather than conflating formal correctness with empirical confirmation.

Three counterarguments deserve continuing scrutiny. First, the apparently emergent precedence may reproduce the reachability order of an ordinary graph and therefore add little explanatory power; the unary full-domain duality makes that risk exact. The additional apparatus earns its place when joint premises, changing admissibility, partial observation, or model uncertainty cause distinct reconstruction judgments that a simple graph representation would conceal. Second, a quotient or threshold can produce mathematically robust comparisons whose interpretation depends strongly on the chosen observational interface, time horizon, or resolution. Margin certificates supply conditional stability, while the selection and physical meaning of the margin remain matters for measurement and theory. Third, a physically accurate effective model may leave the ontology underdetermined: comparable kinetic predictions and invariant observables can support several accounts of underlying entities and dependence. Explanatory success therefore contributes evidence for a representation at the scale tested, while stronger ontological commitments require additional premises.

The controlled baseline comparison in Appendix E makes the redescription objection concrete. For four finite tasks involving observed paths, consumed resources, microscopic kinetics, and weak return transitions, the correct answer is already available from a specialized standard method when that method receives the relevant state and rate information. G1 contributes an explicit audit of the inferential passage between these representations; the benchmark does not demonstrate a new predictor or superior numerical inference. Its next discriminating test is practical: whether matched-input use of such an audit reduces unjustified explanatory claims or improves documented cross-scale interpretation compared with a competent combined baseline. This limitation narrows the scope of the contribution while making its potential research value assessable.

These objections suggest a precise standard for future comparative assessment. For any proposed application, the analyst should publish the state variables and generative rules, identify the alternative baseline method, fix the observations and prediction target in advance, and specify an outcome on which the competing reconstructions disagree. Both successful discrimination and a demonstrated inability to distinguish the models are informative. The present paper provides conditional mathematical constructions and reproducible toy-model checks, while full-text source criticism, experimental testing, and independent examination of the proofs remain outstanding. The accompanying claim audit distinguishes established constructions, model-specific corollaries, numerical checks, and philosophical proposals; it

Table 10: Principal formal warrants and discriminating tests for proposed generative orders. The tests are conditional methodological proposals.

Object of comparison	Conditional warrant	Priority test or countercondition
Quotient generative rule	Saturation under declared equivalence	Joint-input incompatibility among representatives
Reachability precedence	Composable admitted transitions	Hidden resources, cycles, or ill-defined composition
Observed order reconstruction	Verified path lifting or adequate stochastic aggregation	Projected paths lacking concrete realizations
Generative potential strata	Common outcomes, resources, regime, and horizon	Unary duality baseline; changed rankings under plausible conditions
Landscape transformation	Compatible correspondence of states and rules	Changed types, consumption, or lost enabling conditions
Order reorganization	Explicit parameter-dependent rule changes	Threshold artefacts versus physical critical signatures
Physical interpretation	Target-theory correspondence and independent observables	Empirically equivalent models with conflicting orders

is a record of internal scrutiny rather than an external certification.

10.5 Research Programme and Revision Conditions

The immediate research programme is organized around discriminating evidence. For formal reconstruction, benchmark models should specify admissible operations, equivalence classes, resource states, competing reconstructions, and concrete witnesses for claimed paths. For empirical investigation, the target should supply independently measured variables and governing or experimentally constrained dynamics. For changes of scale, a researcher should report both preserved quantities and predictive losses, including memory, correlation, uncertainty, and finite-time approximation error. For philosophical interpretation, alternative ontological readings should be compared by their explanatory commitments and any consequences on which they meaningfully disagree. Table 10 collects the corresponding warrants and counterconditions.

The framework carries explicit revision criteria. A proposed equivalence is inadequate when it erases a compatibility condition needed for composition. An inferred path requires qualification when no concrete lift or empirically justified effective dynamics supports it. A potential ranking must be indexed to the conditions under which its direction holds. A physical ordering claim requires reconsideration if it fails relevant conservation constraints, time-dependent observations, or independent correspondence tests. Such outcomes improve the precision of the inquiry by revealing which generative assumptions carry its conclusions.

The same restraint governs potential applications elsewhere in generative-relational research. Accounts of political economy, institutional power, ethical authority, or evolving norms require domain-specific empirical and normative premises. Formal reachability and potential inclusion may clarify feasibility, yet they do not determine distributive justice or legitimate governance. G1 therefore offers a bounded foundational result: several forms of generative precedence can be derived and compared under explicit assumptions; their stability is relative to the transformations and observables considered; and their physical or ontological standing remains a domain-sensitive question. Independent mathematical review, full source assessment, and experimental tests are needed before the proposal supports stronger conclusions.

A Stoichiometric Feasibility and Kinetic Identification

This appendix records the conservation-aware reaction benchmark used in Section 9.9. Its results separate feasible reaction channels from kinetic discrimination in a deliberately finite, idealized system.

A.1 Stoichiometric Constraints on a Candidate Generative Landscape

The distinction between a formally admissible transition and a physically warranted transformation can be examined in an elementary chemical-reaction system. Chemical reaction network theory provides independently specified descriptions of species, reaction channels, stoichiometric changes, conservation constraints, and rate laws [19]. Stochastic chemical kinetics additionally defines a state-dependent evolution law for molecular counts [23, 1]. The example developed here is an idealized model with declared initial conditions and parameters. It supplies an internally checkable physical-law correspondence rather than experimental evidence for a new ontology of order.

Consider a closed, well-mixed vessel of fixed volume V containing three chemical species with the reversible association reaction A + B $\rightleftharpoons$ C. Let the molecular-count state be $n = (a, b, c) \in \mathbb{N}_0^3$. The forward and reverse stoichiometric increments are $\nu_+ = (-1, -1, 1)$ and $\nu_- = (1, 1, -1)$, respectively. The admissible state space is restricted by nonnegative counts and by the two conserved moieties. Equation 32 states the conservation laws.

$$a + c = M_A, \qquad b + c = M_B, \qquad M_A, M_B \in \mathbb{N}_0. \tag{32}$$

For initial counts $(a, b, c) = (2, 1, 0)$, these laws restrict the reachable count states to $s_0 = (2, 1, 0)$ and $s_1 = (1, 0, 1)$. Each can transform into the other when both reaction channels have positive propensities. The resulting support graph has a single mutual-reachability class. Thus its quotient preorder is a one-element partial order for the entire positive-rate parameter family. That mathematical fact says little about the timescales and likelihoods of the transformations.

The conservation constraints are themselves informative independently of the support graph. In the open-ended descriptive space $\mathbb{N}_0^3$, arbitrary count triples can be drawn, but the laws in Equation 32 select a finite stoichiometric compatibility class for the chosen initial state. The candidate landscape must therefore include an admissibility predicate expressing the conserved quantities. A model that keeps the reaction names while discarding those constraints changes the space of physical possibilities that its derived order purports to describe. The conservation laws are established properties of the declared reaction model and would need chemical justification if applied to a particular experimental system.

A closely related branching example makes the resource issue visible without depending on kinetic rates. Suppose one initial unit each of A and B can form either an AB product C or a different AB product D, through alternative irreversible channels. For count state $m = (a, b, c, d)$, both channels preserve the totals $a + c + d$ and $b + c + d$. Starting from $(1, 1, 0, 0)$, an outcome with $c \geq 1$ is individually attainable, and an outcome with $d \geq 1$ is individually attainable. Yet the joint outcome $c \geq 1$ and $d \geq 1$ violates either conserved total, which equals one. The underlying point can be checked by the inequality in Equation 33.

$$c + d \leq a + c + d = 1. \tag{33}$$

This construction instantiates the consumptive-resource distinction introduced in Section 6: separate existential reachability claims should not be combined into a claim of simultaneous attainability. It also provides a physical interpretation of a specific representation failure. An unrestricted union of attainable products may overstate joint feasibility when the state representation omits an inventory vector and the stoichiometric constraints that govern its evolution.

