A Preliminary Formalization of Braid-Theoretic Trajectory Description in Generative Relational Theory

Wanhong HUANG
huangwanhong@serendip.ngo

Abstract

This paper develops a preliminary formalization of braid-theoretic trajectory description within Generative Relational Theory (GR). The construction distinguishes a generative rule, the trajectory realized under that rule, and a subsequent description of selected structural features of the realized trajectory. Braid theory is placed at the third level. A braid descriptor therefore does not ordinarily specify the mechanism that generated a history; it records a topological organization of a braid-admissible relational trajectory after generation has occurred. Building on an earlier GR metamodel organized around constraint, change, structure, and difference, the paper derives a braid-theoretic instantiation at the descriptive level and separates mechanism-level collapse from information collapsed by a descriptive quotient. The formal analysis then examines compositionality, braid equivalence, topological invariants, deformation robustness, generator underdetermination, and the expressive scope of the construction. A further objective is methodological: the paper proposes preliminary criteria for assessing when a lower-level formal language may count as generalized relative to a declared class of dynamical systems and a declared minimum descriptive resolution. Braid-theoretic trajectory description is used as the first test case. The analysis characterizes its non-generality through level restriction, domain restriction, and resolution restriction while retaining the positive role of topological compression in producing robust structural descriptions. The resulting framework treats braid theory as one trajectory-descriptive formalism within a broader GR architecture rather than as a universal generative language for dynamics.

Keywords: braid groups; trajectory description; dynamical systems; topological invariants; Generative Relational Theory

Discussion Paper Note

This manuscript develops one lower-level formal construction within the broader Generative Relational research programme. Its immediate object is braid-theoretic description of realized relational trajectories. The analysis does not treat braid theory as a universal ontology of dynamics and does not identify braid generators with the generative mechanisms of a dynamical system. The paper instead asks what a braid-theoretic description preserves, what it deliberately quotients out, and under what declared domain and descriptive resolution such a representation is adequate.

The formal starting point is the author's earlier metamodel for ontologically heterogeneous social dynamics, which places constraint, change, structure, and difference at a higher level of abstraction while withholding metric, order, arithmetic, and other lower-level commitments. The present paper works downstream from that metamodel. It introduces additional assumptions only when they are required to extract a braid-admissible trajectory and to assign a braid-theoretic descriptor.

The manuscript is foundational rather than domain-specific. Historical social dynamics, processual justice, interactive justice, trust formation, governance, and law are potential later applications or extensions. Their substantive claims are not assumed here. The present task is to establish a formal vocabulary for trajectory description and to examine its expressive scope, robustness, and limits.

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Notices

Status. This working draft contains all sixteen planned manuscript sections. The present revision is the second global consistency pass over the complete manuscript. It refines the generality criterion so that preservation of required distinctions is separated from optional exact resolution matching, and it propagates that distinction through the braid generality profile, formal examples, introduction, and conclusion. Further release-oriented revision may still refine exposition and bibliography.

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Statement on the use of language models. The exploratory discussions and preparation of this paper involved OpenAI's ChatGPT. ChatGPT supported exploratory dialogue, formal reconstruction, source discovery followed by source verification, argumentative criticism, and drafting in LaTeX. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, and bears sole responsibility for the manuscript, including its definitions, formal constructions, arguments, conclusions, and errors. Authorship credit remains with the human author. Verification records for cited external sources are maintained in the accompanying source ledger.

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1 Introduction

This section locates braid-theoretic trajectory description within a layered account of dynamical representation. Its objective is to distinguish the rule that generates a process, the trajectory realized under that rule, and a subsequent structural description of the realized trajectory. The distinction determines the formal role assigned to braid theory in the present paper and supplies the basis for evaluating its expressive adequacy. The discussion proceeds conceptually before the later sections introduce the full construction: Section 1.1 specifies the representational levels, Section 1.2 identifies the braid-theoretic contribution, Section 1.3 states the formal objectives, and Section 1.4 fixes the scope of the foundational analysis.

1.1 Levels of Dynamical Representation

A dynamical formalism can be asked to perform several logically distinct tasks. It may specify a rule or mechanism that generates change, it may represent a history produced under that rule, and it may provide a further description that preserves selected structural features of the realized history. Classical notation makes the first two levels familiar. A differential equation such as $\dot{x} = f(x)$ specifies a local generative rule, while a solution trajectory records one evolution compatible with that rule and the relevant initial or boundary conditions. The present paper adds a third level explicitly: a trajectory may itself become the input to a structural descriptor.

The three-level relation used throughout the manuscript is stated in Equation 1.

$$G \longmapsto \gamma \longmapsto D(\gamma). \tag{1}$$

Here G denotes a generative specification at an appropriate level of abstraction, γ denotes a realized history or trajectory, and D denotes a description map whose codomain records only the distinctions selected by the descriptive formalism. Equation 1 is a schema rather than an assertion that every dynamical system possesses a single function-valued generator or a manifold-valued trajectory. The notation separates roles that can be instantiated differently across domains.

The distinction matters because a description of generated dynamics need not recover the rule that generated them. Distinct generative specifications can produce histories that receive the same structural description, while one generative specification can produce different descriptions under different initial conditions or contingencies. A trajectory descriptor can therefore be informative without functioning as a generative law. This observation is especially clear in work that uses braids to analyze observed trajectories in fluid dynamics. Boyland, Aref, and Stremler represent stirring protocols as braids in a two-dimensional flow domain with time supplying the third coordinate, thereby using braid structure to characterize properties of realized motion [3]. Thiffeault develops the complementary data-oriented case in which particle trajectories are available while the underlying flow is only partially known, and braid-theoretic tools are used to recover global topological information from those trajectories [8]. These applications do not make braid words substitutes for the full velocity field. They demonstrate that trajectory structure can remain mathematically informative when the generator is unavailable, unnecessary for the local question, or represented at a different level.

Within GR, the distinction also preserves the role of the earlier highest-level metamodel. That metamodel begins with a bare carrier, a semigroup of mechanisms, an action, and a congruence, and it organizes the minimal vocabulary of dynamics around constraint, change, structure, and difference [6]. The present construction does not replace that vocabulary with braid theory. It asks how those four primitives can be instantiated after a realized history has been selected for braid-theoretic description. Braid theory therefore enters as one lower-level formalism with additional assumptions and a determinate descriptive resolution.

The phrase *trajectory-descriptive* is used functionally in this manuscript. It indicates that the formal object takes a realized trajectory or history as its immediate analytical input. The paper does not yet propose a closed class of all trajectory-descriptive languages, nor does it assume that every useful description

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must be topological. The later generality analysis instead asks which properties a lower-level descriptive language would need to satisfy relative to a declared class of systems and a declared equivalence of histories.

1.2 Braid-Theoretic Description of Relational Trajectories

Braid theory is suitable for the present investigation because its standard objects already separate a geometric realization from a topological equivalence class. Artin's braid groups encode the composition of strands that retain their identity through an ordered sequence of crossings, and the standard presentation specifies both local generators and relations among their compositions [1, 2]. The configuration-space interpretation gives the same structure a trajectory-based reading: braids can be identified with homotopy classes of loops in a configuration space of pairwise distinct points, subject to the appropriate ordering convention [2, 4]. These interpretations make braid theory especially useful for asking which distinctions among trajectories survive admissible deformation.

The present paper uses this capacity as a descriptive resource. Let Γ denote, provisionally, a collection of realized histories generated by some underlying dynamical specification. Only a subcollection will ordinarily admit the standard braid construction. For braid-admissible histories, a later section will define a trajectory extraction map and a braid-theoretic description map. The intended relation is summarized in Equation 2.

$$h \longmapsto q_h \longmapsto D_B(h) = [q_h]_B. \tag{2}$$

In Equation 2, h denotes a realized history, q_h denotes the relational trajectory extracted for braid analysis, and $[q_h]_B$ denotes the corresponding braid-equivalence class. The map is descriptive in two senses. It acts after a history has been generated, and it deliberately identifies geometrically different realizations when the selected braid equivalence treats them as the same structural history.

This quotient character is analytically important. A braid descriptor may preserve ordering, intertwining, compositional structure, and topological distinctions while discarding exact metric geometry, duration, velocity, acceleration, or other information that does not alter the braid class. Such information loss is not merely a deficiency. The same quotient can produce robustness: deformations that leave the relevant topological class unchanged leave the braid descriptor unchanged. Applied work on trajectory braids exploits precisely this property when global structural information is extracted from sparse or incomplete trajectory data [8]. The formal task is therefore to state clearly which distinctions are preserved, which are quotiented out, and which assumptions make that quotient meaningful.

The distinction between trajectory description and generative mechanism also resolves a potential tension with the earlier GR metamodel. That paper shows that mechanism-level collapse of distinctions requires noninvertibility under its stated assumptions; a group action compatible with the relevant congruence cannot produce the specified collapse relation [6]. Braid groups, by contrast, are groups. The present paper does not treat this as a contradiction because the two formalisms occupy different levels. A source mechanism may be irreversible while the topological descriptor assigned to one realized trajectory is group-valued. The manuscript will therefore distinguish *path-history*, recorded by a trajectory descriptor, from *collapse-history*, recorded by the distinctions irreversibly lost through generative mechanisms.

1.3 Formal Objectives of the Present Construction

The formal programme has five connected objectives. The first is to derive a braid-theoretic trajectory description downstream from the four meta-primitives already established in the earlier GR metamodel. Constraint will delimit braid-admissible histories; change will be represented at the descriptive level through elementary strand exchanges; structure will include both concatenation and braid-specific composition relations; difference will record the distinctions among full histories that are collapsed by the braid descriptor. This derivation is intended to make the additional commitments of braid language visible rather than importing them into the highest abstract layer.

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The second objective is to distinguish forms of information loss that occur at different levels. Collapse at the mechanism level concerns distinctions that a generative transformation destroys. Collapse at the descriptive level concerns distinctions that remain present in the full history but are identified by a representation. A further invariant map can collapse braid distinctions again. The resulting hierarchy permits descriptive resolution to be analyzed explicitly instead of treating every abstraction as either complete or defective.

The third objective is to formalize the robustness supplied by topological equivalence. Braid classes are stable under admissible deformations that remain within the same topological class. The paper will state this stability without assuming a metric and will separate class-preserving perturbations from perturbations that cross a topology-changing boundary. This distinction provides a precise basis for later applications in domains where exact trajectory reconstruction is uncertain but selected topological characteristics may remain stable.

The fourth objective is to characterize the expressive profile of braid-theoretic trajectory description. The analysis will distinguish three sources of non-generality. *Level restriction* arises because the braid descriptor acts on generated histories rather than supplying the generative rule itself. *Domain restriction* arises because only histories satisfying braid-admissibility conditions enter the standard construction. *Resolution restriction* arises because braid equivalence intentionally identifies histories that differ in features outside the selected topological resolution. The three restrictions are analytically separable and should not be compressed into the undifferentiated claim that braid theory is incomplete.

The fifth objective is to use the braid construction as a first test case for a preliminary criterion of generalized dynamical description. A formal language can be assessed relative to a declared class of systems, a declared baseline equivalence of histories, and a declared set of primitive commitments. Candidate criteria include coverage of the intended history class, preservation of distinctions the theory declares relevant, compositional adequacy, and adequacy with respect to constraint, change, structure, and difference. Exact matching to the declared baseline resolution is treated as a stronger optional condition because a language may remain adequate while retaining additional distinctions. A stronger notion of dynamics-generality additionally requires representation of the generative structure. The paper develops these criteria provisionally. Their purpose is diagnostic: they make explicit what a claim of generality requires before such a claim is attached to a specific formal language.

1.4 Scope of the Foundational Analysis

The present paper is formal and foundational in scope. It does not claim that social, physical, biological, or institutional reality possesses a braid ontology. Braid-theoretic language is treated as a representation whose adequacy depends on the target history class and the distinctions selected for preservation. This position is consistent with the GR commitment to ontological restraint: a useful relational, geometric, or topological representation does not by itself establish the ultimate ontology of the represented domain.

The paper also does not attempt to infer a unique generator from a braid descriptor. Generator underdetermination is part of the formal analysis. A later social-science paper will examine the methodological value of trajectory description when historical trajectories are partially reconstructible but the generative rules of social evolution are unavailable or underdetermined. Processual justice, interactive justice, trust formation, governance, and law constitute further downstream research domains and are excluded from the present manuscript's substantive claims.

The mathematical scope is similarly delimited. The standard planar braid group provides the primary construction because it makes the relation among trajectory, equivalence, composition, and topological invariance especially transparent. Positive braid monoids, braid groupoids, decorated braids, braided categories, and higher-dimensional process structures will be considered only where they clarify an identified limitation of the baseline descriptor. The objective is a controlled derivation of one formal language from the GR meta-primitives, followed by an explicit analysis of its expressive characteristics. Generalization is treated as a question to be earned relative to a declared domain and resolution rather than as a property conferred by mathematical breadth alone.

