

Normatively Structured Interaction Histories: A Preliminary Braid-Theoretic Formalism

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Abstract

This paper develops a preliminary meta-level formalism for normatively structured interaction histories. The substantive normative theory is treated as an input rather than an object to be derived or reconstructed. Given a declared interaction-sensitive evaluation, the paper asks which historical distinctions a representation must preserve, which may be compressed, and when a braid-theoretic trajectory description is adequate for that purpose. The proposed architecture separates interaction history, trajectory topology, independently grounded normative typing, and theory-relative evaluation. Topological equivalence need not coincide with normative equivalence: histories can share the same topology while differing in participation, standing, role, authorization, accountability, burden, or other declared features, while topologically distinct histories may remain equivalent at the selected normative resolution. The analysis develops participant typing, non-circular event annotation, normative quotients, descriptor sufficiency, decorated braid representations, partial composition, continuation equivalence, relational-state sufficiency, institutional specializations, historical and interpretive uncertainty, and cross-context transport. Interactive justice, office accountability, and transitional processes are used only as motivating examples of broader process-sensitive inquiries. The resulting claims concern representational adequacy under stated assumptions; they do not establish a unique substantive theory of justice or the universal adequacy of braid theory.

Keywords: interaction histories; process-sensitive normativity; representation; normative equivalence; braid theory

Discussion Paper Note

This manuscript is the fourth paper in a cumulative research series on trajectory-sensitive social description and justice. Paper I developed braid theory as one lower-level descriptor of realized trajectories within Generative Relational Theory. Paper II inserted historical evidence and set-valued reconstruction between realized history and trajectory description. Paper III developed a general processual-justice architecture in which a substantive evaluation determines which historical distinctions must survive normative compression. The present paper develops a meta-level formalism for normatively structured interaction histories.

The framework is theory-relative. A substantive normative account enters as an input that specifies which differences among interaction histories matter to the inquiry. The paper then asks whether trajectory, topology, annotation, or another descriptor preserves those differences at an appropriate resolution. Interactive justice, office accountability, and transitional-process analysis appear only as examples of domains in which interaction history can matter normatively.

The formal objective is to distinguish interaction history, braid-topological description, independently typed interaction information, and normative equivalence. The paper does not claim that process-sensitive normativity is reducible to braid topology, that every interaction is naturally a braid crossing, or that one substantive theory supplies the universal semantics of the formalism.

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Notices

Status. This manuscript is a release-candidate discussion paper. Revision R20 reframes the paper as a meta-level study of representational adequacy for normatively structured interaction histories. Particular substantive theories appear only as motivating examples or domain-specific inputs; interactive justice is no longer used in the title or as the governing formal vocabulary. No new external sources are introduced.

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Statement on the use of language models. OpenAI’s ChatGPT was used for exploratory dialogue, formal reconstruction, source discovery followed by source verification, argumentative criticism, and drafting in $\text{\LaTeX}$. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, verified the cited sources, and bears sole responsibility for the manuscript, including its definitions, formal constructions, arguments, conclusions, and errors. Verification records for cited external sources are maintained in the accompanying source ledger.

1 Introduction

Many normative inquiries concern not only what outcome is reached, but also how persons, roles, or institutions interact through time. Examples arise in procedural justice, accountability, conflict management, transitional processes, deliberation, and other domains in which opportunities, responses, standing, authorization, burdens, or role relations can matter to evaluation. Once the interaction history itself becomes normatively relevant, a representational problem follows: which historical distinctions must a formal description preserve, and which may be compressed without changing the declared normative evaluation?

This paper studies that problem at a meta-level. Its object is not a particular substantive theory of justice. Instead, it considers a generic *normatively structured interaction history*: a temporally extended interaction equipped with whatever typed features a declared substantive account treats as relevant. The substantive account is taken as an input. The formal task is then to compare the distinctions retained by a proposed representation with the distinctions retained by the declared normative evaluation.

One motivating example comes from the literature on interactive justice. In that literature, Emanuela Ceva develops a proceduralist account concerned with the terms on which parties interact under value conflict, including fair hearing and procedural equality even where disagreement persists [2, 3, 4]. This example is useful because it makes endpoint compression visibly questionable: the same settlement can be reached through interactions that differ in hearing, standing, response, or role relations, while different substantive outcomes can remain compatible with the same procedural evaluation. The present paper does not reconstruct or formalize that theory as a whole. It uses selected interaction-sensitive considerations of this kind only to motivate and test a more general representational framework.

The representational difficulty is two-sided. A description that records only a terminal state may erase distinctions that the substantive account treats as normatively relevant. A description that preserves every microscopic feature may retain far more information than the inquiry requires. The relevant objective is therefore neither maximal compression nor maximal historical detail. It is *resolution relative to a declared normative question*: a descriptor is adequate when it preserves every distinction that the evaluation requires, while additional resolution is analytically optional unless another inquiry needs it.

Trajectory and braid-theoretic descriptions provide one candidate language for studying this problem. Their attraction is limited but precise: they can retain information about ordering, recurrence, relative movement, and interaction structure that endpoint descriptions may omit. Their limitation is equally important. A braid does not determine the normative meaning of a crossing, the standing of a participant, the authorization attached to a role, the burden created by an event, or the responsibility associated with an institutional position. Topology is therefore treated as one descriptive layer among several, never as a source of substantive normativity.

The paper accordingly separates four objects that are often conflated: the interaction history itself, a trajectory or topological description of that history, independently grounded normative typing, and the final theory-relative evaluation. This separation permits two kinds of mismatch. Topologically equivalent histories may be normatively distinct because the topology omits a relevant typed feature. Conversely, topologically distinct histories may be normatively equivalent because the declared evaluation treats the extra topological difference as irrelevant.

The analysis extends this idea beyond static representation. Interaction histories may compose only partially; the significance of a later event can depend on what occurred earlier; and two histories receiving the same present evaluation may support different admissible continuations. The paper therefore distinguishes retrospective normative equivalence from continuation equivalence and asks when a smaller relational state can preserve the continuation structure needed for sequential analysis. Institutional role systems, office accountability, and transitional interaction are treated as domain-specific examples of the general framework rather than as definitions of it.

The relation to the preceding papers in the series is cumulative. Paper I developed braid-theoretic trajectory description as a conditional representation of realized histories rather than a generative ontology. Paper II developed set-valued reconstruction of histories under evidential and classificatory uncertainty.

Paper III formulated processual justice as a question of which historical distinctions a declared justice evaluation permits an analyst to forget [10, 9, 11]. The present paper specializes those representational tools to normatively structured interaction histories while deliberately keeping the substantive normative theory open.

Seven questions organize the analysis. First, how should an interaction-history domain be typed before normative annotation? Second, which local or relational features may be represented without circularly inserting the final judgment into the data? Third, how should historical, endpoint, topological, decorated, and normative equivalence be distinguished? Fourth, when is an ordinary or decorated braid sufficient for a declared interaction-history evaluation? Fifth, when does partial composition preserve normative classes, and when does historical context defeat local compositionality? Sixth, what relational state is sufficient for continuation-sensitive analysis? Seventh, how should uncertainty and cross-context comparison be represented without overstating what the evidence or the normative theory identifies?

The paper proceeds by defining a generic interaction-history architecture, then introducing typed event records, normative equivalence, braid admissibility, partial composition, relational dynamics, institutional specializations, uncertainty, and worked examples. The claims remain deliberately representational. They concern what a formal language must preserve under stated assumptions; they do not establish a unique substantive theory of just interaction, a universal ontology of social processes, or the universal adequacy of braid theory.

2 Normative Architecture of Interaction-History Inquiry

This section states the substantive commitments required by the formal study at a level general enough to accommodate multiple normative theories. Its purpose is methodological: it identifies the kinds of normative sensitivity that can make interaction history relevant before the paper introduces participant types, event records, braid descriptors, or equivalence relations. The section does not define a new moral theory. It specifies an interface through which a declared substantive account can supply the distinctions that the later formalism must preserve.

2.1 Interaction-Sensitive Normative Objects

A normative inquiry is *interaction sensitive* when at least some differences in the way participants interact can alter the evaluation even when a coarser endpoint description is held fixed. The relevant differences may concern opportunities to participate, response relations, role occupancy, authorization, accountability, burden, responsibility, or another domain-specific feature. No universal inventory is assumed.

The resulting evaluative object is therefore neither the terminal state alone nor the complete microscopic history. It is a history at a resolution selected by the substantive account. Two histories may terminate in the same institutional or distributive state while differing normatively because one contains a relevant exclusion, denial of response, misuse of office, or other interactional feature. Conversely, different terminal outcomes may remain normatively equivalent if the declared theory does not treat that difference as part of the process-sensitive question under study.

This architecture keeps three questions separate: what claims or values are at issue; how the parties interact while addressing them; and what substantive outcome follows. A theory may connect these questions, but the representation should not identify them before those connections are stated. In particular, interaction-sensitive evaluation does not entail that outcomes are morally irrelevant; it means only that some properties of the interaction may constitute an additional evaluative object.

2.2 Participation, Standing, and Response

Participation-sensitive theories provide a useful class of examples. Their evaluation may depend not merely on whether a participant appears in the history but on whether that participant has an appropriate

opportunity to present a position, contest a claim, answer another participant, or otherwise exercise a procedurally recognized standing. Such requirements cannot in general be reduced to a count of utterances or to equality of elapsed time.

The interactive-justice literature supplies one concrete illustration. Its account of fair hearing and procedural equality emphasizes an equal chance for parties to have a say in the management of value conflict, while allowing substantive disagreement to persist [3, 4]. For the present paper, the importance of this example is representational: if opportunity, standing, or response matters to the declared evaluation, a descriptor that records only the final settlement can be too coarse. The variables introduced later for opportunity, participation, response, and standing are analytical devices of this paper rather than concepts attributed to that literature.

More generally, asymmetry itself is not a verdict. Judges and litigants, officeholders and members of the public, investigators and witnesses, representatives and represented persons, or parties with different procedural functions may occupy intentionally nonidentical positions. A formal difference in power, role, or authorization acquires normative significance only through a declared substantive criterion.

2.3 Persistent Disagreement and Endpoint Independence

Process-sensitive evaluation need not identify normative success with consensus, compromise, reconciliation, or termination of disagreement. A procedurally acceptable interaction can leave substantive disagreement unresolved, while apparent agreement can be reached through exclusion or coercive conditions that remain normatively relevant. Endpoint agreement and interaction-history evaluation should therefore remain distinct unless the substantive theory explicitly links them.

The same separation applies to dynamic effects. An interaction may improve later possibilities for cooperation, worsen them, or leave them unchanged. Those consequences can matter to a broader inquiry without determining the present normative status of the interaction itself. Later sections therefore keep current evaluation, relational-state change, and continuation structure analytically separate.

2.4 Role-Sensitive and Dynamic Specializations

Some domains require richer participant typing. Institutional analysis may distinguish a person from an office, an office from a role, and role occupancy from action undertaken in institutional capacity. Mandates, permissions, answerability, addressability, and responsibility bases can then become part of the history. Work on institutional accountability and political corruption provides one example of such role-sensitive analysis [6, 5]. The formal point is generic: when normative interpretation depends on an institutional position, tracking only persistent persons can erase a relevant distinction.

Other domains require explicit relational dynamics. Transitional processes, for example, can alter the relations inherited by later stages of interaction; the quality of a process and the relational conditions it generates need not be identical evaluative objects [8]. The formal framework must therefore be able to represent both a present history evaluation and a relational state relevant to future continuation. Again, this is a domain-specific specialization of the generic architecture rather than a universal feature of every interaction-history inquiry.

Taken together, these examples motivate a layered construction. A general interaction-history domain should support participant persistence, role change, event ordering, and context. Domain-specific enrichments may then add hearing, procedural standing, office mandates, accountability relations, burdens, responsibility bases, or relational-state updates when the declared inquiry requires them.

2.5 Formalization Boundary

Table 1 summarizes the requirements that follow from this meta-level architecture. The table states representational obligations rather than substantive moral conclusions.

The first architectural constraint is semantic separation. A map from an untyped braid class directly to a normative value is generally unjustified because the braid alone does not identify what the strands

Table 1: Meta-level requirements for interaction-history representation.

Normative sensitivity	Formal requirement	Not entailed
Endpoint-sensitive interaction	Keep interaction-history evaluation distinct from terminal-state description when the theory requires it.	No claim that outcomes are morally irrelevant.
Participation and standing	Represent opportunities, participation, response, or standing at the resolution required by the domain.	No universal scalar measure of participation or hearing.
Role-sensitive evaluation	Preserve persons, roles, offices, capacities, or authorization when their differences affect the declared evaluation.	No inference that every role or power asymmetry is unjust.
Persistent disagreement	Permit equivalent interaction histories with different substantive settlements when the theory allows it.	No identification of justice with disagreement itself.
Relational indexing	Type interaction features relative to participants and relevant relations.	No claim that every relation must enter the normative object.
Institutional specialization	Permit mandates, accountability relations, and responsibility bases to enrich the general domain.	No reduction of process-sensitive normativity to institutional ethics.
Dynamic specialization	Permit earlier interaction to modify conditions relevant to later interaction.	No inference that beneficial later effects retrospectively justify earlier conduct.

represent, what a crossing means, which role or standing is relevant, or which contextual rule supplies the normative criterion. The domain will therefore be constructed in stages: Section 3 specifies admissible interaction histories and participant types; Section 4 introduces independently interpretable normative typing; Section 5 defines the relevant equivalence relations; and Section 6 asks when braid-based descriptions preserve the required distinctions.

The second constraint is non-circularity. Normative annotations must be interpretable independently of the final history-level evaluation. If an annotation merely records the answer to the final normative question, descriptor sufficiency becomes true by construction and the representation explains nothing. Later event records are therefore built from lower-level properties such as opportunities, responses, roles, permissions, constraints, mandates, burdens, or responsibility bases, with their inventory determined by the domain.

The third constraint is theory relativity. Different substantive accounts can induce different admissible history domains, different annotations, and different equivalence relations on the same underlying social process. The formalism should expose those choices rather than conceal them. Its success is accordingly conditional: it shows what follows once the normative distinctions to be preserved have been declared; it does not establish those distinctions independently.

With these boundaries in place, Section 3 turns to the pre-normative structure of interaction histories: who or what persists, how persons and roles are typed, how role change is represented, how event granularity is selected, and what counts as an admissible history at the chosen analytical resolution.

3 Interaction-History Domains and Participant Typing

The substantive reconstruction in Section 2 determines what the formal model must be capable of distinguishing, but it does not yet determine the mathematical objects on which an interaction-history normative evaluation operates. The role of the present section is to construct that domain at a deliberately pre-evaluative level. The construction identifies the temporal support of an interaction, the bearers that participate in it, the event carriers through which participation becomes historically locatable, and the roles or relational positions under which those bearers appear. It does not yet classify an event as fair,

unfair, authorized, harmful, responsive, or otherwise normatively satisfactory. Those properties are introduced only in Section 4.

This separation serves three purposes. First, it prevents the justice classification from being built into the domain by definition. Second, it allows one persistent participant to occupy different roles at different stages, and different participants to occupy the same office or functional role at different stages. Third, it makes analytical resolution explicit. What counts as one interaction event depends on the research question and evidential resolution; an exchange represented as one event at a coarse scale may contain several normatively relevant subevents at a finer scale. The resulting history domain is therefore indexed by both context and resolution rather than treated as a universal set of social events.

3.1 Interaction-Relevant History Space

An interaction-history normative inquiry normally begins from a richer historical record than the formal representation can retain. The present paper does not identify that record with ultimate social reality. It treats it as a historically warranted record or reconstruction relative to a substantive context. The first formal operation is consequently a selection of the interaction-relevant portion of that record.

Let $c \in \mathcal{C}$ denote a declared interaction context. A context may identify, for example, a political deliberation, an institutional decision process, a transitional-justice process, or another value-conflict setting in which the terms of interaction are part of the justice object. Let $\rho \in \mathcal{R}_c$ denote an analytical resolution admissible for that context. The pair (c, ρ) fixes what kinds of participant distinctions, temporal distinctions, and event boundaries are retained for the inquiry.

Definition 3.1 (Interaction-history domain). For a declared context c and analytical resolution ρ , the *interaction-history domain*

$$\Gamma_{\text{IH}}^{c, \rho} \quad (1)$$

is the set of histories admitted as objects of interaction-history normative evaluation under that context-resolution pair.

Equation (1) is intentionally indexed. A history excluded at one resolution may become admissible at another if, for example, the evidential record is insufficient to distinguish the event structure required at the finer scale. Conversely, a history admissible in one institutional context need not be transportable without modification to another context whose role vocabulary or procedural structure differs.

Each $h \in \Gamma_{\text{IH}}^{c, \rho}$ is represented first by a structural skeleton

$$S(h) = (I_h, A_h, E_h, J_h, \text{part}_h). \quad (2)$$

Here I_h is the temporal support of the history, equipped with the strict temporal order $<_h$ inherited from the temporal convention declared for context c ; A_h is the finite set of participant bearers tracked at resolution ρ ; E_h is a finite or locally finite set of interaction-event carriers; $J_h : E_h \rightarrow \text{TS}(I_h)$ assigns each event carrier a nonempty support in the declared family $\text{TS}(I_h)$ of admissible time-supported subsets of I_h ; and

$$\text{part}_h : E_h \longrightarrow 2^{A_h} \setminus \{\emptyset\} \quad (3)$$

assigns the bearers participating in each event carrier. Equation (3) requires participation to be explicit without yet specifying the normative quality of that participation.

The object in Equation (2) is a skeleton rather than a complete event record. In particular, E_h does not yet encode whether a participant was heard, interrupted, authorized, burdened, benefited, responsive, answerable, or otherwise normatively typed. An event carrier only marks an interactionally relevant occurrence whose more detailed record may later receive such annotations. This ordering of construction implements the non-circularity requirement stated in Section 2.5.

The use of temporal supports $J_h(e)$, rather than an assumed total sequence $e_1, \dots, e_n$, also avoids introducing an artificial order among events that overlap or are only partially ordered by the available evidence. Whenever every admissible time assigned to one event precedes every admissible time assigned

to another, a precedence relation can be derived by

$$e \prec_h e' \iff \forall t \in J_h(e) \forall t' \in J_h(e'), \quad t <_h t'. \quad (4)$$

Equation (4) uses only the declared temporal order and does not require numerical endpoints, suprema, or infima for event supports. It yields only the temporal precedence warranted by the supports. Overlapping or unresolved event supports need not be forced into a total order. A later braid representation may require a stronger parameterization, but that additional representational choice should be stated when it is introduced rather than embedded in the definition of an interaction history.

The domain is also selective in a second sense. Not every fact about the participants belongs to the interaction-history object. A biographical property, institutional affiliation, prior episode, or background relation is retained only when the declared inquiry treats it as relevant to participant identity, role, relational position, event interpretation, or later normative typing. The resulting formal record is therefore a research-relative historical abstraction. Its adequacy depends on what it preserves, not on reproducing every available fact.

3.2 Persons, Roles, Offices, Collectives, and Relational Positions

Process-sensitive evaluation can concern relations among situated participants. The model must therefore distinguish a participant bearer from the role or position under which that bearer acts. The distinction is important whenever normative meaning depends on capacity, office, representation, or another contextually significant position.

For each context c , let $\mathcal{K}_c$ be the vocabulary of participant kinds retained by the inquiry. At minimum, the framework permits the distinction

$$\mathcal{K}_c \supseteq \{\text{person}, \text{collective}\}, \quad (5)$$

Equation (5) leaves the vocabulary open to context-specific extensions without requiring every domain to contain both displayed kinds. A participant-kind map

$$\kappa_h : A_h \longrightarrow \mathcal{K}_c \quad (6)$$

records the kind under which each bearer is represented in the history.

A collective participant is not automatically identified with the set of persons associated with it. When membership is relevant and evidentially available, it may be represented by a time-indexed membership relation

$$M_h(t) \subseteq A_h^{\text{person}} \times A_h^{\text{collective}}, \quad t \in I_h, \quad (7)$$

where the superscripts indicate the subsets of A_h typed by Equation (6). Equation (7) does not imply that an act of a member is an act of the collective, or conversely. Attribution requires an additional domain-specific rule. This restraint is necessary when institutional or collective agency is itself part of the question rather than a background assumption.

Let $\mathcal{L}_c$ denote the context-specific vocabulary of roles. A distinguished subset

$$\mathcal{O}_c \subseteq \mathcal{L}_c \quad (8)$$

contains those roles treated as offices for the purposes of the relevant institutional domain. The inclusion in Equation (8) makes every represented office a role type while avoiding the converse claim that every social role is an office. Role occupancy is recorded by

$$\Lambda_h : A_h \times I_h \longrightarrow 2^{\mathcal{L}_c}. \quad (9)$$

Equation (9) permits multiple simultaneous roles. The map therefore accommodates cases in which the same person is, for example, both a member of an institution and a representative of a constituency, provided the context requires both distinctions.

Table 2: Participant-level distinctions in an interaction history.

Object	Formal role	Distinction preserved
Participant bearer $a \in A_h$	Persistent token tracked through the selected history.	Bearer identity is not identified with a role, office, or justice status.
Participant kind $\kappa_h(a)$	Classifies a represented bearer, for example as person or collective.	Collective participation is not inferred from individual membership alone.
Role $\ell \in \mathcal{L}_c$	Contextual function, status, or capacity occupied by a bearer.	Role occupancy may change while bearer identity persists.
Office $o \in \mathcal{O}_c$	Institutionally specified role included in the domain.	Office persistence does not imply persistence of its occupant.
Relational position $q \in \mathcal{Q}_c$	Position of one bearer relative to another at a specified time.	A relational asymmetry is not, by itself, an interactive injustice.

Roles remain distinct from *relational positions*. A role can be assigned to one bearer without specifying the particular counterpart relative to whom a relation is instantiated. Let $\mathcal{Q}_c$ denote a vocabulary of relational-position types. The relation

$$\Pi_h(a, b, t) \subseteq \mathcal{Q}_c, \quad a, b \in A_h, \quad (10)$$

records positions of a relative to b at time t , where such positions are part of the declared history object. Examples may include claimant relative to adjudicator, officeholder relative to addressee, representative relative to represented party, or other domain-specific relational positions. These examples illustrate possible typings; Equation (10) does not assign any of them a justice value.

Table 2 summarizes the distinctions. They are intentionally non-reductive. A person can persist while their role changes; an office can persist while its occupant changes; a collective can persist while its membership changes; and a relational position can change even when the identities and formal roles of the parties remain fixed.

This architecture allows Section 4 to attach normatively interpretable information to an interaction without asking a topological strand to carry all semantic content. A strand may later track a participant bearer, a role-bearing trajectory, or another typed object depending on the representation. The choice must be declared because the justice-relevant meaning of a crossing can differ substantially depending on what the strands represent.

3.3 Identity Persistence and Role Change

A history-sensitive model requires a rule for determining when appearances at different times belong to the same tracked participant. The present framework handles persistence at the level of participant bearers and treats role occupancy as a time-dependent attribute of those bearers. This avoids two opposite errors: splitting one persistent participant into several agents merely because their role changes, and merging different participants merely because they successively occupy the same role.

For an event carrier $e \in E_h$, define the set of participant appearances

$$\text{App}_h = \{(e, a) : e \in E_h, a \in \text{part}_h(e)\}. \quad (11)$$

Equation (11) records appearances before any inference about role continuity. The persistence projection

$$\iota_h : \text{App}_h \longrightarrow A_h, \quad \iota_h(e, a) = a, \quad (12)$$

identifies event-local appearances that the declared reconstruction treats as appearances of the same participant bearer. Equation (12) is not an ontological theory of personal identity. It records the identification rule adopted for the historical inquiry at resolution ρ .

Role persistence is a separate matter. For a participant a , equality

$$\Lambda_h(a, t) = \Lambda_h(a, t') \quad (13)$$

in Equation (13) means only that the represented role set is unchanged between the two indexed times. It does not establish that every other relational property of the participant is unchanged. Conversely,

$$\Lambda_h(a, t) \neq \Lambda_h(a, t') \quad (14)$$

records a role change without implying a change of bearer identity.

The corresponding distinction for offices is equally important. Suppose $o \in \mathcal{O}_c$. It may be the case that

$$o \in \Lambda_h(a, t), \quad o \in \Lambda_h(b, t'), \quad a \neq b. \quad (15)$$

Equation (15) represents an office that is occupied by different participant bearers at different times. A braid whose strands are labeled only by offices would treat this succession differently from a braid whose strands track persons. Neither representation is automatically preferable; they answer different questions and preserve different historical distinctions.

Proposition 3.2 (Independence of bearer and role persistence). *Within the history structure defined above, persistence of a participant bearer neither entails persistence of role occupancy nor is entailed by persistence of a role or office type.*

Proof. Bearer persistence is represented by equality in the codomain of ι_h , whereas role occupancy is represented by values of Λ_h . Equation (14) permits the same bearer a to have different role sets at different times, so bearer persistence does not entail role persistence. Equation (15) permits distinct bearers $a \neq b$ to occupy the same office o at different times, so persistence of a role or office type does not entail persistence of its bearer. No additional implication follows without a domain-specific identification rule. $\square$

Proposition 3.2 is structurally simple, but it is consequential for process-sensitive evaluation. A formally identical interaction pattern can be normatively interpreted differently when a participant acts in a different capacity, while a stable office can carry obligations across changes of personnel.