A.2 Kinetic Discrimination under Identical Derived Precedence

The reversible association model permits a sharper test of explanatory discrimination. In the chemical master-equation interpretation, the propensity of the forward reaction at state (a, b, c) is $k_+ ab/V$, while

Table 11: Exact predictions for a two-state closed reaction network with identical transition support. Numerical rates use consistent normalized units, $V = 1$; the values are analytic model outputs, not measurements.

Regime	α	β	π_1	τ_{rel}	$p_1(1)$
Fast	2.0	1.0	0.6667	0.3333	0.6335
Slow	0.2	0.1	0.6667	3.3333	0.1728
Shifted ratio	0.4	1.0	0.2857	0.7143	0.2153

the reverse propensity is $k_- c$, with positive rate parameters k_+ and k_- [23, 1]. For the two accessible states s_0 and s_1, set $\alpha = 2k_+/V$ and $\beta = k_-$. Equation 34 defines the generator, using probability row vectors and the state order (s_0, s_1).

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}, \qquad \alpha = \frac{2k_+}{V} > 0, \quad \beta = k_- > 0. \tag{34}$$

The state graph associated with Equation 34 has the same two directed edges for all positive α and β. It follows that the graph-derived preorder and its quotient order are constant throughout this parameter domain. By contrast, the time-resolved occupancy probability, stationary occupancy, mean waiting time, and relaxation timescale depend on the magnitudes of α and β. Starting from s_0 with certainty, the exact solution is given by Equation 35.

$$\begin{aligned} p_1(t) &= \Pr(X_t = s_1 \mid X_0 = s_0) = \frac{\alpha}{\alpha+\beta}\big(1 - e^{-(\alpha+\beta)t}\big), \\ \pi_1 &= \frac{\alpha}{\alpha+\beta}, \qquad \tau_{\text{rel}} = \frac{1}{\alpha+\beta}, \qquad \mathbb{E}_{s_0}[T_{s_1}] = \frac{1}{\alpha}. \end{aligned} \tag{35}$$

These formulas follow from the two-state forward equation and exponential waiting-time law. The stationary probability identifies only the ratio of the two rates; the relaxation rate and the initial-state waiting-time distribution retain additional kinetic information. Thus two models can agree on all graph-support reachability statements, and even on stationary occupancy, while disagreeing on independently measurable time-dependent quantities.

For a transparent numerical comparison, take the vessel volume as one in a consistent system of units. Table 11 reports exact analytic predictions rounded to four decimal places for three hypothetical parameter sets. The fast and slow regimes share the ratio $\alpha/\beta = 2$ and hence share stationary occupancy. Scaling both rates downward by ten changes the characteristic time by a factor of ten. The third regime changes the rate ratio, thereby changing stationary occupancy as well. All three retain exactly the same reachability quotient.

Proposition A.1 (Kinetic underdetermination of support-derived order)**.** *Consider the two-state reversible chemical master-equation model of Equation 34 with $\alpha, \beta > 0$. Its graph reachability and mutual-reachability quotient are independent of the positive values of α and β. The pair of observables $(\pi_1, \tau_{\text{rel}})$ uniquely determines (α, β) within this model class, whereas π_1 alone does not.*

Proof. Positivity fixes the same bidirectional support graph in every model, so both states are mutually reachable and the quotient consists of one class. Let $r = \tau_{\text{rel}}^{-1} = \alpha + \beta$. The stationary probability gives $\pi_1 = \alpha/r$, hence $\alpha = \pi_1/\tau_{\text{rel}}$ and $\beta = (1 - \pi_1)/\tau_{\text{rel}}$. Conversely, keeping α/β fixed while multiplying both rates by an arbitrary positive factor leaves π_1 unchanged and changes τ_{rel}. □

Proposition A.1 is elementary Markov-chain analysis, included as a discriminating baseline rather than offered as an original kinetic theorem. The inference assumes the model class, initial-state preparation, a valid two-state Markov approximation, and observable occupancy across a sufficiently informative range of times. Real measurements introduce finite sampling, instrument resolution, parameter uncertainty, and possibly unmodeled reactions. In particular, if only equilibrium snapshots are available,

Table 12: Physical-law correspondence tests for the same modeled chemical system. Each row states a different representational claim and a possible discriminating observation.

Represented structure	Independent physical commitment	Discriminating observation or failure criterion
Reachability support	Admissible reaction channels and states	Reaction-channel detection; missing or additional intermediates
Conservation-aware feasible states	Stoichiometry, closure, conserved moieties	Count balances and joint-product attainability under a shared inventory
Rate-labelled transitions	Conditional reaction propensities	Time-resolved occupancies, waiting times, rate perturbations
Coarse observable classes	Specified measurement and state-identification map	Failure of path lifting or rate aggregation under the actual observation process

the absolute timescale remains unidentified within the declared model. Conversely, a well-resolved occupancy response can distinguish parameterizations that are identical from the viewpoint of the derived quotient order.

The example suggests a practical test for the additional explanatory value of the generative-landscape formalism. If its chosen equivalence and order retain only transition support, the order is an insufficient statistic for the measured dynamics. Augmenting the represented configuration with rate information, or retaining a rate-labelled transition system alongside the order, makes the description more discriminating; it also increases its physical assumptions. Both choices are intelligible results. The substantive criterion is whether the proposed generative representation records exactly the constraints and observables required for the intended physical comparison, rather than whether the resulting poset appears more elaborate.

A.3 Observational Design and the Physical Scope of the Benchmark

The chemical benchmark permits three explicit levels of model comparison. A *support-level* experiment asks which count states can be reached by permitted reaction channels. A *conservation-level* experiment checks the stoichiometric compatibility class, including whether purported joint products can coexist. A *kinetic-level* experiment measures occupancy as a function of time, waiting-time distributions, or responses after an intervention that changes a rate parameter. Each level may discriminate models that the preceding level leaves indistinguishable. The measured quantities and admissibility tests are specified by the chemical theory, providing an independent check on the meaning assigned to the formal landscape.

A simple intervention design would prepare the same initial counts in multiple trials and change a reaction rate through a controlled modification whose chemical effects were independently characterized. The prediction is a changed $p_1(t)$ as determined by Equation 35, while the stoichiometric invariants in Equation 32 and the support-derived order stay unchanged as long as both rates remain positive. Observed violations of the conservation relations, systematic failures of the kinetic solution, or previously unresolved intermediates would motivate changes to the target model or measurement protocol. The toy model supplies predictions under controlled assumptions; this manuscript reports no experimental trials or empirical estimates.

Table 12 organizes the evidential distinctions. The table is an analytic protocol for potential future work, including negative results, rather than an assertion that the proposed generative-order scheme outperforms conventional chemical modeling.

Several limitations define the scope of these results. The examples assume a well-mixed closed vessel, countable reaction events, idealized reactions, and a stationary environment with constant volume and parameters. Their compatibility classes and rates are conditional on these assumptions, and conservation of molecular moieties in a toy model should be distinguished from a thermodynamic derivation of physically fundamental conserved charges. Stochastic kinetic parameters are part of an effective physi-

cal representation, whose validity depends on the experimental regime. Most importantly, a rate-labelled graph, reaction network, or stoichiometric invariant is an established physical-mathematical construction. G1's additional contribution here is methodological: it identifies the precise kinds of physical information that a proposed generative-order abstraction would preserve, omit, or need to recover. Consequently, the benchmark tests representational adequacy within a chosen physical theory while leaving the paper's broader ontological hypotheses open.

B Exact Aggregation of Conserved Microscopic States

This appendix specifies a microscopic conversion model and establishes when its coarse count projection admits a closed Markov description. It supplies a controlled physical illustration for the discussion in Section 9.9.

B.1 A Conserved Microscopic Model and Its Occupancy Projection

A second physical benchmark addresses the transformation between microscopic and effective descriptions more directly. The two-state association example compares rates within one count-state space; here the starting system has four distinct microscopic states and is mapped to three observed occupation classes. This difference makes preservation under coarse-graining itself an object of analysis. The construction uses two separately labelled reaction sites, each occupied by one molecule that may undergo the reversible conversion $\mathrm{A}_i \rightleftharpoons \mathrm{B}_i$. The site labels identify kinetically distinct environments, while the observable records only the total number of molecules in state B. Such a model is an idealized chemical-kinetic example, not a claim that a particular experiment obeys independent-site dynamics. The use of aggregation to construct reduced stochastic reaction models has established precedents in formal biochemical modelling [39, 7].