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2 Mathematical Background for Braid-Theoretic Trajectory Description

This section supplies the minimum background required for readers who do not routinely work with dynamical systems, topology, or braid groups. Its role is preparatory rather than argumentative. Section 2.1 fixes the classical distinction between an evolution rule and a trajectory. Section 2.2 recapitulates the four GR meta-primitives inherited from the previous paper. Section 2.3 introduces paths, homotopy, and invariants. Section 2.4 then develops configuration spaces and the braid group. Section 2.5 closes by locating braid theory at the interface between topological equivalence and algebraic composition. The section fixes notation only; the paper's new formal construction begins in Section 3.

2.1 Trajectory Formation in Classical Dynamical Systems

A classical dynamical system specifies how a state evolves with time. In a discrete-time formulation, let X denote a state space and let $F : X \to X$ be an evolution map. Starting from an initial state $x_0 \in X$, repeated application of F produces the trajectory stated in Equation 3.

$$x_k = F^k(x_0) \quad \text{for } k \in \mathbb{N}_0, \tag{3}$$

where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and F^k denotes the k-fold composition of F with itself, with F^0 interpreted as the identity map on X. Equation 3 separates the evolution rule F from one trajectory generated by that rule and an initial condition. This rule–trajectory distinction is standard in dynamical-systems theory [7].

A continuous-time flow on X is an evolution map $\Phi : \mathbb{R} \times X \to X$ satisfying the composition law in Equation 4. A forward-time semiflow uses $\mathbb{R}_{\geq 0}$ in place of $\mathbb{R}$ and retains the same composition law whenever $s, t \geq 0$.

$$\Phi(0, x) = x, \qquad \Phi(t + s, x) = \Phi(t, \Phi(s, x)) \,, \tag{4}$$

For a fixed initial state x_0, the corresponding trajectory is the map $t \mapsto \Phi(t, x_0)$. In the familiar differentiable setting, a sufficiently regular vector field f on a manifold generates local trajectories satisfying $\dot{x} = f(x)$ whenever the relevant existence and uniqueness hypotheses hold [7].

The present paper uses these classical constructions only as background. The earlier GR metamodel deliberately avoids assuming that every domain supplies one common manifold, metric, vector space, or differential equation. The paper therefore retains the abstract distinction between a generative specification and a realized history while allowing particular instantiations to recover standard discrete maps, flows, semiflows, or other evolution structures.

2.2 Four GR Meta-Primitives

The braid-theoretic construction in this paper inherits its highest-level vocabulary from *A Metamodel for Ontologically Heterogeneous Social Dynamics, Part I: The Highest Abstract Layer* [6]. That paper introduces a bare carrier S, a semigroup M of mechanisms, an action $\alpha : M \times S \to S$, and an equivalence relation $\sim$ on S. No order, metric, arithmetic, zero element, manifold structure, or vector-space structure is assumed on the carrier. The four primitives are change, structure, constraint, and difference. Their compact formal recap is given in Equation 5.

$$\begin{aligned}
&\text{Change:} && \alpha : M \times S \to S, && (m, x) \mapsto \alpha(m, x), \\
&\text{Structure:} && \alpha(m_1 m_2, x) = \alpha(m_1, \alpha(m_2, x)) \,, && \\
&\text{Constraint:} && x \sim y \implies \alpha(m, x) \sim \alpha(m, y), && \\
&\text{Difference:} && \Delta(m) := \{(x, y) \in S \times S : x \not\sim y \text{ and } \alpha(m, x) \sim \alpha(m, y)\}. &&
\end{aligned} \tag{5}$$

Equation 5 should be read from the top downward. Change states that a mechanism acts on a carrier element and returns a carrier element. Structure states that mechanisms compose associatively and that

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Table 1: The four GR meta-primitives with their roles in the present paper.

Primitive	Highest-layer role	Lower-level braid question
Constraint	Specifies what admissible mechanisms are required to preserve	Which realized histories and deformations are braid-admissible?
Change	Represents a mechanism acting on a carrier element	Which features of a realized trajectory are represented as elementary braid changes?
Structure	Represents composition of mechanisms and compatibility of the action	How do trajectory descriptions compose, and which braid relations identify compositions?
Difference	Records distinctions collapsed by a mechanism	Which distinctions among full histories are preserved or quotiented out by the descriptor?

their product acts consistently with sequential application. Constraint requires every admissible mechanism to preserve the equivalence relation $\sim$. Difference then records the pairs that were distinguishable before the mechanism acted and become equivalent afterward. The earlier paper emphasizes that Δ is derived from α and $\sim$ rather than introduced as an independent numerical residue [6].

Table 1 summarizes the analytical role of the four primitives and anticipates the lower-level questions that the braid formalism must answer.

The distinction between levels is important. Equation 5 concerns the generative metamodel. Later sections construct a descriptive formalism downstream from realized histories. The same four words will therefore reappear with explicitly derived descriptive meanings rather than being silently transferred from mechanisms to braids.

2.3 Topological Equivalence of Paths

Topology provides a language for properties preserved under continuous deformation. Let Y be a topological space. A path in Y is a continuous map $q : [0, 1] \to Y$. Two paths q_0 and q_1 with the same endpoints are homotopic relative to their endpoints when there exists a continuous map $H : [0, 1] \times [0, 1] \to Y$ satisfying the conditions in Equation 6.

$$\begin{aligned} H(s, 0) &= q_0(s), & H(s, 1) &= q_1(s), \\ H(0, u) &= q_0(0) = q_1(0), & H(1, u) &= q_0(1) = q_1(1), \end{aligned} \tag{6}$$

for all $s, u \in [0, 1]$. Equation 6 formalizes a continuous deformation from one path to the other while holding the endpoints fixed. Homotopy relative to endpoints is an equivalence relation on such paths [5].

A loop based at $y_0 \in Y$ is a path q satisfying $q(0) = q(1) = y_0$. Homotopy classes of based loops can be composed by path concatenation. The resulting group is the fundamental group $\pi_1(Y, y_0)$ [5]. The fundamental group is therefore already a two-level construction: geometric paths provide representatives, while homotopy identifies continuously deformable representatives as one algebraic element.

A topological invariant is used here in the modest sense of a quantity or algebraic object whose value is constant on the declared topological equivalence classes. If $\mathcal{Q}$ is a family of paths equipped with an equivalence relation $\simeq$ and $I : \mathcal{Q} \to A$ is a proposed invariant, the invariance condition is stated in Equation 7.

$$q_0 \simeq q_1 \Longrightarrow I(q_0) = I(q_1). \tag{7}$$

Equation 7 does not require A to be numerical. The codomain can be a set, group, polynomial ring, conjugacy class, or another structure suited to the invariant under study. The gain in robustness comes from the equivalence relation: geometric perturbations that remain inside one equivalence class leave

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the invariant unchanged. The corresponding loss of resolution is equally important, because distinctions erased by the equivalence relation cannot be recovered from the invariant alone.

2.4 Configuration-Space Foundations of Braid Groups

Braid groups arise naturally when several persistent points move without collision. Let X be a topological space and let $n \geq 2$. The ordered configuration space of n pairwise distinct points of X is defined in Equation 8.

$$\mathrm{Conf}_n(X) := \{(z_1, \ldots, z_n) \in X^n : z_i \neq z_j \text{ whenever } i \neq j\}. \tag{8}$$

The collision exclusions in Equation 8 remove the diagonals on which two component positions coincide. Configuration spaces of this form are standard objects in the topology underlying braid theory [4, 2].

Let S_n denote the symmetric group on n labels. It acts on $\mathrm{Conf}_n(X)$ by permuting coordinates. The unordered configuration space is the quotient defined in Equation 9.

$$\mathrm{UConf}_n(X) := \mathrm{Conf}_n(X)/S_n. \tag{9}$$

The quotient in Equation 9 records a configuration of n distinct points without fixing which coordinate name is attached to which point in the terminal configuration.

For the plane, the ordered and unordered configuration spaces recover the pure and full braid groups through their fundamental groups. For a chosen base configuration $\mathbf{z}^0 \in \mathrm{Conf}_n(\mathbb{R}^2)$, the standard identifications are stated in Equation 10.

$$P_n \cong \pi_1\big(\mathrm{Conf}_n(\mathbb{R}^2), \mathbf{z}^0\big), \qquad B_n \cong \pi_1\big(\mathrm{UConf}_n(\mathbb{R}^2), [\mathbf{z}^0]\big), \tag{10}$$

where P_n denotes the pure braid group and B_n the full braid group [2, 4]. The distinction is useful for trajectory modeling. A pure braid returns each labeled component to its own terminal label, whereas a full braid permits a permutation of terminal labels while retaining the history of how the strands were intertwined.

The full braid group also has an algebraic presentation. Let σ_i denote the elementary crossing that exchanges the ith and $(i+1)$st adjacent strands according to a fixed positive-crossing convention. Its inverse σ_i^{-1} denotes the opposite crossing. Artin's presentation is stated in Equation 11.

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq n-2 \end{array} \right\rangle. \tag{11}$$

Equation 11 states two kinds of structural equivalence. Crossings involving sufficiently separated pairs commute, while the three-strand braid relation identifies two local crossing sequences that represent the same braid. These relations provide the algebraic form of the topological equivalence classes described geometrically by the configuration-space construction [1, 2].

2.5 Algebraic Encoding of Topological Trajectories

The present paper uses braid theory precisely because it connects a trajectory picture to an algebraic object. A geometric realization consists of strands traced through an admissible configuration space. Homotopy or isotopy identifies realizations that differ by allowed continuous deformation. The resulting equivalence class can then be represented algebraically by an element of B_n and composed by the group operation. The relation is summarized in Equation 12.

$$\text{trajectory representative} \longmapsto \text{topological equivalence class} \longmapsto \text{braid-group element}. \tag{12}$$

Equation 12 is descriptive rather than generative. It does not specify which trajectory will occur from a given state, which crossing is dynamically selected next, or which law governs the underlying system. Those questions belong to the generative layer. Braid theory supplies a structured description of a realized trajectory once a braid-admissible extraction has been made.

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This distinction also explains why topological robustness and descriptive compression arise together. Passing from a detailed trajectory to a braid class removes metric and other non-topological information, but it preserves the distinctions encoded by the braid class under admissible deformation. Later sections will formalize this trade-off through descriptive equivalence, descriptive resolution, and the generality profile of the braid formalism.

3 Descriptive Layer of Dynamical Representation

This section constructs the descriptive layer used by the remainder of the paper. Its role is to derive, from the earlier mechanism-based metamodel, a space of realized histories and then a general form for descriptions defined on those histories. The objective is to keep the generative and descriptive levels formally distinct while making their interface explicit. The derivation proceeds in five stages: Section 3.1 constructs finite realized histories directly from the semigroup action; Section 3.2 defines a trajectory-descriptive map and the equivalence relation that it induces; Section 3.3 derives compositionality at the history level; Section 3.4 formalizes descriptive resolution and coarsening; and Section 3.5 separates underdetermination introduced by generation from underdetermination introduced by description. No topology, metric, manifold, or braid structure is assumed in this section.

3.1 Histories Generated by the Mechanism Action

The starting point is the earlier GR metamodel. For compactness, the present paper writes the data as $\mathfrak{D} = (S, M, \alpha, \sim)$, where S is the bare carrier, M is the semigroup of mechanisms, $\alpha : M \times S \to S$ is the action, and $\sim$ is the congruence preserved by the action [6]. The symbol $\mathfrak{D}$ packages data already fixed at the higher layer and introduces no additional structure on S.

A generated history begins with an initial state together with an ordered finite sequence of mechanisms. Let k be a positive integer and let M^k denote the Cartesian product of k copies of M. An element $w = (m_1, \ldots, m_k) \in M^k$ will be called a *mechanism word of length* k. The word records order and repetition before any trajectory description is introduced.

Because the metamodel does not assume an identity element in M, cumulative mechanisms are defined without an empty product. For a mechanism word $w = (m_1, \ldots, m_k)$, define the prefix products recursively by Equation 13.

$$p_1(w) := m_1, \qquad p_j(w) := m_j\, p_{j-1}(w) \quad \text{for } 2 \le j \le k. \tag{13}$$

Associativity in M makes each $p_j(w)$ independent of parenthesization. The chosen order matches the action convention of the earlier metamodel: $\alpha(m_1 m_2, x) = \alpha(m_1, \alpha(m_2, x))$. Accordingly, $p_j(w) = m_j \cdots m_1$ represents the action of m_1 first and m_j last.

For an initial state $x_0 \in S$, define the states generated by w through Equation 14.

$$x_j(x_0; w) := \alpha(p_j(w), x_0) \quad \text{for } 1 \le j \le k. \tag{14}$$

Equation 14 is equivalent to the stepwise recursion $x_j = \alpha(m_j, x_{j-1})$. For $j = 2$, Equation 13 gives $p_2 = m_2 m_1$, and the action law yields $\alpha(p_2, x_0) = \alpha(m_2, \alpha(m_1, x_0))$. If $x_{j-1} = \alpha(p_{j-1}(w), x_0)$, then Equation 13 and the action law give $\alpha(p_j(w), x_0) = \alpha(m_j, x_{j-1})$. The recursion therefore follows by induction from the semigroup action alone.