Relational positions can change independently as well. Even if a , b , and their role sets remain fixed, one may have

$$\Pi_h(a, b, t) \neq \Pi_h(a, b, t'), \quad (16)$$

when the history contains an event that changes their institutional or social relation. Equation (16) therefore separates relational-position change from both bearer replacement and role change. This possibility anticipates domains in which earlier interaction changes the relational conditions inherited by later stages. Section 8 will formalize such updates. The present section records only that the history domain must be capable of representing the before-and-after distinction.

3.4 Interaction-Event Granularity

The event carrier $e \in E_h$ is not assumed to be a natural atom of social interaction. Event boundaries depend on analytical resolution. A deliberative exchange represented coarsely as one “hearing” may, at a finer resolution, contain an invitation to speak, a restriction on available time, an interruption, a response, an acknowledgment, and a subsequent decision. Whether these distinctions matter depends on the substantive question and the available evidence. The formalism therefore treats granularity as declared rather than discovered once and for all.

Let $\rho' \succeq \rho$ mean that ρ' is at least as fine as ρ within the same context c . For histories that admit both descriptions, refinement is represented by a surjective event coarse-graining map

$$q_{\rho' \rightarrow \rho} : E_h^{\rho'} \longrightarrow E_h^{\rho}. \quad (17)$$

Each coarse event can therefore collect one or more event carriers from the finer description. The map in Equation (17) need not be injective; indeed, non-injectivity is precisely what makes the coarser description a compression.

The event map induces a history-level coarse-graining whenever the participant identifications, temporal supports, and other retained structural data are compatible. Write

$$Q_{\rho' \rightarrow \rho} : \Gamma_{\text{IH}}^{c, \rho'} \rightarrow \Gamma_{\text{IH}}^{c, \rho} \quad (18)$$

for this generally partial map. Partiality is important. A fine history need not coarse-grain to an admissible history at every target resolution if the target representation imposes event forms or participant identifications that are incompatible with that history.

On the actual domain of Equation (18), define the source–target resolution equivalence by

$$h \sim_{\rho' \rightarrow \rho} g \iff h, g \in \text{Dom}(Q_{\rho' \rightarrow \rho}) \text{ and } Q_{\rho' \rightarrow \rho}(h) = Q_{\rho' \rightarrow \rho}(g). \quad (19)$$

The source resolution ρ' is retained in the notation because the same target resolution can be reached from different finer domains with different admissibility conditions. Distinct fine histories may therefore become identical at a coarser resolution. Equation (19) is descriptive: it records information loss under coarse-graining. It does not yet state whether the lost distinctions are normatively dispensable. That question requires the interaction-history normative equivalence relation introduced in Section 5.

Remark 3.3 (Resolution adequacy). A coarse resolution is not inadequate merely because it loses historical detail. It becomes inadequate for a declared interaction-history normative inquiry only when the lost detail includes distinctions that the substantive evaluation requires. The relevant criterion is therefore relational: event resolution must eventually be assessed against the justice distinctions to be preserved, rather than against an unattainable ideal of maximal historical detail.

Remark 3.3 follows the general descriptor discipline developed in the preceding papers of this series [10, 11]. It is especially important here because the language of “interaction” can tempt a model to over-segment ordinary communication into artificially microscopic events. More detail is not automatically more faithful to the normative object. A useful event scale is one at which the independently justified interaction properties can be identified without manufacturing distinctions irrelevant to the inquiry.

Granularity also constrains braid construction. A crossing at one resolution may correspond to several temporally extended interaction events at another, and some normatively relevant events may have no natural interpretation as pairwise crossings at all. Section 6 therefore treats braid admissibility as a property of a selected interaction domain, not as a presumption about every history in $\Gamma_{\text{IH}}^{c, \rho}$.

3.5 Domain Admissibility and Context

The notation $\Gamma_{\text{IH}}^{c, \rho}$ is useful only if admission to the domain is explicit. The context c does more than name a case. It fixes the semantic vocabulary and structural constraints required to interpret the history. Depending on the inquiry, these constraints may include which participant kinds can appear, which roles and offices are recognized, what kinds of events are interactionally relevant, what temporal evidence is sufficient to locate an event, and what background institutional or social relations are treated as part of the record.

The following conditions define the baseline structural admissibility used in this paper.

Definition 3.4 (Structurally admissible interaction history). A candidate history h is structurally admissible for (c, ρ) only if all of the following hold:

1. **Temporal support:** I_h and every $J_h(e)$ are well formed under the temporal convention declared for c , and each event carrier has nonempty support.
2. **Participant traceability:** every participant appearance used by the history is assigned to a bearer in A_h through the declared persistence rule.
3. **Participation support:** every event carrier has at least one represented participant and every attributed participant lies in A_h .

4. **Typing compatibility:** participant kinds, role occupancy, offices, membership relations, and relational positions use vocabularies admitted by c and satisfy any context-specific typing constraints.
5. **Resolution compatibility:** event boundaries and retained participant distinctions are expressible at resolution ρ without claiming distinctions finer than the supporting reconstruction warrants.
6. **Contextual interpretability:** the structural record contains enough context to interpret the participant and event types used by the later normative analysis.

Definition 3.4 is deliberately structural rather than moral. Membership in $\Gamma_{\text{IH}}^{c,\rho}$ does not imply that a history is interactively just, unjust, legitimate, illegitimate, cooperative, antagonistic, or institutionally proper. It means only that the history is sufficiently and consistently represented to be evaluated under the declared inquiry.

Two further closure properties must not be assumed. First, the domain need not be closed under concatenation. Suppose histories h and g have compatible temporal boundaries and participant typings. Their sequence-level concatenation may be well formed while the concatenated object still fails a context-specific admissibility condition. Accordingly,

$$h, g \in \Gamma_{\text{IH}}^{c,\rho} \not\Rightarrow h \cdot g \in \Gamma_{\text{IH}}^{c,\rho}. \quad (20)$$

Equation (20) records this absence of automatic closure. Section 7 will state the stronger conditions required before composition is used in normative arguments.

Second, admissibility is not automatically preserved under transport between contexts. Even if two domains use the same formal tuple in Equation (2), their role vocabularies, relational positions, event semantics, or evidential standards may differ. For $c \neq c'$, there is therefore no default map

$$\Gamma_{\text{IH}}^{c,\rho} \longrightarrow \Gamma_{\text{IH}}^{c',\rho'}. \quad (21)$$

Equation (21) records the absence of such a default transport. A cross-context transport must be constructed and justified separately. This restriction anticipates the comparative problem in Section 10.4 and prevents formal similarity from being mistaken for semantic equivalence.

The same caution applies to institutional enrichment. When offices, mandates, or accountability relations are absent from a context, they need not be added merely because the general framework permits them. Conversely, when the justice question depends on office occupancy or status-sensitive interaction, omitting those distinctions can make the domain too coarse for the intended evaluation. Institutional and transitional applications illustrate why participant status and changing relationships can matter; they do not establish that every interaction-history domain requires the same enrichment.

The construction in this section provides the typed substrate required by the remainder of the paper. It identifies who or what is tracked, when interaction carriers occur, under which roles and relational positions participants appear, and at what resolution the history is represented. It still withholds the properties that carry explicit normative meaning. Section 4 now introduces those properties as event-level records derived from the substantive requirements of Section 2, while preserving the distinction between local event typing and whole-history interaction-history normative evaluation.

4 Normatively Typed Interaction Events

Section 3 constructed an interaction-history domain without assigning justice values to its event carriers. The present section adds the intermediate semantic layer required before interaction-history normative equivalence can be defined. Its objective is to record properties that may be normatively relevant to interaction while preserving a strict distinction between local typing and whole-history evaluation. The construction proceeds from event incidence to participation and hearing, then to role-sensitive power and authorization, and finally to burdens, benefits, responsibility-relevant bases, and local context. The

closing subsection assembles these components into an annotation map and states a non-circularity requirement for its construction.

The resulting records are deliberately heterogeneous. Fair hearing, procedural standing, authorization, power, burdens, and responsibility do not naturally occupy one common scalar scale, and the substantive reconstruction in Section 2 gives no basis for forcing them onto one. A typed record therefore preserves their distinct semantic roles. Later sections may identify which components are actually required by a declared interaction-history normative theory and which can be omitted without loss of justice-relevant information.

4.1 Primitive Interaction Records

The structural skeleton in Equation (2) supplies event carriers and participating bearers. Normative typing requires a finer local carrier that associates a participant with an event without yet assigning that participation a moral value. For each history $h \in \Gamma_{\text{IH}}^{c,\rho}$, define the participant-event incidence set by Equation (22).

$$\mathcal{I}_h = \{(e, a) \in E_h \times A_h : a \in \text{part}_h(e)\}. \quad (22)$$

Equation (22) provides the carrier for participant-relative properties such as opportunity, standing, burden, or office occupancy during a represented event.

Some interaction properties are directed from one participant toward another. Define the directed participant-event incidence set by Equation (23).

$$\mathcal{D}_h = \{(e, a, b) \in E_h \times A_h \times A_h : a, b \in \text{part}_h(e), a \neq b\}. \quad (23)$$

A member $(e, a, b) \in \mathcal{D}_h$ states only that a and b are represented as co-participants in event carrier e and that a directed feature from a toward b may be meaningful in the declared context. It does not assert that a acted on b , exercised power over b , owed a duty to b , or wronged b . Those claims require independently typed information.

The semantic interpretation of an event may also matter before a justice judgment is made. Let $\mathcal{E}_c$ be a context-specific vocabulary of interaction-event types. When the evidence and selected resolution support such typing, use the partial map in Equation (24).

$$\tau_h : E_h \rightharpoonup \mathcal{E}_c. \quad (24)$$

The partiality of Equation (24) allows an event carrier to remain unclassified when the available reconstruction warrants the occurrence but not a more specific interpretation. Depending on c , admissible event types may include presenting a claim, requesting reasons, responding, interrupting, excluding, deciding, implementing, reviewing, or another context-specific category. The vocabulary is descriptive and semantic rather than a list of justice outcomes.

A general feature schema can now be stated without presupposing one universal inventory of interaction properties. Let $\mathfrak{F}_c^E$ contain event-level feature types, $\mathfrak{F}_c^I$ participant-event feature types, and $\mathfrak{F}_c^D$ directed participant-event feature types. For each feature f in one of these families, let $V_{c,f}$ be its declared codomain. A feature valuation is a partial function of the form appropriate to its carrier, as summarized in Equation (25).

$$\begin{aligned} \nu_{h,f}^E : E_h &\rightharpoonup V_{c,f}, & f &\in \mathfrak{F}_c^E, \\ \nu_{h,f}^I : \mathcal{I}_h &\rightharpoonup V_{c,f}, & f &\in \mathfrak{F}_c^I, \\ \nu_{h,f}^D : \mathcal{D}_h &\rightharpoonup V_{c,f}, & f &\in \mathfrak{F}_c^D. \end{aligned} \quad (25)$$

Equation (25) separates feature identity from feature value. A context may therefore represent several qualitatively different properties without coercing them into a single score. Partiality also separates absence of a warranted value from a substantively meaningful negative value. Section 10 will treat uncertainty in the values themselves; the present section requires only that an omitted valuation not be silently interpreted as evidence that the corresponding property is absent.

Definition 4.1 (Normatively typed interaction feature). A feature $f \in \mathfrak{F}_c^E \cup \mathfrak{F}_c^I \cup \mathfrak{F}_c^D$ is *normatively typed* when its semantic interpretation specifies a property that can enter an interaction-history normative assessment under the declared theory, while its value can be identified independently of the final interaction-history normative classification of the whole history.

Definition 4.1 permits a feature to carry normative relevance without making it a local verdict of justice. A represented exclusion, authorization, burden, or office duty may be highly relevant to evaluation, yet the significance of that feature can depend on its relation to other events and to the history as a whole. This distinction implements the substantive caution in Section 2.2: local records provide material for evaluation; they do not automatically compose into an interaction-history normative judgment.

4.2 Hearing, Participation, and Procedural Standing

Some process-sensitive theories treat opportunities to participate, realized participation, response, and procedural standing as normatively relevant. The decomposition introduced below is an analytic device of this paper, not a universal theory of fair hearing or participation. Its purpose is to test whether independently interpretable opportunity, participation, response, and standing records can preserve distinctions required by a declared evaluation without reducing participation to the number of utterances or equality to an unexamined numerical identity.

Let $\mathcal{U}_c$ denote the vocabulary of participatory functions recognized in context c . A participatory function describes what a participant can do within the procedure, such as present a claim, answer an objection, contest a characterization, request reasons, submit evidence, question another participant, or make a reply. The opportunity record and realized-participation record are defined on $\mathcal{I}_h$ by Equations (26) and (27).

$$\text{Opp}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{U}_c} \quad (26)$$

$$\text{Act}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{U}_c}. \quad (27)$$

Equation (26) records functions made meaningfully available to the participant at the represented event, while Equation (27) records the functions actually exercised when the history supports that classification. The two maps need not coincide. An available opportunity can go unused, and an action can occur despite lacking authorization or recognized standing. Their separation prevents realized speech from being treated as evidence that an adequate opportunity was provided.

A hearing relation may also depend on what becomes possible after a contribution. Let $\mathcal{Y}_c$ be a context-specific vocabulary of response modes, and define the directed response record in Equation (28).

$$\text{Rsp}_h : \mathcal{D}_h \rightarrow 2^{\mathcal{Y}_c}. \quad (28)$$

Equation (28) can record, for example, whether a contribution by a is answered, acknowledged, contested, referred for decision, or otherwise taken up by b in a manner recognized by the context. It does not by itself state that the response is adequate or that a has been fairly heard. A fair-hearing judgment may require a temporally extended pattern involving several opportunities, contributions, responses, and later acts.

Procedural standing requires a further distinction. Let $\mathcal{S}_c$ denote the vocabulary of procedural statuses recognized in c . The standing map is defined in Equation (29).

$$\text{Stand}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{S}_c}. \quad (29)$$

Possible statuses may include eligibility to initiate a claim, entitlement to answer, recognized party status, decision authority, review entitlement, or another domain-specific position. A status in Equation (29) is not equivalent to a role in Equation (9). Roles describe how a bearer is situated in the interaction or institution; procedural standing records what the declared procedure recognizes that bearer as entitled or competent to do at the relevant event.

The hearing-related information associated with one incidence can therefore be collected without assigning a fair-hearing verdict. Equation (30) defines the local hearing profile when the required components are available.

$$H_h(e, a) = (\text{Opp}_h(e, a), \text{Act}_h(e, a), \text{Stand}_h(e, a)). \quad (30)$$

Equation (30) deliberately omits a scalar “hearing score.” Directed response information from Equation (28) can be linked to the profile when the substantive criterion requires it. This architecture is compatible with a theory under which fair hearing is evaluated over a segment or complete interaction history rather than event by event.

Remark 4.2 (Procedural equality as a theory-relative relation). The existence of the maps in Equations (26)–(29) does not define procedural equality. A declared interaction-history normative theory must specify which opportunity, standing, response, or other differences count as relevant to equal participatory terms in context c . Equal cardinalities of opportunity sets are neither necessary nor sufficient in general.

Remark 4.2 preserves the substantive interpretation established in Section 2.2. Two participants may have differently structured opportunities that are nevertheless judged procedurally equal under a justified rule, while formally identical opportunity counts may conceal differences in timing, standing, responsiveness, or practical accessibility. The equivalence relation induced by the substantive interaction-history normative standard is therefore introduced only in Section 5.

4.3 Role, Power, Authorization, and Institutional Position

Participant-relative interaction is shaped by role and institutional position, but the formalism should avoid treating every asymmetry as an injustice. Institutional applications can require interrelated roles, office responsibilities, answerability, or other accountability relations. The present subsection therefore records role-sensitive capacities and authorization as typed relations whose normative significance remains theory dependent.

The role map Λ_h from Equation (9) is time indexed. For event-relative analysis, define the represented office occupancy set by Equation (31).

$$\text{Off}_h(e, a) = \{o \in \mathcal{O}_c : \exists t \in J_h(e) \text{ with } o \in \Lambda_h(a, t)\}, \quad (e, a) \in \mathcal{I}_h. \quad (31)$$

Equation (31) records offices occupied at some point during the event support. If an office-sensitive evaluation depends on changes within $J_h(e)$, the union implicit in this representation may be too coarse. In that case the event should be refined or the time-varying role restriction $\Lambda_h(a, \cdot)|_{J_h(e)}$ retained directly.

Power is represented through contextually typed capacities rather than a universal scalar. Let $\mathcal{P}_c$ be a vocabulary of power channels or interaction-relevant capacities. Define the directed power record by Equation (32).

$$\text{Pwr}_h : \mathcal{D}_h \rightarrow 2^{\mathcal{P}_c}. \quad (32)$$

Depending on the domain, values of Equation (32) may represent capacities to set an agenda, admit or exclude participation, control information, issue binding decisions, allocate resources, impose sanctions, initiate review, or otherwise alter the interaction conditions of another participant. The record states which power channels are historically relevant at the selected resolution; it does not infer domination, corruption, unfairness, or injustice from their mere presence.

Authorization is conceptually distinct from power. A participant may possess the capacity to perform an act while lacking authorization to do so, or may hold formal authorization while lacking effective capacity. Let $\mathcal{Z}_c$ be a vocabulary of authorization bases, such as office mandate, delegated authority, procedural rule, consent, statutory competence, or another domain-specific source. The authorization record is given in Equation (33).

$$\text{Auth}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{Z}_c}. \quad (33)$$

A nonempty value of Equation (33) records represented authorization bases applicable to the participant-event incidence. An empty represented set and an undefined value must remain distinct: the former can express a warranted finding that no admitted basis applies, whereas the latter expresses absence of a warranted classification at the selected resolution.

Institutional position can also involve directed accountability relations. Let $\mathcal{K}_c^{\text{acc}}$ be a vocabulary of accountability-relation types admitted by an institutional context. Equation (34) defines an optional directed record.

$$\text{Acc}_h : \mathcal{D}_h \rightarrow 2^{\mathcal{K}_c^{\text{acc}}}. \quad (34)$$

The relation in Equation (34) can encode contextually specified forms of answerability, addressability, oversight, reporting, or another accountability relation. It is an institutional specialization and need not appear in noninstitutional interaction domains.

The separation among Equations (32), (33), and (34) is analytically important. Effective power, valid authorization, and accountability can covary, but they are not interchangeable. A justice theory may treat the relation among them as significant—for example, a use of power outside an authorization basis or a failure to answer within an accountability relation—without implying that any one component independently determines the justice of the complete history.

4.4 Burden, Benefit, Responsibility, and Context

Interaction histories can distribute burdens and benefits, generate effects, and create grounds for responsibility that are not exhausted by hearing conditions or institutional position. These features require separate records because their normative roles differ. A benefit need not cancel a burden, and causal involvement need not by itself establish responsibility. The present framework therefore keeps effects and responsibility-relevant bases distinct.

Let $\mathcal{B}_c$ and $\mathcal{G}_c$ be context-specific vocabularies of burden and benefit types. The participant-relative records are defined by Equations (35) and (36).

$$\text{Bur}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{B}_c} \quad (35)$$

$$\text{Ben}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{G}_c}. \quad (36)$$

The vocabularies can distinguish material, procedural, expressive, institutional, reputational, or other effect types when those distinctions are warranted by the inquiry. Equations (35) and (36) do not assume commensurability, additivity, or a common metric. They therefore permit a history to contain several qualitatively different burdens and benefits without presupposing a balancing rule.

Responsibility is treated with additional caution. The paper does not infer responsibility directly from participation, causal contribution, role occupancy, or benefit. Let $\mathcal{V}_c^{\text{resp}}$ be a vocabulary of responsibility-relevant bases recognized by the declared theory. The responsibility-basis record is defined in Equation (37).

$$\text{RB}_h : \mathcal{I}_h \rightarrow 2^{\mathcal{V}_c^{\text{resp}}}. \quad (37)$$

A basis in Equation (37) may concern authorship, contribution, control, role duty, omission, benefit from an institutional process, answerability, or another ground admitted by the substantive theory. The presence of such a basis identifies information that a responsibility judgment may use; it does not state the final degree, kind, or allocation of responsibility. This distinction is especially important in institutional settings, where interdependent roles can make responsibility relationally distributed rather than reducible to one isolated act.

Local context can alter the interpretation of otherwise similar events. Let $\mathcal{X}_c$ denote a vocabulary of context conditions retained inside the declared global context c . The event-context map is stated in Equation (38).

$$\chi_h : E_h \rightarrow 2^{\mathcal{X}_c}. \quad (38)$$

Possible values may concern an emergency rule, a stage of a procedure, a transition period, evidential constraints, a declared institutional mandate, or another condition whose presence changes the interpretation

Table 3: Principal components of normatively typed interaction records.

Component	Carrier	Recorded information	Inference not licensed by the component alone
Opportunity / participation	$\mathcal{I}_h$	Available and exercised participatory functions	Fair hearing or procedural equality
Response	$\mathcal{D}_h$	Directed forms of uptake, answer, or contestation	Adequate consideration or agreement
Procedural standing	$\mathcal{I}_h$	Recognized procedural status or competence	Equal standing across participants
Role / office	$A_h \times I_h, \mathcal{I}_h$	Role occupancy and event-relative office position	Justice or injustice of the role relation
Power	$\mathcal{D}_h$	Contextually typed capacities affecting another participant or the interaction	Domination, corruption, or injustice
Authorization	$\mathcal{I}_h$	Represented bases for permission or competence	Moral permissibility of the complete act or history
Accountability	$\mathcal{D}_h$	Directed institutional accountability relations	Fulfilled accountability or institutional justice
Burden / benefit	$\mathcal{I}_h$	Qualitatively typed effects on participants	Net welfare or compensating balance
Responsibility bases	$\mathcal{I}_h$	Grounds potentially relevant to responsibility	Final responsibility attribution
Local context	E_h	Event-specific conditions needed for interpretation	A new justice standard without further argument

of event-level features. Equation (38) does not reproduce all background facts. It retains only context variables that the declared interaction theory or evidential protocol treats as semantically necessary.

The distinction between global context c and local context $\chi_h(e)$ permits one global interaction domain to contain varying local circumstances while preserving event-specific conditions that alter interpretation. In dynamic process domains, relationship conditions and institutional circumstances can evolve over time; local contextual typing provides one place to record changes whose normative interpretation depends on the stage or conditions of the process.

Table 3 summarizes the principal components introduced in Sections 4.2–4.4. The final column states a corresponding non-inference that the later justice analysis should preserve.

The table makes explicit that normative relevance and normative conclusion are different levels of analysis. A typed record can carry information required by a justice judgment while remaining insufficient to determine that judgment independently of the history and the substantive evaluation rule.

4.5 Non-Circular Construction of Normative Decoration

The components above can now be assembled into the interaction annotation anticipated in the Introduction. The purpose of the assembly is representational: it gives later sections a typed object whose information content can be compared with interaction-history normative equivalence. The assembly should remain independent of the final justice class so that descriptor sufficiency is not obtained by construction.

For a fixed context-resolution pair (c, ρ) , let $\mathcal{A}_{c,\rho}$ denote the space of admissible annotation records built from a declared subset of the component maps introduced in this section. Define the annotation map by Equation (39).

$$A_{c,\rho} : \Gamma_{\text{IH}}^{c,\rho} \longrightarrow \mathcal{A}_{c,\rho}. \quad (39)$$

For a history h , the value $A_{c,\rho}(h)$ contains the selected event types and feature valuations together with

enough carrier information to identify which event, participant-event incidence, or directed incidence each value belongs to. The exact inventory is domain specific. A value-conflict deliberation may require hearing and standing information while omitting office accountability; an institutional-corruption inquiry may require office, authorization, power, and responsibility bases in addition to hearing information.

The assembly should satisfy a provenance condition. Let $\mathcal{W}_{c,\rho}$ denote the space of admissible pre-judgment records: structural history information, evidence-supported semantic classifications, and declared interpretive rules that do not contain the final interaction-history normative result. The annotation construction rule is stated in Equation (40).

$$\alpha_{c,\rho} : \mathcal{W}_{c,\rho} \longrightarrow \mathcal{A}_{c,\rho} \quad (40)$$

produces the annotation from that record. If $W_{c,\rho} : \Gamma_{\text{IH}}^{c,\rho} \rightarrow \mathcal{W}_{c,\rho}$ denotes the corresponding record map, the assembled annotation is required to satisfy Equation (41).

$$A_{c,\rho} = \alpha_{c,\rho} \circ W_{c,\rho}. \quad (41)$$

Equation (41) makes the construction path explicit. It does not imply that the underlying record is value neutral: interpreting an exclusion, office mandate, burden, or responsibility basis can require normative concepts. The narrower requirement is that those concepts have independently stated semantics and are not assigned by first looking up the complete interaction-history normative classification of h .

Research Obligation 4.3 (Judgment-independent annotation). Before a component is used in $A_{c,\rho}$, its carrier, codomain, semantic interpretation, and identification rule should be specifiable without taking the final interaction-history normative value of the same history as an input. A direct recoding of the final justice class therefore fails the annotation requirement even when it would make later descriptor sufficiency trivial.