Write a microscopic state as $x = (x_1, x_2) \in X = \{0,1\}^2$, where $x_i = 0$ denotes A_i and $x_i = 1$ denotes B_i. Conversion $0 \to 1$ at site i occurs with rate $\alpha_i > 0$, and conversion $1 \to 0$ with rate $\beta_i > 0$. The sites are conditionally independent given these fixed rates, and their occupancies are constrained by the conserved local inventories $N_{A_i} + N_{B_i} = 1$. Consequently, the global molecular total $N_A + N_B = 2$ is preserved by every permitted transition. The effective measurement is the occupancy projection $\pi : X \to Y = \{0,1,2\}$ defined by Equation 36.

$$\pi(x_1, x_2) = x_1 + x_2 =: n_B, \qquad N_A = 2 - n_B. \tag{36}$$

The microscopic continuous-time Markov generator Q is specified by its off-diagonal rates in Equation 37, with each diagonal entry chosen to make the row sum zero. Here $x^{(i)}$ is the configuration obtained by flipping site i; these rules fully define the four-state generator and the physical assumptions supporting it.

$$Q(x, x^{(i)}) = \begin{cases} \alpha_i, & x_i = 0, \\ \beta_i, & x_i = 1, \end{cases} \qquad Q(x,x) = -\sum_{z \neq x} Q(x,z). \tag{37}$$

Equation 36 specifies which physical quantities are retained. The effective count n_B and the total inventory remain well defined at the coarser scale, while the distinction between configurations $(1,0)$ and $(0,1)$ disappears. More generally, a microscopic observable $I : X \to \mathbb{R}$ has an exact representation $\bar{I} : Y \to \mathbb{R}$ with $I = \bar{I} \circ \pi$ precisely when I is constant on every fibre of π. This elementary factorization requirement is independent of whether the effective dynamics are Markovian. In the present case the count n_B satisfies the criterion by construction; the site-resolved conversion status x_1 does not. The conserved total $N_A + N_B$ is also preserved, although its constant value makes it a relatively weak discriminator of competing rate models.

For every choice of positive rates, the microscopic support graph connects all four states by single-site conversions, and its observable image connects neighbouring count classes in both directions. All configurations are mutually reachable; the corresponding mutual-reachability quotient is a single element.

Thus a derived order based on unweighted transition support carries the same information in the symmetric and heterogeneous parameter regimes below. The question of physically faithful coarse-graining remains substantive despite this agreement in order structure.

B.2 Exact Dynamic Closure and Representative Dependence

An effective count process is particularly useful if its future transition law can be determined from the present count alone, for every choice of microscopic initial state compatible with that count. For a finite continuous-time Markov chain, this is the strong-lumpability problem. Let C be the four-by-three class-indicator matrix, $C_{x,n} = 1$ when $\pi(x) = n$ and zero otherwise. An exact effective Markov generator $\bar{Q}$ independent of microscopic initial preparation exists when the intertwining relation in Equation 38 holds. The equality concerns time-resolved stochastic evolution, a stronger demand than agreement of reachable classes.

$$QC = C\bar{Q}. \tag{38}$$

Proposition B.1 (Exact count-level kinetic closure for two conversion sites)**.** *For the microscopic generator in Equation 37 and occupancy projection in Equation 36, an effective continuous-time Markov generator $\bar{Q}$ satisfying Equation 38 exists if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. In that case, writing the common rates as α and β, the effective count process has upward rate $(2-n)\alpha$ and downward rate $n\beta$ from count n.*

Proof. The observed class $n = 1$ contains exactly two microscopic states, $(1, 0)$ and $(0, 1)$. From $(1, 0)$ the aggregate rate into class $n = 2$ equals α_2, and the aggregate rate into class $n = 0$ equals β_1. From $(0, 1)$ the corresponding rates are α_1 and β_2. Equality of aggregated exit rates to each other class is necessary for the rows of QC to agree within the same observed class, hence $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. These equalities are sufficient because the classes $n = 0$ and $n = 2$ each contain one microscopic state and the shared rates on the $n = 1$ fibre yield an effective generator with the asserted transition rates. Diagonal entries follow from zero row sums. □

Proposition B.1 is a direct application of established continuous-time Markov lumpability theory [7, 39]. Its role in G1 is to identify a precise point at which physical information lost by a representation becomes dynamically consequential. In the symmetric case the effective generator is Equation 39, with states ordered as $n = 0, 1, 2$.

$$\bar{Q} = \begin{pmatrix} -2\alpha & 2\alpha & 0 \\ \beta & -(\alpha+\beta) & \alpha \\ 0 & 2\beta & -2\beta \end{pmatrix}. \tag{39}$$

The result separates observational indistinguishability from kinetic equivalence. The configurations $(1, 0)$ and $(0, 1)$ have the same value of the measured count, and both permit transitions to $n = 0$ and $n = 2$. With heterogeneous site rates, however, their conditional hazards of reaching those classes differ. The count equivalence therefore preserves outgoing transition support while failing to preserve its quantitative law. The same failure is compatible with complete microscopic and effective graph reachability. Structural identification at one scale and dynamical closure at that scale answer different questions.

Strong lumpability is an intentionally demanding standard. A specific initial ensemble, fast internal equilibration, or a justified time-scale separation may support an approximate or specially initialized effective model even when Proposition B.1's equality fails. Such approximations require an error analysis over a stated time horizon and specified observables. In a model with hidden variables, an accurate alternative may be to retain a conditional distribution over microscopic configurations or to introduce a memory kernel. The measured count alone cannot automatically be assigned a time-homogeneous Markov generator merely because it admits a three-node graph. Established work on stochastic thermodynamics under coarse-graining further shows that additional thermodynamic accounting may be required when mesostates fail to internally equilibrate [17]; the present example makes no claim of measured entropy production or a nonequilibrium steady-state flux.

Table 13: Observable distinctions across two microscopic kinetic models with the same support-derived order and stationary occupancy distribution. The covariance values are model predictions at $t = 1/2$.

Diagnostic	Symmetric model U	Heterogeneous model H
Positive-rate count support	$0 \leftrightarrow 1 \leftrightarrow 2$	$0 \leftrightarrow 1 \leftrightarrow 2$
Global molecular inventory	$N_A + N_B = 2$	$N_A + N_B = 2$
Stationary count law	$(4/9, 4/9, 1/9)$	$(4/9, 4/9, 1/9)$
Strong count lumpability	Holds	Fails
$C(1/2)$	$\frac{4}{9}e^{-3} \approx 0.02213$	$\frac{2}{9}(e^{-3/2} + e^{-9/2}) \approx 0.05205$
Effective count description	Exact count-level Markov process	Hidden-state or memory model needed

B.3 Identical Equilibrium Counts and Distinct Temporal Predictions

A stronger discriminating comparison retains the same effective count support *and* the same equilibrium distribution while changing the microscopic timescales. Consider model U with identical sites $\alpha_1 = \alpha_2 = 2$ and $\beta_1 = \beta_2 = 4$. Compare model H, whose site-specific rates are $(\alpha_1, \beta_1) = (1, 2)$ and $(\alpha_2, \beta_2) = (3, 6)$. All rates have units of inverse time. Each site in either model has equilibrium $\Pr(x_i = 1) = \alpha_i/(\alpha_i + \beta_i) = 1/3$, and the two sites are independent in their stationary distribution. Their observable equilibrium count probabilities consequently agree exactly, as stated in Equation 40.

$$\Pr_{\mathrm{U}}(n_B = n) = \Pr_{\mathrm{H}}(n_B = n) = \binom{2}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^{2-n}, \qquad n \in \{0, 1, 2\}. \tag{40}$$

Equation 40 gives the shared count distribution $(4/9, 4/9, 1/9)$ and the same mean count $2/3$. Nevertheless, the heterogeneous model fails the criterion of Proposition B.1. In model H, a count-one state prepared microscopically as $(1, 0)$ has initial transition rate 3 toward count two and 2 toward count zero. The alternative microscopic preparation $(0, 1)$ has the respective rates 1 and 6. Thus short-time probabilities of the same observed destination depend on which microscopic representative was prepared. Equilibrium count statistics alone conceal this difference.