The generative specification (x_0, w) and the history generated by that specification are kept distinct. The mechanism word belongs to the generative side of the construction. The realized history records the ordered states that result. This separation is stated in Definition 3.1.

Definition 3.1 (Finite realized history)**.** For $x_0 \in S$ and $w = (m_1, \ldots, m_k) \in M^k$, the *finite realized history generated by* (x_0, w) is the state sequence defined by Equation 15.

$$h(x_0; w) := (x_0, x_1(x_0; w), \ldots, x_k(x_0; w)). \tag{15}$$

Definition 3.1 deliberately omits the mechanism labels from the realized-history object. Two different mechanism words can therefore generate the same realized history. This is the formal expression of the distinction introduced in Section 1.1: a generated trajectory need not uniquely reveal the rule or mechanism sequence that generated it.

The set of all length-k realized histories generated by $\mathfrak{D}$ is defined in Equation 16.

$$\Gamma_k(\mathfrak{D}) := \left\{ (x_0, \ldots, x_k) \in S^{k+1} : \begin{array}{l} \text{there exists } (m_1, \ldots, m_k) \in M^k \\ \text{such that } x_j = \alpha(m_j, x_{j-1}) \text{ for } 1 \le j \le k \end{array} \right\}. \tag{16}$$

The finite-history collection used in the present paper is then the disjoint union in Equation 17.

$$\Gamma(\mathfrak{D}) := \coprod_{k \ge 1} \Gamma_k(\mathfrak{D}). \tag{17}$$

The disjoint union in Equation 17 retains the length of a history as part of its formal identity and avoids accidental identification of sequences belonging to different lengths.

The mechanism action determines a realization map from generative specifications to histories. For each $k \ge 1$, define the map in Equation 18.

$$\mathrm{Gen}_k : S \times M^k \longrightarrow \Gamma_k(\mathfrak{D}), \qquad \mathrm{Gen}_k(x_0, w) := h(x_0; w). \tag{18}$$

Equation 16 makes Gen_k surjective by construction. Injectivity is not required. If two different mechanism words act identically along the realized sequence from a given initial state, then the corresponding distinct inputs of Equation 18 have the same image. The resulting generator underdetermination is analyzed explicitly in Section 3.5.

3.2 Induced Equivalence of Trajectory-Descriptive Maps

A descriptive formalism need not apply to every history generated by $\mathfrak{D}$. The domain of a particular descriptor is therefore represented explicitly. Let $\Gamma_D \subseteq \Gamma(\mathfrak{D})$ be the histories for which a proposed descriptor is defined, and let Ω_D be the set of descriptive objects produced by that formalism. Neither Γ_D nor Ω_D is assumed to possess additional structure at this stage.

Definition 3.2 (Trajectory-descriptive map)**.** A *trajectory-descriptive map* on $\mathfrak{D}$ is a function defined by Equation 19.

$$D : \Gamma_D \longrightarrow \Omega_D. \tag{19}$$

The adjective *trajectory-descriptive* indicates the logical position of D: its immediate input is a realized history selected from $\Gamma(\mathfrak{D})$. Definition 3.2 does not require D to recover or encode the generative specification that produced that history.

Every map of the form in Equation 19 determines an equivalence relation on its domain. Define the relation $\equiv_D$ through Equation 20.

$$h_1 \equiv_D h_2 \quad \Longleftrightarrow \quad D(h_1) = D(h_2), \qquad h_1, h_2 \in \Gamma_D. \tag{20}$$

The symbol $\equiv_D$ is derived from D rather than introduced as an independent primitive. Equality in Ω_D is reflexive, symmetric, and transitive; Equation 20 therefore makes $\equiv_D$ an equivalence relation on Γ_D.

The equivalence relation in Equation 20 defines a quotient set $\Gamma_D/\equiv_D$. Let $\pi_D : \Gamma_D \to \Gamma_D/\equiv_D$ denote the canonical projection that sends each history to its equivalence class. The descriptor factors through this quotient as stated in Proposition 3.3.

Proposition 3.3 (Canonical factorization of a descriptor)**.** *For every trajectory-descriptive map $D : \Gamma_D \to \Omega_D$, there is a unique bijection $\overline{D} : \Gamma_D/\equiv_D \to \mathrm{im}(D)$ satisfying Equation 21.*

$$D = \iota_D \circ \overline{D} \circ \pi_D, \tag{21}$$

where $\mathrm{im}(D) := \{D(h) : h \in \Gamma_D\}$ and $\iota_D : \mathrm{im}(D) \hookrightarrow \Omega_D$ is the inclusion map.

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Proof. Define $\overline{D}([h]_D) := D(h)$, where $[h]_D$ denotes the $\equiv_D$-equivalence class of h. If $[h_1]_D = [h_2]_D$, then $h_1 \equiv_D h_2$, and Equation 20 gives $D(h_1) = D(h_2)$. Hence $\overline{D}$ is well defined. If $\overline{D}([h_1]_D) = \overline{D}([h_2]_D)$, then $D(h_1) = D(h_2)$; Equation 20 gives $[h_1]_D = [h_2]_D$, so $\overline{D}$ is injective. Every element of $\mathrm{im}(D)$ equals $D(h)$ for some $h \in \Gamma_D$, and therefore equals $\overline{D}([h]_D)$; hence $\overline{D}$ is surjective. The definition gives Equation 21. Uniqueness follows from surjectivity of π_D: any map satisfying Equation 21 must send $[h]_D$ to $D(h)$. □

Proposition 3.3 formalizes the quotient character of description without presupposing topology. A descriptor partitions its admissible histories into classes that it cannot distinguish. The braid construction developed later will add a particular topological meaning to those classes.

3.3 Compositional Structure across Histories and Descriptors

The ordered state histories in $\Gamma(\mathfrak{D})$ have a natural partial composition. For a history $h = (x_0, \ldots, x_k) \in \Gamma_k(\mathfrak{D})$, define its initial state by $s(h) := x_0$ and its terminal state by $t(h) := x_k$. Two histories h_1 and h_2 are called *composable* when $t(h_1) = s(h_2)$.

Suppose $h_1 = (x_0, \ldots, x_k)$ and $h_2 = (x_k, y_1, \ldots, y_\ell)$ are composable. Their concatenation is defined by Equation 22.

$$h_2 \star h_1 := (x_0, \ldots, x_k, y_1, \ldots, y_\ell). \tag{22}$$

The symbol $\star$ therefore denotes concatenation of realized histories, with the right-hand history h_1 occurring first.

Equation 22 produces another generated history. To see this, choose a mechanism word $(m_1, \ldots, m_k)$ witnessing $h_1 \in \Gamma_k(\mathfrak{D})$ and a mechanism word $(n_1, \ldots, n_\ell)$ witnessing $h_2 \in \Gamma_\ell(\mathfrak{D})$. The concatenated mechanism word $(m_1, \ldots, m_k, n_1, \ldots, n_\ell)$ generates exactly the state sequence in Equation 22. Hence $h_2 \star h_1 \in \Gamma_{k+\ell}(\mathfrak{D})$. Associativity of $\star$ follows from associativity of finite sequence concatenation whenever all required endpoint equalities hold.

A descriptive codomain may also carry a composition law. Let $\odot_D$ denote a binary operation on those pairs of elements of Ω_D that correspond to composable histories. The operation is not assumed to be globally defined or invertible.

Definition 3.4 (Compositional descriptor)**.** A trajectory-descriptive map $D : \Gamma_D \to \Omega_D$ is *compositional* when, for every pair $h_1, h_2 \in \Gamma_D$ such that $h_2 \star h_1 \in \Gamma_D$, the compatibility condition in Equation 23 holds.

$$D(h_2 \star h_1) = D(h_2) \odot_D D(h_1). \tag{23}$$

Definition 3.4 places a substantive requirement on a descriptive language: composition of realized histories is represented by composition of their descriptions. It does not require that the descriptor preserve every other feature of the histories.

The induced equivalence of a compositional descriptor is compatible with history concatenation in the following sense.

Proposition 3.5 (Compatibility of descriptive equivalence with composition)**.** *Let D be compositional. Suppose $h_1 \equiv_D h_1'$ and $h_2 \equiv_D h_2'$, and suppose both $h_2 \star h_1$ and $h_2' \star h_1'$ belong to Γ_D. Then the conclusion in Equation 24 holds.*

$$h_2 \star h_1 \equiv_D h_2' \star h_1'. \tag{24}$$

Proof. The assumptions $h_1 \equiv_D h_1'$ and $h_2 \equiv_D h_2'$ give $D(h_1) = D(h_1')$ and $D(h_2) = D(h_2')$ by Equation 20. Applying Equation 23 to both concatenations yields the same value $D(h_2) \odot_D D(h_1)$. Equation 20 then gives Equation 24. □

Proposition 3.5 is the general compositional fact that the later braid relations will instantiate more specifically. The proposition does not yet impose the Artin relations or any topology on the history space.

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3.4 Factorization of Descriptive Resolution

The quotient in Proposition 3.3 permits descriptive resolution to be compared without assigning a metric to histories or descriptions. Consider two descriptors $D_1 : \Gamma_* \to \Omega_1$ and $D_2 : \Gamma_* \to \Omega_2$ defined on a common history domain Γ_*. Let $\equiv_{D_1}$ and $\equiv_{D_2}$ be the equivalence relations induced by Equation 20.

Definition 3.6 (Relative descriptive resolution)**.** The descriptor D_1 is *at least as fine as* D_2, written $D_1 \preceq D_2$, when the relation in Equation 25 holds.

$$\equiv_{D_1} \subseteq \equiv_{D_2} . \tag{25}$$

Thus Equation 25 means that every pair identified by D_1 is also identified by D_2. The descriptor D_2 may collapse additional distinctions and is therefore weakly coarser.

The symbol $\preceq$ in Definition 3.6 is local to this paper and refers only to relative descriptive resolution. It is not an order on the carrier S and therefore does not introduce the commensurability assumptions excluded at the highest metamodel layer.

The relation in Equation 25 has a factorization characterization.

Proposition 3.7 (Factorization criterion for descriptive coarsening)**.** *For descriptors D_1 and D_2 with common domain Γ_*, the condition $D_1 \preceq D_2$ holds if and only if there exists a unique map $\varphi_{21} : \mathrm{im}(D_1) \to \mathrm{im}(D_2)$ satisfying Equation 26.*

$$D_2 = \iota_2 \circ \varphi_{21} \circ D_1^{\mathrm{im}}, \tag{26}$$

where $D_1^{\mathrm{im}} : \Gamma_* \to \mathrm{im}(D_1)$ is D_1 with its codomain restricted to its image and $\iota_2 : \mathrm{im}(D_2) \hookrightarrow \Omega_2$ is inclusion.

Proof. Assume first that $D_1 \preceq D_2$. For $z \in \mathrm{im}(D_1)$, choose $h \in \Gamma_*$ with $D_1(h) = z$ and define $\varphi_{21}(z) := D_2(h)$. If h' is another history with $D_1(h') = z$, then $h \equiv_{D_1} h'$. Equation 25 gives $h \equiv_{D_2} h'$, so $D_2(h) = D_2(h')$. The definition of φ_{21} is therefore independent of the chosen representative and hence well defined. Substitution gives Equation 26. Uniqueness follows because every $z \in \mathrm{im}(D_1)$ has the form $D_1(h)$.

Conversely, suppose a map φ_{21} satisfying Equation 26 exists. If $h \equiv_{D_1} h'$, then $D_1(h) = D_1(h')$. Applying φ_{21} to the common image and using Equation 26 gives $D_2(h) = D_2(h')$. Hence $h \equiv_{D_2} h'$, proving Equation 25. □

Proposition 3.7 will later allow braid invariants to be represented as a further coarsening of braid classes. If D_B is the braid descriptor and I is an invariant, then $I \circ D_B$ is formally downstream from D_B because its value is determined entirely by the braid description.

3.5 Generator Underdetermination at the Descriptive Layer

The distinction between generation and description can now be stated without relying on the informal grammar analogy. For each fixed length k, Equation 18 maps generative specifications to histories, while Equation 19 maps selected histories to descriptions. Define the subset $U_{k,D} \subseteq S \times M^k$ to consist of precisely those generative specifications u for which $\mathrm{Gen}_k(u) \in \Gamma_D$. The observable descriptive composite is the map in Equation 27.

$$D \circ \mathrm{Gen}_k : U_{k,D} \longrightarrow \Omega_D. \tag{27}$$

Equation 27 makes explicit that a descriptive object is downstream from both a generative specification and a realized history.

Two distinct sources of underdetermination can now be separated. They are stated in Equation 28.

$$\begin{aligned} u_1 \neq u_2 \text{ and } \mathrm{Gen}_k(u_1) = \mathrm{Gen}_k(u_2) \quad &\text{(generative underdetermination)}, \\ h_1 \neq h_2 \text{ and } D(h_1) = D(h_2) \quad &\text{(descriptive underdetermination)}. \end{aligned} \tag{28}$$

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The first line of Equation 28 concerns the map from mechanism-bearing generative specifications to realized histories. The second concerns the map from realized histories to descriptions. Neither form of underdetermination implies the other.