Research Obligation 4.3 is methodological rather than a theorem derivable from set-theoretic structure alone. A circular annotation can have the same extensional values as a legitimately derived one. The difference lies in how the values are warranted. The accompanying evidence and interpretation therefore remain part of the source discipline of the formalization even when the final mathematical object records only the resulting types.

The construction also permits redundancy. Two independently meaningful components may covary across a particular dataset or history domain. Their coexistence does not violate non-circularity. Minimality is a different question: Section 5 will ask which components are needed to preserve interaction-history normative distinctions and whether some can be removed without changing the induced justice-relevant resolution.

The typed architecture now supplies the annotation map used by later decorated descriptors. It also clarifies what a later decorated braid can and cannot mean. A decoration may attach selected components of $A_{c,\rho}(h)$ to strands, crossings, intervals, or other admissible braid features, but topology alone does not supply those components. Conversely, some annotations may concern temporally extended or multilateral structures that cannot be localized naturally to a single crossing. Section 6 will therefore treat decoration as a conditional representation of independently typed interaction information rather than as a source of normative semantics.

Section 5 now compares the equivalence relations induced by these representational layers and uses them to state descriptor sufficiency, underspecification, and surplus resolution precisely.

5 Normative Equivalence on Interaction Histories

This section specifies the equivalence architecture used to evaluate interaction-history representations. Its role is to keep five forms of sameness analytically distinct: identity of the admitted history object, equality of a declared endpoint description, equality of a topological description, equality of an independently constructed interaction annotation, and equality under the substantive declared normative evaluation. The objective is to determine when a representation preserves every distinction required by the

declared normative inquiry and when it either collapses a normatively relevant difference or retains distinctions that the declared evaluation permits an analyst to forget. The argument is formal. It specializes the descriptor-sufficiency architecture developed for processual justice [11] to the context- and resolution-indexed interaction domains introduced in Section 3. The section first defines the relevant equivalence relations on common domains, then constructs the normative quotient, identifies topological underspecification and surplus resolution, and concludes with descriptor-level and annotation-component notions of sufficiency and minimality.

5.1 Historical, Endpoint, Topological, Decorated, and Normative Equivalence

Fix a context-resolution pair (c, ρ) and write $X := \Gamma_{\text{IH}}^{c, \rho}$ for the corresponding nonempty interaction-history domain. Whenever a descriptor $D : X_D \rightarrow \Omega_D$ is defined on a nonempty subset $X_D \subseteq X$, its equality-induced equivalence relation is defined by Equation 42.

$$h \sim_D g \iff D(h) = D(g), \quad h, g \in X_D. \quad (42)$$

For equivalence relations $\approx_1$ and $\approx_2$ on a common domain, the refinement convention in Equation 43 will be used throughout the section.

$$\approx_1 \preceq \approx_2 \iff \approx_1 \subseteq \approx_2. \quad (43)$$

Equation 43 means that $\approx_1$ is at least as fine as $\approx_2$: every pair identified by the finer relation is also identified by the coarser relation. The notation orders relations by retained distinctions and does not assign normative superiority to finer resolution.

Complete historical identity is the finest equivalence available on the declared formal history domain. It is defined in Equation 44.

$$h \sim_H^{c, \rho} g \iff h = g, \quad h, g \in X. \quad (44)$$

The equality in Equation 44 concerns the formal objects admitted to X . It does not assert that empirical inquiry can uniquely identify those objects from evidence.

An endpoint description requires an explicit endpoint convention. Let $\mathcal{E}_{c, \rho}$ denote the codomain of the terminal interaction information retained by the inquiry, and define the endpoint descriptor in Equation 45.

$$E_{c, \rho} : X \longrightarrow \mathcal{E}_{c, \rho}. \quad (45)$$

The value $E_{c, \rho}(h)$ may record, for example, a terminal decision, the final configuration of declared roles, the terminal state of disagreement, or another explicitly selected endpoint profile. Equation 45 does not presuppose that the endpoint profile is a complete physical or social state. Endpoint equivalence is induced by Equation 46.

$$h \sim_E^{c, \rho} g \iff E_{c, \rho}(h) = E_{c, \rho}(g). \quad (46)$$

Because $E_{c, \rho}$ is a function on X , historical identity refines endpoint equivalence.

The annotation map $A_{c, \rho}$ was defined in Equation 39. Its equality-induced relation is stated in Equation 47.

$$h \sim_A^{c, \rho} g \iff A_{c, \rho}(h) = A_{c, \rho}(g). \quad (47)$$

Equation 47 identifies histories whose selected event, participant-event, directed-relation, and contextual annotations agree. It does not imply equality of the underlying history or equality of declared normative evaluation.

Topology requires a more restricted domain. Let $X_{\text{top}}^{c, \rho} \subseteq X$ be the histories for which a declared trajectory-topology descriptor is defined, and let Equation 48 denote that descriptor.

$$\Theta_{c, \rho} : X_{\text{top}}^{c, \rho} \longrightarrow \Omega_{\text{top}}^{c, \rho}. \quad (48)$$

The symbol $\Theta_{c,\rho}$ is intentionally generic. Section 6 will instantiate it with braid-theoretic descriptions on braid-admissible sectors, while other interaction domains may require a groupoid, a typed process structure, or a non-braid descriptor. Topological equivalence is defined in Equation 49.

$$h \sim_{\text{top}}^{c,\rho} g \iff \Theta_{c,\rho}(h) = \Theta_{c,\rho}(g), \quad h, g \in X_{\text{top}}^{c,\rho}. \quad (49)$$

On the same topological domain, combine the topology with the restriction of the annotation map. The decorated descriptor is defined in Equation 50.

$$D_{\text{dec}}^{c,\rho}(h) := (\Theta_{c,\rho}(h), A_{c,\rho}(h)). \quad (50)$$

Its equality-induced equivalence satisfies Equation 51.

$$\sim_{\text{dec}}^{c,\rho} = \sim_{\text{top}}^{c,\rho} \cap \sim_A^{c,\rho} \big|_{X_{\text{top}}^{c,\rho}}. \quad (51)$$

Equation 51 gives both refinement relations in Equation 52.

$$\sim_{\text{dec}}^{c,\rho} \preceq \sim_{\text{top}}^{c,\rho}, \quad \sim_{\text{dec}}^{c,\rho} \preceq \sim_A^{c,\rho} \big|_{X_{\text{top}}^{c,\rho}}. \quad (52)$$

Independent annotation can preserve topological distinctions while separating histories that topology alone identifies. Equation 52 makes no claim that every added distinction is normatively required.

Finally, declare the substantive evaluation on the fixed context-resolution domain as Equation 53.

$$\mathcal{J}_{c,\rho} : X \longrightarrow \Omega_{\text{IH}}^{c,\rho}. \quad (53)$$

The codomain in Equation 53 may be categorical, partially ordered, structured, or otherwise non-cardinal. Normative equivalence on interaction histories is defined by Equation 54.

$$h \sim_{\text{IH}}^{c,\rho} g \iff \mathcal{J}_{c,\rho}(h) = \mathcal{J}_{c,\rho}(g). \quad (54)$$

Equation 54 records the distinctions preserved by the declared substantive specification of the declared normative inquiry. A different admissible specification can induce a different equivalence relation even on the same history domain.

The relations defined above should not be arranged into a universal chain. Historical identity refines every equality-induced descriptor relation on the relevant common domain, and Equation 52 follows from construction. Further inclusions require additional factorization or substantive assumptions. End-point equality can forget a normatively relevant exclusion; annotation equality can omit an unrecorded temporal ordering; topological equality can ignore a role or authorization difference; and normative equivalence can intentionally ignore a topological variation. The task of the remaining subsections is to state these possibilities as conditions rather than intuitions.

5.2 Normative Quotients of Interaction Histories

Normative equivalence on interaction histories determines an abstract compression of the history domain that retains exactly the distinctions represented by the declared evaluation. Define the quotient map by Equation 55.

$$q_{\text{IH}}^{c,\rho} : X \longrightarrow X / \sim_{\text{IH}}^{c,\rho}, \quad q_{\text{IH}}^{c,\rho}(h) := [h]_{\text{IH}}^{c,\rho}. \quad (55)$$

Histories in the same class under Equation 55 can differ historically, topologically, or in annotated features that the selected declared normative evaluation treats as irrelevant to its final classification. Conversely, histories in distinct quotient classes differ in at least one respect preserved by that evaluation.

The quotient in Equation 55 is an abstract normative partition. It is not itself an empirical measurement procedure or a claim that an available symbolic language realizes exactly that partition. Its formal role can be stated through the set of normative-preserving equivalence relations. Define that set in Equation 56.

$$\mathcal{E}_{\text{IH}}(X) := \{ \approx : \approx \text{ is an equivalence relation on } X, \quad \approx \subseteq \sim_{\text{IH}}^{c,\rho} \}. \quad (56)$$

Proposition 5.1 (Normative interaction-history quotient as the coarsest normative-preserving partition). *The relation $\sim_{\text{IH}}^{c,\rho}$ is the greatest element of $\mathcal{E}_{\text{IH}}(X)$ under set inclusion.*

Proof. The relation $\sim_{\text{IH}}^{c,\rho}$ belongs to $\mathcal{E}_{\text{IH}}(X)$ because it is an equivalence relation and is contained in itself. Every $\approx \in \mathcal{E}_{\text{IH}}(X)$ satisfies $\approx \subseteq \sim_{\text{IH}}^{c,\rho}$ by Equation 56. Hence no normative-preserving equivalence relation is strictly coarser than $\sim_{\text{IH}}^{c,\rho}$. $\square$

Proposition 5.1 identifies the formal boundary of permissible compression relative to the declared evaluation. A finer relation may preserve additional distinctions. Equality with $\sim_{\text{IH}}^{c,\rho}$ preserves exactly the declared normative partition. A coarser relation necessarily identifies at least one pair that the evaluation distinguishes.

The quotient also clarifies why the annotation layer should remain independently interpretable. If an annotation simply contained $q_{\text{IH}}^{c,\rho}(h)$ as one of its components, a decorated descriptor could attain exact normative resolution by recoding the target partition. Research Obligation 4.3 excludes that shortcut. The analytical problem is to determine whether independently warranted information about interaction suffices to recover the quotient class.

5.3 Topological Underspecification of Normative Difference

Topological underspecification occurs when a trajectory-topology descriptor identifies histories that the substantive declared normative evaluation distinguishes. The relevant defect set is defined on the common topological domain by Equation 57.

$$\Delta_{\text{IH}|\text{top}}^{c,\rho} := \left\{ (h, g) \in (X_{\text{top}}^{c,\rho})^2 : h \sim_{\text{top}}^{c,\rho} g, \quad h \not\sim_{\text{IH}}^{c,\rho} g \right\}. \quad (57)$$

A pair in Equation 57 is a topological alias with respect to the declared normative inquiry. The two histories have the same declared topology while differing under the declared justice evaluation.

Proposition 5.2 (Topological sufficiency criterion). *The topological descriptor $\Theta_{c,\rho}$ is normatively sufficient on $X_{\text{top}}^{c,\rho}$ if and only if Equation 58 holds.*

$$\Delta_{\text{IH}|\text{top}}^{c,\rho} = \emptyset. \quad (58)$$

Equivalently, sufficiency holds exactly when $\sim_{\text{top}}^{c,\rho} \subseteq \sim_{\text{IH}}^{c,\rho}|_{X_{\text{top}}^{c,\rho}}$.

Proof. Equation 57 is empty exactly when every pair identified by $\sim_{\text{top}}^{c,\rho}$ is also identified by the restricted normative equivalence. This is the stated inclusion. $\square$

Proposition 5.2 makes topological adequacy conditional rather than categorical. The paper does not claim that topology is always insufficient. Topological sufficiency must instead be established for a specified domain and substantive evaluation.

The annotation layer has a precise corrective role. Restrict $A_{c,\rho}$ to $X_{\text{top}}^{c,\rho}$ and define the disagreement relation in Equation 59.

$$h \not\sim_A^{c,\rho} g \iff A_{c,\rho}(h) \neq A_{c,\rho}(g). \quad (59)$$

Proposition 5.3 (Annotation separation of topological normative aliases). *The decorated descriptor $D_{\text{dec}}^{c,\rho}$ is normatively sufficient on $X_{\text{top}}^{c,\rho}$ if and only if Equation 60 holds.*

$$\Delta_{\text{IH}|\text{top}}^{c,\rho} \subseteq \left\{ (h, g) \in (X_{\text{top}}^{c,\rho})^2 : h \not\sim_A^{c,\rho} g \right\}. \quad (60)$$

Proof. Assume the decorated descriptor is sufficient. Let (h, g) belong to $\Delta_{\text{IH}|\text{top}}^{c,\rho}$. The histories are topologically equivalent and justice-distinct. If they were also annotation-equivalent, Equation 51 would make them decorated-equivalent, contradicting sufficiency. Hence the annotation separates the pair, which proves Equation 60.

Conversely, assume Equation 60. If two histories are decorated-equivalent, Equation 51 makes them both topologically equivalent and annotation-equivalent. If they were justice-distinct, they would belong to $\Delta_{\text{IH}|\text{top}}^{c,\rho}$ and therefore would have to be annotation-distinct by Equation 60, a contradiction. Thus decorated equivalence implies normative equivalence. $\square$

Proposition 5.3 gives a selective interpretation of normative decoration. The annotation need not reconstruct the complete history and need not make every topologically distinct history annotation-distinct. It must separate the topological collisions that cross an interaction-history normative boundary. An annotation may preserve additional distinctions for interpretive, evidential, or later analytical purposes, but those additions are not required by sufficiency alone.

The result also explains why one universal list of decorations is unlikely to be minimal across all domains. In one context, procedural standing may be the feature that separates two topologically identical histories; in another, office authorization, an accountability relation, or a responsibility-relevant basis may carry the distinction. The required annotation depends jointly on the topology, the history domain, and the substantive evaluation.

5.4 Surplus Topological Resolution

A descriptor can also be finer than the declared justice evaluation. Such surplus resolution is different from underspecification. Define the surplus-topology set by Equation 61.

$$\Sigma_{\text{top}|\text{IH}}^{c,\rho} := \left\{ (h, g) \in (X_{\text{top}}^{c,\rho})^2 : h \sim_{\text{IH}}^{c,\rho} g, \quad h \not\sim_{\text{top}}^{c,\rho} g \right\}. \quad (61)$$

A pair in Equation 61 is normatively equivalent under the declared evaluation while remaining topologically distinguishable. The topology therefore retains information that the justice classification does not require.

Surplus resolution is not, by itself, a defect. A topological descriptor may be useful for causal analysis, historical explanation, comparative research, or another inquiry even when some of its distinctions are irrelevant to the selected interaction-history normative classification. For the narrower representation question, surplus resolution means only that the descriptor is more discriminating than necessary.

The two defect sets organize the possible relation between topology and normative equivalence.

Proposition 5.4 (Four relation regimes for topology and the declared normative inquiry). *On $X_{\text{top}}^{c,\rho}$, the following cases hold.*

1. *If both $\Delta_{\text{IH}|\text{top}}^{c,\rho}$ and $\Sigma_{\text{top}|\text{IH}}^{c,\rho}$ are empty, then $\sim_{\text{top}}^{c,\rho} = \sim_{\text{IH}}^{c,\rho}|_{X_{\text{top}}^{c,\rho}}$.*
2. *If $\Delta_{\text{IH}|\text{top}}^{c,\rho}$ is empty and $\Sigma_{\text{top}|\text{IH}}^{c,\rho}$ is nonempty, then topology is strictly finer than normative equivalence and is sufficient with surplus resolution.*
3. *If $\Delta_{\text{IH}|\text{top}}^{c,\rho}$ is nonempty and $\Sigma_{\text{top}|\text{IH}}^{c,\rho}$ is empty, then normative equivalence is strictly finer than topology and topology is insufficient.*
4. *If both sets are nonempty, then the two equivalence relations are incomparable under inclusion.*

Proof. The set $\Delta_{\text{IH}|\text{top}}^{c,\rho}$ is empty exactly when $\sim_{\text{top}}^{c,\rho} \subseteq \sim_{\text{IH}}^{c,\rho}$ on the common domain. The set $\Sigma_{\text{top}|\text{IH}}^{c,\rho}$ is empty exactly when the reverse inclusion holds. The four cases follow from whether each inclusion holds strictly, as equality, or fails. $\square$

Table 4 summarizes Proposition 5.4. The table is representational rather than moral: it classifies the relation between two equivalence structures for a fixed context, resolution, and substantive evaluation.

The possibility of incomparability is particularly important for decorated braid analysis. Adding more topological detail does not necessarily repair a missing normative distinction if the omitted information concerns standing, authorization, responsibility, or another non-topological feature. Conversely,

Table 4: Equivalence regimes for topology and declared normative evaluation.

Relation regime	Equivalence relation	Representational status	Interpretation
Exact resolution	$\sim_{\text{top}} = \sim_{\text{IH}}$	Sufficient, no surplus relative to the declared evaluation	Topology preserves exactly the declared justice partition
Surplus resolution	$\sim_{\text{top}} \subsetneq \sim_{\text{IH}}$	Sufficient, finer than necessary	Some topological distinctions are ignored by the justice evaluation
Underspecification	$\sim_{\text{IH}} \subsetneq \sim_{\text{top}}$	Insufficient	Some justice-distinct histories are topologically aliased
Incomparability	Neither inclusion holds	Insufficient topology with additional surplus distinctions	Topology both collapses some justice distinctions and preserves other differences the justice evaluation ignores

adding normative annotation does not make every topological distinction normatively important. The representation should therefore be evaluated by the induced equivalence relation rather than by an informal notion of mathematical richness.

5.5 Descriptor Sufficiency and Minimality

The preceding analysis can be generalized beyond topology. Let $Y \subseteq X$ be a nonempty common domain and let $D : Y \rightarrow \Omega_D$ be any interaction-history descriptor. Its role may be endpoint-based, topological, annotation-based, decorated, or otherwise domain specific.

Definition 5.5 (Normatively sufficient descriptor). The descriptor D is *normatively sufficient* on Y when Equation 62 holds.

$$\sim_D \subseteq \sim_{\text{IH}}^{c,p}|_Y. \quad (62)$$

It has *exact interaction-history normative resolution* when equality holds in Equation 62.

Definition 5.5 treats sufficiency as preservation of the normative partition. A sufficient descriptor may retain surplus history. Exact resolution is therefore stronger than sufficiency and should not be imposed when the available representation legitimately preserves additional distinctions.

Sufficiency can be expressed as recoverability of the justice quotient. Let $q_{\text{IH}}^Y : Y \rightarrow Y / (\sim_{\text{IH}}^{c,p}|_Y)$ be the quotient map for the restricted equivalence relation, defined by $q_{\text{IH}}^Y(h) := [h]_{\sim_{\text{IH}}|_Y}$. This is the quotient associated with the restricted relation on Y ; it should not be confused with a literal codomain-preserving restriction of Equation 55.

Proposition 5.6 (Normative recovery criterion). A descriptor $D : Y \rightarrow \Omega_D$ is *normatively sufficient* if and only if there exists a unique map r_D satisfying Equation 63.

$$r_D : \text{im}(D) \longrightarrow Y / (\sim_{\text{IH}}^{c,p}|_Y), \quad q_{\text{IH}}^Y = r_D \circ D. \quad (63)$$

Proof. Assume D is sufficient. For $d \in \text{im}(D)$, choose any $h \in Y$ with $D(h) = d$ and define $r_D(d) := [h]_{\text{IH}}$. If another history g has $D(g) = d$, then $h \sim_D g$. Sufficiency gives $h \sim_{\text{IH}}^{c,p} g$, so the quotient class is independent of the chosen representative. Hence r_D is well defined and Equation 63 holds. Its value on each d is forced by any preimage of d , so the map is unique.

Conversely, suppose Equation 63 holds. If $D(h) = D(g)$, applying r_D gives $q_{\text{IH}}^Y(h) = q_{\text{IH}}^Y(g)$. Thus $h \sim_{\text{IH}}^{c,p} g$, proving Equation 62. $\square$

Proposition 5.6 states the representational objective without demanding reconstruction of the complete history. A sufficient descriptor need only determine the interaction-history normative class represented by the declared evaluation.

Minimality requires a declared comparison family. Let $\mathcal{D}_Y$ be a family of descriptors on the common domain Y , and define its sufficient subfamily by Equation 64.

$$\mathcal{D}_Y^{\text{IH}} := \{D \in \mathcal{D}_Y : \sim_D \subseteq \sim_{\text{IH}}^{c,\rho}|_Y\}. \quad (64)$$

Definition 5.7 (Family-minimal sufficient interaction descriptor). A descriptor $D_\star \in \mathcal{D}_Y^{\text{IH}}$ is *family-minimal sufficient* when Equation 65 holds.

$$\forall D \in \mathcal{D}_Y^{\text{IH}}, \quad \sim_D \subseteq \sim_{D_\star}. \quad (65)$$

Definition 5.7 selects a coarsest sufficient descriptor within the declared family, and therefore one that retains no information that can be discarded by moving to another sufficient member of that family. Such a descriptor need not exist. Several sufficient descriptors can be incomparable, and an available descriptor family need not contain a representation with exact normative resolution.

The abstract justice quotient remains the lower-information boundary. If $\mathcal{D}_Y^{\text{IH}}$ contains an exact descriptor and a family-minimal sufficient descriptor exists, that family-minimal descriptor must also have exact interaction-history normative resolution. The argument follows immediately from Equation 65: the exact descriptor's equivalence is the restricted justice equivalence, while sufficiency prevents $D_\star$ from being coarser than that relation.

A second notion of minimality is useful for normative decoration. Let K be the declared index set of independently interpretable annotation components available for a fixed topological descriptor. For each $S \subseteq K$, let A_S denote the projection of the full annotation to the components indexed by S , and define the partially decorated descriptor in Equation 66.

$$D_S(h) := (\Theta_{c,\rho}(h), A_S(h)), \quad h \in X_{\text{top}}^{c,\rho}. \quad (66)$$

If $S \subseteq S'$, then $D_{S'}$ retains every component retained by D_S and possibly more. Their induced equivalences therefore satisfy Equation 67.

$$S \subseteq S' \implies \sim_{D_{S'}} \subseteq \sim_{D_S}. \quad (67)$$

Definition 5.8 (Inclusion-minimal sufficient annotation set). A component set $S_\star \subseteq K$ is *inclusion-minimal sufficient relative to $\Theta_{c,\rho}$* when $D_{S_\star}$ is normatively sufficient and Equation 68 holds.

$$\forall S \subsetneq S_\star, \quad \sim_{D_S} \not\subseteq \sim_{\text{IH}}^{c,\rho}|_{X_{\text{top}}^{c,\rho}}. \quad (68)$$

Definition 5.8 addresses component economy rather than the abstract information ordering in Definition 5.7. Two different inclusion-minimal annotation sets can exist, and they can encode different kinds of information while inducing equally sufficient or incomparable decorated relations. The formalism therefore does not identify one universal list of normatively necessary labels. It asks which independently justified features are necessary relative to a declared topology, domain, resolution, and substantive declared normative evaluation.

Section 6 can now introduce braid-specific representations without assigning normative content to braid generators by fiat. The task there is conditional: identify braid-admissible interaction domains, specify how selected components of $A_{c,\rho}$ can be attached to braid-level structures, and determine which braid relations remain valid under the chosen typing. The sufficiency and minimality criteria developed here then provide the standard by which those decorated representations can be evaluated.

6 Decorated Braid Representation of Interaction Histories

Sections 3–5 established the historical domain, the independently grounded annotation layer, and the equivalence criterion by which a descriptor should be judged. The role of the present section is to specialize the generic topological descriptor of Section 5 to braid-admissible interaction histories. The specialization proceeds conditionally. It first declares which persistent components are represented as strands and how their trajectories are extracted. It then distinguishes an ordinary braid descriptor from an internally decorated braid representation. The final two subsections examine which standard braid relations remain admissible after normative typing and identify the circumstances in which a groupoid or a more general typed process structure is preferable to one ordinary braid group.

The construction follows the trajectory-descriptive discipline developed in Paper I [10]. A braid is used here as a descriptor of selected realized interaction trajectories. It is not a generative mechanism, a complete social ontology, or a source of normative meaning. In particular, the crossing generators introduced below inherit no moral interpretation from braid theory itself. Any hearing, authorization, accountability, burden, benefit, power, or responsibility information attached to a braid must originate in the independently specified records of Section 4.

6.1 Braid-Admissible Interaction Domains

An interaction history does not become braid-admissible merely because it contains several participants. Ordinary braid description requires a fixed number of persistent tracked components, a collision-free relational chart, and a declared endpoint convention. Interaction-history analysis adds a further requirement: the semantics of each tracked component must be stated explicitly. A strand that follows a person through role changes represents a different object from a strand that follows an office through occupant changes, even when the two traces happen to coincide over part of a history.