Time correlations distinguish the models even under stationary preparation. For one reversible two-state site with rates α_i, β_i and stationary occupancy $p = 1/3$, its stationary autocovariance equals $p(1 - p)e^{-(\alpha_i+\beta_i)t}$. Independence of the sites makes the covariance of their sum the sum of the individual covariances. The two exact predictions are reported in Equation 41.

$$\begin{aligned} C_{\mathrm{U}}(t) &= \mathrm{Cov}\big(n_B(t), n_B(0)\big) = \frac{4}{9}e^{-6t}, \\ C_{\mathrm{H}}(t) &= \mathrm{Cov}\big(n_B(t), n_B(0)\big) = \frac{2}{9}\left(e^{-3t} + e^{-9t}\right). \end{aligned} \tag{41}$$

Both functions agree at $t = 0$, giving $\mathrm{Var}(n_B) = 4/9$. Their different exponential components lead to different nonzero-lag correlations, although the stationary one-time distribution, conservation constraints, and unweighted reachability information coincide. The non-Markovian character of model H's aggregated process is an exact statement about closure for arbitrary microscopic preparations; the correlation expression is independently an observable prediction under stationary preparation. One need not infer the stronger failure solely from a stationary autocovariance fit, because additional effective models or measurement processes could reproduce a selected correlation function.

Table 13 gives the quantities used in the explicit comparison. The numerical covariance values at $t = 1/2$ are rounded analytical evaluations, not fitted measurements. All rates and samples are hypothetical. The independent executable check accompanying v1.7 computes the four-state generator, tests exact aggregation, evaluates both predicted covariances by matrix exponentiation, and verifies the analytic expressions.

The example also clarifies the evidential difference between an invariant and a complete effective dynamics. Equation 40 retains an exact equilibrium observable across the projection, while Proposition B.1 specifies when the dynamical law descends for all compatible microscopic preparations. If an

investigation observes only equilibrium counts, the two models remain indistinguishable on those data. Time-resolved correlations, preparation-sensitive initial hazards, or perturbations that isolate site-specific rates could separate them. These possible experiments arise from the independently specified stochastic dynamics, not from the quotient order considered alone.

B.4 Criteria for Cross-Scale Explanatory Adequacy

The two-site construction gives a concrete interpretation of cross-scale preservation. A transformation $\pi : X \to Y$ can retain a conserved observable, preserve which effective count states are reachable, reproduce the complete equilibrium law of a selected measured quantity, and still fail to preserve Markovian dynamics. Each retained property requires a distinct mathematical warrant. The preservation of a microscopic invariant is a fibrewise factorization condition. The preservation of path-level possibilities needs a compatible relation between projected transitions and concrete paths. Exact dynamical closure requires equality of aggregate transition rates for representatives of the same coarse class; practical approximations require quantitative error bounds. A defensible physical interpretation should report the intended preservation level before interpreting a derived order as stable across scales.

This analysis provides a controlled extension of the reconstruction problem in Section 5. A class-based description can be sound for one observational objective while inadequate for another. A researcher may reasonably use the count process as an effective equilibrium description in model H, yet require site labels or a conditional microscopic distribution to predict time-resolved correlations. An observation scheme with enough time resolution could reveal that discrepancy and motivate a refined landscape. Conversely, limited temporal resolution might make the same coarse description adequate over a restricted range of tasks. Adequacy is conditional on the question and error tolerance, rather than a binary property of the coarse-graining map considered in isolation.

The construction should also be distinguished from renormalization in a field-theoretic sense. The map π aggregates a finite collection of discrete microscopic states; it supplies neither a continuum limit nor a renormalization-group flow or critical scaling laws. It therefore establishes a tractable case of exact and inexact stochastic aggregation, which can serve as a benchmark for more demanding effective theories. Its philosophical relevance to G1 is the need to explain why a claimed effective order remains meaningful after physically consequential information has been discarded. The fact that both models lead to the same one-element mutual-reachability quotient makes that limitation especially transparent.

A productive next study could vary the observation map, initial ensemble, site-coupling rules, and sampling interval while assessing which effective descriptions reproduce independently observed count statistics within declared errors. It could then compare three reconstructions: the support-only quotient, an exact or approximate count-level stochastic model, and a history-dependent or belief-state representation. The evaluation would measure what each description preserves and how its extra assumptions affect predictability, identifiability, and interpretability. No empirical claim about molecular systems follows solely from the analytic example, and the benchmark leaves open the ontological status of any structure obtained through the chosen projection. Its contribution is a reproducible method for locating the point where representational simplification ceases to preserve a specified physical explanation.

C Approximate Aggregation and Observation Resolution

This appendix examines quantitative approximation when exact microscopic-to-effective closure is unavailable. Its bounds are finite-horizon statements about distributions, and their measurement interpretation depends on the stated observation channel.

C.1 Quantitative Errors in Approximate Aggregation

Exact kinetic closure provides an important consistency test, yet an effective representation may remain scientifically serviceable when microscopic rates differ modestly within an observed class. An assess-

ment of approximate coarse-graining therefore requires both a declared observation map and a quantitative tolerance for the observable predictions that the reduced description is intended to preserve. The analysis in this subsection gives a finite-state diagnostic for that purpose. It first connects the microscopic and effective generators through an aggregation residual, then bounds the discrepancy in their transient one-time laws. The subsequent subsections apply that diagnostic to weakly heterogeneous conversion rates and finite-resolution measurements. These calculations belong to established stochastic aggregation theory: earlier work on near lumpability and more general error bounds provides the relevant mathematical lineage [10, 25, 34]. The particular residual estimate below is proved directly for the finite model under consideration.

Let Q be a generator on a finite microscopic state set X, let C be the indicator matrix of a deterministic projection onto a smaller state set Y, and let $\bar{Q}$ be a proposed effective generator on Y. Both generators have nonnegative off-diagonal entries and zero row sums. Define the aggregation residual and its maximal row discrepancy by Equation 42.

$$E := QC - C\bar{Q}, \qquad \delta := \max_{x \in X} \sum_{y \in Y} |E_{xy}|. \tag{42}$$

The residual is a generator-level error measured in inverse-time units. It compares aggregated instantaneous rates from each microscopic representative with those specified by the proposed effective generator. As $E\mathbf{1} = 0$, each row describes a signed redistribution of effective transition rates. The residual vanishes under exact strong lumpability with a matching effective generator. Small residuals provide a finite-horizon approximation criterion; their magnitude depends on both the chosen partition and the proposed effective rates. In particular, selecting $\bar{Q}$ from an equilibrium-weighted average, a representative state, or an independently calibrated effective experiment can yield different values of δ. Such a choice should be reported rather than concealed within a label such as "coarse scale."

Proposition C.1 (Finite-horizon aggregation error bound)**.** *For finite-state Markov generators Q and $\bar{Q}$, a deterministic class-indicator matrix C, and δ as in Equation 42, every microscopic initial probability row vector p and every $t \geq 0$ satisfy the total-variation estimate in Equation 43. Here $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2}\sum_y |\mu_y - \nu_y|$.*

$$\left\| p e^{tQ} C - p C e^{t\bar{Q}} \right\|_{\mathrm{TV}} \leq \min\left\{ 1, \frac{t\delta}{2} \right\}. \tag{43}$$

Proof. Differentiating the interpolating product of the two stochastic semigroups gives the identity in Equation 44.

$$e^{tQ} C - C e^{t\bar{Q}} = \int_0^t e^{(t-s)Q} E e^{s\bar{Q}} \, \mathrm{d}s. \tag{44}$$

For each s, the row $p e^{(t-s)Q}$ is a probability distribution. Its product with E has ℓ^1 norm at most δ, by convexity of the row norm. Right multiplication by the stochastic matrix $e^{s\bar{Q}}$ is an ℓ^1 contraction on signed rows. Integrating and dividing by two yields $t\delta/2$; the total variation between probability distributions is at most one. □

Proposition C.1 gives a uniform estimate over microscopic initial preparations, including those that assign different weights to representatives sharing one observed class. Its scope is deliberately narrow. The inequality controls a transient marginal distribution at a fixed time; it does not establish that the projected process has the Markov property, reproduce its multi-time joint law, or bound its likelihood for an entire observed path. The linear estimate may also be conservative as t grows, since it disregards relaxation and possible cancellation of positive and negative residual contributions. An application concerned with path-dependent causal claims, correlation functions, or waiting-time distributions requires additional analysis of those quantities. Exact observation of an invariant quantity can coexist with substantial dynamical approximation error.