The two layers can also be derived at the level of fibers. For a history $h \in \Gamma_k(\mathfrak{D})$, define its *generation fiber* by $\mathcal{G}_k(h) := \mathrm{Gen}_k^{-1}(\{h\})$. For a descriptive object $\omega \in \mathrm{im}(D)$, define its *description fiber* by $\mathcal{F}_D(\omega) := D^{-1}(\{\omega\})$. These sets contain, respectively, the generative specifications compatible with one history and the histories compatible with one description.

For ω that lies in the image of the composite map in Equation 27, the complete set of compatible generative specifications is derived in Equation 29.

$$(D \circ \mathrm{Gen}_k)^{-1}(\{\omega\}) = \bigcup_{h \in \mathcal{F}_D(\omega) \cap \Gamma_k(\mathfrak{D})} \mathcal{G}_k(h). \tag{29}$$

To verify Equation 29, take u in the left-hand side. Then $D(\mathrm{Gen}_k(u)) = \omega$. Setting $h := \mathrm{Gen}_k(u)$ gives $h \in \mathcal{F}_D(\omega) \cap \Gamma_k(\mathfrak{D})$ and $u \in \mathcal{G}_k(h)$, so u belongs to the right-hand side. Conversely, if u belongs to a set $\mathcal{G}_k(h)$ appearing in the union, then $\mathrm{Gen}_k(u) = h$ and $D(h) = \omega$, so u belongs to the left-hand side.

Equation 29 gives a precise decomposition of backward ambiguity. A description can be compatible with several histories because the descriptor is noninjective, and each of those histories can itself be compatible with several generative specifications because the generation map is noninjective. Recovering a unique generator from a descriptive object therefore requires assumptions strong enough to collapse both kinds of fiber to singletons.

Remark 3.8 (Scope of the descriptive layer)**.** The construction in this section does not yet assert that any particular descriptor is topological, robust, complete, or appropriate to a substantive domain. It establishes the formal location at which those properties can be evaluated. The braid-theoretic descriptor introduced later will be one specific choice of (Γ_D, Ω_D, D), together with additional structure on its admissible histories and descriptive codomain.

Remark 3.8 fixes the scope of this bridge: no topology, robustness, or domain adequacy is asserted at this stage. The section therefore supplies the bridge required for the remainder of the manuscript. The earlier meta-language determines how mechanisms act and compose; realized histories in $\Gamma(\mathfrak{D})$ record particular generated state sequences; a trajectory-descriptive map D assigns a lower-resolution structural object to selected histories. The next section returns to the four GR meta-primitives and states how constraint, change, structure, and difference are to be interpreted when a lower-level trajectory descriptor is derived from that hierarchy.

4 Braid-Theoretic Instantiation of the GR Meta-Primitives

This section instantiates the four GR meta-primitives at the trajectory-descriptive level. Its role is to derive a braid-theoretic descriptor from the general descriptive architecture of Section 3 while keeping the mechanism layer formally separate. The construction proceeds in five stages. Section 4.1 defines the additional data required to extract a braid-admissible trajectory from a realized history. Section 4.2 derives the braid descriptor as a composition of trajectory extraction, homotopy quotient, and algebraic identification. Sections 4.3 and 4.4 then derive the descriptive meanings of change, structure, and constraint. Section 4.5 derives the corresponding notion of descriptive difference and separates it from the mechanism-level collapse operator of the earlier metamodel. The derivation uses ordinary Artin braid groups as the baseline case; later sections examine the expressive restrictions created by that choice.

4.1 Braid-Admissible Extraction Data

The abstract history space $\Gamma(\mathfrak{D})$ of Section 3.1 contains only generated state sequences. A braid description requires additional lower-level structure because braid theory is defined on trajectories in a configuration space. The additional commitments are therefore introduced explicitly rather than attributed to the highest-layer carrier S.

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Fix an integer $n \geq 2$. Let $\mathcal{U}_n := \mathrm{UConf}_n(\mathbb{R}^2)$ denote the unordered configuration space introduced in Equation 9, and choose a base configuration $c_0 \in \mathcal{U}_n$. The based loop space used in the present construction is defined in Equation 30.

$$\Omega(\mathcal{U}_n, c_0) := \{q : [0,1] \to \mathcal{U}_n : q \text{ is continuous and } q(0) = q(1) = c_0\}. \tag{30}$$

The symbol $\Omega(\mathcal{U}_n, c_0)$ in Equation 30 denotes a set of based loops. No topology on the loop space itself is required for the arguments in this section.

A realized history need not admit any canonical element of $\Omega(\mathcal{U}_n, c_0)$. The construction therefore specifies both the admissible history domain and the rule used to extract a loop.

Definition 4.1 (Braid-admissible extraction datum)**.** A *braid-admissible extraction datum* for $\mathfrak{D}$ is the tuple $\mathcal{E}_B := (n, c_0, \Gamma_B, \mathcal{T}_B)$, where:

1. $n \geq 2$ is the number of persistent components represented by the descriptor;
2. $c_0 \in \mathcal{U}_n$ is the selected base configuration;
3. $\Gamma_B \subseteq \Gamma(\mathfrak{D})$ is the set of realized histories declared braid-admissible;
4. $\mathcal{T}_B$ is the extraction map defined in Equation 31.

$$\mathcal{T}_B : \Gamma_B \longrightarrow \Omega(\mathcal{U}_n, c_0). \tag{31}$$

For $h \in \Gamma_B$, the loop $\mathcal{T}_B(h)$ is the relational trajectory selected from the full realized history by the chosen instantiation.

Definition 4.1 separates three commitments that can otherwise be conflated. The history h is generated by the mechanism layer. Membership in Γ_B states that the history satisfies the chosen descriptive admissibility conditions. The map $\mathcal{T}_B$ states which persistent relational components are traced and how those traces are represented in $\mathcal{U}_n$. In particular, Definition 4.1 does not require an embedding of the full carrier S into $\mathbb{R}^{2n}$ or into any common manifold. Only the selected trajectory extracted from a history is required to take values in $\mathcal{U}_n$.

The use of $\mathcal{U}_n$ makes the collision constraint explicit. Every value of a loop $q \in \Omega(\mathcal{U}_n, c_0)$ is an unordered configuration of n pairwise distinct points, so no two represented components coincide along the extracted trajectory. The fixed value of n records persistence of the represented components throughout the descriptive interval. The base-loop condition fixes a common endpoint convention suitable for the group-valued descriptor used below. Histories that require changing component number, branching, fusion, or incompatible endpoint conventions are not represented by Definition 4.1; these exclusions will be analyzed as domain restrictions rather than treated as failures of the underlying GR metamodel.

4.2 Derivation of the Braid Descriptor

The braid descriptor is derived in two quotient-preserving steps after trajectory extraction. First, loops are identified under homotopy relative to the basepoint. Let $Q_B : \Omega(\mathcal{U}_n, c_0) \to \pi_1(\mathcal{U}_n, c_0)$ denote the canonical quotient map that sends a loop to its based homotopy class. The map Q_B is stated explicitly in Equation 32.

$$Q_B(q) := [q]_\partial, \tag{32}$$

where $[q]_\partial$ denotes the homotopy class of q relative to its endpoints. Because every loop in Equation 30 has both endpoints equal to c_0, endpoint-preserving homotopy is equivalent here to based-loop homotopy.

For the planar configuration space, Equation 10 gives an isomorphism between $\pi_1(\mathcal{U}_n, c_0)$ and the Artin braid group B_n [4, 2]. Fix one such isomorphism and denote it by $\kappa_n : \pi_1(\mathcal{U}_n, c_0) \to B_n$. The choice of κ_n identifies the topological homotopy class with the conventional algebraic braid-group representation; it introduces no further quotient because κ_n is bijective.

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Definition 4.2 (Braid-theoretic trajectory descriptor)**.** Given a braid-admissible extraction datum $\mathcal{E}_B$, the *braid-theoretic trajectory descriptor* is the composite map defined in Equation 33.

$$D_B := \kappa_n \circ Q_B \circ \mathcal{T}_B : \Gamma_B \longrightarrow B_n. \tag{33}$$

Definition 4.2 and Equation 33 give the full derivation of the central descriptor from objects already defined. A realized history h is first mapped to the selected relational loop $\mathcal{T}_B(h)$. The quotient map Q_B then removes distinctions among endpoint-preserving homotopic representatives. The isomorphism κ_n finally encodes the resulting class as an element of B_n. The information-reducing step is therefore Q_B together with any prior selectivity in $\mathcal{T}_B$; the final map κ_n is only an algebraic relabeling of the same class.

The equivalence relation induced by D_B follows from Equation 20. It is convenient to name it explicitly.

Definition 4.3 (Braid-descriptive equivalence)**.** For $h_1, h_2 \in \Gamma_B$, define the relation $\equiv_B$ by Equation 34.

$$h_1 \equiv_B h_2 \quad \Longleftrightarrow \quad D_B(h_1) = D_B(h_2). \tag{34}$$

The induced equivalence in Definition 4.3 has an exact topological characterization.

Proposition 4.4 (Topological characterization of braid-descriptive equivalence)**.** *For $h_1, h_2 \in \Gamma_B$, the equivalence in Equation 34 holds if and only if the extracted loops are based-homotopic. The equivalence is stated in Equation 35.*

$$h_1 \equiv_B h_2 \quad \Longleftrightarrow \quad \mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2). \tag{35}$$

Proof. Assume first that $h_1 \equiv_B h_2$. By Definition 4.3, $D_B(h_1) = D_B(h_2)$. Equation 33 therefore gives Equation 36.

$$\kappa_n(Q_B(\mathcal{T}_B(h_1))) = \kappa_n(Q_B(\mathcal{T}_B(h_2)))\,. \tag{36}$$

The map κ_n is injective because it is an isomorphism, so $Q_B(\mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2))$. By Equation 32, equality of these classes means $\mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2)$.

Conversely, if $\mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2)$, then Equation 32 gives $Q_B(\mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2))$. Applying κ_n to both sides and using Equation 33 yields $D_B(h_1) = D_B(h_2)$, which is exactly $h_1 \equiv_B h_2$. □

Proposition 4.4 identifies the descriptive resolution of the baseline braid construction: two admissible histories are indistinguishable precisely when the trajectories selected by $\mathcal{T}_B$ lie in the same based-homotopy class. Differences in the full histories that do not alter this class are absent from the braid descriptor.

4.3 Compositional Structure of Braid-Descriptive Change

The change primitive at the descriptive level concerns the structure assigned to an extracted trajectory after the trajectory has occurred. The Artin presentation in Equation 11 provides elementary algebraic symbols for local crossing descriptions. Define the braid alphabet by Equation 37.

$$\Sigma_n := \{\sigma_i^\varepsilon : 1 \le i \le n-1,\ \varepsilon \in \{-1, +1\}\}\,. \tag{37}$$

An element of Σ_n is called an *elementary descriptive crossing*. The exponent ε records the chosen crossing orientation. Equation 37 is an alphabet for describing local features of a braid trajectory; it is not a set of mechanisms acting on the GR carrier S.

Let Σ_n^* denote the free monoid of finite words over the alphabet Σ_n, including the empty word **1**. A word has the form $w = a_r \cdots a_1$ with $a_j \in \Sigma_n$. The order convention is again right-to-left: a_1 records the first elementary crossing in the represented sequence.

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The algebraic braid structure is obtained by identifying words according to the relations that define B_n. Let $\approx_A$ denote the smallest monoid congruence on Σ_n^* containing the relations stated in Equation 38.

$$\begin{aligned} \sigma_i\sigma_j &\approx_A \sigma_j\sigma_i && \text{for } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i &\approx_A \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } 1 \leq i \leq n-2, \\ \sigma_i\sigma_i^{-1} &\approx_A \mathbf{1}, \qquad \sigma_i^{-1}\sigma_i \approx_A \mathbf{1} && \text{for } 1 \leq i \leq n-1. \end{aligned} \tag{38}$$

A monoid congruence means an equivalence relation compatible with concatenation on both sides. The phrase "smallest monoid congruence" in Equation 38 therefore means that two words are identified only when their equivalence follows from repeated use of the displayed relations together with reflexivity, symmetry, transitivity, and compatibility with word concatenation.

Artin's presentation implies the quotient identification stated in Equation 39.

$$B_n \cong \Sigma_n^*/\approx_A. \tag{39}$$

Equation 39 supplies the descriptive realization of the structure primitive: local crossing symbols compose by word concatenation, while the Artin and inverse relations determine which composed descriptions represent the same braid [1, 2].

Let $\mathrm{ev}_B : \Sigma_n^* \to B_n$ denote the canonical evaluation map that sends each word to its equivalence class under $\approx_A$. The evaluation map is surjective by Equation 39. Hence every braid descriptor $D_B(h)$ has at least one word representative $w \in \Sigma_n^*$ satisfying Equation 40.

$$\mathrm{ev}_B(w) = D_B(h). \tag{40}$$

The word w in Equation 40 is generally nonunique. The nonuniqueness is not an ambiguity in the braid element; it is the algebraic expression of the equivalences encoded by Equation 38.