For a fixed context-resolution pair (c, ρ) , let τ denote a *strand-tracking convention*. The convention declares an integer $n_\tau \geq 2$, a strand registry

$$S_\tau := \{s_1, \dots, s_{n_\tau}\}, \quad (69)$$

and a semantic typing rule for the registry. The typing rule states what each s_i tracks in the declared inquiry: for example, a participant bearer from A_h , a collective participant, an office in $\mathcal{O}_c$, a role-bearing track derived from Λ_h , or another explicitly constructed persistent relational object. The list is illustrative rather than exhaustive. The admissibility of a tracking target depends on whether its persistence criterion can be specified from the history primitives already declared.

The strand registry in Equation (69) is an analytical registry rather than a new social ontology. Two histories may use the same abstract registry while realizing its strands through different bearers or role occupants if the tracking convention is office-based; conversely, a bearer-based convention may follow the same persons while their role annotations change. The choice must therefore be held fixed when braid descriptors are compared.

Let $\mathcal{U}_{n_\tau} := \text{UConf}_{n_\tau}(\mathbb{R}^2)$ be the unordered configuration space of n_τ pairwise distinct points in the declared relational chart, and choose a base configuration $b_\tau \in \mathcal{U}_{n_\tau}$. Define the associated based-loop space by Equation (70).

$$\Omega(\mathcal{U}_{n_\tau}, b_\tau) := \{q : [0, 1] \rightarrow \mathcal{U}_{n_\tau} : q \text{ is continuous and } q(0) = q(1) = b_\tau\}. \quad (70)$$

The use of a based-loop codomain in Equation (70) is the baseline fixed-base case at the level of the *unordered* configuration. It does not by itself require each semantically tracked strand to return to its own initial chart position. If the tracking convention requires labelwise endpoint return, the extraction must lie in the corresponding pure-braid sector, equivalently in a loop of the ordered configuration space, and the resulting braid lies in the pure-braid subgroup $P_{n_\tau} := \ker(B_{n_\tau} \rightarrow \mathfrak{S}_{n_\tau}) \subseteq B_{n_\tau}$. If endpoint permutations of the tracked components are admissible, the full braid group remains appropriate. The endpoint semantics must therefore be declared together with τ . More general endpoint variation is deferred to Subsection 6.5.

Definition 6.1 (Braid-admissible interaction sector). For (c, ρ, τ) , a *braid-admissible interaction sector* is a subset

$$X_B^{c,\rho,\tau} \subseteq \Gamma_{\text{IH}}^{c,\rho} \quad (71)$$

for which there exists an extraction map

$$\mathcal{T}_B^{c,\rho,\tau} : X_B^{c,\rho,\tau} \longrightarrow \Omega(\mathcal{U}_{n_\tau}, b_\tau). \quad (72)$$

Equation (71) identifies the history domain on which the chosen braid construction is claimed to apply, while Equation (72) supplies the selected trajectory representation. Membership in $X_B^{c,\rho,\tau}$ requires the declared strand registry to be trackable throughout the represented interval, the extracted components to remain pairwise distinct in the selected chart, and the endpoint convention required by Equation (70) to be satisfied.

Definition 6.1 makes braid admissibility relative to the tracking convention. A history with one participant entering midway may fail a bearer-based fixed-strand convention while remaining describable under another representation that tracks persistent offices. The reverse can also occur: office creation or dissolution may violate an office-based fixed-strand convention even when the relevant persons persist. Failure of one braid sector therefore establishes only a domain mismatch for that extraction rule.

Let Q_B be the based-homotopy quotient on Equation (70). As in Paper I [10], fix the isomorphism in Equation (73).

$$\kappa_{n_\tau} : \pi_1(\mathcal{U}_{n_\tau}, b_\tau) \longrightarrow B_{n_\tau}. \quad (73)$$

The ordinary braid descriptor for the sector is defined by Equation (74).

$$D_B^{c,\rho,\tau} := \kappa_{n_\tau} \circ Q_B \circ \mathcal{T}_B^{c,\rho,\tau} : X_B^{c,\rho,\tau} \longrightarrow B_{n_\tau}. \quad (74)$$

For the braid-admissible sector, the generic topological descriptor of Section 5 may therefore be instantiated by Equation (75).

$$\Theta_{c,\rho}|_{X_B^{c,\rho,\tau}} := D_B^{c,\rho,\tau}. \quad (75)$$

Equation (75) is an instantiation on a declared sector, not an identification of every topological descriptor with a braid descriptor.

The induced braid equivalence is given by Equation (76).

$$h \sim_B^{c,\rho,\tau} g \iff D_B^{c,\rho,\tau}(h) = D_B^{c,\rho,\tau}(g), \quad h, g \in X_B^{c,\rho,\tau}. \quad (76)$$

By the construction inherited from Paper I, Equation (76) identifies histories whose extracted trajectories are based-homotopic under the selected tracking convention. It does not identify histories merely because the same persons, offices, or roles appear in both.

Remark 6.2 (Tracking dependence). The superscript τ in Equations (74) and (76) is substantive. Changing from bearer-tracking to office-tracking can change the braid domain, the extracted trajectory, the endpoint convention, and the induced equivalence relation. Statements about “the braid of an interaction” are therefore incomplete unless the tracked components are declared.

6.2 Typed and Decorated Braid Descriptors

The braid descriptor in Equation (74) records selected trajectory topology but does not yet retain the independently typed information constructed in Section 4. There are two distinct ways to combine the layers. The first keeps topology and annotation as external coordinates. The second localizes selected annotations on strand segments, crossing occurrences, or other braid-level features. The external construction is formally simpler and should remain available even when localization is impossible.

Let $S \subseteq K$ be a declared set of annotation components in the sense of Section 5.5, and let A_S denote their projection. The externally decorated braid descriptor is defined by Equation (77).

$$D_{B,S}^{\text{ext}}(h) := (D_B^{c,\rho,\tau}(h), A_S(h)), \quad h \in X_B^{c,\rho,\tau}. \quad (77)$$

Because both coordinates in Equation (77) are defined at the history level, the construction does not require every annotation to be localized to a single crossing. A temporally extended hearing condition, a multilateral accountability relation, or a context-wide procedural constraint may remain in $A_S(h)$ as a nonlocal record.

The second construction requires additional localization data. For internal event anchoring, choose a word representative produced by the declared diagram, projection, and parameterization of the extracted trajectory; an arbitrary algebraically equivalent word is not automatically eligible for historical anchoring. Let $w = x_1 \cdots x_m$ be such an admissible word representative of the braid $D_B^{c,\rho,\tau}(h)$, with each letter $x_k = \sigma_{i_k}^{\varepsilon_k}$, $\varepsilon_k \in \{+1, -1\}$. The set of word occurrences is

$$\text{Occ}(w) := \{1, \dots, m\}. \quad (78)$$

Equation (78) indexes occurrences rather than generator types, so repeated appearances of the same Artin generator remain distinguishable at the word level. A word occurrence is not yet an interaction event. To relate the word to the event structure of Section 3, introduce a partial anchoring map

$$\eta_{h,w} : \text{Occ}(w) \rightarrow E_h. \quad (79)$$

When $\eta_{h,w}(k) = e$, the k th crossing occurrence is interpreted as geometrically associated, under the declared extraction and interpolation rule, with the event carrier e . Undefined values in Equation (79) are permitted because a geometric crossing can be an artefact of the chosen relational chart or interpolation without constituting a substantively identified interaction event.

Conversely, Equation (79) need not be surjective. Some interaction events can matter normatively without generating a crossing in the chosen braid chart. A denied speaking opportunity, an institutional refusal, or a background authorization relation can be historically and normatively relevant while leaving the represented strand order unchanged.

Let $\mathcal{L}_S$ denote the codomain of the selected localizable annotation records. A localization rule

$$\lambda_{h,w} : \text{dom}(\eta_{h,w}) \rightarrow \mathcal{L}_S \quad (80)$$

is required to be derived from $A_S(h)$ and the anchored event, rather than from the final interaction-history normative value. Equation (80) therefore localizes only those selected records for which an anchoring rule has been declared. Any selected annotation that is not naturally localizable is retained in a residual record

$$R_S(h) \in \mathcal{R}_S^{\text{nonloc}}. \quad (81)$$

Equation (81) prevents the localization step from discarding selected information merely because it lacks a natural crossing carrier. The symbol $\mathcal{R}_S^{\text{nonloc}}$ denotes only the declared codomain of residual nonlocal annotation; it does not reuse the resolution notation $\mathcal{R}_c$ of Section 3.

Definition 6.3 (Anchored decorated braid word). For a history $h \in X_B^{c,\rho,\tau}$ and a chosen braid-word representative w of $D_B^{c,\rho,\tau}(h)$, an *anchored decorated braid word* is the tuple

$$\tilde{w}_h := (w, \eta_{h,w}, \lambda_{h,w}, R_S(h)). \quad (82)$$

Definition 6.3 does not yet define a decorated braid *element*. A single braid element has many word representatives. An internal decoration is therefore well defined only after specifying how decorated words transform under the relations that identify those representatives. Subsection 6.4 addresses that quotient problem.

The externally decorated descriptor in Equation (77) is consequently more general than a crossing-local decoration. The internal construction is useful when interaction information can be anchored to braid-level features in a stable manner; the external construction remains preferable when the normatively relevant object is distributed across intervals, multiple participants, or the history as a whole.

6.3 Interpretation of Crossing Generators

The ordinary Artin generator σ_i and its inverse σ_i^{-1} encode oriented adjacent-strand exchanges in a chosen braid representation. Their mathematical meaning is topological. Normative meaning enters only through the tracking convention, the event anchoring in Equation (79), and the independently grounded annotation records.

This separation matters because three distinct statements can otherwise be conflated. First, two tracked components exchange relative position in the selected chart. Second, two participants interact in a substantively recognized event. Third, that interaction satisfies or violates a condition relevant to interaction-history normativity. The first statement can be supplied by the braid descriptor. The second requires an anchoring rule. The third requires substantive evaluation using the typed records developed in Section 4.

Research Obligation 6.4 (Crossing-interpretation obligation). Before a braid crossing is used as a carrier of interaction meaning, the representation should declare: (i) what the two strands track; (ii) why the crossing occurrence is associated with the identified event carrier or interaction interval; (iii) which annotation components are transported to the crossing; and (iv) which parts of the interaction remain outside the crossing-local representation.

Research Obligation 6.4 blocks direct readings such as “positive crossing means cooperative interaction” or “negative crossing means domination” unless those meanings are independently defined and empirically or interpretively justified in the relevant domain. The sign of an Artin generator records orientation in the braid chart; it is not a valence scale for justice.

A second restraint concerns multiplicity. One interaction event can induce several geometric crossings under a fine interpolation, and several historically distinct events can be compressed into one crossing under a coarser representation. Accordingly, neither the number of braid letters nor the algebraic exponent sum provides, by itself, a count of morally significant interactions. Such quantities may become meaningful only after a domain-specific interpretation establishes the required correspondence.

A third restraint concerns strand-local information. Suppose a braid word contains a crossing between strands s_i and s_{i+1} . If the tracking convention is bearer-based, role and office information must be read from the relevant time-indexed records, because the strand identity alone does not determine the role under which the bearer acted. If the tracking convention is office-based, occupant identity may need to be carried as annotation because office persistence does not imply bearer persistence. The participant-typing distinctions of Section 3.2 therefore remain active inside the braid representation rather than disappearing once strands have been chosen.

The resulting interpretation is deliberately asymmetric between topology and semantics. Topology supplies a stable descriptive skeleton when its admissibility assumptions hold. Semantic and normative typing can refine that skeleton, but they cannot be inferred from it without an additional mapping. This asymmetric dependence is what makes the decorated-braid construction compatible with the non-circularity obligation of Section 4.5.

6.4 Braid Relations under Normative Typing

An ordinary braid element is independent of the chosen word because the Artin relations identify different word representatives. Internal decoration creates a new question: should every ordinary braid relation continue to identify the decorated processes? The answer depends on what the decoration preserves. If a relation changes the order, provenance, or semantic status of anchored events, quotienting by that relation can erase distinctions that the decoration was introduced to retain.

Let $\widetilde{\mathcal{W}}_{n_\tau, S}$ denote the set of anchored decorated words of the form in Equation (82) that satisfy the declared typing rules. Let

$$U : \widetilde{\mathcal{W}}_{n_\tau, S} \longrightarrow \mathcal{W}_{n_\tau} \quad (83)$$

Table 5: Ordinary braid relations and additional conditions for decorated interaction representation.

Ordinary relation	Topological role	Additional decorated-process question
Distant commutation $\sigma_i \sigma_j = \sigma_j \sigma_i,$ $ i - j \geq 2$	Reorders crossings involving disjoint adjacent strand pairs.	Are the anchored events semantically independent so that their order can be exchanged without changing retained context, role state, obligation, or other selected annotation?
Braid relation $\sigma_i \sigma_{i+1} \sigma_i =$ $\sigma_{i+1} \sigma_i \sigma_{i+1}$	Identifies the standard three-strand local deformation.	Does the local correspondence preserve participant typing, event anchoring, and every selected annotation through the three-event rewrite?
Cancellation $\sigma_i \sigma_i^{-1} = e$ and $\sigma_i^{-1} \sigma_i = e$	Removes a topologically inverse crossing pair.	Is the paired interaction record explicitly neutral or reversible at the retained semantic level, rather than merely topologically invertible?

forget the anchoring and annotation and retain only the underlying Artin word. Equation (83) is a word-level forgetful map; whether it descends to quotient classes depends on the soundness condition introduced below. Here $\mathcal{W}_{n_\tau}$ denotes the ordinary word set on the generators $\sigma_1^{\pm 1}, \dots, \sigma_{n_\tau-1}^{\pm 1}$.

The decorated representation needs its own equivalence relation. Rather than importing the full Artin relation set automatically, let $\approx_{B,S}^{c,\rho,\tau}$ be the equivalence relation generated by those lifted word moves that the declared typing and anchoring rules treat as semantics-preserving. The minimum soundness condition is stated in Equation (84).

$$\tilde{w} \approx_{B,S}^{c,\rho,\tau} \tilde{v} \implies \text{ev}_B(U\tilde{w}) = \text{ev}_B(U\tilde{v}), \quad (84)$$

where ev_B is the ordinary braid-word evaluation map. Equation (84) requires every decorated identification to be topologically sound. It does not require every topologically sound Artin move to remain admissible after decoration.

Definition 6.5 (Decorated braid-process quotient). The *decorated braid-process quotient* for (c, ρ, τ, S) is

$$\tilde{\mathcal{B}}_{c,\rho,\tau,S} := \tilde{\mathcal{W}}_{n_\tau,S} / \approx_{B,S}^{c,\rho,\tau}. \quad (85)$$

Equation (84) ensures that forgetting the decoration descends to the quotient.

Proposition 6.6 (Well-defined topological forgetful map). *If Equation (84) holds, there exists a unique map*

$$F_{B,S} : \tilde{\mathcal{B}}_{c,\rho,\tau,S} \longrightarrow B_{n_\tau} \quad (86)$$

that sends each decorated equivalence class to the ordinary braid represented by its underlying word.

Proof. For a decorated class $[\tilde{w}]$, define $F_{B,S}([\tilde{w}]) := \text{ev}_B(U\tilde{w})$. If $\tilde{w} \approx_{B,S}^{c,\rho,\tau} \tilde{v}$, Equation (84) gives equal ordinary braid values, so the definition is independent of the representative. Uniqueness follows because the value is fixed on every quotient class by its representatives. $\square$

Proposition 6.6 formalizes the sense in which the decorated structure refines rather than replaces the underlying braid description. It also makes the relation to Section 5 explicit: the decorated quotient can preserve distinctions that B_{n_τ} erases, while the ordinary braid remains recoverable through Equation (86).

The effect of normative typing is easiest to see by examining the three standard forms of Artin word relation. Table 5 records the relevant issue for each relation. The table states conditions for retaining an ordinary relation in the decorated quotient; it does not assert that those conditions hold in every domain.

Distant commutation illustrates why geometry and historical meaning can diverge. Two crossings on disjoint strands commute in B_n , but their associated events need not be independent if, for example,

the first event changes authorization, standing, available evidence, or institutional context for the second. In such a case, retaining ordinary commutation inside the decorated quotient would erase historical dependence that the selected annotations were intended to preserve.

The three-strand braid relation raises a related issue. A topological deformation can preserve the ordinary braid while changing which anchored occurrence is associated with which event carrier unless a transport rule explicitly preserves the local typing. A decorated version of the braid relation should therefore specify a correspondence among the involved strand segments and anchored records. Equality of the undecorated braid words alone is insufficient.

Cancellation is the sharpest case. The equality $\sigma_i \sigma_i^{-1} = e$ holds in the ordinary braid group, but an interaction followed by a geometrically inverse interaction need not erase the first interaction's burden, exclusion, commitment, responsibility basis, or institutional consequence. The decorated quotient should retain cancellation only when the declared semantic composition rule treats the paired records as cancellable. This condition is stronger than topological invertibility.

Proposition 6.7 (Refinement by relation restriction). *Assume the decorated equivalence satisfies Equation (84). Let $D_{B,S}^{\text{int}}$ be a history-level descriptor into the quotient in Equation (85), constructed from anchored decorated words and independent of the chosen admissible representative. If*

$$D_{B,S}^{\text{int}}(h) = D_{B,S}^{\text{int}}(g), \quad (87)$$

then

$$h \sim_B^{c,\rho,\tau} g. \quad (88)$$

Proof. Apply the forgetful map in Proposition 6.6 to both sides of Equation (87). By construction of the history-level descriptor, the resulting values are $D_B^{c,\rho,\tau}(h)$ and $D_B^{c,\rho,\tau}(g)$. Their equality is exactly Equation (88) by Equation (76). $\square$

Proposition 6.7 gives the information-order relation expected of a sound decoration: decorated equality may be finer than ordinary braid equality, but it cannot identify histories whose underlying braid descriptors differ. Normative sufficiency still requires the independent criterion of Section 5.5. A highly refined decorated quotient can remain normatively insufficient if it omits a justice-relevant distinction, and it can retain surplus information even when it is sufficient.

6.5 Groupoid and Typed-Process Generalizations

The fixed-base braid group is appropriate only when the represented histories share one compatible endpoint convention and the relevant process identifications remain invertible. Several interaction domains considered in this paper violate one or both conditions. Role states can change between the beginning and end of an interaction, different officeholders can occupy the same persistent office, participant sets can change, and a semantically retained event may be non-cancellable even when its underlying crossing has a topological inverse. These cases motivate a more general process architecture.

Let $\text{Bnd}_{c,\rho,\tau,S}$ denote a declared set of typed boundary states. A boundary state records the information required to determine whether two represented interaction segments can be composed. Depending on the domain, it may include the ordered or unordered strand registry, endpoint positions in the relational chart, relevant role or office typing, and any retained contextual state required by the decoration. No universal boundary inventory is imposed.

A decorated braid-like process from boundary state b_0 to boundary state b_1 is written

$$p : b_0 \longrightarrow b_1. \quad (89)$$

Equation (89) makes the source and target boundary states part of the process type rather than auxiliary metadata. Composition of $p : b_0 \rightarrow b_1$ and $q : b_1 \rightarrow b_2$ is permitted only when the shared boundary data agree and the underlying history concatenation is admissible in the sense already emphasized in

Section 3.5. When the declared quotient relations respect this composition and associativity holds at the quotient level, the represented processes form a category

$$\mathcal{P}_{c,\rho,\tau,S}^B \quad (90)$$

The superscript B in Equation (90) records that the processes retain a braid-like trajectory layer; it does not assert that every morphism is an element of one braid group.

The ordinary braid group appears as a special case. If the boundary set has one relevant base object, all admissible morphisms are invertible, the full ordinary Artin relation set is retained, and the decoration is forgotten, then the endomorphisms recover the corresponding fixed-base braid group. If multiple typed boundary objects are permitted but every morphism remains invertible, the resulting structure is groupoid-like. If some retained semantic processes are not invertible, a general category or another typed process structure is more appropriate than a groupoid.

This distinction is particularly important for interaction histories. Topological inversion reverses an extracted path class; it does not guarantee restoration of a previous normative or relational state. A denied hearing followed later by an opportunity to speak can produce an interaction path whose geometric component resembles a reversal without making the earlier exclusion historically absent. Likewise, a transfer of office followed by a later return of the original officeholder does not imply that the intervening institutional interactions are erased. The algebraic structure should therefore inherit invertibility only when the represented semantics warrant it.

Changing strand number requires an additional extension. If participant or office creation, exit, fusion, or division belongs to the justice-relevant history, the fixed- n_τ configuration space of Definition 6.1 is no longer adequate. The present paper does not select a unique replacement. A variable-boundary process category, a tangle-like construction, or another domain-specific trajectory language may be required. The methodological point is narrower: such histories should not be forced into a fixed-strand braid model by deleting the very entry, exit, or transformation that may matter to the justice inquiry.

The generalization in Equation (90) also prepares the compositional analysis of Section 7. There the relevant question is not merely whether trajectory pieces can be concatenated geometrically. It is whether the interaction-history normative evaluation behaves congruentially under admissible composition and, when it does not, which contextual or historical dependencies explain the failure. The braid, decorated-braid, groupoid, and typed-process representations developed here provide alternative descriptive carriers for that question; the choice among them remains subordinate to the substantive interaction domain and the equivalence structure of interaction-history normativity.

7 Context-Dependent Composition of Interaction Histories

This section examines whether interaction-history normative evaluation is stable under the composition of admissible interaction histories. Its role is to connect the equivalence architecture of Section 5 with the typed-process perspective of Section 6. The objective is narrower than constructing a full dynamics. The section first defines partial concatenation on the context- and resolution-indexed history domain, then separates evaluation of component histories from evaluation of their composition, states exact congruence conditions, identifies context-sensitive failures of substitution, and finally derives a continuation profile that records when current interaction-history normative equivalence is insufficient to determine the normative consequences of later interaction. The method is formal and theory-relative. No general compositionality of justice is assumed.

7.1 Concatenation of Interaction Histories

Fix a context-resolution pair (c, ρ) and write $X := \Gamma_{\text{IH}}^{c,\rho}$ as in Section 5.1. Equation (20) already records that X need not be closed under concatenation. Composition therefore requires both a boundary-matching rule and a separate admissibility test.

Let $\text{Bnd}_{c,\rho}$ be the declared space of boundary records relevant to composition at (c, ρ) . A boundary record can contain participant registries, role or office occupancy, unresolved procedural status, local contextual state, or other information needed to decide whether one represented segment may follow another. Define initial and terminal boundary maps by Equation (91).

$$\partial_{c,\rho}^-, \partial_{c,\rho}^+ : X \longrightarrow \text{Bnd}_{c,\rho}. \quad (91)$$

The inventory of $\text{Bnd}_{c,\rho}$ is domain specific. Equation (91) therefore does not require every history domain to preserve the same kinds of boundary information.

Boundary equality can be stronger than necessary for stitching. Let $\bowtie_{c,\rho}$ be a declared compatibility relation on $\text{Bnd}_{c,\rho}$. A pair (h, g) is structurally stitchable when Equation (92) holds.

$$\partial_{c,\rho}^+(h) \bowtie_{c,\rho} \partial_{c,\rho}^-(g). \quad (92)$$

Equation (92) is only a boundary condition. It does not imply that the combined history satisfies the structural, evidential, contextual, or resolution constraints of Definition 3.4.

Let $\text{Comp}_{c,\rho} \subseteq X \times X$ be the set of pairs for which the declared concatenation rule is defined and produces another member of X . The partial composition map is written in Equation (93).

$$\diamond_{c,\rho} : \text{Comp}_{c,\rho} \longrightarrow X, \quad (h, g) \longmapsto h \diamond g, \quad (93)$$

where $h \diamond g$ means that h occurs first and g follows it. Membership in $\text{Comp}_{c,\rho}$ entails Equation (92), but the converse need not hold.

The distinction between stitchability and admissible composition prevents a boundary match from silently creating facts that the reconstruction does not warrant. Two segments can have compatible end-point descriptions while their concatenation introduces an unresolved identity conflict, exceeds the declared resolution, violates an institutional typing condition, or otherwise falls outside X . For the same reason, Equation (93) is a partial operation rather than a binary operation on all of X .

Associativity also belongs to the declared composition system rather than following from notation. Let $\text{Comp}_{c,\rho}^{(3)}$ contain the triples (h, g, k) for which both bracketings are defined. The composition is associative on admissible triples when Equation (94) holds.

$$(h \diamond g) \diamond k = h \diamond (g \diamond k) \quad \text{for all } (h, g, k) \in \text{Comp}_{c,\rho}^{(3)}. \quad (94)$$

Equation (94) is required if the history representation is later treated categorically. The present section does not require it for every proposition below, because the central issue is whether justice equivalence survives admissible substitution.

7.2 Compositional Sufficiency of Local Normative Values

A composition can be evaluated at two distinct levels. The component histories h and g each receive their own values under $\mathcal{J}_{c,\rho}$, while the admissible composition $h \diamond g$ receives a whole-history value. Equality of component evaluations does not by itself determine the evaluation of the composition.