C.2 Weakly Broken Site Symmetry and Finite-Horizon Predictions

The two-site model permits a controlled interpolation between the exactly lumpable regime and the heterogeneous kinetic regime. Keep the independent-site state space and count projection of Equation 36, and set the rate parameters according to Equation 45, where $0 \le \varepsilon < 2$. The baseline effective generator $\bar{Q}_0$ is Equation 39 evaluated at $\alpha = 2$ and $\beta = 4$.

$$\begin{aligned}(\alpha_1, \alpha_2) &= (2 - \varepsilon, 2 + \varepsilon),\\ (\beta_1, \beta_2) &= (4 - 2\varepsilon, 4 + 2\varepsilon).\end{aligned} \tag{45}$$

The parameterization changes the site-specific kinetic speeds while fixing each equilibrium conversion ratio $\alpha_i/\beta_i = 1/2$. Both sites therefore have stationary occupation probability $1/3$, and the equilibrium count distribution remains $(4/9, 4/9, 1/9)$ for every admitted ε. The conserved total molecular inventory remains two, and all positive-rate microscopic states remain mutually reachable. Accordingly, the count-level support graph, its derived mutual-reachability quotient, and the stationary count law coincide throughout the parameter family. Only the detailed rates change. This controlled variation isolates the influence of symmetry breaking on effective dynamics from changes in support, material inventory, or equilibrium occupancy.

Computing $Q_\varepsilon C - C\bar{Q}_0$ shows that its rows vanish for counts zero and two; the two microscopic representatives of count one have residual vectors $(\mp 2\varepsilon, \pm\varepsilon, \pm\varepsilon)$ in count order, with the signs reversed between representatives. Hence $\delta = 4\varepsilon$, and Proposition C.1 yields the concrete estimate in Equation 46.

$$\sup_p \left\| p e^{tQ_\varepsilon} C - p C e^{t\bar{Q}_0} \right\|_{\mathrm{TV}} \le \min\{1, 2\varepsilon t\}. \tag{46}$$

The estimate gives an explicit sufficient condition for a requested finite-time total-variation tolerance: if $2\varepsilon t \le \tau$ over the declared observational horizon, the reduced generator meets tolerance τ for every microscopic initial distribution. This is a conservative guarantee, rather than an equivalence between rate heterogeneity and approximation failure. Even when the bound exceeds a selected tolerance, the actual error can be substantially smaller. When the physical task needs a preparation-specific prediction, the measured or known initial distribution may provide a sharper assessment than the uniform worst-case bound.

Time-correlations furnish an additional diagnostic of the weak symmetry breaking. At equilibrium, independence of the sites and their two distinct relaxation rates give Equation 47.

$$C_\varepsilon(t) = \frac{4}{9} e^{-6t} \cosh(3\varepsilon t), \qquad t \ge 0. \tag{47}$$

Equation 47 agrees exactly with the symmetric reference when $\varepsilon = 0$; its deviation from that reference is quadratic in ε at fixed t near zero, because the first-order speed changes cancel between the sites. The transient distribution discrepancy controlled by Equation 46, by contrast, can be first order in ε under a prepared microscopic initial state. This difference matters for experimental design: stationary two-time covariance can be less sensitive to weak kinetic heterogeneity than a preparation-sensitive aggregate transition measurement. It also illustrates why an approximation that is accurate for one chosen statistic need not be accurate for all admissible questions.

Table 14 compares the exact worst-case transient errors calculated from the microscopic matrix exponential with the conservative bound and the equilibrium covariance. All values are numerical evaluations of the specified idealized model; time is in the inverse units of the declared rates. The table also shows why the smallest available δ is only one aspect of choosing an effective representation. A useful approximation can have a loose global guarantee while making sufficiently accurate predictions for an explicitly limited task.

The construction is intentionally small enough to permit direct verification. Positivity of the site rates is preserved under the declared range of ε; row-sum checks establish that the matrices remain Markov generators. The accompanying executable test computes the residual, verifies the estimate at multiple

Table 14: Weak site heterogeneity with a fixed symmetric effective generator. The exact maximum total-variation error and the conservative bound are reported at $t = 1/4$; the equilibrium count autocovariance is evaluated at $t = 1/2$. Values are analytic or matrix-exponential model predictions.

ε	Exact maximum TV at $t = 1/4$	Upper bound at $t = 1/4$	$C_\varepsilon(1/2)$
0	0.00000	0.00000	0.022128
0.2	0.02296	0.10000	0.023131
0.4	0.04755	0.20000	0.026231
0.8	0.10390	0.40000	0.040065

horizons, checks the shared equilibrium law, and compares matrix-exponential covariance predictions with Equation 47. The finite test supports the calculations within this model. It neither estimates physical error from measured trajectories nor licenses extrapolation to large reaction networks with interactions, rare events, or additional unobserved conservation constraints.

C.3 Observation Resolution and Measurement-Induced Indistinguishability

Dynamic adequacy is jointly conditioned by the physical process and the observation apparatus. A model might resolve distinct kinetic mechanisms in principle while a finite-resolution measurement attenuates their observable differences. To express this point without assigning an unspecified experimental noise distribution, let a measurement channel W map effective count categories to recorded outcomes. For a stochastic matrix W, contraction of total variation gives an immediate post-measurement consequence of Proposition C.1. Any error bounded by $t\delta/2$ before measurement remains bounded by the same quantity after applying the same measurement channel to both candidate predictions. More strongly, an explicit channel can quantify how much distinguishability it removes.

Consider the stylized replacement-noise channel in Equation 48. With probability $1 - \eta$ the instrument reports the underlying count category; with probability η it reports an independent draw from a fixed distribution v, where $0 \le \eta \le 1$. This measurement is an analytical construction, not a description of a particular detector.

$$W_\eta = (1-\eta)I + \eta \mathbf{1} v^{\mathsf{T}}, \qquad v \ge 0, \quad \sum_{y \in Y} v_y = 1. \tag{48}$$

For any two probability rows r, s on the effective categories, $(r - s)W_\eta = (1 - \eta)(r - s)$, because their total masses agree. Thus the measured total-variation distance is exactly $1 - \eta$ times the distance before the channel. Applied to the two-site approximation, this identity yields Equation 49.

$$\left\| (p e^{tQ_\varepsilon} C) W_\eta - (p C e^{t\bar{Q}_0}) W_\eta \right\|_{\mathrm{TV}} \le (1-\eta) \min\{1, 2\varepsilon t\}. \tag{49}$$

Equation 49 concerns the discrepancies of predicted single-time measured distributions, not the epistemic value of noise itself. A smaller observable difference can make two models harder to distinguish; it does not improve the underlying effective dynamics. If independent replacement draws are made at two separated measurement times, and the replacement law has the same mean count as the stationary physical model, then for strictly positive lag t the measured count autocovariance is $C_\varepsilon^{\mathrm{obs}}(t) = (1-\eta)^2 C_\varepsilon(t)$. For $\varepsilon = 0.4$, $\eta = 0.35$, and $t = 1/2$, the physical-model covariance difference from the symmetric reference is approximately 0.004104, while the corresponding measured-model difference is approximately 0.001734. Both differences are predictions conditional on the stipulated replacement channel. Real detectors may instead show state-dependent confusion, temporal correlation in noise, missed events, finite exposure windows, or nonlinear readout, each requiring an explicit observation model.