The connection between history composition and braid multiplication requires one further compatibility condition on the extraction map. For based loops $q_1, q_2 \in \Omega(\mathcal{U}_n, c_0)$, define their sequential concatenation $q_2 * q_1$ by Equation 41.

$$(q_2 * q_1)(t) := \begin{cases} q_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ q_2(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{41}$$

The right-hand factor q_1 in Equation 41 is traversed first, matching the history convention of Equation 22.

Definition 4.5 (Composition-compatible braid extraction)**.** A braid extraction map $\mathcal{T}_B$ is *composition-compatible* when, for every pair of composable histories $h_1, h_2 \in \Gamma_B$ with $h_2 \star h_1 \in \Gamma_B$, the relation in Equation 42 holds.

$$\mathcal{T}_B(h_2 \star h_1) \simeq_\partial \mathcal{T}_B(h_2) * \mathcal{T}_B(h_1). \tag{42}$$

The homotopy in Equation 42 permits harmless reparameterization or other admissible deformation introduced by the extraction procedure. Exact equality of parameterized loops is therefore unnecessary.

Proposition 4.6 (Compositionality of the braid descriptor)**.** *If $\mathcal{T}_B$ is composition-compatible, then the braid descriptor is compositional. For every admissible composable pair, Equation 43 holds.*

$$D_B(h_2 \star h_1) = D_B(h_2)\, D_B(h_1). \tag{43}$$

Proof. By Equation 33 and Definition 4.5, Equation 44 holds.

$$D_B(h_2 \star h_1) = \kappa_n(Q_B(\mathcal{T}_B(h_2 \star h_1))) = \kappa_n(Q_B(\mathcal{T}_B(h_2) * \mathcal{T}_B(h_1)))\,. \tag{44}$$

The group operation in the fundamental group is induced by loop concatenation, which gives Equation 45.

$$Q_B(\mathcal{T}_B(h_2) * \mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2))\, Q_B(\mathcal{T}_B(h_1)). \tag{45}$$

Because κ_n is a group isomorphism, it preserves multiplication. Substituting Equation 33 for the two factors gives Equation 43. □

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Proposition 4.6 supplies the exact descriptive counterpart of the general compositionality condition in Equation 23. The construction also clarifies the level distinction: multiplication in B_n composes descriptions of realized trajectory segments; it does not determine which mechanism in M produces the next segment.

4.4 Constraint Realization in Braid Description

The constraint primitive enters the braid construction in two related forms. The first is *admissibility*: only histories in Γ_B and loops in $\Omega(\mathcal{U}_n, c_0)$ enter the descriptor. The second is *equivalence preservation*: distinctions already declared irrelevant at the full-history level should not be reintroduced merely by the descriptive map.

To state the second form, let $\approx_\Gamma$ be an equivalence relation on Γ_B. The relation $\approx_\Gamma$ represents a baseline history equivalence declared before braid description. Its substantive content is left to the instantiation; for example, a domain may decide that certain changes of coordinates, observational encodings, or other presentation differences do not constitute different histories at the resolution under study.

Definition 4.7 (Braid-compatible equivalence)**.** The baseline relation $\approx_\Gamma$ is *braid-compatible* when the implication in Equation 46 holds.

$$h_1 \approx_\Gamma h_2 \quad \Longrightarrow \quad \mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2). \tag{46}$$

Equation 46 is the descriptive analogue of an equivalence-preservation constraint. It does not specify which histories should be regarded as equivalent; it requires only that, once that equivalence has been declared, the braid extraction and homotopy quotient do not split an equivalence class.

The consequence for the braid descriptor is immediate.

Proposition 4.8 (Soundness with respect to baseline equivalence)**.** *If $\approx_\Gamma$ is braid-compatible in the sense of Definition 4.7, then Equation 47 holds.*

$$h_1 \approx_\Gamma h_2 \quad \Longrightarrow \quad D_B(h_1) = D_B(h_2). \tag{47}$$

Proof. Equation 46 gives $\mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2)$. Proposition 4.4 then yields $D_B(h_1) = D_B(h_2)$. □

Proposition 4.8 permits D_B to descend to the quotient by $\approx_\Gamma$. Let $\pi_\Gamma : \Gamma_B \to \Gamma_B/\!\approx_\Gamma$ denote the canonical projection. Because Equation 47 makes D_B constant on each $\approx_\Gamma$-class, there is a unique induced map $\overline{D}_B$ satisfying Equation 48.

$$D_B = \overline{D}_B \circ \pi_\Gamma, \qquad \overline{D}_B : \Gamma_B/\!\approx_\Gamma \longrightarrow B_n. \tag{48}$$

Equation 48 is derived by defining $\overline{D}_B([h]_\Gamma) := D_B(h)$. Proposition 4.8 guarantees that this definition is independent of the chosen representative.

The two forms of constraint should therefore be kept distinct. Admissibility determines the domain and allowed deformations of the braid descriptor. Equivalence preservation determines whether the descriptor respects distinctions already fixed at the baseline history level. Both are commitments of the lower-level instantiation and neither is supplied by the abstract carrier S alone.

4.5 Braid-Level Descriptive Collapse

The earlier metamodel defines mechanism-level difference by the distinctions a mechanism collapses on the carrier. The braid construction requires a different operator because D_B acts on histories rather than on carrier states. The baseline equivalence $\approx_\Gamma$ introduced in Section 4.4 provides the reference against which braid-level loss of discrimination can be stated.

Definition 4.9 (Braid-descriptive collapse)**.** Assume $\approx_\Gamma$ is braid-compatible. The *braid-descriptive collapse relation* is defined by Equation 49.

$$\Delta_B^{\text{desc}} := \{(h_1, h_2) \in \Gamma_B \times \Gamma_B : h_1 \not\approx_\Gamma h_2 \text{ and } D_B(h_1) = D_B(h_2)\}. \tag{49}$$

Definition 4.9 records distinctions retained by the baseline history equivalence and erased by the braid descriptor. It is relation-valued for the same reason as the earlier mechanism-level operator: no metric, arithmetic, or magnitude of descriptive loss is assumed.

Proposition 4.4 gives an equivalent path-level form of Definition 4.9. The derivation is stated in Equation 50.

$$\Delta_B^{\text{desc}} = \{(h_1, h_2) \in \Gamma_B \times \Gamma_B : h_1 \not\approx_\Gamma h_2 \text{ and } \mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2)\}. \tag{50}$$

Equation 50 follows because equality of braid descriptors is equivalent to based homotopy of the extracted loops.

The induced map in Equation 48 provides an exact criterion for the absence of descriptive collapse.

Proposition 4.10 (Injectivity criterion for braid-descriptive collapse)**.** *Assume $\approx_\Gamma$ is braid-compatible. Then the equivalence in Equation 51 holds.*

$$\Delta_B^{\text{desc}} = \varnothing \quad \Longleftrightarrow \quad \overline{D}_B \text{ is injective}. \tag{51}$$

Proof. Suppose first that $\Delta_B^{\text{desc}} = \varnothing$. If $\overline{D}_B([h_1]_\Gamma) = \overline{D}_B([h_2]_\Gamma)$, then Equation 48 gives $D_B(h_1) = D_B(h_2)$. If $[h_1]_\Gamma \neq [h_2]_\Gamma$, then $h_1 \not\approx_\Gamma h_2$, and Definition 4.9 would imply $(h_1, h_2) \in \Delta_B^{\text{desc}}$, contradicting emptiness. Hence $[h_1]_\Gamma = [h_2]_\Gamma$, so $\overline{D}_B$ is injective.

Conversely, suppose $\overline{D}_B$ is injective and assume for contradiction that $(h_1, h_2) \in \Delta_B^{\text{desc}}$. Definition 4.9 gives $h_1 \not\approx_\Gamma h_2$ and $D_B(h_1) = D_B(h_2)$. The first condition implies $[h_1]_\Gamma \neq [h_2]_\Gamma$, while Equation 48 and the second condition give $\overline{D}_B([h_1]_\Gamma) = \overline{D}_B([h_2]_\Gamma)$. This contradicts injectivity. Therefore $\Delta_B^{\text{desc}} = \varnothing$. □

Proposition 4.10 parallels the injectivity criterion of the earlier HUAAMF metamodel while changing the logical level. The earlier collapse operator $\Delta(m)$ is associated with a mechanism $m \in M$ acting on carrier states. The present operator Δ_B^{desc} is associated with a descriptor acting on realized histories. To make the distinction explicit, write the earlier operator as $\Delta^{\text{mech}}(m)$ when both appear in the same discussion. Their domains and interpretations are summarized in Equation 52.

$$\begin{aligned} \Delta^{\text{mech}}(m) &\subseteq S \times S, && \text{distinctions collapsed by a generating mechanism,} \\ \Delta_B^{\text{desc}} &\subseteq \Gamma_B \times \Gamma_B, && \text{distinctions collapsed by a trajectory descriptor.} \end{aligned} \tag{52}$$

Equation 52 resolves an apparent tension between the earlier irreversibility result and the present use of a group-valued descriptor. The earlier result applies when a group itself acts as the mechanism family while preserving the carrier congruence [6]. In the present construction, B_n is the codomain of D_B and need not act as the mechanism semigroup M. A noninjective map from histories into a group can therefore collapse descriptive distinctions even though each group element has an inverse. Group invertibility at the descriptor level does not restore information discarded before or during the map D_B. Table 2 summarizes the four derived descriptive roles and the additional commitments introduced by the braid instantiation.

4.6 Combined Primitive Profile of the Braid Instantiation

The preceding derivation gives each GR meta-primitive a specific trajectory-descriptive realization. The subsection consolidates those results before the manuscript turns to the internal stages of the braid-description pipeline.

The braid construction therefore instantiates all four meta-primitives while adding substantial lower-level commitments. Constraint selects a restricted history domain and deformation class. Change is represented through a relational trajectory and, algebraically, through crossing generators. Structure is strengthened from generic associative composition to braid-specific relations. Difference becomes a quotient-sensitive relation on histories. These commitments are the source of both the descriptive power and the restricted generality of the braid formalism.

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Table 2: Braid-theoretic instantiation of the four GR meta-primitives at the trajectory-descriptive level.

Primitive	Braid-theoretic realization	Added commitment
Constraint	The domain Γ_B, collision-free based loops in $\mathcal{U}_n$, and baseline-equivalence preservation through Equation 46	Fixed represented component number, persistent traceability, selected endpoint convention, and a declared admissible deformation class
Change	Extracted loop $\mathcal{T}_B(h)$ together with elementary descriptive crossings in Σ_n	Relational trajectory change is described through braid-compatible crossing structure rather than arbitrary state transformation
Structure	Loop concatenation, braid multiplication, and the word relations in Equation 38	Compositions related by the Artin and inverse relations are identified at the braid level
Difference	The induced equivalence $\equiv_B$ and the descriptive collapse relation Δ_B^{desc}	Baseline-distinct histories become descriptively identical whenever their extracted loops are based-homotopic

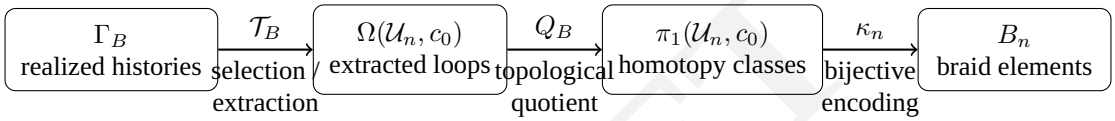

Figure 1: Stages of braid-theoretic trajectory description. Resolution may be reduced by the extraction map $\mathcal{T}_B$ and by the homotopy quotient Q_B. The isomorphism κ_n changes representation without further quotienting.

The next section develops the descriptor itself in greater detail as a trajectory-description pipeline. It examines what information enters through $\mathcal{T}_B$, what is removed by the homotopy quotient Q_B, what is preserved by the algebraic identification κ_n, and how these stages determine the descriptive resolution of the resulting braid representation.

5 Information Flow in Braid-Theoretic Trajectory Description

This section analyzes the internal information structure of the braid descriptor derived in Section 4. Its role is to distinguish the information removed by trajectory extraction from the information removed by the topological quotient, while separating both from the algebraic encoding of the resulting class. The objective is to identify exactly where descriptive resolution changes in the composite map $D_B = \kappa_n \circ Q_B \circ \mathcal{T}_B$. Section 5.1 defines three successive descriptive stages and derives their induced equivalence relations. Section 5.2 decomposes braid-descriptive collapse into extraction-stage and homotopy-stage components. Section 5.3 shows that the isomorphism κ_n adds no further collapse, while braid words can remain syntactically nonunique. Section 5.4 states precise recovery conditions for histories, extracted trajectories, and homotopy classes. Section 5.5 consolidates the resulting resolution profile.

Figure 1 summarizes the formal pipeline before the stage-by-stage derivation.