Define the component-evaluation descriptor on composable pairs by Equation (95).

$$L_{c,\rho} : \text{Comp}_{c,\rho} \longrightarrow \Omega_{\text{IH}}^{c,\rho} \times \Omega_{\text{IH}}^{c,\rho}, \quad L_{c,\rho}(h, g) := (\mathcal{J}_{c,\rho}(h), \mathcal{J}_{c,\rho}(g)). \quad (95)$$

Define the whole-composition evaluation by Equation (96).

$$W_{c,\rho} : \text{Comp}_{c,\rho} \longrightarrow \Omega_{\text{IH}}^{c,\rho}, \quad W_{c,\rho}(h, g) := \mathcal{J}_{c,\rho}(h \diamond g). \quad (96)$$

The two maps keep component classification separate from whole-history classification. Their common codomain need not support addition, averaging, or any other numerical aggregation.

Definition 7.1 (Componentwise compositional evaluation). The interaction-history normative evaluation is *componentwise compositional* on $\text{Comp}_{c,\rho}$ when there exists a map of the form stated in Equation (97).

$$\mu_{c,\rho} : L_{c,\rho}(\text{Comp}_{c,\rho}) \longrightarrow \Omega_{\text{IH}}^{c,\rho} \quad (97)$$

such that Equation (98) holds.

$$W_{c,\rho} = \mu_{c,\rho} \circ L_{c,\rho}. \quad (98)$$

Definition 7.1 does not require $\mu_{c,\rho}$ to be defined on every pair of values in the justice codomain. Its domain is only the set of value pairs actually realized by admissibly composable histories. The definition therefore avoids treating substantive justice values as elements of a universally composable algebra.

Proposition 7.2 (Factorization criterion for local sufficiency). *The factorization in Equation (98) exists if and only if, for any $(h, g), (h', g') \in \text{Comp}_{c,\rho}$,*

$$\mathcal{J}_{c,\rho}(h) = \mathcal{J}_{c,\rho}(h'), \quad \mathcal{J}_{c,\rho}(g) = \mathcal{J}_{c,\rho}(g') \quad (99)$$

imply Equation (100).

$$\mathcal{J}_{c,\rho}(h \diamond g) = \mathcal{J}_{c,\rho}(h' \diamond g'). \quad (100)$$

Proof. If Equation (98) holds, equal values of $L_{c,\rho}$ give equal values of $W_{c,\rho}$, which is Equation (100). Conversely, assume the stated implication. For a realized pair of component values $\omega = L_{c,\rho}(h, g)$, define $\mu_{c,\rho}(\omega) := W_{c,\rho}(h, g)$. The implication from Equation (99) to Equation (100) makes this value independent of the chosen realizing pair. Hence $\mu_{c,\rho}$ is well defined and Equation (98) follows. $\square$

Proposition 7.2 states exactly what would be required for the justice values of the parts to determine the justice value of the whole. It should not be assumed merely because each part has been evaluated. A history segment can alter the conditions under which a later segment is interpreted, and those continuation-relevant effects can be omitted by the current justice classification of the first segment. In that case, the pair of local justice values is an insufficient descriptor of the composed history.

The distinction also prevents an additive reading. Even if $\Omega_{\text{IH}}^{c,\rho}$ is numerically represented in a particular application, Equation (98) does not imply $\mathcal{J}(h \diamond g) = \mathcal{J}(h) + \mathcal{J}(g)$. Any such aggregation rule would require an additional substantive argument.

7.3 Conditions for Normative Congruence

The factorization problem can be expressed directly at the level of interaction-history normative equivalence. Write $\sim_{\text{IH}}$ for $\sim_{\text{IH}}^{c,\rho}$ within this subsection. Because composition is partial, right and left substitution should be stated separately.

Definition 7.3 (Right and left interaction-history normative congruence). The relation $\sim_{\text{IH}}$ is *right-congruent* on $\text{Comp}_{c,\rho}$ when Equation (101) holds.

$$h \sim_{\text{IH}} h', \quad (h, g), (h', g) \in \text{Comp}_{c,\rho} \implies h \diamond g \sim_{\text{IH}} h' \diamond g. \quad (101)$$

It is *left-congruent* when Equation (102) holds.

$$g \sim_{\text{IH}} g', \quad (h, g), (h, g') \in \text{Comp}_{c,\rho} \implies h \diamond g \sim_{\text{IH}} h \diamond g'. \quad (102)$$

Right congruence concerns replacement of the earlier history while holding the continuation fixed. Left congruence concerns replacement of the later segment while holding the prefix fixed. The names follow the position of the fixed continuation or prefix in the chronological product rather than a claim about causal direction.

For simultaneous replacement, partial composition requires a separate condition.

Definition 7.4 (Pairwise interaction-history normative congruence). The relation $\sim_{\text{IH}}$ is *pairwise congruent* on $\text{Comp}_{c,\rho}$ when Equation (103) holds.

$$\begin{aligned} (h, g), (h', g') \in \text{Comp}_{c,\rho}, \quad h \sim_{\text{IH}} h', \quad g \sim_{\text{IH}} g' \\ \implies h \diamond g \sim_{\text{IH}} h' \diamond g'. \end{aligned} \quad (103)$$

Proposition 7.5 (Factorization and pairwise congruence). *The factorization in Equation (98) exists if and only if $\sim_{\text{IH}}$ is pairwise congruent in the sense of Equation (103).*

Proof. By Equation (54), equality of the two component justice values is exactly the conjunction $h \sim_{\text{IH}} h'$ and $g \sim_{\text{IH}} g'$. Equality of the composed justice values is exactly $h \diamond g \sim_{\text{IH}} h' \diamond g'$. Proposition 7.2 therefore gives the equivalence directly. $\square$

Pairwise congruence implies both one-sided congruence conditions by taking $g' = g$ or $h' = h$. The converse does not follow on an arbitrary partial-composition domain, because the intermediate pair needed for two successive substitutions can be inadmissible even when the initial and final pairs are admissible. This domain issue must remain separate from the normative substitution rule.

For quotient-level composition, admissibility itself must be stable under replacement by justice-equivalent representatives. Define this property by Equation (104).

$$h \sim_{\text{IH}} h', \quad g \sim_{\text{IH}} g' \implies [(h, g) \in \text{Comp}_{c,\rho} \iff (h', g') \in \text{Comp}_{c,\rho}]. \quad (104)$$

Equation (104) is stronger than congruence. It requires the possibility of composition, not only the value of an already admissible composition, to depend solely on justice classes. Under this condition, right and left congruence together imply pairwise congruence: if (h, g) is admissible and the components are replaced by justice-equivalent histories, the intermediate pairs (h', g) and (h', g') are also admissible, so the two one-sided substitutions can be composed.

Proposition 7.6 (Justice-quotient composition). *Assume class-stable composability in Equation (104) and the componentwise implication of Proposition 7.2. Then the partial operation in Equation (105) is well defined on the interaction-history normative quotient.*

$$[h]_{\text{IH}} \odot [g]_{\text{IH}} := [h \diamond g]_{\text{IH}}, \quad (105)$$

whenever one, and therefore every, pair of representatives from the two classes is admissibly composable.

Proof. Equation (104) makes the domain of Equation (105) independent of the chosen representatives. If $h \sim_{\text{IH}} h'$ and $g \sim_{\text{IH}} g'$, the componentwise implication gives $h \diamond g \sim_{\text{IH}} h' \diamond g'$. Hence the output class is also independent of the representatives. $\square$

Proposition 7.6 identifies the precise conditions under which interaction-history normative classes themselves form a compositional state description. Those conditions can fail even when each individual history has a well-defined justice evaluation. When they fail, the quotient $X / \sim_{\text{IH}}$ remains a valid classification of completed histories, but it is too coarse to serve as the sole interface for sequential composition.

7.4 Context-Sensitive Failure of Congruence

Failure of congruence is not a mathematical anomaly. It is a formal way to state that the normative significance of a segment can depend on interaction conditions established by another segment. The failure can occur on either side of composition.

Define the right-congruence defect set by Equation (106).

$$\begin{aligned} \Delta_{\text{IH};c,\rho}^R := \{ (h, h', g) : h \sim_{\text{IH}}^{c,\rho} h', (h, g), (h', g) \in \text{Comp}_{c,\rho}, \\ h \diamond g \not\sim_{\text{IH}}^{c,\rho} h' \diamond g \}. \end{aligned} \quad (106)$$

Table 6: Illustrative mechanisms through which local justice equivalence can fail to be congruential.

Retained feature	Possible inter-segment dependence	Resulting compositional risk
Procedural standing	An earlier event creates, removes, or contests a recognized status relevant to a later hearing.	Locally equivalent prefixes can make the same later participation opportunity normatively different.
Authorization	Delegation, consent, mandate, or revocation occurs in one segment.	A later exercise of the same capacity can differ in authorization despite matching local topology.
Role or office occupancy	Bearers enter, leave, or change institutionally relevant roles.	Accountability or duty attached to a later act can depend on the earlier role transition.
Burden and benefit	Effects accumulate, persist, or alter later exposure.	Two locally equivalent events can differ once cumulative burden or benefit is retained by the substantive evaluation.
Responsibility basis	Contribution, control, omission, or answerability is distributed across segments.	Responsibility-relevant meaning may arise only from the relation among events rather than from either event alone.
Local context	Procedure stage, evidential status, emergency condition, or transition state changes through history.	The same event type can receive a different interpretation after a different admissible prefix.

Similarly, define the left-congruence defect set by Equation (107).

$$\Delta_{\text{IH};c,\rho}^L := \{(h, g, g') : g \sim_{\text{IH}}^{c,\rho} g', (h, g), (h, g') \in \text{Comp}_{c,\rho}, h \diamond g \not\sim_{\text{IH}}^{c,\rho} h \diamond g'\}. \quad (107)$$

The right defect set is empty exactly when right congruence holds, and the left defect set is empty exactly when left congruence holds.

The components introduced in Section 4 provide several possible carriers of such context dependence. These are analytical possibilities rather than universal sources of injustice. A prior segment can alter standing, authorization, role occupancy, unresolved burden, responsibility bases, or the local contextual conditions under which a later event is interpreted. Conversely, the significance of an earlier segment can depend on whether a later segment answers, repairs, ratifies, contests, or otherwise transforms an interactional condition left open by the first.

Table 6 should not be read as a checklist that every application must encode. The substantive theory decides which dependencies belong to the justice object. The formal consequence is narrower: if a retained dependency changes the composed justice class while leaving a component's current justice class unchanged, that dependency cannot be recovered from the current class alone.

A transitional-process application provides one useful example: institutional processes can affect relationship dynamics over time [8]. The present formalism does not infer a particular update law from that account. It uses the dynamic possibility to motivate a structural distinction: a segment can be locally classified in one way while also changing the relational or institutional conditions under which a later segment must be evaluated.

Context sensitivity can also invalidate ordinary braid identifications discussed in Section 6.4. Distant crossings can commute topologically while their anchored events fail to commute semantically because the first changes the conditions of the second. Likewise, a topological inverse can fail to erase a retained historical condition. The congruence defect sets in Equations (106) and (107) provide an evaluation-level counterpart to those representation-level restrictions.

7.5 Historical Dependence of Interaction Meaning

The congruence defect sets identify particular failures. A more global construction asks what information about a completed prefix is required to determine its admissible continuations and the justice evaluations produced by those continuations. This construction provides an abstract notion of continuation-relevant normative memory without yet selecting a substantive relational state model.

For $h \in X$, define its admissible continuation set by Equation (108).

$$\text{Cont}_{c,\rho}(h) := \{g \in X : (h, g) \in \text{Comp}_{c,\rho}\}. \quad (108)$$

The associated continuation-response function is defined in Equation (109).

$$R_h^+ : \text{Cont}_{c,\rho}(h) \longrightarrow \Omega_{\text{IH}}^{c,\rho}, \quad R_h^+(g) := \mathcal{J}_{c,\rho}(h \diamond g). \quad (109)$$

Equation (109) is behavioral and theory-relative. It does not assert that a social relation literally is a function over all possible continuations. It records which future segments are admitted by the model and how the declared interaction-history normative evaluation classifies the resulting complete histories.

Definition 7.7 (Continuation equivalence). Two histories $h, h' \in X$ are *continuation-equivalent*, written $h \approx_{c,\rho}^+ h'$, when Equation (110) holds.

$$h \approx_{c,\rho}^+ h' \iff \text{Cont}_{c,\rho}(h) = \text{Cont}_{c,\rho}(h') \text{ and } R_h^+ = R_{h'}^+. \quad (110)$$

Continuation equivalence is an equivalence relation because it is equality of a continuation domain together with equality of the corresponding response function. It is generally different from current interaction-history normative equivalence. Two completed histories can receive the same current justice value while leaving different admissible future conditions or producing different evaluations under a shared continuation.

Define the continuation-memory defect set by Equation (111).

$$\mathcal{M}_{\text{IH};c,\rho}^+ := \{(h, h') \in X^2 : h \sim_{\text{IH}}^{c,\rho} h', h \not\approx_{c,\rho}^+ h'\}. \quad (111)$$

A nonempty set in Equation (111) means that the current interaction-history normative class is not a continuation-complete state description. The phrase “memory defect” refers to insufficiency of the current classification for future composition; it does not imply that the substantive theory is defective.

Proposition 7.8 (Continuation-completeness criterion). *The defect set in Equation (111) is empty if and only if Equation (112) holds.*

$$\sim_{\text{IH}}^{c,\rho} \subseteq \approx_{c,\rho}^+. \quad (112)$$

When Equation (112) holds, justice-equivalent histories have the same admissible continuations and the same justice response to every such continuation.

Proof. Equation (111) is empty precisely when there is no pair related by $\sim_{\text{IH}}^{c,\rho}$ that is separated by $\approx_{c,\rho}^+$. This is exactly the inclusion in Equation (112). The second statement follows from Definition 7.7. $\square$

Proposition 7.8 gives a precise form to historical dependence. If Equation (112) fails, a justice class that is adequate for classifying completed histories is too coarse to determine the future normative behavior of those histories under composition. A model intended only for retrospective classification need not repair that coarseness. A model intended for sequential analysis, institutional design, or dynamic simulation requires additional continuation-relevant information if it is to reproduce those future distinctions.

One abstract enrichment is the joint state descriptor in Equation (113).

$$S_{\text{IH};c,\rho}^+(h) := ([h]_{\text{IH}}, [h]_{\approx+}). \quad (113)$$

In Equation (113), $[h]_{\approx+}$ denotes the continuation-equivalence class of h . The joint descriptor separates current normative classification from continuation-relevant response type. It is an abstract construction

rather than a proposed empirical state variable. In a substantive application, a much smaller interpretable state may suffice if role occupancy, standing, unresolved obligation, accumulated burden, relationship condition, or another domain-specific variable explains the continuation differences.

This observation sets the task for Section 8. The next section asks when interaction itself can be represented as changing an interpretable relational state and how that state alters later interaction conditions. Equation (113) supplies the formal benchmark: a proposed dynamic state is adequate for sequential interaction-history normative analysis only if it preserves the continuation distinctions required by the intended evaluation.

8 Relational Dynamics Generated by Interaction

Section 7 showed that a completed interaction history can have two analytically distinct properties: a current interaction-history normative classification and a continuation profile that determines which later segments are admissible and how the resulting compositions are evaluated. The present section asks whether those continuation-relevant differences can be represented by a smaller and substantively interpretable relational state. Its objective is not to construct a universal social dynamics. The section defines an interaction-induced relational-state map, gives an exact criterion for sequential adequacy, states when relational change admits a state-update representation, distinguishes facilitative and constraining dynamics without identifying either with justice, develops a transitional-justice specialization, and finally marks the boundary between the present construction and the trust-formation dynamics reserved for the next paper in the series.

The method remains theory-relative and representation-relative. A relational state is introduced only when an application can identify variables that are meaningful independently of the final justice classification. The state may summarize role occupancy, standing, unresolved obligations, accumulated burdens, institutional conditions, or other continuation-relevant features, but no universal inventory is assumed. In particular, the existence of historical dependence does not by itself prove that a low-dimensional state representation exists.

8.1 Interaction-Induced Relational State Change

Fix a context-resolution pair (c, ρ) and again write $X := \Gamma_{\text{IH}}^{c, \rho}$. Let $\mathfrak{R}_{c, \rho}$ be a declared relational-state space. Its elements are not complete interaction histories. They are summaries of selected conditions that can affect the admissibility or normative interpretation of later interaction. Define the relational-state extractor by Equation (114).

$$\zeta_{c, \rho} : X \longrightarrow \mathfrak{R}_{c, \rho}. \quad (114)$$

The semantic content of $\mathfrak{R}_{c, \rho}$ must be stated by the substantive application. Equation (114) therefore does not license interpreting an arbitrary latent vector, numerical embedding, or equivalence-class identifier as a social relation.

The extractor induces the state equivalence in Equation (115).

$$h \sim_{\zeta}^{c, \rho} h' \iff \zeta_{c, \rho}(h) = \zeta_{c, \rho}(h'). \quad (115)$$

State equality and interaction-history normative equality answer different questions. The former concerns the represented continuation-relevant condition after a history; the latter concerns the current justice classification of that history. Neither relation is assumed to refine the other.

A candidate state becomes useful for sequential analysis only when it preserves the continuation distinctions identified in Definition 7.7.

Definition 8.1 (Sequentially sufficient relational state). The extractor $\zeta_{c, \rho}$ is *sequentially sufficient* for the declared interaction domain and interaction-history normative evaluation when Equation (116) holds.

$$\sim_{\zeta}^{c, \rho} \subseteq \approx_{c, \rho}^+. \quad (116)$$

Definition 8.1 addresses a different requirement from present-justice sufficiency. Without an additional relation between $\sim_{\text{IH}}^{c,\rho}$ and $\approx_{c,\rho}^+$, neither requirement implies the other. Two histories can receive the same relational state while admitting different future segments or yielding different justice responses to a shared continuation, making the state too coarse for the sequential task even if it preserves their present justice distinction. Conversely, two histories can share a continuation profile while remaining distinct under the current interaction-history normative evaluation. If one state representation is intended to determine both present justice class and continuation profile, the appropriate joint requirement is $\sim_{\zeta}^{c,\rho} \subseteq \sim_{\text{IH}}^{c,\rho} \cap \approx_{c,\rho}^+$. The present section keeps the two outputs separate unless such a joint requirement is explicitly imposed.

Let the continuation profile map be defined by Equation (117).

$$C_{c,\rho}^+(h) := (\text{Cont}_{c,\rho}(h), R_h^+). \quad (117)$$

Equality of the values in Equation (117) is exactly continuation equivalence.

Proposition 8.2 (Factorization criterion for sequential sufficiency). *The relational-state extractor $\zeta_{c,\rho}$ is sequentially sufficient if and only if there exists a map*

$$\text{Dec}_{c,\rho}^+ : \zeta_{c,\rho}(X) \longrightarrow C_{c,\rho}^+(X) \quad (118)$$

The decoder in Equation (118) is defined only on realized relational states. It is not required to assign a continuation profile to states that never occur in X . The required factorization is stated in Equation (119).

$$C_{c,\rho}^+ = \text{Dec}_{c,\rho}^+ \circ \zeta_{c,\rho}. \quad (119)$$

Proof. If Equation (119) holds, equal relational states imply equal continuation profiles and hence continuation equivalence, giving Equation (116). Conversely, suppose Equation (116) holds. For any realized state $r = \zeta_{c,\rho}(h)$, define $\text{Dec}_{c,\rho}^+(r) := C_{c,\rho}^+(h)$. Sequential sufficiency makes the value independent of the chosen realizing history, so the decoder is well defined and Equation (119) follows. $\square$

Proposition 8.2 gives the formal role of an interpretable relational state. The abstract continuation class in Equation (113) remains a maximally direct benchmark. A substantive state earns its place when a smaller or more interpretable representation factors the continuation profile with adequate resolution for the intended inquiry.

The state should also remain non-circular in the sense established for normative annotation in Research Obligation 4.3. A state component can be normatively typed, but its identification rule should not be “look up the complete future response function and assign a code.” The point of the state construction is to identify relational conditions that explain or preserve continuation differences, not to rename those differences after they have already been computed.

8.2 Dynamic Update of Interaction Conditions

A state representation becomes dynamic when an admissible continuation changes the relational condition. For $(h, g) \in \text{Comp}_{c,\rho}$, the interaction-induced state transition is the ordered pair stated in Equation (120).

$$\zeta_{c,\rho}(h) \longmapsto \zeta_{c,\rho}(h \diamond g). \quad (120)$$

Equation (120) records a change associated with the appended interaction segment g . It does not yet assert that the new state can be computed from the old state and a history-independent description of g .

To express such a computation, let $\mathcal{I}_{c,\rho}$ be a declared space of *interaction increments*. An increment extractor is a map on admissibly composable pairs,

$$\delta_{c,\rho} : \text{Comp}_{c,\rho} \longrightarrow \mathcal{I}_{c,\rho}. \quad (121)$$

The use of a pair-level map in Equation (121) is intentional. The same nominal continuation can be interpreted differently after different prefixes because standing, authorization, role occupancy, or local

context may have changed. A prefix-independent increment descriptor $\delta(g)$ is therefore a stronger specialization rather than a default assumption. The increment must nevertheless have an independently stated interpretation. Choosing $\delta_{c,\rho}(h, g)$ merely to encode the already computed next state $\zeta_{c,\rho}(h \diamond g)$ would make the update factorization formally true but analytically empty, just as direct recoding of a justice verdict would trivialize normative decoration.

Definition 8.3 (State-update representation). A pair $(\zeta_{c,\rho}, \delta_{c,\rho})$ admits a *state-update representation* when there exists a partial map

$$U_{c,\rho} : \text{Dom}(U_{c,\rho}) \subseteq \mathfrak{R}_{c,\rho} \times \mathfrak{I}_{c,\rho} \longrightarrow \mathfrak{R}_{c,\rho} \quad (122)$$

The partial domain in Equation (122) contains the realized state-increment pairs on which the update is warranted. Every composable pair must then satisfy Equation (123).

$$\zeta_{c,\rho}(h \diamond g) = U_{c,\rho}(\zeta_{c,\rho}(h), \delta_{c,\rho}(h, g)). \quad (123)$$

Definition 8.3 is a representational condition. It does not imply that the social process is Markovian in an ontological sense. It says only that the information retained in the chosen state and increment is sufficient to determine the next represented state for the declared domain.

The exact obstruction is captured by the update-defect set in Equation (124).

$$\mathcal{D}_{c,\rho}^U := \left\{ ((h, g), (h', g')) \in \text{Comp}_{c,\rho}^2 : \begin{array}{l} \zeta_{c,\rho}(h) = \zeta_{c,\rho}(h'), \delta_{c,\rho}(h, g) = \delta_{c,\rho}(h', g'), \\ \zeta_{c,\rho}(h \diamond g) \neq \zeta_{c,\rho}(h' \diamond g') \end{array} \right\}. \quad (124)$$

Proposition 8.4 (Update factorization criterion). A state-update map satisfying Equation (123) exists if and only if the defect set in Equation (124) is empty.

Proof. If the factorization exists, equal current states and equal increments give equal arguments to $U_{c,\rho}$ and therefore equal next states, so the defect set is empty. Conversely, if the defect set is empty, define $U_{c,\rho}(r, i)$ using any composable pair (h, g) realizing $r = \zeta_{c,\rho}(h)$ and $i = \delta_{c,\rho}(h, g)$. Emptiness of the defect set makes the resulting next state independent of the realizing pair. The map is therefore well defined on the realized state-increment pairs. $\square$

Proposition 8.4 separates two sources of modeling failure. The state may omit a continuation-relevant historical condition, or the increment descriptor may omit a property of the appended interaction that changes the next state. Either omission can make Equation (123) fail. Adding more state variables is therefore not the only possible repair; the interaction increment itself may need finer typing.

When the state is sequentially sufficient, its update can alter both what interactions are possible and how later interactions are evaluated. By Proposition 8.2, a realized state r determines a continuation profile $\text{Dec}_{c,\rho}^+(r)$. The transition in Equation (120) can therefore change the admissible continuation set, the continuation-response function, or both. This is the precise sense in which an interaction can modify the conditions of subsequent interaction within the present formalism.

8.3 Antagonistic and Cooperative Interaction Dynamics

The terms *cooperative* and *antagonistic* are often used too quickly in relational analysis. They should not be treated as primitive moral labels. A cooperative interaction can sustain an unjust arrangement, while antagonistic conduct can be a justified response to exclusion or domination. The present framework therefore defines directional relational tendencies only relative to a declared interaction-capacity criterion.

Let $\preceq_{c,\rho}^{\text{cap}}$ be a preorder on $\mathfrak{R}_{c,\rho}$ whose interpretation is specified by the application. The intended reading is that $r \preceq_{c,\rho}^{\text{cap}} r'$ means that r' preserves at least as much of a declared capacity for the relevant

form of reciprocal or institutionally admissible interaction as r . The criterion might concern accessible hearing opportunities, stable channels of answerability, capacity to participate, or another explicitly defined relational affordance. No universal capacity order is assumed.