Observation frequency constitutes a second choice. Sampling only at a few widely separated times can erase precisely the short-time preparation dependence that violates exact count-level lumpability.

Conversely, sufficiently resolved measurements with controlled initial preparation could reveal the different instantaneous count-one hazards. A scientific model may therefore be adequate for equilibrium counts and a declared finite observation interval while failing for event-resolved kinetic inference. The experimental question must identify which conclusions use information that survives the instrument's temporal and state resolution. The mathematically true conservation law and support-derived order remain invariant in this example; the detectability of kinetic heterogeneity changes with observation design.

C.4 Cross-Scale Error Budgets and Physical Interpretation

The approximate construction supplies a more graded assessment than a binary test of whether a reduced model exists. It distinguishes four statements that can be evaluated separately: conservation-compatible state identification, preservation of permitted transition support, bounded error for a selected distribution and horizon, and detectability of model differences through a declared observation process. These statements may hold to different degrees in the same physical system. A researcher evaluating a proposed emergent order should report their respective warrants, because the coexistence of a robust support quotient and a poor effective kinetic model is entirely possible.

The residual in Equation 42 depends on the selected microscopic generator and the effective generator against which it is compared. The bound in Proposition C.1 controls one-time distributions and has a declared time dependence. The covariance in Equation 47 tests a different dynamical property under stationary preparation. The measurement channel in Equation 48 explicitly mediates what may be discriminated empirically. Together these objects form an error budget whose terms should be connected to measured or independently constrained quantities before any physical conclusion is drawn. A small error over one finite horizon alone is insufficient to identify microscopic reality or settle whether the effective ordering is fundamental.

Approximate aggregation can be improved through several distinct and testable revisions: retaining a dynamically relevant hidden variable, fitting an effective generator for a specified initial ensemble, refining the observation partition, or adopting an effective model with memory. Each choice changes the relation between data requirements and predictive scope. The present residual bound supplies a conservative sufficient guarantee; sharper bounds or calibrated model selection may be available when an application supplies mixing information, time-scale separation, or measurement error estimates. The literature on quantitative coarse-graining makes clear that such analyses are an established domain of stochastic process theory [25, 34]. G1 uses their methodological role to clarify when a proposed generative representation can remain meaningful while sacrificing exact microscopic closure.

For the ontological inquiry, the consequence is a limited but constructive one. Stability of a derived ordering under perturbation supports a mathematically specified form of robustness. Accurate effective predictions strengthen the case that the representation captures a reproducible physical regularity at its chosen scale. Neither finding establishes the ultimate status of precedence as a primitive constituent of the world. That stronger interpretation would require the additional correspondence arguments, independently motivated laws, and evidential discrimination developed elsewhere in this section. The value of the approximate benchmark is that it makes these obligations measurable: the quality of an effective generative order is assessed through the physical claims it actually supports, the errors those claims tolerate, and the observable contexts in which they survive.

D Stability Certificates for Effective Generative Order

The analysis concerns support reachability, thresholded transitions, and finite-horizon probabilistic potential. The propositions supply conditional stability certificates for specified finite models.

Table 15: A conserved three-form kinetic model: exact total-variation error starting from form C, compared with the finite-time theoretical bound at $t = 1$. Every positive return rate merges the three support classes into one.

Return rate ε	Support classes	Exact TV error	Upper bound
0	3	0	0
0.001	1	0.000896	0.001
0.010	1	0.008919	0.010
0.050	1	0.043727	0.050

D.1 Discontinuous Reachability under Weak Physical Perturbations

The finite-horizon aggregation estimates permit a further evaluation of the relation between predictive accuracy and derived generative precedence. Reachability identifies the existence of an admissible path, whereas transient probabilities describe how likely a realized path is over a specified interval. An arbitrarily weak transition can affect these assessments in sharply different ways. The purpose of this subsection is to establish that separation with a conserved microscopic example and then specify conditions under which a limited ordering claim remains stable. The distinction between qualitative and quantitative verification is also familiar in probabilistic model checking, where step-bounded target probabilities are treated separately from support reachability [28, 18].

Consider three mutually exclusive molecular forms A, B, and C of one conserved particle. The base reaction system has transitions A $\to$ B and B $\to$ C, both with unit rates, while a perturbed system also permits C $\to$ A at rate $\varepsilon > 0$. The corresponding continuous-time generators, in that order of states, are given in Equation 50.

$$Q_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_\varepsilon = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ \varepsilon & 0 & -\varepsilon \end{pmatrix}. \tag{50}$$

Both generators conserve total particle number: a trajectory changes molecular form while retaining exactly one particle. For Q_0, the directed support yields three distinct mutual-reachability classes in the chain $[\mathrm{A}] \prec [\mathrm{B}] \prec [\mathrm{C}]$. Every $\varepsilon > 0$ closes the directed cycle and merges these classes into a single strongly connected component. This discrete change occurs for arbitrarily small positive ε. It is a graph-theoretic transition in the derived order; the example supplies neither a thermodynamic limit nor evidence of a physical phase transition.

The finite-time predictions, however, remain close. In Equation 51, p is any initial distribution, $t \geq 0$, and the total-variation distance is computed on the common three-form state space.

$$\left\| p e^{tQ_\varepsilon} - p e^{tQ_0} \right\|_{\mathrm{TV}} \leq \min\{1, \varepsilon t\}. \tag{51}$$

Proposition D.1 (Support-order instability under small generator perturbations)**.** *For the generators in Equation 50, their mutual-reachability quotient orders differ for every $\varepsilon > 0$, while Equation 51 holds uniformly over initial distributions at each finite time.*

Proof. For Q_0, the only positive off-diagonal rates form a directed chain. Adding the positive edge from C to A creates a directed cycle containing every state, establishing the quotient claim. Moreover $Q_\varepsilon - Q_0$ has only one nonzero row, $(\varepsilon, 0, -\varepsilon)$, whose ℓ^1 norm is 2ε. The finite-state Duhamel estimate used in Proposition C.1, with the identity observation map, yields the total-variation bound $t\varepsilon$; the universal bound of one completes Equation 51. □

The numerical calculations in Table 15 illustrate the distinction at $t = 1$ when the initial molecule is in form C. The errors are computed by exact finite-matrix exponentiation, not estimated from sampling.

Proposition D.1 isolates an important inference limit. Small finite-horizon predictive discrepancy places no general restriction on the topology of a support-derived quotient order. The reverse inference

also fails: exact agreement in support order permits substantial differences in transition rates, as already demonstrated in Subsection B.3. A declaration that an order is *stable* should therefore state the comparison topology, the quantitative norm, and the observational horizon. Infinite-horizon reachability is particularly sensitive to events with very small positive rates, although their practical detectability may be limited within finite observation windows.

D.2 Threshold Margins and Conditional Order Stability

A practical reconstruction may use a declared significance threshold on transition rates or probabilities. This creates a different mathematical object from positive-support reachability. Let $W = (w_{ij})$ be a finite matrix of nonnegative transition weights on a common configuration set C, and fix $\theta > 0$. Define the threshold edge relation by Equation 52. Self-loops may be omitted because reachability is reflexively closed.

$$T_\theta(W) = \{(i,j) \in C^2 : i \neq j,\ w_{ij} > \theta\}. \tag{52}$$

Proposition D.2 (Margin certificate for threshold-order stability)**.** *Let W and W' be nonnegative finite matrices on the same labelled state set. If $\max_{i\neq j} |w'_{ij} - w_{ij}| \leq \eta$ and $|w_{ij} - \theta| > \eta$ for all $i \neq j$, then $T_\theta(W') = T_\theta(W)$. Their reflexive–transitive closures, mutual-reachability equivalence classes, and induced quotient partial orders coincide. In the full-domain unary setting, their unrestricted potential profiles coincide as well.*

Proof. A weight strictly above $\theta + \eta$ remains above θ under the perturbation; a weight strictly below $\theta - \eta$ remains below it. The assumed margin exhausts all off-diagonal entries, giving equality of threshold edge relations. Every subsequent assertion follows from applying the same reachability, quotient, and unary potential constructions to equal edge relations. □

The strict margin in Proposition D.2 is a sufficient certificate for the particular threshold convention, with the state identities and regime held fixed. A threshold is a declared analytical choice; its numerical value does not become a law of nature by virtue of the proposition. When a transition lies within the error band of the threshold, a separate uncertainty description is needed. The theorem does not cover changes of the configuration space, resource semantics, joint-input compatibility, or the target physical interpretation.