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5.1 Induced Equivalences across Descriptor Stages

The composite in Equation 33 contains three maps with different analytical roles. To compare their resolutions, define the stage descriptors in Equation 53.

$$\begin{aligned} D_T &:= \mathcal{T}_B, \\ D_H &:= Q_B \circ \mathcal{T}_B, \\ D_B &:= \kappa_n \circ Q_B \circ \mathcal{T}_B. \end{aligned} \tag{53}$$

The subscript T in D_T refers to the extracted trajectory, while the subscript H in D_H refers to its homotopy class. Equation 53 introduces no new map beyond the three maps already defined in Section 4; it only names the intermediate composites so that their descriptive resolutions can be compared.

Each descriptor in Equation 53 induces an equivalence relation by Equation 20. The three induced relations are written explicitly in Equation 54.

$$\begin{aligned} h_1 \equiv_T h_2 \quad &\iff \quad \mathcal{T}_B(h_1) = \mathcal{T}_B(h_2), \\ h_1 \equiv_H h_2 \quad &\iff \quad Q_B(\mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2)), \\ h_1 \equiv_B h_2 \quad &\iff \quad D_B(h_1) = D_B(h_2). \end{aligned} \tag{54}$$

The relation $\equiv_T$ identifies histories that produce exactly the same extracted parameterized loop. The relation $\equiv_H$ permits distinct extracted loops when they lie in the same based-homotopy class. The relation $\equiv_B$ is the braid-descriptive equivalence already introduced in Definition 4.3.

The three relations have a fixed inclusion structure.

Proposition 5.1 (Resolution relations among the descriptor stages)**.** *The equivalence relations in Equation 54 satisfy Equation 55.*

$$\equiv_T \subseteq \equiv_H = \equiv_B . \tag{55}$$

Proof. If $h_1 \equiv_T h_2$, then $\mathcal{T}_B(h_1) = \mathcal{T}_B(h_2)$ by Equation 54. Applying Q_B to equal loops gives equal homotopy classes, so $h_1 \equiv_H h_2$. Hence $\equiv_T \subseteq \equiv_H$.

For the second relation, assume first that $h_1 \equiv_H h_2$. Equation 54 gives equality of the two elements of $\pi_1(\mathcal{U}_n, c_0)$. Applying κ_n to both sides and using Equation 33 gives $D_B(h_1) = D_B(h_2)$, hence $h_1 \equiv_B h_2$. Conversely, if $h_1 \equiv_B h_2$, then $\kappa_n(D_H(h_1)) = \kappa_n(D_H(h_2))$. The map κ_n is injective, so $D_H(h_1) = D_H(h_2)$, which is $h_1 \equiv_H h_2$. Therefore $\equiv_H = \equiv_B$. □

Proposition 5.1 has two consequences. First, passing from exact extracted loops to homotopy classes can only preserve or coarsen descriptive resolution. Second, passing from the fundamental-group class to the corresponding braid element changes notation and algebraic representation while preserving exactly the same partition of Γ_B.

The resolution order of Definition 3.6 provides an equivalent formulation. Proposition 5.1 implies the relations stated in Equation 56.

$$D_T \preceq D_H, \qquad \equiv_H = \equiv_B . \tag{56}$$

Equation 56 means that D_T is at least as fine as D_H, while D_H and D_B have identical descriptive resolution.

5.2 Stage-Specific Descriptive Collapse

The braid-descriptive collapse relation of Definition 4.9 can now be decomposed according to the first stage at which two histories that remain distinct under the baseline relation become indistinguishable. The symbol $\approx_\Gamma$ continues to denote the braid-compatible baseline equivalence introduced in Section 4.4.

The extraction-stage collapse relation records histories that are already identified by the extraction map. Define it by Equation 57.

$$\Delta^{\text{desc}}_{B,\text{ext}} := \{(h_1, h_2) \in \Gamma_B \times \Gamma_B : h_1 \not\approx_\Gamma h_2 \text{ and } \mathcal{T}_B(h_1) = \mathcal{T}_B(h_2)\} . \tag{57}$$

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The subscript ext indicates that the loss occurs before the homotopy quotient is applied. Equation 57 can be nonempty whenever $\mathcal{T}_B$ intentionally selects only part of the information contained in a realized history.

The homotopy-stage collapse relation records the additional distinctions removed by Q_B. Define it by Equation 58.

$$\Delta^{\text{desc}}_{B,\text{hom}} := \{(h_1, h_2) \in \Gamma_B \times \Gamma_B : h_1 \not\approx_\Gamma h_2,\ \mathcal{T}_B(h_1) \neq \mathcal{T}_B(h_2),\ \text{and } Q_B(\mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2))\}\,. \tag{58}$$

The explicit inequality of the extracted loops in Equation 58 assigns to this second relation only those collapses that were not already present at the extraction stage.

The two relations provide a disjoint decomposition of the total braid-descriptive collapse.

Proposition 5.2 (Stage decomposition of braid-descriptive collapse)**.** *The braid-descriptive collapse relation satisfies Equation 59.*

$$\Delta^{\text{desc}}_B = \Delta^{\text{desc}}_{B,\text{ext}} \,\dot\cup\, \Delta^{\text{desc}}_{B,\text{hom}}, \tag{59}$$

where $\dot\cup$ *denotes disjoint union.*

Proof. Take $(h_1, h_2) \in \Delta^{\text{desc}}_B$. By Definition 4.9, the histories are baseline-distinct and satisfy $D_B(h_1) = D_B(h_2)$. Proposition 4.4 therefore gives $Q_B(\mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2))$.

There are two exhaustive cases. If $\mathcal{T}_B(h_1) = \mathcal{T}_B(h_2)$, then Equation 57 gives $(h_1, h_2) \in \Delta^{\text{desc}}_{B,\text{ext}}$. If the extracted loops are unequal, then Equation 58 gives $(h_1, h_2) \in \Delta^{\text{desc}}_{B,\text{hom}}$. Hence the left-hand side of Equation 59 is contained in the union on the right.

Conversely, any pair in $\Delta^{\text{desc}}_{B,\text{ext}}$ has equal extracted loops and therefore equal homotopy classes, while any pair in $\Delta^{\text{desc}}_{B,\text{hom}}$ has equal homotopy classes by definition. Applying κ_n in either case gives equal braid descriptors. Both relations also require baseline distinction. Definition 4.9 therefore places every pair from the right-hand side in Δ^{desc}_B.

Finally, the two component relations are disjoint because Equation 57 requires equality of the extracted loops whereas Equation 58 requires their inequality. □

Proposition 5.2 makes the descriptive compression analytically separable. The extraction map can discard information because only selected relational trajectories are retained. The homotopy quotient can then discard further geometric information because distinct extracted loops may belong to the same topological class. The isomorphism κ_n contributes no third collapse term.

The decomposition also identifies two distinct ways to refine a braid description. Refining $\mathcal{T}_B$ can reduce extraction-stage collapse by retaining more information from the full history. Refining the equivalence imposed after extraction can reduce homotopy-stage collapse, although doing so would generally move the descriptor away from ordinary braid topology and would also weaken the deformation robustness supplied by the homotopy quotient.

5.3 Word-Level Nonuniqueness in Algebraic Encoding

The final map κ_n in Figure 1 is bijective. Consequently, a braid element contains exactly the information contained in the corresponding fundamental-group class. The algebraic presentation of that braid element nevertheless admits multiple word representatives. These two facts concern different levels and should be distinguished.

Let $b \in B_n$. The set of Artin words representing b is the fiber of the evaluation map ev_B introduced before Equation 40. Define this word-representation fiber in Equation 60.

$$\mathcal{W}_B(b) := \text{ev}_B^{-1}(\{b\}) \subseteq \Sigma_n^*. \tag{60}$$

The set $\mathcal{W}_B(b)$ can contain several syntactically distinct words because the relations in Equation 38 identify different words with the same braid element.

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Word-level nonuniqueness therefore differs from trajectory-level descriptive collapse. If $w_1, w_2 \in \mathcal{W}_B(b)$ with $w_1 \neq w_2$, the two words are distinct syntactic representatives of one already-fixed descriptive object b. By contrast, a pair $(h_1, h_2) \in \Delta_B^{\text{desc}}$ consists of baseline-distinct histories that the descriptor maps to one braid element. The former concerns redundancy inside the notation used for the codomain; the latter concerns information removed from the domain of histories.

The absence of additional collapse under κ_n can be stated directly.

Proposition 5.3 (Resolution preservation under algebraic braid encoding)**.** *For all $h_1, h_2 \in \Gamma_B$, the equivalence in Equation 61 holds.*

$$D_H(h_1) = D_H(h_2) \quad \Longleftrightarrow \quad D_B(h_1) = D_B(h_2). \tag{61}$$

Proof. Equation 53 gives $D_B = \kappa_n \circ D_H$. If $D_H(h_1) = D_H(h_2)$, applying κ_n yields equality of the corresponding braid descriptors. Conversely, if the braid descriptors are equal, injectivity of the isomorphism κ_n gives equality of the fundamental-group classes. □

Proposition 5.3 shows that the algebraic braid group is a faithful encoding of the topological quotient already taken by Q_B. The descriptive reduction in the baseline construction occurs before the braid group element is algebraically named.

5.4 Recovery Conditions at Successive Resolutions

The stage factorization makes several recovery questions precise. Three different objects may be sought from a braid description: the homotopy class of the extracted loop, the exact extracted loop, or the full realized history. These require increasingly strong conditions.

The homotopy class is always recoverable from the braid element because κ_n is an isomorphism. Its inverse provides the recovery map stated in Equation 62.

$$D_H(h) = \kappa_n^{-1}(D_B(h)) \quad \text{for every } h \in \Gamma_B. \tag{62}$$

Equation 62 recovers the class $Q_B(\mathcal{T}_B(h))$ and does not select a unique loop representative within that class.

Exact-loop recovery depends on the restriction of the homotopy quotient to the trajectories actually produced by the extraction map. Define the restricted quotient map in Equation 63.

$$Q_B^{\mathcal{T}} := Q_B\big|_{\text{im}(\mathcal{T}_B)} : \text{im}(\mathcal{T}_B) \longrightarrow \pi_1(\mathcal{U}_n, c_0). \tag{63}$$

A unique extracted loop can be recovered from its braid element precisely when the map in Equation 63 is injective.

Proposition 5.4 (Exact-loop recovery criterion)**.** *There exists a map $R_T : \text{im}(D_B) \to \text{im}(\mathcal{T}_B)$ satisfying Equation 64 for every $h \in \Gamma_B$ if and only if $Q_B^{\mathcal{T}}$ is injective.*

$$R_T(D_B(h)) = \mathcal{T}_B(h). \tag{64}$$

Proof. Assume first that $Q_B^{\mathcal{T}}$ is injective. For $b \in \text{im}(D_B)$, Equation 62 gives the class $\kappa_n^{-1}(b)$ in the image of $Q_B^{\mathcal{T}}$. Injectivity of $Q_B^{\mathcal{T}}$ gives a unique $q \in \text{im}(\mathcal{T}_B)$ with $Q_B^{\mathcal{T}}(q) = \kappa_n^{-1}(b)$. Define $R_T(b) := q$. Substituting $b = D_B(h)$ yields Equation 64.

Conversely, suppose a map R_T satisfying Equation 64 exists. Let $q_1, q_2 \in \text{im}(\mathcal{T}_B)$ satisfy $Q_B^{\mathcal{T}}(q_1) = Q_B^{\mathcal{T}}(q_2)$. Choose histories $h_1, h_2 \in \Gamma_B$ with $\mathcal{T}_B(h_i) = q_i$. Applying κ_n to the equality of classes gives $D_B(h_1) = D_B(h_2)$. Equation 64 then gives $q_1 = R_T(D_B(h_1)) = R_T(D_B(h_2)) = q_2$. Hence $Q_B^{\mathcal{T}}$ is injective. □

Recovery of a full history requires an additional condition on the extraction map. If exact histories rather than baseline-equivalence classes are sought, define $\mathcal{T}_B$ to be *history-faithful* when it is injective on Γ_B. Under that condition, exact-loop recovery can be followed by the inverse of $\mathcal{T}_B$ on its image.

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Table 3: Resolution stages in braid-theoretic trajectory description.

Stage	Equality at the stage	Descriptive effect
Extraction $D_T = \mathcal{T}_B$	$\mathcal{T}_B(h_1) = \mathcal{T}_B(h_2)$	Can collapse baseline-distinct histories when the selected relational trajectory omits other historical information
Homotopy $D_H = Q_B \circ \mathcal{T}_B$	$Q_B(\mathcal{T}_B(h_1)) = Q_B(\mathcal{T}_B(h_2))$	Further identifies distinct extracted loops that are connected by an admissible based homotopy
Algebraic braid encoding $D_B = \kappa_n \circ D_H$	$D_B(h_1) = D_B(h_2)$	Preserves the homotopy-class resolution exactly because κ_n is bijective
Artin word representation	$\mathrm{ev}_B(w_1) = \mathrm{ev}_B(w_2)$	Permits multiple syntactic representatives of one braid element without further collapsing histories

Corollary 5.5 (Exact-history recovery criterion)**.** *A braid descriptor determines a unique history in Γ_B if both $\mathcal{T}_B$ and $Q_B^{\mathcal{T}}$ are injective. Conversely, if either map is noninjective, the braid descriptor cannot determine a unique history without additional information.*

Proof. If both maps are injective, Proposition 5.4 recovers a unique extracted loop, and injectivity of $\mathcal{T}_B$ identifies the unique history that produced that loop. If $\mathcal{T}_B$ is noninjective, two histories share one extracted loop and therefore one braid descriptor. If $Q_B^{\mathcal{T}}$ is noninjective, two distinct extracted loops in the image of $\mathcal{T}_B$ share one homotopy class and therefore one braid descriptor. □

Corollary 5.5 concerns exact histories. The weaker requirement relevant to a declared baseline equivalence is already given by Proposition 4.10: the braid descriptor is faithful to baseline history classes precisely when the induced map $\overline{D}_B : \Gamma_B/\approx_\Gamma \to B_n$ is injective. The distinction matters because a descriptor can be intentionally noninjective on exact histories while remaining fully faithful at the resolution declared relevant by the analysis.