For a state transition $r \mapsto r'$, define the directional classes in Equation (125).

$$\begin{aligned} r \rightsquigarrow_+ r' &\iff r \preceq_{c,\rho}^{\text{cap}} r' \text{ and } r' \not\preceq_{c,\rho}^{\text{cap}} r, \\ r \rightsquigarrow_- r' &\iff r' \preceq_{c,\rho}^{\text{cap}} r \text{ and } r \not\preceq_{c,\rho}^{\text{cap}} r', \\ r \rightsquigarrow_0 r' &\iff r \preceq_{c,\rho}^{\text{cap}} r' \text{ and } r' \preceq_{c,\rho}^{\text{cap}} r. \end{aligned} \tag{125}$$

If neither state is comparable under the declared preorder, the transition is directionally indeterminate relative to that criterion. Let $\mathfrak{D}_{c,\rho}^{\text{cap}} := \{+, -, 0, \parallel\}$, where $\parallel$ denotes incomparability. The directional classifier is the map defined in Equation (126).

$$\text{dir}_{\text{cap}} : \mathfrak{R}_{c,\rho} \times \mathfrak{R}_{c,\rho} \longrightarrow \mathfrak{D}_{c,\rho}^{\text{cap}}, \tag{126}$$

with values assigned by Equation (125) and $\parallel$ assigned to incomparable pairs. Equation (126) records only the direction relative to the declared capacity preorder.

Equation (125) permits a careful use of cooperative and antagonistic language. A transition of type $\rightsquigarrow_+$ may be described as *cooperation-supporting relative to the declared capacity criterion*; a transition of type $\rightsquigarrow_-$ may be described as *interaction-constraining* or, where the substantive interpretation warrants it, *antagonism-supporting*. These are relational-dynamic classifications, not justice verdicts.

The distinction can be made explicit by comparing the state transition with the current justice evaluation. Define the joint transition report in Equation (127).

$$\mathsf{K}_{c,\rho}(h, g) := \left(\mathcal{J}_{c,\rho}(h \diamond g), \text{dir}_{\text{cap}}(\zeta_{c,\rho}(h), \zeta_{c,\rho}(h \diamond g)) \right), \tag{127}$$

where dir_{cap} returns the appropriate class from Equation (125), including an incomparable case when necessary. Equation (127) preserves the difference between evaluating the interaction and describing the relational direction it generates.

This separation rules out several unwarranted implications. Interaction-history normativity does not require every interaction to increase the declared cooperation capacity. A procedurally fair hearing can leave substantive disagreement intact, as emphasized by the conflict-management architecture reconstructed in Section 2.3. Conversely, an interaction that expands future coordination can remain unjust if the expansion is produced through coercive exclusion, unequal standing, or another feature condemned by the declared substantive evaluation. The direction of relational change and the justice of the process should therefore be reported separately unless a specific theory explicitly links them.

Repeated interaction yields a state trajectory

$$r_0, r_1, \dots, r_m, \quad r_k = \zeta_{c,\rho}(h_{0:k}), \tag{128}$$

where $h_{0:k}$ denotes the admissible history prefix through stage k . The sequence in Equation (128) can contain facilitative, constraining, neutral, and incomparable transitions. Nothing in the formalism requires monotonic improvement, cumulative deterioration, or convergence. Those would be additional hypotheses about a particular interaction system.

8.4 Transitional-Justice Processes as Dynamic Interaction Systems

Transitional justice provides a natural specialization because the substantive literature already motivates a dynamic view. The transitional-justice account characterizes the domain through institutional processes and changing relationship dynamics rather than as a collection of isolated measures [8]. The present formalization does not infer a specific causal law from that account. It uses the dynamic orientation to show how a sequence of institutional processes can be represented without reducing each stage to an independent justice score.

Let the represented transitional-process history be defined recursively by Equation (129).

$$H_1 := g_1, \quad H_{k+1} := H_k \diamond g_{k+1}, \quad (H_k, g_{k+1}) \in \text{Comp}_{c,\rho}, \quad 1 \leq k < m. \quad (129)$$

The recursive form in Equation (129) does not presuppose global associativity of $\diamond_{c,\rho}$; it records one declared chronological bracketing of the admissible process sequence. For each prefix H_k , define the relational state

$$r_k := \zeta_{c,\rho}(H_k), \quad 1 \leq k \leq m. \quad (130)$$

Equation (130) keeps the process prefix distinct from the state extracted from that prefix. When the state-update representation exists, consecutive states satisfy Equation (131).

$$r_{k+1} = \mathcal{U}_{c,\rho}(r_k, \delta_{c,\rho}(H_k, g_{k+1})). \quad (131)$$

The prefix dependence retained in the increment of Equation (131) permits the same nominal institutional measure to have a different represented meaning when undertaken after different prior processes.

A dynamic transitional-justice analysis can therefore ask at least three distinct questions at each stage. First, what is the interaction-history normative evaluation of the realized prefix? Second, what relational state has the process generated? Third, what continuations and future justice responses are compatible with that state? The three questions are represented respectively by

$$\mathcal{J}_{c,\rho}(H_k), \quad r_k, \quad \text{Dec}_{c,\rho}^+(r_k) \quad (132)$$

when $\zeta_{c,\rho}$ is sequentially sufficient. Equation (132) prevents a favorable evaluation of one institutional process from being treated as a complete description of the relational conditions it leaves for later processes.

The framework also permits dynamic comparison without assuming that all transitional measures share a common scalar effect. Two process sequences H_m and H'_n can be compared separately with respect to their present justice classes, their relational states, and their continuation profiles. Equality at one level need not imply equality at another. For example, two histories can be interaction-history normative equivalent at the present stage while occupying different relational states if the current justice evaluation is continuation-incomplete; conversely, different justice evaluations can lead to the same later relational state if the represented relational effects converge.

The latter possibility is important for historical interpretation. Relational convergence does not erase processual difference. If

$$\zeta_{c,\rho}(H_m) = \zeta_{c,\rho}(H'_n) \quad (133)$$

while $\mathcal{J}_{c,\rho}(H_m) \neq \mathcal{J}_{c,\rho}(H'_n)$, Equation (133) shows only that the declared relational-state representation has converged. It does not imply that the histories, their justice evaluations, or responsibility-relevant pasts have become identical. The distinction parallels the historical non-erasure discipline established in Paper III and prevents dynamic state models from laundering earlier injustice through later convergence.

8.5 Boundary with Trust-Formation Dynamics

The relational-state construction is intentionally broader and thinner than a theory of trust. It records whichever interaction conditions are needed for the declared sequential interaction-history normative task. Those conditions can include recognized standing, available channels of answerability, unresolved obligations, cumulative burdens, role occupancy, or other relational features. The state is not identified with trust, confidence, legitimacy, willingness to cooperate, or any single psychological or institutional quantity.

The present paper also restricts its dynamic claim to *interaction-induced* change. Equation (123) updates the state when a represented interaction segment is appended. It does not introduce an autonomous evolution law for periods in which no represented interaction occurs, a latent trust variable with its own persistence dynamics, or a stochastic formation process operating beneath the interaction record.

Several stronger phenomena are therefore reserved for the subsequent paper. These include asymmetric trust damage and repair, hysteresis under apparently similar current conditions, path-dependent trust accumulation, endogenous persistence after the initiating interaction has ended, and self-sustaining or self-undermining relational fields. Such phenomena may eventually require non-Markovian kernels, stochastic transitions, semigroup or operator-algebraic descriptions, or other machinery that is unnecessary for the present objective.

A useful boundary can be stated formally. Paper IV asks whether an interpretable state $\zeta_{c,\rho}(h)$ is sufficient for the continuation profile relevant to interaction-history normativity and whether appended interaction increments induce a well-defined update on that state. The next paper may introduce an additional trust-bearing state T and ask whether its evolution depends on more than the current interaction-justice state and the immediately represented increment. The present formalism neither presupposes such a variable nor rules it out.

This separation also prevents explanatory inversion. If a later trust model shows that two interaction histories produce different trust trajectories, that result can motivate a finer relational state or a richer continuation analysis, but it does not retroactively make trust part of the definition of interaction-history normativity. Interaction-history normativity remains the substantive normative object reconstructed in Sections 1–2; relational dynamics describe consequences and continuation conditions generated by interaction.

Section 9 will specialize the resulting architecture to institutional role systems and office accountability. That specialization can use relational states to track office occupancy, answerability channels, institutional cooperation conditions, or other role-mediated changes while preserving the distinctions developed here between current justice evaluation, relational state, and future continuation profile.

9 Institutional Roles and Office Accountability

The general framework developed in Sections 3–8 treats roles, offices, authorization, accountability relations, and responsibility bases as typed components of interaction histories. The substantive institutional specialization draws on the cited account of public ethics of office and office accountability, supplemented by later work on institutional accountability [6, 5]. The typed carriers and equivalence structures below are limited analytical devices for testing how selected institutional distinctions survive representation. The section represents office accountability as one institutional specification of interaction-history normativity while preserving four distinctions that are easily collapsed in formal models: an office and its occupant, institutional power and its mandate, accountability and retrospective responsibility, and political corruption and other forms of interactive injustice.

9.1 Interrelated Institutional Role Systems

The institutional specialization begins with a restricted history domain rather than with a new ontology. Let

$$\Gamma_{\text{inst}}^{c,\rho} \subseteq \Gamma_{\text{IH}}^{c,\rho} \quad (134)$$

contain those interaction histories for which the declared context identifies institutional offices, role-mediated powers, and at least one accountability-relevant relation. Equation (134) preserves all of the participant and event structure of the general interaction domain; the institutional model adds typing appropriate to office-based action.

The cited institutional account describes public institutions through the interrelated conduct of officeholders and treats institutional action as enacted through officeholders occupying interdependent roles rather than as the operation of rules independently of their human enactment [6, 5]. The distinction between office and occupant from Section 3.3 is therefore retained explicitly. For $h \in \Gamma_{\text{inst}}^{c,\rho}$, define the set of represented office-action incidences by Equation (135).

$$\Xi_h^{\text{off}} := \{(e, a, o) : (e, a) \in \mathcal{I}_h, o \in \text{Off}_h(e, a)\}. \quad (135)$$

Equation (135) indexes an event, the participant bearer involved in it, and the office occupied during the represented event support. The triple is an analytical carrier. It does not imply that every act occurring while an office is occupied is performed in the officeholder's institutional capacity; that further classification must be supplied by the declared institutional reconstruction.

To preserve that distinction, let $\mathcal{C}_c^{\text{cap}}$ be a context-specific vocabulary of capacities in which conduct may be performed. Define the capacity record by Equation (136).

$$\text{Cap}_h : \Xi_h^{\text{off}} \rightarrow 2^{\mathcal{C}_c^{\text{cap}}}. \quad (136)$$

A value of Equation (136) may distinguish, for example, conduct undertaken in an institutional capacity from conduct classified as personal or as belonging to another simultaneous role. The vocabulary and identification rule are domain specific. This record prevents office occupancy alone from determining the institutional character of every event involving the occupant.

The institutional role system also contains relations among offices and officeholders. The directed accountability record Acc_h from Equation (34) already provides one such layer, while the relational-position map Π_h from Equation (10) can represent other contextually relevant role relations. The resulting picture is therefore relational rather than office-isolated: the unit of analysis can include how several officeholders jointly interpret, exercise, question, and revise institutionally mediated action. This relational structure is important for later corruption analysis because an institutional dysfunction can be constituted by a pattern of interrelated conduct even when no single event provides a complete description of the institutional failure [6].

9.2 Office Accountability as a Domain-Specific Evaluation

Office accountability is treated here as a substantive specialization of interaction-history normativity, not as the definition of interaction-history normativity in general. The institutional account characterizes office accountability through the requirement that officeholders' institutional conduct be answerable to the terms of the power mandate associated with their offices, while later work emphasizes the interpretive and interdependent character of role-based institutional action [6, 5]. The formal specialization therefore needs to represent both the exercise of office and the rationale under which that exercise is institutionally vindicated.

Let $\mathcal{G}_c^{\text{rat}}$ be a context-specific vocabulary of institutional agenda or rationale types. For the office-action incidences in Equation (135), define the reconstructed rationale record by Equation (137).

$$\text{Rat}_h : \Xi_h^{\text{off}} \rightarrow 2^{\mathcal{G}_c^{\text{rat}}}. \quad (137)$$

Equation (137) records the rationale or agenda attributed to the represented office-based conduct at the selected historical resolution. In applications this attribution can be uncertain or contested; Section 10 will therefore treat it as one possible locus of set-valued reconstruction rather than as an infallible observation.

The corresponding mandate interpretation is represented by Equation (138).

$$\text{Mand}_h : \Xi_h^{\text{off}} \rightarrow 2^{\mathcal{G}_c^{\text{rat}}}. \quad (138)$$

A value of Equation (138) records the rationale types that the declared institutional interpretation treats as supportable under the relevant office mandate and local context. The construction is intentionally incidence sensitive: an office can persist while applicable rules, delegated authority, procedural stage, emergency conditions, or other mandate-relevant circumstances change.

Simple set equality between Equations (137) and (138) would impose an unnecessarily strong interpretation of mandate coherence. Let

$$\bowtie_c^{\text{mand}} \subseteq 2^{\mathcal{G}_c^{\text{rat}}} \times 2^{\mathcal{G}_c^{\text{rat}}} \quad (139)$$

be a declared compatibility relation whose semantics specify when a reconstructed rationale is institutionally coherent with the relevant mandate interpretation. Equation (139) may encode permissive, defeasible, or multi-ground compatibility without forcing all institutional mandates into a common numerical scale.

Mandate coherence is one input to office accountability, but office accountability is a history-level evaluative object. Let

$$\mathcal{J}_{\text{OA}}^{c,\rho} : \Gamma_{\text{inst}}^{c,\rho} \longrightarrow \Omega_{\text{OA}}^{c,\rho} \quad (140)$$

be the declared office-accountability evaluation. Equation (140) may use mandate coherence, accountability relations, role interdependence, responsibility-relevant bases, sequences of answerability or addressability, and other independently typed information. It is not defined as a sum of incidence-level compatibility judgments.

The equivalence relation induced by Equation (140) is stated in Equation (141).

$$h \sim_{\text{OA}}^{c,\rho} g \iff \mathcal{J}_{\text{OA}}^{c,\rho}(h) = \mathcal{J}_{\text{OA}}^{c,\rho}(g). \quad (141)$$

Equation (141) makes office-accountability resolution explicit. A declared application treats office accountability as a *complete* interaction-history normative specialization on the institutional subdomain only when Equation (142) holds.

$$\sim_{\text{IH}}^{c,\rho} |_{\Gamma_{\text{inst}}^{c,\rho}} = \sim_{\text{OA}}^{c,\rho}. \quad (142)$$

Equation (142) is a substantive declaration rather than a theorem of the general framework. A broader institutional-justice inquiry can preserve additional distinctions. In that setting, office accountability supplies one evaluative coordinate without exhausting the interaction-history normative quotient.

9.3 Political Corruption as a Domain-Specific Classification

The account characterizes political corruption as a deficit of office accountability and argue that its wrong is interactive: corrupt institutional conduct alters the terms on which interrelated officeholders act together within the public institution [6]. The present formalism can represent that thesis without treating every institutional wrong, every abuse of power, or every harmful consequence as corruption.

Let

$$\Omega_{\text{OA,corr}}^{c,\rho} \subseteq \Omega_{\text{OA}}^{c,\rho} \quad (143)$$

be the subset of office-accountability values that the declared institutional specialization classifies as corruption-relevant deficits. Equation (143) therefore declares the evaluative region associated with the corruption diagnosis rather than deriving that region from topology or event frequency. The corresponding corruption locus is defined in Equation (144).

$$\text{Corr}_{c,\rho} := \left\{ h \in \Gamma_{\text{inst}}^{c,\rho} : \mathcal{J}_{\text{OA}}^{c,\rho}(h) \in \Omega_{\text{OA,corr}}^{c,\rho} \right\}. \quad (144)$$

Equation (144) makes the dependence on the declared office-accountability theory explicit. It also leaves room for the contextual and pro tanto qualifications emphasized in the later clarification of corruption as an interactive wrong: identifying corruption need not settle every other moral assessment of the institution or its surrounding political order [7].

A binary corruption-status descriptor can be written as Equation (145) when the application requires that coarse distinction.

$$C_{c,\rho}(h) := \begin{cases} 1, & h \in \text{Corr}_{c,\rho}, \\ 0, & h \notin \text{Corr}_{c,\rho}. \end{cases} \quad (145)$$

Equation (145) is an output classification. It should not be inserted back into the primitive annotation layer as an input, because doing so would violate the judgment-independent annotation obligation in Research Obligation 4.3.

Proposition 9.1 (Office-accountability invariance of corruption status). *For the corruption locus defined by Equation (144), office-accountability equivalence implies equal corruption status:*

$$h \sim_{\text{OA}}^{c,\rho} g \implies C_{c,\rho}(h) = C_{c,\rho}(g). \quad (146)$$

Proof. If $h \sim_{\text{OA}}^{c,\rho} g$, Equation (141) gives equal values of $\mathcal{J}_{\text{OA}}^{c,\rho}$. Membership in the subset $\Omega_{\text{OA},\text{corr}}^{c,\rho}$ is therefore identical for the two histories, so Equation (145) assigns them the same status. $\square$

Equation (146) states the formal consequence proved in Proposition 9.1. The proposition gives an invariance internal to the declared specialization. It does not establish that office-accountability equivalence captures every morally relevant distinction among corrupt histories. Two histories can both fall inside $\text{Corr}_{c,\rho}$ while differing in magnitude, affected parties, consequences, responsibility allocation, or the standing of the institution itself.

The distinction between topology and institutional meaning is especially important here. A braid descriptor that preserves who interacted and in what trajectory order can still omit whether an officeholder acted in an institutional capacity, which mandate applied, what rationale guided the conduct, or what accountability relation connected the officeholders. For corruption classification, a descriptor D is adequate only if Equation (147) holds on the institutional subdomain.

$$\sim_D \subseteq \sim_{C_{c,\rho}}. \quad (147)$$

Equation (147) is the specialized form of the descriptor-sufficiency criterion from Section 5.5. Ordinary braid topology may satisfy it in a specially constrained domain, but the present framework supplies no general reason to expect that result when mandate, capacity, and accountability information are normatively relevant.

9.4 Institutional Accountability Modes

The recent institutional-accountability account distinguishes two potentially integrated modes of institutional accountability. Answerability is oriented toward oversight, assessment, correction, and possible sanction, whereas addressability concerns reciprocal reason-giving, mutual recognition, and co-authorship among interdependent officeholders; the latter can support cooperation without becoming identical to trust or institutional success [5]. The distinction can be represented using the accountability-relation record already defined in Equation (34).

Assume that the context-specific vocabulary $\mathcal{K}_c^{\text{acc}}$ contains distinguished mode labels `ans` and `adr`. For $k \in \{\text{ans}, \text{adr}\}$, define the mode-specific accountability incidences in Equation (148).

$$\mathcal{E}_h^k := \{(e, a, b) \in \mathcal{D}_h : k \in \text{Acc}_h(e, a, b)\}. \quad (148)$$

Equation (148) retains event-local directionality. A coarser participant-level projection is given by Equation (149).

$$\mathbf{G}_h^k := (A_h, \mathbf{E}_h^k), \quad \mathbf{E}_h^k := \{(a, b) : \exists e \in E_h \text{ with } (e, a, b) \in \mathcal{E}_h^k\}. \quad (149)$$

The graph in Equation (149) is a descriptive projection only. It discards the timing, content, quality, and sequence of accountability interactions, so graph equality cannot by itself establish equality of office accountability or interaction-history normativity.

The two mode layers should also remain formally distinct. Define their joint mode profile by Equation (150).

$$\mathbf{M}_h^{\text{acc}} := (\mathcal{E}_h^{\text{ans}}, \mathcal{E}_h^{\text{adr}}). \quad (150)$$

Equation (150) does not impose an ordering between answerability and addressability or require one mode to substitute for the other. An institutional theory may require both, may assign them different tasks, or may recognize settings in which one mode is inapplicable.

The dynamic framework of Section 8 supplies an additional reason to preserve the distinction. Accountability practices can alter later interaction conditions by changing recognized standing, available

Table 7: Adequacy requirements for the institutional specialization of interaction-history normativity.

Requirement	Formal object	Distinction preserved
Bearer–office separation	$A_h, \mathcal{O}_c, \Lambda_h, \Xi_h^{\text{off}}$	Persistent office versus changing occupant and simultaneous nonoffice roles.
Institutional-capacity typing	Cap_h	Office occupancy versus conduct performed in the relevant institutional capacity.
Mandate traceability	$\text{Rat}_h, \text{Mand}_h, \bowtie_c^{\text{mand}}$	Exercise of power versus rationale coherent with the represented mandate.
Accountability-mode typing	$\text{Acc}_h, \text{M}_h^{\text{acc}}$	Answerability, addressability, and other accountability relations.
Evaluative separation	$\mathcal{J}_{\text{OA}}, \mathcal{J}_{\text{IH}}, C_{c,\rho}$	Office-accountability evaluation, broader interaction-history normativity, and corruption classification.

channels for challenge, officeholder expectations, unresolved obligations, or the capacity for cooperative institutional work. If the declared relational state $\zeta_{c,\rho}$ retains such information, the mode-sensitive component can be represented by a projection

$$\pi_{\text{acc}} : \mathfrak{R}_{c,\rho} \longrightarrow \mathfrak{R}_{c,\rho}^{\text{acc}}, \quad r_h^{\text{acc}} := \pi_{\text{acc}}(\zeta_{c,\rho}(h)). \quad (151)$$

Equation (151) is warranted only when the chosen relational-state representation contains accountability-relevant coordinates. It permits analysis of how answerability or addressability practices alter later institutional interaction while keeping trust formation outside the present state definition. The stronger dynamics of trust remain reserved for the subsequent paper.

9.5 Domain-Specific Formal Specialization

The institutional application is complete only when its additional objects are tied back to the general interaction-history architecture. A convenient way to state the specialization is through the institutional analysis packet in Equation (152).

$$\mathfrak{S}_{c,\rho}^{\text{inst}} := \left(\Gamma_{\text{inst}}^{c,\rho}, \mathcal{O}_c, \text{Cap}, \text{Rat}, \text{Mand}, \bowtie_c^{\text{mand}}, \text{Acc}, \mathcal{J}_{\text{OA}}^{c,\rho} \right). \quad (152)$$

Equation (152) is a bookkeeping object rather than an ontological claim. It records which additional structures an application must make explicit when office accountability is part of the justice inquiry.

Table 7 summarizes five corresponding adequacy requirements. Each requirement prevents a distinct form of representational collapse.

The first four rows of Table 7 concern information that can enter a descriptor; the final row concerns evaluations that should remain outputs unless a declared higher-level analysis uses them as prior data. This directionality preserves the non-circular construction developed in Section 4.5.

Braid representation requires one further declaration. If strands track participant bearers, occupant changes are visible while office continuity must be retained by decoration. If strands track offices, office continuity is visible while occupant succession can disappear unless bearer information is separately encoded. If strands track role-bearing trajectories, both the identity and transition rules for those trajectories must be declared. The strand-tracking convention τ from Section 6.1 therefore remains part of the institutional specification rather than being chosen for graphical convenience.

A domain-specific institutional descriptor D_{inst} is adequate for office-accountability analysis when Equation (153) holds.

$$\sim_{D_{\text{inst}}} \subseteq \sim_{\text{OA}}^{c,\rho}. \quad (153)$$

If the application additionally declares the complete specialization in Equation (142), Equation (153) is also sufficient for the corresponding interaction-history normative classification on $\Gamma_{\text{inst}}^{c,\rho}$. Without that declaration, the broader criterion $\sim_{D_{\text{inst}}} \subseteq \sim_{\text{IH}}^{c,\rho}$ must be checked separately.

The institutional specialization therefore adds structure without revising the general theory retrospectively. Office accountability identifies a role-mediated normative object within institutional interaction; political corruption identifies one class of office-accountability deficit; answerability and addressability distinguish modes through which institutional agents hold one another to account; and braid-theoretic description remains a conditional trajectory representation whose adequacy depends on the distinctions preserved by the declared application. Section 10 will relax the assumption that the relevant history, office capacity, mandate interpretation, and normative typing are uniquely identified by the available evidence.

10 Normative Evaluation under Historical and Interpretive Uncertainty

The preceding sections defined normative evaluation on histories whose structural reconstruction, participant typing, role occupancy, event classification, and context-sensitive interpretation were treated as fixed within a declared application. Empirical and historical inquiry often leaves several of those elements unresolved. The present section therefore propagates uncertainty through the interaction-history normative architecture rather than selecting one preferred reconstruction or one preferred interpretation prematurely. Its role is to distinguish uncertainty about what interaction history occurred from uncertainty about how a fixed substantive theory applies to that history. The section first defines set-valued interaction reconstruction, then introduces admissible interpretive regimes for normatively typed interaction features, derives criteria for complete and partial normative identification, states conditions for semantically disciplined cross-context transport, and concludes with a maximally warranted reporting rule. The construction inherits the set-valued historical methodology of Paper II and the normative-identification architecture of Paper III while retaining the participant-, role-, office-, and context-specific structure developed in the present paper [9, 11].

The uncertainty analyzed here should remain distinct from uncertainty about which moral theory is correct. The substantive interaction-history normative framework, its declared evaluation codomain, and the admissible kinds of interaction-relevant reason remain fixed within each analysis regime. Interpretive variation concerns the application of that framework to incomplete or contestable evidence: for example, whether a participant acted in an institutional capacity, which office or role was occupied during an event, how a mandate should be reconstructed, whether an opportunity counted as substantively available, or which responsibility-relevant relation is supported by the record. A later inquiry could add theory uncertainty as another layer, but the present section does not treat alternative justice theories as interchangeable with evidential or interpretive ambiguity.