For a family of uncertain edge weights, let $L_{ij} \leq w_{ij} \leq U_{ij}$ be jointly valid entrywise intervals. Define *certain* and *possible* threshold edges, respectively, by Equation 53.

$$E_\theta^- = \{(i,j) : i \neq j,\ L_{ij} > \theta\}, \qquad E_\theta^+ = \{(i,j) : i \neq j,\ U_{ij} > \theta\}. \tag{53}$$

For every admissible W, the containment $E_\theta^- \subseteq T_\theta(W) \subseteq E_\theta^+$ implies the reachability sandwich in Equation 54.

$$(E_\theta^-)^* \ \subseteq\ T_\theta(W)^* \ \subseteq\ (E_\theta^+)^*. \tag{54}$$

A path in the lower graph is therefore certified across the declared family, whereas a path absent from the upper graph is excluded across the family. The intermediate cases require additional information. In particular, a path in the upper graph can combine edges that never coexist in one admissible model; the upper relation is a conservative envelope rather than a claim of joint realizability. Furthermore, inclusion of reachability relations need not yield inclusion of the *quotient posets*: adding edges may merge strongly connected components and remove previously distinct order elements. These qualifications connect the interval construction to the model-uncertainty analysis in Section 5.

D.3 Finite-Horizon Probabilistic Potential Profiles

Thresholding a transition rate gives a transparent sensitivity diagnostic, but its significance depends on the time unit and the observation protocol. A complementary comparison uses probabilities of reaching an outcome within a declared number of steps. For a finite discrete-time stochastic matrix P, an integer $h \geq 0$, and a target y, define the hitting probability $H_{P,h}(x,y)$ as the probability, starting from x, of

Table 16: Different meanings of stability for a physically interpreted generative ordering. Each certificate is conditional on its stated state space, observation regime, and model assumptions.

Comparison	Available certificate	Unresolved information
Positive-support order	Exact edge information, or an assumption excluding new positive edges	Arbitrarily rare transitions can merge quotient classes
Threshold-weighted order	Entrywise error below the distance to a declared threshold	Dependence on threshold choice and unit calibration
Interval-envelope reachability	Lower-graph guaranteed paths and upper-graph excluded paths	Upper paths may combine incompatible candidate models
Finite-horizon target potential	Coupling bound and a margin around a probability criterion	Long-horizon behavior and history-dependent observations
Approximate state aggregation	Residual-based one-time total-variation bound	Multi-time observables, memory, and support order

visiting y at one of the times $0, \dots, h$. For $0 < p_0 < 1$, define the finite-horizon probabilistic potential profile by Equation 55.

$$\mathcal{P}^{[h]}_{P,p_0}(x) = \{y \in C : H_{P,h}(x, y) \geq p_0\}. \tag{55}$$

The definition assigns an operationally interpretable outcome set to each starting configuration. Its inclusion relation is again a preorder, with a quotient poset under profile equality, by the elementary argument of Proposition 6.2. Finite-horizon hitting relations themselves need not be transitive: a one-step path $a \to b$ followed by a one-step path $b \to c$ need not give one-step access from a to c. Profile inclusion and hitting reachability should accordingly be kept distinct.

Proposition D.3 (Finite-horizon robustness of probabilistic profiles)**.** *Let P and P' be discrete-time stochastic matrices on the same finite state set, and suppose $\sup_x \|P(x,\cdot) - P'(x,\cdot)\|_{\mathrm{TV}} \leq \eta$. For every x, y and integer $h \geq 0$, Equation 56 holds. If, moreover, $|H_{P,h}(x, y) - p_0| > h\eta$ for every x, y, then $\mathcal{P}^{[h]}_{P,p_0}(x) = \mathcal{P}^{[h]}_{P',p_0}(x)$ for every x; consequently the profile-equivalence quotient and its inclusion order are identical.*

$$|H_{P,h}(x, y) - H_{P',h}(x, y)| \leq \min\{1, h\eta\}. \tag{56}$$

Proof. Couple both chains from the same starting state and, whenever their states agree, use a maximal coupling of their next-step distributions. The conditional probability of disagreement at any such step is at most η. A union bound through h steps gives a probability of path disagreement bounded by $h\eta$, capped at one. The two hitting indicators agree whenever the coupled paths agree, establishing Equation 56. The strict uniform margin ensures that every hitting probability remains on its original side of p_0. Thus all target memberships and the resulting inclusion comparisons are preserved. □

The coupling bound is a finite-horizon result of standard stochastic-process reasoning, not a novel general perturbation theorem. It complements the generator-level marginal bound of Proposition C.1: that earlier bound concerns one-time observations, while Proposition D.3 explicitly controls an event depending on an entire finite path. Its assumptions require a common state space and one-step kernel, and the certified horizon becomes more restrictive as $h\eta$ grows. A small one-step error alone generally provides insufficient grounds for an unbounded-horizon certificate.

D.4 Cross-Scale Stability Certificates and Evidential Limits

The three constructions distinguish support topology, threshold order, and finite-horizon probabilistic potential. They answer different questions and need separate empirical calibration. Table 16 identifies the relevant observable, mathematical guarantee, and principal limitation. It also clarifies how the physical correspondence programme may connect the reconstructed generative landscapes of Sections 5 and 6 to measurements of reaction kinetics or other domain-specific processes.

A physical application should specify its conserved quantities and state representation, identify which transition rates or target probabilities are measured, and report measurement uncertainty together with the chosen horizon and threshold. Its conclusion should distinguish *certified order*, *excluded comparison*, and *unresolved comparison* where an interval model is used. A conservation-compatible reaction network can support an identifiable transition graph under sufficiently accurate observation; other domains may require independent equations of motion, hidden-state inference, or experimental interventions. Changing the observation map or the generative regime may invalidate a previously established certificate even if every calculation within the original model is correct.

The resulting philosophical position is deliberately circumscribed. An approximately correct effective dynamics can be scientifically useful without preserving exact support order. Conversely, a robust thresholded or probabilistic profile order obtains its stability from specified observational and inferential choices. These examples do not establish generative precedence as a fundamental physical relation. They provide means of specifying what an inquiry into physically emergent order must preserve and what types of evidence could refute a proposed correspondence.

E Comparative Explanatory Benchmarks

The comparative benchmark evaluates the explanatory role of the generative-landscape framework against methods already adequate for finite transition and reaction models. Its objective is to determine which kinds of conclusions are supported by each representation, with the admissible information and the target question declared before evaluation. Four controlled diagnostics examine projected paths, consumptive resources, stochastic aggregation, and support stability. Each diagnostic uses a small specified model that can be exhaustively evaluated. The accompanying script `comparative_baseline_checks.py` reproduces the claims and exports a machine-readable results record.

E.1 Comparison Protocol and Information Budget

The comparison separates a *question* from an *information budget*. A reachability graph whose vertices include the full resource inventory already encodes resource consumption through its admissible edges. Likewise, a full generator matrix specifies information about stochastic rates that an unweighted graph deliberately discards. Comparing a richly annotated framework with an impoverished baseline would conflate the information supplied with the method used to infer a result. The present tests therefore state when a baseline has adequate information and when a proposed summary loses information relevant to the specified question.

The reference methods are ordinary finite-state reachability and strong-component analysis, partition refinement or bisimulation-style behavioral comparison, marked Petri-net reachability, and finite-state continuous-time Markov-chain (CTMC) aggregation. These are established approaches with different objects and tasks [37, 35, 36, 10]. The G1 approach is evaluated as a cross-representational *warrant audit*: it specifies the rules, feasible configurations, observations, equivalences, and physical interpretation relevant to a claim. It invokes an appropriate established method to solve the underlying finite calculation. Consequently, its role here is methodological rather than that of a distinct numerical inference algorithm competing on accuracy or runtime.