5.5 Resolution Profile of the Braid Descriptor

The results above identify where information is retained and where it is discarded. Table 3 summarizes the stages, their induced equivalences, and their possible contribution to descriptive collapse.

The stage analysis also clarifies the source of topological robustness. Homotopy-stage compression identifies geometric variations that remain inside one admissible class. The same operation that removes those geometric distinctions is therefore responsible for the stability of the descriptor under admissible deformation. A finer descriptor could retain more geometric information, but that added discrimination would no longer be invariant under the full homotopy relation used by Q_B.

This tradeoff will be formalized in the later sections on descriptive equivalence and topological invariants. The present section establishes the prior result needed for that analysis: braid-theoretic trajectory description is a staged information transformation. Extraction determines which part of a realized history becomes a trajectory object; the homotopy quotient determines which trajectory differences remain visible; and the braid-group isomorphism provides a compositional algebraic encoding of the resulting topological class without introducing an additional loss of descriptive resolution.

6 Equivalence Structure Across Trajectory-Descriptive Levels

This section organizes the equivalence relations that have appeared in the construction and derives their logical relations. Its role is to prevent equality of histories, equality of extracted trajectories, topological

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equivalence, braid equivalence, and endpoint agreement from being treated as interchangeable notions. The objective is to identify which implications follow from the formalism itself, which require an additional compatibility assumption, and which generally fail. Section 6.1 defines the relevant relations on realized histories. Section 6.2 derives the inclusion structure forced by the descriptor pipeline. Section 6.3 relates the baseline history quotient to the braid quotient and identifies descriptive collapse as the exact excess identification introduced by the braid descriptor. Section 6.4 places endpoint information within the same hierarchy. Section 6.5 states conditions under which the successive equivalence levels are genuinely distinct.

6.1 History-Level Equivalence Relations

All relations in this subsection are defined on the common set Γ_B of braid-admissible realized histories. The finest relation is exact history identity. Define $\equiv_{\text{id}}$ by Equation 65.

$$h_1 \equiv_{\text{id}} h_2 \quad \Longleftrightarrow \quad h_1 = h_2. \tag{65}$$

Equation 65 records equality of the complete state sequences introduced in Definition 3.1. It does not identify different histories merely because they share endpoints or a trajectory description.

The baseline history relation $\approx_\Gamma$ was introduced in Section 4.4. It is supplied by the lower-level instantiation and records distinctions that the analysis has already declared irrelevant before braid description is applied. Because $\approx_\Gamma$ is an equivalence relation, exact identity always implies baseline equivalence. The inclusion is stated in Equation 66.

$$\equiv_{\text{id}} \subseteq \approx_\Gamma . \tag{66}$$

Equation 66 follows because every equivalence relation is reflexive: if $h_1 = h_2$, then $h_1 \approx_\Gamma h_2$.

The extraction-stage, homotopy-stage, and braid-stage equivalences were defined in Equation 54. The relation $\equiv_T$ compares exact extracted loops, $\equiv_H$ compares their homotopy classes, and $\equiv_B$ compares braid elements. These phrases introduce no additional relations; they name the three relations already derived from D_T, D_H, and D_B.

The baseline relation and the extraction relation need not be comparable. A domain may declare two full histories equivalent even when $\mathcal{T}_B$ retains different geometric loop representatives, provided those loops remain homotopic. Conversely, $\mathcal{T}_B$ may omit a baseline-relevant distinction and map two baseline-distinct histories to the same extracted loop. The formalism therefore derives no inclusion between $\approx_\Gamma$ and $\equiv_T$ without an additional assumption.

The braid-compatibility condition of Definition 4.7 does, however, place the baseline relation below braid equivalence. Proposition 4.8 gives the implication $h_1 \approx_\Gamma h_2 \Rightarrow h_1 \equiv_B h_2$. In relation notation, this is Equation 67.

$$\approx_\Gamma \subseteq \equiv_B . \tag{67}$$

Combining Equations 66 and 67 yields the baseline-to-braid chain in Equation 68.

$$\equiv_{\text{id}} \subseteq \approx_\Gamma \subseteq \equiv_B . \tag{68}$$

Equation 68 is conditional only in its second inclusion: $\approx_\Gamma \subseteq \equiv_B$ requires braid compatibility of the baseline relation. The first inclusion follows from equivalence alone.

6.2 Inclusions Forced by the Descriptor Pipeline

The internal descriptor stages impose an inclusion structure independently of the chosen baseline relation. Proposition 5.1 established $\equiv_T \subseteq \equiv_H = \equiv_B$. Exact history identity also implies extraction equality because $\mathcal{T}_B$ is a function. If $h_1 = h_2$, then applying $\mathcal{T}_B$ to both sides gives $\mathcal{T}_B(h_1) = \mathcal{T}_B(h_2)$. Hence Equation 69 holds.

$$\equiv_{\text{id}} \subseteq \equiv_T . \tag{69}$$

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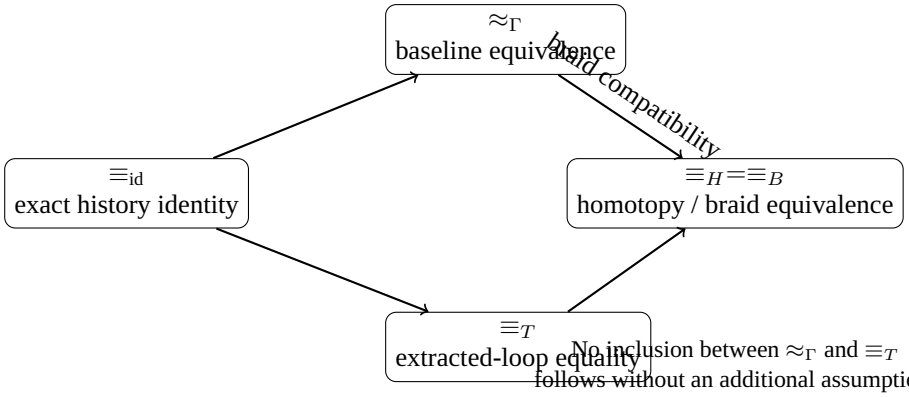

Figure 2: Implication structure among exact identity, baseline history equivalence, extracted-loop equality, and braid equivalence. Arrows indicate inclusion of equivalence relations. The baseline-to-braid arrow uses the braid-compatibility assumption of Definition 4.7.

Combining Equation 69 with Proposition 5.1 gives the descriptor-stage chain in Equation 70.

$$\equiv_{\text{id}} \subseteq \equiv_T \subseteq \equiv_H = \equiv_B . \tag{70}$$

The two chains in Equations 68 and 70 share their finest and coarsest displayed relations but can differ in the middle. Figure 2 records precisely the implications forced by the construction.

The absence of a general relation between $\approx_\Gamma$ and $\equiv_T$ is analytically important. Baseline equivalence is chosen according to the resolution that a substantive analysis regards as relevant. Extraction equality is determined by the particular data retained in $\mathcal{T}_B$. Either can be finer than the other in a concrete instantiation. The formal braid construction therefore requires their compatibility only at the coarser homotopy level, not at the level of exact extracted loops.

6.3 Canonical Baseline-to-Braid Quotient Map

When $\approx_\Gamma$ is braid-compatible, Equation 67 allows a canonical map from baseline history classes to braid-descriptive classes. Let $\Gamma_B/\approx_\Gamma$ denote the set of baseline equivalence classes and let $\Gamma_B/\equiv_B$ denote the set of braid-descriptive equivalence classes.

Define the map in Equation 71.

$$q_{\Gamma B} : \Gamma_B/\approx_\Gamma \longrightarrow \Gamma_B/\equiv_B, \qquad q_{\Gamma B}([h]_\Gamma) := [h]_B. \tag{71}$$

The symbols $[h]_\Gamma$ and $[h]_B$ denote, respectively, the $\approx_\Gamma$-class and the $\equiv_B$-class of the same history h.

Proposition 6.1 (Baseline-to-braid quotient map)**.** *If $\approx_\Gamma$ is braid-compatible, then the map in Equation 71 is well defined and surjective.*

Proof. Suppose $[h_1]_\Gamma = [h_2]_\Gamma$. Then $h_1 \approx_\Gamma h_2$. Equation 67 gives $h_1 \equiv_B h_2$, so $[h_1]_B = [h_2]_B$. Hence the value of $q_{\Gamma B}$ is independent of the chosen representative and the map is well defined.

For surjectivity, take any class $[h]_B \in \Gamma_B/\equiv_B$. The baseline class $[h]_\Gamma$ belongs to $\Gamma_B/\approx_\Gamma$ and Equation 71 sends it to $[h]_B$. Therefore every braid class lies in the image. □

Proposition 6.1 makes $q_{\Gamma B}$ the canonical map from baseline classes to braid classes. This map isolates the additional equivalence introduced by braid description beyond the baseline resolution. Its failure of injectivity is exactly the existence of braid-descriptive collapse.

Proposition 6.2 (Descriptive collapse as excess equivalence)**.** *Assume $\approx_\Gamma$ is braid-compatible. Then the braid-descriptive collapse relation is the set difference stated in Equation 72.*

$$\Delta_B^{\text{desc}} = \equiv_B \setminus \approx_\Gamma, \tag{72}$$

where both equivalence relations are regarded as subsets of $\Gamma_B \times \Gamma_B$.

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Proof. A pair (h_1, h_2) belongs to the right-hand side of Equation 72 exactly when $h_1 \equiv_B h_2$ and $h_1 \not\approx_\Gamma h_2$. By Definition 4.3, the first condition is equivalent to $D_B(h_1) = D_B(h_2)$. Together the two conditions are precisely Definition 4.9. □

Equation 72 gives a direct relation-theoretic interpretation of descriptive loss. The braid descriptor does not erase a quantity attached to a history. It enlarges the equivalence relation from $\approx_\Gamma$ to $\equiv_B$, and Δ_B^{desc} records the newly identified pairs.

The quotient map in Equation 71 is injective precisely when no such new pairs exist.

Corollary 6.3 (Faithfulness of the quotient coarsening)**.** *The map $q_{\Gamma B}$ is injective if and only if $\Delta_B^{\text{desc}} = \varnothing$.*

Proof. If $q_{\Gamma B}$ is injective and $h_1 \equiv_B h_2$, then $q_{\Gamma B}([h_1]_\Gamma) = q_{\Gamma B}([h_2]_\Gamma)$. Injectivity gives $[h_1]_\Gamma = [h_2]_\Gamma$, so $h_1 \approx_\Gamma h_2$. Equation 72 then gives $\Delta_B^{\text{desc}} = \varnothing$.

Conversely, if $\Delta_B^{\text{desc}} = \varnothing$, Equation 72 and Equation 67 imply $\equiv_B = \approx_\Gamma$. Hence two baseline classes have the same braid class only when they are the same baseline class, and $q_{\Gamma B}$ is injective. □

Corollary 6.3 is equivalent in content to Proposition 4.10, but it exposes the quotient geometry of the same fact. The induced map $\overline{D}_B$ in Equation 48 can be decomposed into a possible coarsening followed by a bijective encoding. Let $\psi_B : \Gamma_B/\equiv_B \to \text{im}(D_B)$ be the bijection defined by $\psi_B([h]_B) := D_B(h)$ and supplied by Proposition 3.3. Equation 73 then holds.

$$\overline{D}_B = \psi_B \circ q_{\Gamma B}. \tag{73}$$

Equation 73 shows that all loss relative to the declared baseline equivalence occurs in $q_{\Gamma B}$; the subsequent map ψ_B is bijective onto the image of the braid descriptor.

6.4 Resolution of Endpoint Equivalence

Trajectory description is often compared with endpoint-only description. The present based-loop construction makes that comparison particularly clear because every extracted loop begins and ends at the same selected configuration c_0.