10.1 Set-Valued Interaction Reconstruction

Fix a context-resolution pair (c, ρ) , an evidential record Y , and a structural reconstruction regime η . The regime η specifies the historical assumptions required to map the record into candidate interaction histories: fact classification, identity reconstruction, event delimitation, temporal ordering, admissibility constraints, and the representational commitments relevant to the declared resolution. Let $\mathcal{R}_\eta(Y)$ denote the resulting set of evidence-compatible histories in the broader historical universe. The interaction-relevant reconstruction set is defined in Equation (154).

$$\mathcal{R}_{\text{IH}}^{c,\rho}(Y; \eta) := \mathcal{R}_\eta(Y) \cap \Gamma_{\text{IH}}^{c,\rho}. \quad (154)$$

Equation (154) separates historical compatibility from the domain of the interaction-history normative question. A history may be compatible with the evidential record while lying outside the declared context, resolution, participant registry, or admissibility conditions of the present inquiry.

The reconstruction in Equation (154) is allowed to preserve ambiguity in participant identity, event boundaries, role or office occupancy, relational position, and other structural objects defined in Section 3. A unique structural reconstruction is therefore a special case rather than a prerequisite for justice evaluation. Define the evidence-compatible structural-history set by Equation (155).

$$\mathcal{S}_{\text{IH}}^{c,\rho}(Y; \eta) := \{S(h) : h \in \mathcal{R}_{\text{IH}}^{c,\rho}(Y; \eta)\}, \quad (155)$$

where S is the pre-normative history skeleton from Section 3. Equation (155) permits an analyst to distinguish uncertainty in the complete history object from uncertainty that already appears in the structural interaction skeleton.

Definition 10.1 (Structural interaction identifiability). The interaction structure is *identified from Y under η* when Equation (156) holds.

$$\mathcal{R}_{\text{IH}}^{c,\rho}(Y; \eta) \neq \emptyset, \quad |\mathcal{S}_{\text{IH}}^{c,\rho}(Y; \eta)| = 1. \quad (156)$$

Definition 10.1 is intentionally weaker than uniqueness of the complete history. Distinct histories can induce the same interaction skeleton while differing in background facts that the declared interaction-history normative analysis does not structurally retain. Conversely, a singleton complete-history reconstruction automatically gives a singleton structural image.

Historical nonidentification should not be repaired by silently narrowing the reconstruction set until a preferred narrative remains. The evidential record supports the set in Equation (154) only relative to the declared reconstruction regime. If a later evidential constraint removes histories, the smaller set can support a stronger conclusion; if the analyst changes the taxonomy, identity rule, event resolution, or admissibility assumptions, the resulting difference should be recorded as a regime change rather than described as mere evidential strengthening.

For later robustness analysis, let $\mathcal{U}_H$ be a nonempty family of structural reconstruction regimes for the same context-resolution pair. The aggregate interaction reconstruction is defined in Equation (157).

$$\mathcal{R}_{\text{IH}}^{c,\rho}(Y; \mathcal{U}_H) := \bigcup_{\eta \in \mathcal{U}_H} \mathcal{R}_{\text{IH}}^{c,\rho}(Y; \eta). \quad (157)$$

Equation (157) does not claim that the regimes are equally plausible or probabilistically weighted. It records the histories admitted by the uncertainty class that the inquiry has chosen to test.

10.2 Uncertain Normative Typing

Historical reconstruction does not exhaust uncertainty in an interaction-history normative analysis. Even after a structural history h has been fixed, the available evidence may support more than one admissible interpretation of the interaction-relevant features developed in Section 4. The same event sequence can leave open whether an opportunity was genuinely available, whether conduct occurred in an institutional capacity, which mandate interpretation applies, whether a burden is attributable to a given interaction, or which responsibility-relevant relation is supported. This subsection represents those possibilities without making the annotation map itself circular.

Let $\Xi_{c,\rho}$ be a declared set of *interpretive regimes*. An element $\xi \in \Xi_{c,\rho}$ fixes the application-level rules needed to construct the selected normative annotations and evaluate them under the substantive interaction-history normative framework. Every admissible ξ must respect the judgment-independent provenance requirement of Research Obligation 4.3. In particular, an interpretive regime may resolve contested typing rules, but it may not define those rules by consulting the final normative value of the same history.

For each $\xi \in \Xi_{c,\rho}$, let $\text{Dom}(\xi) \subseteq \Gamma_{\text{IH}}^{c,\rho}$ be the histories on which the regime yields a complete application-level interpretation. The regime-specific annotation map is stated in Equation (158).

$$A_{c,\rho}^{\xi} : \text{Dom}(\xi) \longrightarrow \mathcal{A}_{c,\rho}. \quad (158)$$

Equation (158) uses the same annotation codomain as Section 4; uncertainty concerns which admissible annotation value is supported, rather than a change in the meaning of the codomain itself. If two interpretive regimes use genuinely different semantic vocabularies, they should first be transported to a declared common reporting vocabulary before their outputs are compared.

Evidence compatibility can restrict which regimes apply to which reconstructed histories. Define the admissible typed-completion set in Equation (159).

$$\tilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) := \left\{ (h, \xi) : \begin{array}{l} h \in \mathcal{R}_{\text{IH}}^{c,\rho}(Y; \eta), \\ \xi \in \Xi \subseteq \Xi_{c,\rho}, \\ h \in \text{Dom}(\xi), \\ \xi \text{ is compatible with } (Y, h) \end{array} \right\}. \quad (159)$$

Equation (159) is the basic uncertainty object for the remainder of the section. The first coordinate records uncertainty over reconstructed history; the second records uncertainty over the interpretation applied to that history.

For a fixed reconstructed history h , define its evidence-compatible annotation set in Equation (160).

$$\mathcal{V}_A(h; Y, \eta, \Xi) := \left\{ A_{c,\rho}^\xi(h) : (h, \xi) \in \tilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) \right\}. \quad (160)$$

Equation (160) isolates interpretive uncertainty conditional on one structural history. A singleton value means that the declared interpretive regimes agree on the complete selected annotation packet for that history. Several annotation packets can remain compatible even when the structural history is uniquely reconstructed.

Complete annotation agreement is often stronger than the inquiry requires. Let $G : \mathcal{A}_{c,\rho} \rightarrow \Omega_G$ be a declared annotation feature, such as an office-capacity status, a hearing-relevant feature, a mandate-coherence class, or another independently interpretable projection. The history-conditioned feature set is defined in Equation (161).

$$\mathcal{V}_G(h; Y, \eta, \Xi) := \{G(a) : a \in \mathcal{V}_A(h; Y, \eta, \Xi)\}. \quad (161)$$

Definition 10.2 (Conditional typing identifiability). For a fixed reconstructed history h , the feature G is *interpretively identified* under (Y, η, Ξ) when Equation (162) holds.

$$\mathcal{V}_A(h; Y, \eta, \Xi) \neq \emptyset, \quad |\mathcal{V}_G(h; Y, \eta, \Xi)| = 1. \quad (162)$$

Definition 10.2 permits a narrow interaction feature to be identified while the complete annotation packet remains contested. This is particularly useful in institutional applications. The evidence can support, for example, that an officeholder acted in the relevant institutional capacity while leaving the finer interpretation of mandate rationale unresolved; or it can identify that a participant lacked a meaningful opportunity to respond while leaving responsibility allocation open.

The distinction between historical and interpretive uncertainty is analytical rather than metaphysical. In one application, office occupancy may be treated as part of the reconstructed structural history; in another, the record may identify the bearer and event sequence while leaving office capacity as an interpretive classification. The requirement is consistency: the same uncertainty should not be counted twice at both the history and interpretation layers without an explicit reason.

10.3 Robust Normative Identification

A complete interpretive regime determines how the fixed substantive theory is applied to histories in its domain. Write the corresponding evaluation as Equation (163).

$$\mathcal{J}_{c,\rho}^\xi : \text{Dom}(\xi) \longrightarrow \Omega_{\text{IH}}^{c,\rho}. \quad (163)$$

The common codomain in Equation (163) records that the same substantive interaction-history normative question is being applied across the admissible interpretations. The maps may differ in their values because disputed evidence or classification rules are resolved differently, but they do not represent different moral theories merely by virtue of that variation.

Define the augmented evaluation on admissible typed completions by Equation (164).

$$\widehat{\mathcal{J}}_{c,\rho}(h, \xi) := \mathcal{J}_{c,\rho}^\xi(h). \quad (164)$$

The evidence-compatible normative value set is then given by Equation (165).

$$\mathcal{V}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) := \left\{ \widehat{\mathcal{J}}_{c,\rho}(h, \xi) : (h, \xi) \in \widetilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) \right\}. \quad (165)$$

Equation (165) propagates both historical and interpretive uncertainty into the justice evaluation.

Definition 10.3 (Normative identifiability under one analysis regime). The normative evaluation is *identified under* (η, Ξ) when Equation (166) holds.

$$\widetilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) \neq \emptyset, \quad |\mathcal{V}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi)| = 1. \quad (166)$$

Definition 10.3 does not require either the history or the annotation packet to be uniquely identified. Several history–interpretation pairs may remain compatible with the record while still belonging to one interaction-history normative class.

For $\omega \in \Omega_{\text{IH}}^{c,\rho}$, define the augmented justice fiber in Equation (167).

$$\widehat{\mathcal{F}}_{\text{IH}}^{c,\rho}(\omega) := \left\{ (h, \xi) : h \in \text{Dom}(\xi), \widehat{\mathcal{J}}_{c,\rho}(h, \xi) = \omega \right\}. \quad (167)$$

Proposition 10.4 (Fiber criterion for normative identification). Assume the typed-completion set in Equation (159) is nonempty. The normative value is identified as ω under (η, Ξ) if and only if Equation (168) holds.

$$\widetilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) \subseteq \widehat{\mathcal{F}}_{\text{IH}}^{c,\rho}(\omega). \quad (168)$$

Proof. If Equation (168) holds, every admissible typed completion receives the value ω . Equation (165) is therefore the singleton $\{\omega\}$. Conversely, if that value set is $\{\omega\}$, every admissible typed completion lies in the fiber defined by Equation (167). $\square$

The augmented fiber criterion makes the source of robustness explicit. A justice judgment can be identified because historically different reconstructions and interpretively different typings all converge normatively. Such convergence should not be reported as historical agreement or interpretive agreement; it is agreement at the declared interaction-history normative resolution.

A coarser feature can remain identified when the complete justice value is not. Let $\Phi : \Omega_{\text{IH}}^{c,\rho} \rightarrow \Omega_\Phi$ be a declared reporting feature. Define its compatible value set in Equation (169).

$$\mathcal{V}_\Phi^{c,\rho}(Y; \eta, \Xi) := \{\Phi(\omega) : \omega \in \mathcal{V}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi)\}. \quad (169)$$

Definition 10.5 (Partial normative identifiability). The analysis has *partial normative identifiability through* Φ when Equation (170) holds.

$$|\mathcal{V}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi)| > 1, \quad |\mathcal{V}_\Phi^{c,\rho}(Y; \eta, \Xi)| = 1, \quad (170)$$

with a nonempty typed-completion set understood.

Definition 10.5 supports conclusions such as stable identification of an unacceptable hearing condition, stable office-capacity failure, or stable membership in a declared accountability-deficit class while finer justice distinctions remain unresolved. The feature must be defined independently of the uncertainty result; it should not be invented after inspecting the compatible values merely to force a singleton report.

Robustness across alternative analysis regimes requires one further layer. Let $\mathfrak{U}_{\text{IH}}$ be a nonempty family of pairs $v = (\eta_v, \Xi_v)$, each representing a reconstruction-and-interpretation regime admitted by the uncertainty analysis. Define the aggregate justice-value set in Equation (171).

$$\mathcal{V}_{\text{IH}}^{c,\rho}(Y; \mathfrak{U}_{\text{IH}}) := \bigcup_{v \in \mathfrak{U}_{\text{IH}}} \mathcal{V}_{\text{IH}}^{c,\rho}(Y; \eta_v, \Xi_v). \quad (171)$$

Regimes that admit no typed completion are recorded by Equation (172).

$$\mathcal{E}_{\text{IH}}^{\emptyset}(Y; \mathfrak{U}_{\text{IH}}) := \left\{ v \in \mathfrak{U}_{\text{IH}} : \tilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta_v, \Xi_v) = \emptyset \right\}. \quad (172)$$

Definition 10.6 (Robust normative identification). A value $\omega \in \Omega_{\text{IH}}^{c,\rho}$ is *robustly identified over* $\mathfrak{U}_{\text{IH}}$ when Equation (173) holds.

$$\mathcal{E}_{\text{IH}}^{\emptyset}(Y; \mathfrak{U}_{\text{IH}}) = \emptyset, \quad \mathcal{V}_{\text{IH}}^{c,\rho}(Y; \mathfrak{U}_{\text{IH}}) = \{\omega\}. \quad (173)$$

Definition 10.6 treats an empty regime as a failure of applicability rather than evidence of agreement. It also makes robustness explicitly relative to the declared uncertainty family. Expanding the family can destroy robustness; narrowing it can create apparent robustness. The report should therefore identify the regime family over which the conclusion has been tested.

Two diagnostic slices help locate the source of nonidentification. For a fixed interpretive regime ξ , variation of $\mathcal{J}_{c,\rho}^{\xi}(h)$ across reconstructed histories measures historical sensitivity under that interpretation. For a fixed reconstructed history h , variation of $\mathcal{J}_{c,\rho}^{\xi}(h)$ across admissible ξ measures interpretive sensitivity conditional on that history. The two forms can coexist, and the full set in Equation (165) should be retained when their interaction matters.

10.4 Cross-Context Comparison and Transport

The context index c has been retained throughout the paper because interactive meaning can depend on institutional structure, role vocabulary, procedural setting, and the relation under examination. Cross-context comparison therefore requires an explicit transport protocol. A common label or numerical codomain does not by itself establish semantic comparability.

Let $\mathcal{C}$ be a declared family of contexts. For each $c \in \mathcal{C}$, let $\Omega_{\text{IH},c}$ be the context-specific interaction-history normative codomain after any resolution dependence has been suppressed for readability. Form the disjoint union in Equation (174).

$$\Omega_{\text{IH}}^{\sqcup} := \coprod_{c \in \mathcal{C}} \Omega_{\text{IH},c}. \quad (174)$$

A comparison protocol first declares an equivalence relation $\approx_{\text{sem}}^{\text{IH}}$ on $\Omega_{\text{IH}}^{\sqcup}$ whose classes state which context-tagged values are treated as semantically equivalent at the intended comparative resolution. This relation is substantive comparative input. The formalism does not infer it from identical terminology, identical braid descriptors, or equality of numerical labels.

For each context c , let $\psi_c : \Omega_{\text{IH},c} \rightarrow \Omega_{\text{cmp}}$ be a declared transport to a common comparative codomain. The transports define the collective map in Equation (175).

$$\Psi_{\text{IH}} : \Omega_{\text{IH}}^{\sqcup} \longrightarrow \Omega_{\text{cmp}}, \quad \Psi_{\text{IH}}(c, \omega) := \psi_c(\omega). \quad (175)$$

Let $\sim_{\Psi_{\text{IH}}}$ be the equality-induced relation of Equation (175). The transport is semantically sound relative to the declared protocol when Equation (176) holds.

$$\sim_{\Psi_{\text{IH}}} \subseteq \approx_{\text{sem}}^{\text{IH}}. \quad (176)$$

Equation (176) prevents the common codomain from identifying values that the comparison protocol still treats as meaningfully distinct. Equality of the two relations would mean exact semantic resolution relative to that protocol; strict inclusion means that the transport preserves distinctions finer than the declared cross-context equivalence.

The criterion in Equation (176) does not require each ψ_c to be injective. A comparative inquiry may deliberately retain only one feature of a richer context-specific justice evaluation. If complete within-context justice distinctions are intended to survive transport, injectivity of the corresponding ψ_c is an additional requirement. The comparison resolution should therefore be stated before transported equality is interpreted.

Historical and interpretive uncertainty propagate through the transport. For a context c with analysis regime (η_c, Ξ_c) , define the transported compatible-value set in Equation (177).

$$\mathcal{V}_{\text{cmp}}(c) := \{\psi_c(\omega) : \omega \in \mathcal{V}_{\text{IH}}^{c, \rho_c}(Y_c; \eta_c, \Xi_c)\} \subseteq \Omega_{\text{cmp}}. \quad (177)$$

Definition 10.7 (Transported comparative identifiability). The comparative value for context c is *identified under the declared transport* when Equation (178) holds.

$$\tilde{\mathcal{R}}_{\text{IH}}^{c, \rho_c}(Y_c; \eta_c, \Xi_c) \neq \emptyset, \quad |\mathcal{V}_{\text{cmp}}(c)| = 1. \quad (178)$$

Definition 10.7 permits a common comparative feature to be identified even when the complete context-specific justice value remains uncertain. This is legitimate only at the resolution declared by the transport. A shared transported value across two contexts does not establish identity of their histories, institutional meanings, participant roles, or complete normative evaluations.

If several transport protocols are themselves considered admissible, transport uncertainty should also be propagated. Let $\mathfrak{P}_c$ be a nonempty family of transports from $\Omega_{\text{IH}, c}$ to comparison codomains that have already been aligned to a common reporting space. Define the transport-robust value set in Equation (179).

$$\mathcal{V}_{\text{cmp}}(c; \mathfrak{P}_c) := \bigcup_{\psi \in \mathfrak{P}_c} \{\psi(\omega) : \omega \in \mathcal{V}_{\text{IH}}^{c, \rho_c}(Y_c; \eta_c, \Xi_c)\}. \quad (179)$$

A singleton in Equation (179) supports a comparison that is stable across the declared transport family. A nonsingleton requires the report to retain transport sensitivity rather than selecting the mapping that produces the preferred comparison.

Cross-context comparison can therefore fail at several distinct levels: the histories may be nonidentified, normative typing may be nonidentified, the context-specific justice value may be nonidentified, or the semantic transport itself may be contested. These failures should not be collapsed into one generic statement of uncertainty because each suggests a different evidential or methodological remedy.

10.5 Maximally Warranted Reporting

Set-valued reconstruction changes the reporting problem. The analyst should report the finest conclusion supported by every admissible typed completion at the declared uncertainty level rather than select one compatible history or one interpretive regime. The relevant resolution can be formalized through a family of candidate report maps.

Let $V \subseteq \Omega_{\text{IH}}^{c, \rho}$ be the nonempty compatible justice-value set produced by the analysis regime or robustness family currently under consideration. Let $\mathcal{P}$ be a declared family of reporting maps $P : \Omega_{\text{IH}}^{c, \rho} \rightarrow \Omega_P$. The family can contain the identity map, acceptability indicators, office-accountability status, hearing-related features, corruption status in the institutional specialization, or another independently justified collection of reportable features. For $P \in \mathcal{P}$, define the report-value set in Equation (180).

$$\mathcal{V}_P(V) := \{P(\omega) : \omega \in V\}. \quad (180)$$

The report map P is identified on V when $|\mathcal{V}_P(V)| = 1$.

Information ordering among report maps is induced by their equality relations. For $P, Q \in \mathcal{P}$, say that P is at least as informative as Q on the relevant justice domain when Equation (181) holds.

$$\sim_P \subseteq \sim_Q. \quad (181)$$

Equation (181) means that equality under P never collapses a distinction retained by Q ; P can therefore support at least the distinctions supported by Q .

Definition 10.8 (Maximally warranted report). An identified report map $P_\star \in \mathcal{P}$ is *maximally warranted on V relative to $\mathcal{P}$* when there is no identified $Q \in \mathcal{P}$ satisfying Equation (182).

$$\sim_Q \subsetneq \sim_{P_\star} . \quad (182)$$

Definition 10.8 does not guarantee uniqueness. Two identified report maps can be incomparable in information content, and an analysis can therefore have several maximally warranted conclusions. If the candidate family is finite and contains at least one identified report map, at least one maximally warranted report exists.

Proposition 10.9 (Existence in a finite reporting family). *Let $\mathcal{P}$ be finite and suppose at least one member of $\mathcal{P}$ is identified on V . Then the set of identified members contains at least one maximally warranted report in the sense of Definition 10.8.*

Proof. Restrict the strict refinement relation induced by Equation (181) to the nonempty finite set of identified report maps. Every finite strict partial order has at least one minimal element with respect to inclusion of the induced equivalence relation. Such an element admits no strictly finer identified member and therefore satisfies Definition 10.8. $\square$

The identity report is maximally informative when the complete justice value is identified. When the complete value is not identified, a coarser feature can still be maximally warranted. Several incomparable features may survive simultaneously; in that case the analyst can report them jointly if their joint map is itself independently meaningful and identified, or report the separate stable features without claiming a uniquely privileged compression.

Table 8 summarizes the major reporting states. The table distinguishes absence of an admissible typed completion, complete identification, partial identification, source-specific sensitivity, robustness across analysis regimes, and transport-level identification.

The table implements a simple discipline: disagreement at a finer level should not erase agreement at a coarser justified level, and agreement at a coarse level should not be inflated into a finer claim. The same principle applies to braid-theoretic description. A braid class can be robustly identified while normative evaluation remains uncertain; conversely, justice can be robustly identified while several braid, role, or event reconstructions remain compatible. The representation and the normative object should therefore be reported at their separately warranted resolutions.

The uncertainty architecture also applies to the relational-state analysis of Section 8. A state $\zeta_{c,\rho}(h)$ may be uncertain because either the reconstructed history or its admissible interpretation is uncertain. Sequential analysis should then propagate the compatible relational states or continuation profiles rather than select one state silently.

The main methodological consequence is therefore modest but important. Formalization of interaction histories does not eliminate historical or interpretive uncertainty. It makes the location of uncertainty explicit, preserves the distinction between structural reconstruction and normative application, and permits the analyst to state the strongest conclusion that remains invariant across the declared uncertainty class. Section 11 uses this discipline in worked examples where equal topology, unequal hearing conditions, role reversal, institutional accountability failure, and dynamic transitional interaction generate different patterns of what can and cannot be identified.

Table 8: Reporting states for interaction-history normative analysis under historical and interpretive uncertainty.

Analysis state	Formal condition	Warranted report
No admissible typed completion	$\tilde{\mathcal{R}}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) = \emptyset$	The declared reconstruction–interpretation regime does not support an normative evaluation at the stated domain.
Complete normative identification	$ \mathcal{V}_{\text{IH}}^{c,\rho}(Y; \eta, \Xi) = 1$ with nonempty completion set	Report the identified normative value or corresponding justice class while retaining any unresolved historical or interpretive variation.
Partial identification	$ \mathcal{V}_{\text{IH}} > 1$ and $ \mathcal{V}_{\Phi} = 1$	Report the identified feature Φ and preserve uncertainty about the finer justice value.
Historical sensitivity	Justice values vary across h under a fixed admissible ξ	Identify the conclusion as reconstruction-sensitive and state which historical distinctions are material.
Interpretive sensitivity	Justice values vary across admissible ξ for a fixed h	Identify the conclusion as interpretation-sensitive and state which typing or application rule generates the variation.
Robust identification	Equation (173) holds	Report the value as robust over the declared family $\mathfrak{U}_{\text{IH}}$.
Transport-level identification	$ \mathcal{V}_{\text{cmp}}(c) = 1$ while the underlying context-specific set is finer or nonidentified	Report only the identified comparative value at the declared transport resolution.
Transport sensitivity	$ \mathcal{V}_{\text{cmp}}(c; \mathfrak{P}_c) > 1$	Retain the set of compatible comparative values and identify the transport protocol as a source of uncertainty.

11 Formal Worked Examples

This section tests the formal architecture developed in Sections 3–10 on five deliberately small constructed examples. Its role is diagnostic rather than empirical: each example fixes a narrow interaction domain, varies one representational or normative feature, and checks which equivalence relation or identification criterion changes. The objective is to show that the distinctions among trajectory topology, independently grounded annotation, interaction-history normative evaluation, institutional accountability, relational dynamics, and uncertainty are operational rather than merely terminological. The examples are idealized formal constructions; they neither describe observed cases nor claim that the selected justice classifications are uniquely correct outside the declared toy specifications. Table 9 summarizes the formal function of the five examples before the subsections develop them in detail.

11.1 Equal Interaction Topology with Unequal Hearing Conditions

The first example isolates the topological-aliasing problem from Section 5.3. Fix a context-resolution pair (c_H, ρ_H) with two participant bearers a and b , one represented interaction event e , and a braid-admissible sector in which the chosen topology descriptor assigns the same value to two histories h^+ and h^- . The common topology is stated in Equation (183).

$$\Theta_{c_H, \rho_H}(h^+) = \Theta_{c_H, \rho_H}(h^-) = \theta_H. \quad (183)$$

Table 9: Formal role of the five worked examples.