A diagnostic supports an incremental explanatory claim only when an explicit baseline supplied with comparable input fails on a prespecified target and the integrated analysis supplies a reproducible, physically or logically meaningful distinction. An error due solely to removing information from one baseline's input is evidence about the adequacy of that representation, rather than evidence of a new computational capability. None of the following tests is an empirical predictive contest; all rates, maps, and inventories are stipulated.

E.2 Projected Paths and Behavioral Compatibility

Let the concrete states be a, b, c, d, with edges $a \to b$ and $c \to d$. Define an observation map with $\pi(a) = A$, $\pi(b) = \pi(c) = B$, and $\pi(d) = C$. The existentially projected edge graph contains $A \to B$ and $B \to C$, so its abstract transitive closure includes $A \to C$. The concrete transition system contains no path from a to d. Moreover, the image of concrete reachability under π contains the ordered pairs (A, B) and (B, C), but lacks (A, C); transitivity is therefore lost under relational image.

The full concrete graph answers the concrete path question correctly. A partition-refinement test also detects the behavioral incompatibility within the proposed class B: state b has no transition to C, whereas state c does. The G1 audit identifies the incorrect inference obtained when independent concrete representatives are silently joined. It makes explicit the reconstruction warrant, including the path-lifting condition of Proposition 5.3, without supplying a capability unavailable to the appropriate full-state baseline.

E.3 Consumptive Inventories and Joint Feasibility

Consider two permitted reactions $A \to B$ and $A \to C$ with one initial unit of A and no replenishment. The relevant states are inventories (n_A, n_B, n_C), with the initial marking $(1, 0, 0)$. The complete reachable set consists of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Individually attainable outcomes include a unit of B and a unit of C; a marking containing both products is unattainable under this conserved inventory.

A Boolean summary listing which products can occur individually loses the joint-feasibility question. A marked Petri net retains it, as does an ordinary directed graph generated on full inventory markings. These correct results are available without introducing G1-specific mathematics. The generative-landscape vocabulary makes the analytical choice visible by distinguishing the set of outcomes appearing somewhere from the set of jointly attainable resource states. The benchmark therefore favors a consumptive model whenever the target concerns compatible outcomes; reusable-premise closure is suitable for a different question.

E.4 Kinetic Information and Effective-State Aggregation

The third diagnostic reuses the two-site conversion model of Appendix B. Both sites have stationary product occupancy $1/3$. The symmetric model U uses forward rates $(2, 2)$ and reverse rates $(4, 4)$; the heterogeneous model H uses forward rates $(1, 3)$ and reverse rates $(2, 6)$. Their microscopic support graphs coincide, and their equilibrium distributions of the observable total product count are identical: $\Pr(N = 0) = 4/9$, $\Pr(N = 1) = 4/9$, and $\Pr(N = 2) = 1/9$. Their count dynamics, however, differ.

Strong lumpability holds for the count partition in U: the two microstates with count one have equal total rates into every count block. In H, the two count-one microstates have respective exit rates 2 and 6 to count zero, and 3 and 1 to count two. Consequently, the same partition fails strong lumpability in H. At lag $t = 0.5$, stationary count autocovariance equals approximately 0.022128 for U and 0.052053 for H. These are exact-expression evaluations of the stipulated CTMCs, reproduced in the benchmark script. Support-only reachability does not contain the rate data required for this comparison; a complete CTMC and standard aggregation analysis do.

This diagnostic yields a direct test of the warrant attached to an effective representation. Agreement on reachability and equilibrium count statistics supports those limited claims. Exact Markov closure and time-dependent correlation predictions require further kinetic conditions. The cross-scale G1 interpretation labels the claims and their failures, while the numerical distinction is provided by established stochastic-process methods.

E.5 Weak Transitions and Support Sensitivity

The fourth diagnostic starts with three states and transitions $A \to B \to C$. Introducing a return reaction $C \to A$ with arbitrarily small positive rate ε changes the mutual-reachability quotient from three singleton classes to one class. For a specified finite time horizon, the total-variation discrepancy induced

Table 17: Controlled diagnostic tasks, adequate established methods, and the claim-scope contribution of the G1 audit. Each task is solved within its stipulated finite model; no algorithmic superiority or empirical generalization is claimed.

Prespecified diagnostic	Established baseline with adequate information	Warrant clarified by G1
Projected $A \to C$ path	Concrete-state reachability or behavioral partition refinement	Abstract path requires compatible representative composition
Simultaneous B, C products	Marked Petri net or full-inventory state graph	Individual potential differs from jointly feasible inventory
Count-level Markov closure and $t = 0.5$ covariance	Full CTMC, strong lumpability test, and correlation calculation	Equilibrium and support agreement give limited cross-scale warrants
Tiny return rate and merged strong components	Graph strong components plus finite-time Markov perturbation	Support-order change differs from finite-time kinetic discrepancy

by the added transition admits the coupling bound $\min\{1, \varepsilon t\}$, as developed in Appendix D. Standard graph analysis detects the change in strong components, and standard perturbation analysis bounds the kinetic discrepancy. The G1 audit identifies the interpretation error of treating topological sensitivity as evidence of a commensurate finite-horizon dynamical change.

E.6 Comparative Outcomes and Explanatory Scope

Table 17 records the predeclared target for each diagnostic and the specialized baseline that fully addresses it when given the necessary input. The distinction between an insufficient summary and an insufficient analytical method is central to interpreting the table.

All four diagnostics are resolved by established methods with suitable state variables, resources, and rate information. The present examples therefore provide *no demonstrated predictive superiority* for G1 over the union of those methods. They do exhibit four recurrent classes of invalid inference that a coordinated cross-representational review can diagnose: unliftable projected paths, conflation of possible outcomes with joint feasibility, kinetic predictions inferred from support alone, and physical interpretation inferred from unstable graph topology. Whether the auditing framework has practical added value requires a separate study of analysts, workflows, annotation costs, error detection, and independently selected domain cases.

The comparison does not measure computational time, sample efficiency, or predictive performance on observed systems. Its four toy tasks are intentionally small, and their selection reflects the manuscript's prior theoretical questions. A future preregistered evaluation should supply both the specialized-method suite and G1-guided workflows with identical data and target questions, score correctness and calibration of stated warrants, report the effort needed to construct models and audit assumptions, and include cases where the proposed synthesis detects no additional error. Such a test would decide whether the framework offers practical explanatory discrimination beyond existing disciplinary practice.

Research Ethics and Source Provenance

Scope. The present release contains ten substantively drafted sections and five technical appendices, with detailed benchmark derivations separated from the central philosophical argument. The manuscript is released as an unreviewed exploratory preprint and has undergone focused formal-consistency, philosophical-lineage, cross-construction, and finite-benchmark, uncertainty, consumptive-resource, hidden-memory, and physical-correspondence revisions. The paper offers a conceptual research programme, with elementary formal propositions proved under their stated assumptions and stoi-

chiometrically constrained, analytically solved toy chemical-kinetic benchmarks, including an explicit microscopic-to-effective stochastic aggregation and a finite-horizon approximate aggregation bound and conditional stability certificates for ordering claims. It provides neither experimental validation of physical generativity nor independent measurements. Its references to causal-set physics and renormalization are bounded comparisons rather than a derivation of their results within the proposed framework.

Sources. Initial bibliography metadata and the stated abstract-level claims were checked against authoritative publisher, institutional, or author materials before use. The accompanying verification ledger records access limitations. A v2.1 claim audit, v2.2 comparative-baseline audit, v2.3 editorial audit, and v2.4 preprint audit distinguish elementary proofs, reproducible model checks, and interpretive proposals; source-level full-text evaluation and independent scholarly verification remain outstanding.

Authorship and status. This unreviewed preprint was prepared under human scholarly direction with AI assistance. It has received neither independent peer review nor independent certification of the mathematical proofs. Public availability should not be interpreted as external validation.

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