Example	Held fixed	Varied information	Formal point tested
Equal topology / hearing	Interaction topology	Hearing opportunity and standing	Topological aliasing of a justice-relevant distinction
Distinct topology / equal justice	Declared justice value	Braid-topological trajectory	Surplus topological resolution
Role reversal	Bearer-tracked crossing structure	Role, office, and authorization typing	Dependence on strand convention and normative decoration
Recurrent office interaction	Recurrent interaction pattern	Mandate coherence and accountability record	Office-accountability information beyond recurrence topology
Dynamic transitional process	Stable reported feature	Reconstructed history, relational state, and full justice value	Partial identification under historical and interpretive uncertainty

Equation (183) can be realized, for example, by two bearer-tracked interaction histories with the same crossing sequence and the same participant incidences. The example therefore does not rely on a change in who interacts or on a change in the declared braid word.

Let u_{pres} and u_{rep} be two participatory functions in the vocabulary $\mathcal{U}_{c_H}$, interpreted respectively as presenting a claim and making a reply. Let s_{party} and s_{reply} be procedural-standing labels in $\mathcal{S}_{c_H}$. The two histories differ only in the hearing-related record for participant a . Their local hearing profiles from Equation (30) are specified by Equation (184).

$$\begin{aligned} H_{h^+}(e, a) &= (\{u_{\text{pres}}, u_{\text{rep}}\}, \{u_{\text{pres}}\}, \{s_{\text{party}}, s_{\text{reply}}\}), \\ H_{h^-}(e, a) &= (\{u_{\text{pres}}\}, \{u_{\text{pres}}\}, \{s_{\text{party}}\}). \end{aligned} \quad (184)$$

Equation (184) is deliberately modest. Both histories contain realized participation by a ; the difference concerns the opportunity and standing to reply. Realized speech therefore cannot by itself erase the procedural distinction.

Suppose the declared interaction-history normative specification treats the reply opportunity and reply standing as relevant in this context and assigns distinct justice values ω_H^+ and ω_H^- . The evaluation is stated in Equation (185).

$$\mathcal{J}_{c_H, \rho_H}(h^+) = \omega_H^+, \quad \mathcal{J}_{c_H, \rho_H}(h^-) = \omega_H^-, \quad \omega_H^+ \neq \omega_H^-. \quad (185)$$

Equations (183) and (185) imply the membership statement in Equation (186).

$$(h^+, h^-) \in \Delta_{\text{IH}|\text{top}}^{c_H, \rho_H}. \quad (186)$$

Equation (186) is a concrete instance of the defect set defined in Equation (57). Ordinary interaction topology is therefore insufficient on any domain containing this pair under the declared evaluation.

The annotation layer repairs this alias when it retains the hearing information in Equation (184). The resulting separation is stated in Equation (187).

$$A_{c_H, \rho_H}(h^+) \neq A_{c_H, \rho_H}(h^-). \quad (187)$$

Equation (187) supplies exactly the separation required for this pair by Proposition 5.3. The example does not establish that every hearing difference is justice relevant or that the full annotation is minimal. It shows only that, once a declared theory treats the difference in Equation (184) as normatively relevant, a topology-only descriptor cannot recover the resulting justice distinction.

11.2 Distinct Topologies with Equal Normative Evaluation

The second example tests the converse possibility: trajectory topology can retain distinctions that the declared interaction-history normative evaluation permits an analyst to forget. Fix a two-strand braid-admissible sector in context (c_S, ρ_S) and let $g^{(1)}$ and $g^{(3)}$ be histories whose braid descriptors are the elements stated in Equation (188).

$$D_B^{c_S, \rho_S, \tau}(g^{(1)}) = \sigma_1, \quad D_B^{c_S, \rho_S, \tau}(g^{(3)}) = \sigma_1^3. \quad (188)$$

Within the ordinary two-strand braid group, the values in Equation (188) are distinct. In the toy interaction interpretation, $g^{(3)}$ contains two additional structurally retained exchanges relative to $g^{(1)}$.

Assume that the declared interaction-history normative evaluation treats those additional exchanges as normatively redundant because both histories satisfy the same hearing, standing, authorization, and other criteria that the inquiry has declared relevant. Their equal justice value is specified in Equation (189).

$$\mathcal{J}_{c_S, \rho_S}(g^{(1)}) = \mathcal{J}_{c_S, \rho_S}(g^{(3)}) = \omega_S. \quad (189)$$

Equations (188) and (189) give the surplus-resolution relation in Equation (190).

$$(g^{(1)}, g^{(3)}) \in \Sigma_{\text{top|IH}}^{c_S, \rho_S}. \quad (190)$$

Equation (190) instantiates the surplus-topology set introduced in Section 5.4. The additional topological resolution is genuine information about the represented trajectory, but it does not improve resolution of the selected justice classification.

This example also clarifies why representation adequacy should be stated relative to a task. A researcher studying interaction frequency or historical choreography may need the difference between σ_1 and σ_1^3 . A researcher whose target is only the quotient class $[g]_{\text{IH}}^{c_S, \rho_S}$ may legitimately compress the pair if the equality in Equation (189) is substantively justified. Finer topology therefore carries potentially useful historical information without becoming automatically superior for the declared normative inquiry.

11.3 Role Reversal under Fixed Crossing Structure

The third example tests participant typing and the choice of strand-tracking convention. Fix two bearers a and b , two roles r_D and r_C interpreted respectively as a decision role and a claimant role, and an event e at which the bearers interact. Let h^{DC} and h^{CD} have identical bearer-tracked braid descriptors, as stated in Equation (191).

$$D_B^{c_R, \rho_R, \tau_{\text{bearer}}}(h^{DC}) = D_B^{c_R, \rho_R, \tau_{\text{bearer}}}(h^{CD}) = \beta_R. \quad (191)$$

The role occupancy differs. For a represented time $t_e \in J_h(e)$, define the role assignments by Equation (192).

$$\begin{aligned} r_D &\in \Lambda_{h^{DC}}(a, t_e), & r_C &\in \Lambda_{h^{DC}}(b, t_e), \\ r_C &\in \Lambda_{h^{CD}}(a, t_e), & r_D &\in \Lambda_{h^{CD}}(b, t_e). \end{aligned} \quad (192)$$

Equation (192) changes the role-bearing structure without replacing either participant bearer. It is therefore a concrete instance of the bearer-role independence established in Section 3.3.

Assume further that the context associates a decision authorization label k_D with the participant occupying r_D . The event-relative authorization records in Equation (193) follow the role reversal.

$$\begin{aligned} k_D &\in \text{Auth}_{h^{DC}}(e, a), & k_D &\notin \text{Auth}_{h^{DC}}(e, b), \\ k_D &\notin \text{Auth}_{h^{CD}}(e, a), & k_D &\in \text{Auth}_{h^{CD}}(e, b). \end{aligned} \quad (193)$$

Equation (193) does not itself classify either history as just or unjust. It records a role-sensitive difference that a declared theory may use.

Suppose the represented decision act is performed by bearer a in both histories and the declared evaluation treats role-based authorization as relevant to that act. The resulting justice values may then satisfy Equation (194).

$$\mathcal{J}_{c_R, \rho_R}(h^{DC}) = \omega_R^{\text{auth}}, \quad \mathcal{J}_{c_R, \rho_R}(h^{CD}) = \omega_R^{\text{unauth}}, \quad \omega_R^{\text{auth}} \neq \omega_R^{\text{unauth}}. \quad (194)$$

Equations (191) and (194) reproduce topological aliasing under the bearer-tracking convention, while Equations (192) and (193) identify the information that a role-sensitive decoration can preserve.

The example also shows why the tracking convention τ in Section 6 is substantive. Bearer tracking makes the two histories topologically identical by construction. A role-tracking representation would follow r_D and r_C rather than a and b , and therefore records a different persistence relation. Neither convention is universally preferable. The appropriate convention depends on whether the research object is the continuity of persons, offices, roles, or another relational bearer.

11.4 Office-Accountability Failure in a Recurrent Interaction Pattern

The fourth example tests the institutional specialization from Section 9. Fix an office o , an officeholder x , another institutional participant y , and a recurrent sequence of $m \geq 2$ office interactions $e_1, \dots, e_m$. Let h_m^{coh} and h_m^{def} have the same bearer-tracked recurrence topology, as specified in Equation (195).

$$\Theta_{c_O, \rho_O}(h_m^{\text{coh}}) = \Theta_{c_O, \rho_O}(h_m^{\text{def}}) = \theta_O^{(m)}. \quad (195)$$

For each k , assume $(e_k, x, o) \in \Xi_h^{\text{off}}$ and that the action is classified as occurring in institutional capacity under the record introduced in Section 9.

The mandate-coherence relation in Equation (139) allows a local compatibility indicator to be derived without making it the complete office-accountability evaluation. Define the indicator in Equation (196).

$$d_k(h) := \begin{cases} 0, & (\text{Rat}_h(e_k, x, o), \text{Mand}_h(e_k, x, o)) \in \bowtie_{c_O}^{\text{mand}}, \\ 1, & \text{otherwise.} \end{cases} \quad (196)$$

Equation (196) records only whether the reconstructed rationale is compatible with the interpreted mandate according to the declared relation. It does not determine the complete evaluation $\mathcal{J}_{\text{OA}}^{c_O, \rho_O}$.

Let the recurrent-defect locus be the index set in Equation (197).

$$F_O(h) := \{k \in \{1, \dots, m\} : d_k(h) = 1\}. \quad (197)$$

The two histories are specified so that the coherent history has no detected mandate mismatch while the defective history contains recurrent mismatches. The condition is stated in Equation (198).

$$F_O(h_m^{\text{coh}}) = \emptyset, \quad |F_O(h_m^{\text{def}})| \geq 2. \quad (198)$$

Equation (198) is one possible institutional difference. Additional accountability-mode information from Equation (148) could also distinguish the histories if repeated failures of answerability or addressability are part of the declared inquiry.

Suppose the office-accountability evaluation assigns distinct values to the two recurrent histories. Equation (199) records that declaration.

$$\mathcal{J}_{\text{OA}}^{c_O, \rho_O}(h_m^{\text{coh}}) = \omega_O^{\text{coh}}, \quad \mathcal{J}_{\text{OA}}^{c_O, \rho_O}(h_m^{\text{def}}) = \omega_O^{\text{def}}, \quad \omega_O^{\text{coh}} \neq \omega_O^{\text{def}}. \quad (199)$$

Equations (195) and (199) show that recurrent interaction geometry does not by itself determine office accountability. The institutional descriptor must retain enough information about office capacity, mandate, rationale, accountability relations, or other declared inputs to recover the office-accountability class.

The example also preserves the distinction between office-accountability failure and a later political-corruption classification. Suppose the declared specialization maps ω_O^{def} into the corruption-relevant region introduced in Equation (143). That placement occurs only after the history-level evaluation in Equation (140). The local defect indicator d_k in Equation (196) is therefore not a primitive corruption label and cannot be substituted for the derived classification in Equation (145).

11.5 Dynamic Transitional Interaction with Stable Normative Features

The fifth example combines the relational-dynamics machinery of Section 8 with the uncertainty architecture of Section 10. Fix an evidential record Y_T for a transitional process and suppose that, under one declared reconstruction-interpretation regime (η_T, Ξ_T) , exactly two typed completions remain admissible. The completion set is stated in Equation (200).

$$\tilde{\mathcal{R}}_{\text{IH}}^{c_T, \rho_T}(Y_T; \eta_T, \Xi_T) = \{(H_m, \xi), (G_n, \xi')\}. \quad (200)$$

The histories H_m and G_n are themselves recursively constructed process prefixes of the form introduced in Section 8. Their lengths need not be equal, and Equation (200) does not assume that their topologies coincide.

Let the two completions produce distinct terminal relational states and distinct complete justice values. These differences are stated in Equations (201) and (202).

$$\zeta_{c_T, \rho_T}(H_m) \neq \zeta_{c_T, \rho_T}(G_n). \quad (201)$$

$$\hat{\mathcal{J}}_{c_T, \rho_T}(H_m, \xi) = \omega_T^{(1)}, \quad \hat{\mathcal{J}}_{c_T, \rho_T}(G_n, \xi') = \omega_T^{(2)}, \quad \omega_T^{(1)} \neq \omega_T^{(2)}. \quad (202)$$

The two-completion set in Equation (200), together with the distinct evaluations in Equation (202), implies that complete interaction-history normative identification fails under the chosen regime.

Now let $\Phi_T : \Omega_{\text{IH}}^{c_T, \rho_T} \rightarrow \mathcal{Z}_T$ be an independently declared reporting feature that retains only a normatively relevant aspect shared by the two evaluations—for example, a feature concerning preservation of a specified hearing entitlement across the process. Suppose the two complete values agree under that feature, as stated in Equation (203).

$$\Phi_T(\omega_T^{(1)}) = \Phi_T(\omega_T^{(2)}) = \phi_T^*. \quad (203)$$

The compatible complete-value set therefore has cardinality two while the compatible feature set is a singleton. Equation (204) records both facts.

$$|\mathcal{V}_{\text{IH}}^{c_T, \rho_T}(Y_T; \eta_T, \Xi_T)| = 2, \quad \mathcal{V}_{\Phi_T}^{c_T, \rho_T}(Y_T; \eta_T, \Xi_T) = \{\phi_T^*\}. \quad (204)$$

Equation (204) is a concrete case of partial interaction-history normative identifiability from Section 10.3. The warranted report is the stable feature ϕ_T^* rather than either complete value $\omega_T^{(1)}$ or $\omega_T^{(2)}$.

The dynamic and normative distinctions remain separate. Equation (201) allows the two reconstructions to imply different future interaction capacities even though Equation (203) supports one common normative report. Conversely, equality of relational states would not by itself establish equality of complete justice values. The example therefore illustrates why transitional-process analysis may need a joint report containing current justice classification, generated relational state, and uncertainty-qualified stable features, while leaving trust-specific persistence and hysteresis to the subsequent paper.

11.6 Cross-Example Diagnostic Implications

The five constructed cases establish a compact stress test for the paper's representational claims. The result is summarized by the implication pattern in Equation (205).

$$\begin{aligned} \text{equal topology} &\not\Rightarrow \text{equal interaction-history normative value,} \\ \text{equal interaction-history normative value} &\not\Rightarrow \text{equal topology,} \\ \text{equal bearer-tracked braid} &\not\Rightarrow \text{equal role-sensitive annotation,} \\ \text{equal recurrent interaction pattern} &\not\Rightarrow \text{equal office-accountability value,} \\ \text{non-identified complete justice value} &\not\Rightarrow \text{absence of identified normative features.} \end{aligned} \quad (205)$$

Every line of Equation (205) is witnessed by one of the preceding examples under its declared assumptions. The equation should therefore be read as a summary of possibilities admitted by the formal architecture, not as a universal empirical regularity about social interaction.

The examples also clarify the division of labor among the paper’s representational layers. Interaction topology records selected trajectory structure; annotation records independently grounded interaction features; interaction-history normative evaluation specifies the normative partition; institutional specialization adds office-sensitive evaluative structure; relational-state representation addresses future continuation behavior; and uncertainty analysis limits reporting to what remains stable across admissible reconstructions and interpretations. Keeping these layers distinct allows a descriptor to be judged by the information needed for the declared question rather than by an undifferentiated demand to preserve every available historical detail.

12 Scope Boundaries and Methodological Interfaces

The formal architecture developed above concerns a limited object: the representation of interaction histories under a declared process-sensitive normative evaluation. This section states the boundaries of that object. The framework is not a complete theory of procedural justice, relational equality, legitimacy, trust, or institutional design, and braid-theoretic description is not treated as a privileged ontology of social interaction. The central methodological rule is narrower: a descriptor is useful only relative to the distinctions that the declared inquiry requires it to preserve.

12.1 Procedural Justice and Process-Sensitive Evaluation

Process-sensitive interaction-history evaluation can overlap with procedural justice without exhausting it. Procedural theories may evaluate participation, accuracy, impartiality, cost, legitimacy-relevant features, or other properties that extend beyond the interaction-history distinctions selected in a particular application. Solum’s treatment of procedural justice in adjudication, for example, illustrates a broader procedural space in which participation and accuracy both matter [13].

Let $X \subseteq \Gamma_{\text{IH}}^{c,\rho}$ be a common history domain on which both the declared interaction-history evaluation and another procedural evaluation are defined. Write the latter as Equation 206.

$$\mathcal{J}_{\text{P}}^{c,\rho} : X \longrightarrow \Omega_{\text{P}}^{c,\rho}. \quad (206)$$

Equation 206 induces an equivalence relation $\sim_{\text{P}}^{c,\rho}$ by equality of procedural values. No universal refinement relation between $\sim_{\text{P}}^{c,\rho}$ and $\sim_{\text{IH}}^{c,\rho}$ is assumed.

Where two evaluative schemes share a feature, the overlap should be stated explicitly rather than inferred from common vocabulary. Let Φ_{IH} and Φ_{P} transport their outputs to a common feature codomain Ω_{ov} . Agreement at that feature resolution is stated in Equation 207.

$$\Phi_{\text{IH}}(\mathcal{J}_{c,\rho}(h)) = \Phi_{\text{P}}(\mathcal{J}_{\text{P}}^{c,\rho}(h)). \quad (207)$$

Equation 207 establishes agreement only at the declared feature resolution. It does not identify the theories or their complete codomains.

12.2 Relational Equality and Interaction-Specific Evaluation

Relational egalitarian theories address broader questions of social standing, domination, hierarchy, exclusion, and equal membership. Anderson’s account of democratic equality is a foundational statement of this orientation [1]. An interaction-history evaluation can overlap with such concerns while remaining narrower in analytical unit and temporal scope.

Let $\mathfrak{R}_{c,\rho}$ be the relational-state space introduced in Section 8, and let a domain-specific relational evaluation be given by Equation 208.

$$\mathcal{E}_{\text{rel}}^{c,\rho} : \mathfrak{R}_{c,\rho} \longrightarrow \Omega_{\text{rel}}^{c,\rho}. \quad (208)$$

Equation 208 evaluates represented relational states rather than complete interaction histories. Composing it with $\zeta_{c,\rho}$ yields a history-indexed relational descriptor while remaining analytically distinct from $\mathcal{I}_{c,\rho}$.

The two induced equivalence relations can be incomparable. When both witness types exist on the declared domain, Equation 209 records that possibility.

$$\sim_{\text{IH}}^{c,\rho} \not\subseteq \sim_{\mathcal{E}_{\text{rel}} \circ \zeta}^{c,\rho}, \quad \sim_{\mathcal{E}_{\text{rel}} \circ \zeta}^{c,\rho} \not\subseteq \sim_{\text{IH}}^{c,\rho}. \quad (209)$$

Equation 209 prevents a local process evaluation from silently becoming a total theory of social relations.

12.3 Representation Limits of Braid-Theoretic Description

The braid construction applies only to a braid-admissible sector with a justified tracking convention. Earlier papers in the series treat braid structure as a trajectory descriptor rather than a generator of social history and permit changes of representational language when object registry, identity persistence, or topology changes [10, 9]. The same discipline applies here.

Let $\mathfrak{D}_{\text{IH}}^{c,\rho}$ be a declared family of descriptors, with domain $\Gamma_D \subseteq \Gamma_{\text{IH}}^{c,\rho}$ for each $D \in \mathfrak{D}_{\text{IH}}^{c,\rho}$. The covered domain is defined in Equation 210.

$$\Gamma_{\text{IH},\mathfrak{D}}^{c,\rho} := \bigcup_{D \in \mathfrak{D}_{\text{IH}}^{c,\rho}} \Gamma_D. \quad (210)$$

The uncovered histories are defined in Equation 211.

$$\mathcal{O}_{\mathfrak{D}}^{c,\rho} := \Gamma_{\text{IH}}^{c,\rho} \setminus \Gamma_{\text{IH},\mathfrak{D}}^{c,\rho}. \quad (211)$$

A nonempty set in Equation 211 indicates incompleteness of the declared representation family, not absence of normative relevance.

Braid representation may fail because tracked-entity number changes, identity becomes unreconstructable, role succession matters more than bearer persistence, branching or merging occurs, relevant information is temporally nonlocal, or semantic noninvertibility invalidates ordinary group cancellation. In such cases representation retreat is appropriate. Let $X \subseteq \Gamma_{\text{IH}}^{c,\rho}$ and let $D_\star : X \rightarrow \Omega_\star$ be an alternative descriptor. Sufficiency is preserved when Equation 212 holds.

$$\sim_{D_\star} \subseteq \sim_{\text{IH}}^{c,\rho}|_X. \quad (212)$$

Equation 212 makes braid availability irrelevant once another justified descriptor preserves the distinctions required by the declared inquiry.

12.4 Semantic and Ontological Restraint

Formal structure does not interpret itself. Work in the philosophy of scientific representation similarly treats the relation between a representation and its target as a substantive problem [12]. The present framework therefore requires explicit tracking, anchoring, provenance, and evaluation rules.

A strand is not intrinsically a person, office, role, or institution; a crossing is not intrinsically co-operation, domination, recognition, or harm; and a normative-equivalence class is not an ontological partition of social reality. For a descriptor $D : X \rightarrow \Omega_D$, let the interpretation protocol be represented by Equation 213.

$$\mathcal{I}_D : \Omega_D \longrightarrow \mathcal{C}_D, \quad (213)$$

where $\mathcal{C}_D$ is the declared space of claims licensed by the representation. Equality under D licenses only equality at the declared descriptive resolution; equality under $\sim_{\text{IH}}^{c,\rho}$ licenses only equality under the declared normative evaluation.

12.5 Boundary with Trust, Legitimacy, and Institutional Design

Trust, legitimacy, and institutional design require additional theory. The relational-state machinery permits interactions to alter future continuation profiles, but it does not define a trust variable, a trust-specific update law, asymmetric damage and repair, hysteresis, or autonomous persistence. Legitimacy may depend on authorization, democratic pedigree, public justification, effectiveness, or other considerations not exhausted by an interaction-history evaluation. Institutional design additionally requires objectives, feasibility constraints, causal assumptions, empirical evidence, resource conditions, and trade-offs.

The non-derivation statement in Equation 214 summarizes this boundary.

$$\mathcal{J}_{c,\rho}(h) \not\Rightarrow (T(h), L(h), \Pi^*(h)), \quad (214)$$

where $T(h)$ denotes a trust conclusion, $L(h)$ a legitimacy conclusion, and $\Pi^*(h)$ an optimal institutional-design recommendation. Equation 214 denies logical determination without additional premises; it does not deny substantive relations among these neighboring inquiries.

The methodological consequence is a typed interface architecture. A process-sensitive interaction-history evaluation can supply one normative coordinate to trust analysis, legitimacy theory, transitional-process evaluation, or institutional design, while those neighboring inquiries can supply causal, institutional, or relational information relevant to interpreting the same history. The connection is one of coupling rather than theoretical absorption.

13 Conclusion

This paper has developed a preliminary meta-level formalism for normatively structured interaction histories. Its central question is representational rather than doctrinal: once a substantive account declares which differences among interaction histories matter normatively, what information must a formal descriptor preserve, and when may historical detail be compressed without changing the declared evaluation?

The framework separates the interaction history, its trajectory or topology, independently grounded normative typing, and the final theory-relative evaluation. That separation makes two forms of mismatch explicit. A topological descriptor can be too coarse when it identifies histories that the normative evaluation distinguishes, and it can be more discriminating than necessary when it preserves differences that remain within one normative class. Descriptor adequacy is therefore relative to an induced equivalence relation rather than to descriptive richness alone.

The braid-theoretic part of the paper is deliberately conditional. Strand tracking, crossing interpretation, endpoint convention, and semantic admissibility must be declared before braid structure receives substantive meaning. Decorated representations may recover distinctions omitted by topology, but their annotations must be independently interpretable rather than direct recodings of the final judgment. When fixed-strand braid assumptions fail, another descriptor can replace the braid whenever its induced equivalence preserves the distinctions required by the inquiry.

The paper also extends the representation problem beyond static histories. Composition is partial, local evaluations need not determine whole-history evaluation, and two histories receiving the same present normative value can support different admissible continuations. Continuation equivalence and relational-state sufficiency provide a way to represent those differences without identifying present evaluation with future relational dynamics.

Institutional role systems, office accountability, political corruption, transitional interaction, and interactive justice appear in the manuscript as domain-specific examples of this broader architecture. They illustrate how different substantive theories can require different participant registries, role structures, event annotations, contextual rules, or dynamic states. None of those examples defines the general framework, and the formalism does not purport to reconstruct any one substantive theory as a whole.

Historical and interpretive uncertainty remain explicit. Set-valued reconstruction can leave several histories compatible with the evidence, and several admissible normative interpretations can remain available for the same reconstructed history. The appropriate report is then the strongest conclusion invariant

across the declared uncertainty class, which may be a partial feature rather than a complete normative value. Cross-context comparison likewise requires an independently justified transport rather than a common numerical codomain alone.

The resulting division of labor is limited but useful. Substantive normative theory determines which distinctions matter. Historical reconstruction determines which histories remain evidentially admissible. Formal representation determines which of those distinctions survive compression. Braid theory supplies one possible trajectory language within that representational layer. Keeping these roles separate permits formal analysis of interaction histories without moralizing topology, overclaiming the scope of a borrowed normative concept, or treating one mathematical language as a universal ontology of social process.

The next paper in the series can build on this architecture by adding a substantive theory of trust formation. That step requires new assumptions about persistence, asymmetric damage and repair, hysteresis, and possibly non-Markovian or stochastic dynamics. Those questions are therefore left outside the present formalism rather than being inferred from interaction-history evaluation alone.

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