Define the extracted-endpoint descriptor D_∂ in Equation 74.

$$D_\partial : \Gamma_B \longrightarrow \mathcal{U}_n \times \mathcal{U}_n, \qquad D_\partial(h) := (\mathcal{T}_B(h)(0), \mathcal{T}_B(h)(1)) \,. \tag{74}$$

Because $\mathcal{T}_B(h)$ belongs to the based loop space of Equation 30, Equation 74 simplifies to Equation 75.

$$D_\partial(h) = (c_0, c_0) \quad \text{for every } h \in \Gamma_B. \tag{75}$$

Let $\equiv_\partial$ be the equivalence relation induced by D_∂. Equation 75 implies that every pair of admissible histories is endpoint-equivalent. Thus Equation 76 holds.

$$\equiv_\partial = \Gamma_B \times \Gamma_B. \tag{76}$$

The braid descriptor is consequently at least as fine as the extracted-endpoint descriptor in the sense of Definition 3.6. The relation is stated in Equation 77.

$$\equiv_B \subseteq \equiv_\partial, \qquad D_B \preceq D_\partial. \tag{77}$$

Equation 77 formalizes the elementary but important fact that a based braid description can distinguish histories that endpoint-only description necessarily identifies.

The endpoint result should be interpreted at the level at which it is defined. The descriptor D_∂ records endpoints of the extracted relational trajectory, not the complete initial and terminal states in the heterogeneous carrier S. If a later application requires full-state endpoint comparison, it can introduce the separate map $h = (x_0, \ldots, x_k) \mapsto (x_0, x_k)$. No implication between that full-state endpoint descriptor and D_B follows without assumptions connecting full carrier states to the extraction map $\mathcal{T}_B$.

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6.5 Strictness Conditions Across Equivalence Levels

The inclusions derived above need not all be equalities. Their strictness identifies which kinds of information are genuinely present at one level and absent at the next.

The inclusion $\equiv_T \subseteq \equiv_B$ is strict exactly when the extraction image contains two distinct loops in the same based-homotopy class.

Proposition 6.4 (Strictness of the trajectory-to-braid quotient)**.** *The inclusion $\equiv_T \subseteq \equiv_B$ is strict if and only if there exist histories $h_1, h_2 \in \Gamma_B$ satisfying Equation 78.*

$$\mathcal{T}_B(h_1) \neq \mathcal{T}_B(h_2) \quad \textit{and} \quad \mathcal{T}_B(h_1) \simeq_\partial \mathcal{T}_B(h_2). \tag{78}$$

Proof. If histories satisfying Equation 78 exist, they are not $\equiv_T$-equivalent by Equation 54, while Proposition 4.4 makes them $\equiv_B$-equivalent. Hence the inclusion is strict.

Conversely, if $\equiv_T \subseteq \equiv_B$ is strict, there exists a pair with $h_1 \equiv_B h_2$ and $h_1 \not\equiv_T h_2$. The second condition gives unequal extracted loops, while the first gives homotopy equivalence by Proposition 4.4. This is Equation 78. □

Proposition 6.4 characterizes strictness between exact extracted-loop equality and braid equivalence. The inclusion $\equiv_B \subseteq \equiv_\partial$ is strict whenever the braid descriptor realizes at least two braid elements.

Proposition 6.5 (Strictness of braid information beyond endpoints)**.** *If there exist $h_1, h_2 \in \Gamma_B$ with $D_B(h_1) \neq D_B(h_2)$, then $\equiv_B \subsetneq \equiv_\partial$.*

Proof. Equation 76 gives $h_1 \equiv_\partial h_2$ for every pair of admissible histories. The assumed inequality $D_B(h_1) \neq D_B(h_2)$ means that this particular pair does not satisfy $h_1 \equiv_B h_2$. Hence $\equiv_B$ is a proper subset of $\equiv_\partial$. □

For $n \geq 2$, the braid group itself contains distinct elements, including the identity and a standard generator σ_1 [1, 2]. Proposition 6.5 therefore applies whenever the extraction domain realizes histories whose braid descriptors include two such distinct elements. The result does not claim that every concrete instantiation realizes all elements of B_n; strictness is a property of the actual image of D_B.

The baseline inclusion $\approx_\Gamma \subseteq \equiv_B$ is strict precisely when descriptive collapse is nonempty, by Proposition 6.2. Thus Equation 79 holds.

$$\approx_\Gamma \subsetneq \equiv_B \quad \Longleftrightarrow \quad \Delta_B^{\text{desc}} \neq \varnothing. \tag{79}$$

Equation 79 makes descriptive collapse equivalent to a strict enlargement of the declared baseline equivalence.

Table 4 consolidates the relations derived in this section.

The resulting hierarchy is more precise than a binary distinction between "same history" and "different history." Exact identity, baseline equivalence, extracted-loop equality, homotopy equivalence, braid equivalence, and endpoint equivalence answer different questions. The braid formalism is informative precisely because it occupies an intermediate resolution: it preserves topological distinctions that endpoint description erases while intentionally identifying geometric variations within a homotopy class. The next section develops the invariant structures that can be defined after this quotient has been taken and distinguishes invariants that preserve the full braid class from those that coarsen it further.

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Table 4: Equivalence relations used in the braid-theoretic trajectory-descriptive formalism.

Relation	Criterion	Analytical role
$\equiv_{\text{id}}$	Exact equality of realized histories	Finest history-level distinction used in the present formalism
$\approx_\Gamma$	Domain-supplied baseline equivalence	Declares which historical distinctions are already irrelevant before braid description
$\equiv_T$	Equality of extracted loops $\mathcal{T}_B(h)$	Measures the resolution retained immediately after trajectory extraction
$\equiv_H$	Equality of based-homotopy classes	Removes geometric distinctions within the admissible homotopy class
$\equiv_B$	Equality of braid descriptors $D_B(h)$	Algebraic encoding of the same equivalence as $\equiv_H$
$\equiv_\partial$	Equality of extracted endpoints	Universal relation in the present based-loop construction and therefore coarser than braid equivalence

7 Derived Invariants of Braid Descriptions

This section studies information that can be derived after a realized history has already been represented by a braid element. Its role is to distinguish the braid descriptor itself from secondary quantities or quotient descriptions defined on its codomain. The objective is to make explicit which derived constructions preserve the full braid-level resolution and which introduce an additional coarsening. Section 7.1 defines invariant maps and their induced history equivalences. Section 7.2 derives two elementary algebraic invariants directly from the Artin presentation. Section 7.3 formalizes joint invariant families and a criterion for completeness relative to the realized braid image. Section 7.4 develops conjugacy-level description for settings in which a cyclic change of the observation cut is irrelevant. Section 7.5 derives the further quotient associated with braid closure and Markov equivalence. Section 7.6 states the interpretive scope of these constructions and relates them to later complexity-sensitive applications.

7.1 Descriptive Coarsening by Invariant Maps

Let $\mathcal{A}$ be a set of values and let $I : B_n \to \mathcal{A}$ be a map defined on braid elements. In the present paper, the term *derived braid invariant* is used functionally for such a map when the value $I(\beta)$ depends only on the braid element β and therefore is independent of the chosen Artin word representing that element. This usage is deliberately broad. Additional invariance requirements, such as invariance under conjugacy or under Markov moves, will be introduced separately when needed.

Given the braid descriptor $D_B : \Gamma_B \to B_n$, the derived history descriptor associated with I is the composite map defined in Equation 80.

$$D_I := I \circ D_B : \Gamma_B \longrightarrow \mathcal{A}. \tag{80}$$

Equation 80 is downstream from the complete braid descriptor. It therefore cannot distinguish two histories already identified by D_B.

Let $\equiv_I$ denote the history equivalence induced by D_I. Its definition is stated in Equation 81.

$$h_1 \equiv_I h_2 \quad \Longleftrightarrow \quad I(D_B(h_1)) = I(D_B(h_2)). \tag{81}$$

The braid equivalence relation is always contained in the invariant-induced relation.

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Proposition 7.1 (Derived invariants weakly coarsen braid description)**.** *For every map $I : B_n \to \mathcal{A}$, the inclusion in Equation 82 holds.*

$$\equiv_B \subseteq \equiv_I . \tag{82}$$

Equivalently, $D_B \preceq D_I$ in the descriptive-resolution order of Definition 3.6.

Proof. If $h_1 \equiv_B h_2$, then Definition 4.3 gives $D_B(h_1) = D_B(h_2)$. Applying I to equal braid elements yields $I(D_B(h_1)) = I(D_B(h_2))$. Equation 81 therefore gives $h_1 \equiv_I h_2$. □

The additional information loss introduced by I can be isolated from the collapse already introduced by D_B. Define the *invariant-stage additional collapse* by Equation 83.

$$\Delta^{\text{desc}}_{I|B} := \{(h_1, h_2) \in \Gamma_B \times \Gamma_B : D_B(h_1) \neq D_B(h_2) \text{ and } I(D_B(h_1)) = I(D_B(h_2))\} . \tag{83}$$

The notation $I \mid B$ in Equation 83 indicates collapse introduced by I relative to the braid-level resolution. It does not denote a restriction of I in the set-theoretic sense.

For the remainder of this subsection, assume that the baseline relation is braid-compatible in the sense of Definition 4.7. The total descriptive collapse of the derived descriptor D_I is defined in Equation 84.

$$\Delta^{\text{desc}}_I := \{(h_1, h_2) \in \Gamma_B \times \Gamma_B : h_1 \not\approx_\Gamma h_2 \text{ and } h_1 \equiv_I h_2\} . \tag{84}$$

The collapse from baseline histories to invariant values decomposes into the collapse already present at the braid stage and the new collapse introduced by I.

Proposition 7.2 (Stage decomposition for a derived invariant)**.** *If $\approx_\Gamma$ is braid-compatible, then Equation 85 holds.*

$$\Delta^{\text{desc}}_I = \Delta^{\text{desc}}_B \dot\cup \Delta^{\text{desc}}_{I|B}, \tag{85}$$

where $\dot\cup$ denotes disjoint union.

Proof. Take $(h_1, h_2) \in \Delta^{\text{desc}}_I$. Equation 84 gives baseline distinction and equality of invariant values. If $D_B(h_1) = D_B(h_2)$, Definition 4.9 places the pair in Δ^{desc}_B. If the braid descriptors are unequal, Equation 83 places the pair in $\Delta^{\text{desc}}_{I|B}$. These cases are exhaustive and disjoint.

Conversely, a pair in Δ^{desc}_B is baseline-distinct and has equal braid descriptors; applying I gives equal invariant values, so the pair belongs to Δ^{desc}_I. A pair in $\Delta^{\text{desc}}_{I|B}$ has unequal braid descriptors with equal invariant values. Braid compatibility implies that baseline-equivalent histories must have equal braid descriptors by Proposition 4.8; hence the unequal braid descriptors force baseline distinction. The pair therefore also belongs to Δ^{desc}_I. □

Proposition 7.2 gives a general accounting rule for downstream invariants. A derived invariant is analytically useful because it can compress a braid into a simpler feature, while that same compression can identify braid-distinct histories. The relevant question is therefore not whether an invariant loses information, but which information it preserves for the declared analytical purpose.

7.2 Elementary Algebraic Invariants

Two elementary invariants can be derived directly from the Artin presentation and illustrate different kinds of descriptive compression. The first retains the permutation of strand labels induced by a braid. The second retains the total signed exponent of its Artin generators.

Let S_n denote the symmetric group on $\{1, \ldots, n\}$. For $1 \leq i \leq n-1$, let $s_i \in S_n$ denote the adjacent transposition exchanging i and $i+1$. Define the generator assignment in Equation 86.

$$\sigma_i \longmapsto s_i, \qquad \sigma_i^{-1} \longmapsto s_i. \tag{86}$$

The same transposition appears for both crossing orientations because a permutation records only which two labels exchange positions, not the over-crossing or under-crossing orientation.

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Proposition 7.3 (Natural permutation homomorphism)**.** *The assignment in Equation 86 extends uniquely to a surjective group homomorphism* $\pi_{\text{perm}} : B_n \to S_n$.

Proof. The adjacent transpositions satisfy the relations $s_i s_j = s_j s_i$ when $|i - j| \geq 2$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. They also satisfy $s_i^2 = e$. Consequently, the images of the generators respect every relation in Equation 38: the Artin commutation and braid relations hold, and the inverse relations hold because $s_i^{-1} = s_i$. The universal property of the presented group therefore gives a unique homomorphism from B_n to S_n. Surjectivity follows because the adjacent transpositions generate S_n [2]. □

The permutation invariant of a history is the composite $D_{\text{perm}} := \pi_{\text{perm}} \circ D_B$. It records the terminal permutation of persistent strand labels represented by the braid while discarding pure braiding that leaves the permutation unchanged.

The second invariant is derived from signed exponent count. For a word $w = a_r \cdots a_1 \in \Sigma_n^*$, write each letter as $a_j = \sigma_{i_j}^{\varepsilon_j}$ with $\varepsilon_j \in \{-1, +1\}$. Define the word exponent sum by Equation 87.

$$E(w) := \sum_{j=1}^{r} \varepsilon_j, \qquad E(\mathbf{1}) := 0. \tag{87}$$

To obtain an invariant of braid elements, the quantity in Equation 87 must be independent of the chosen word representative.

Proposition 7.4 (Exponent-sum homomorphism)**.** *The word exponent sum descends through the quotient in Equation 39 to a well-defined surjective homomorphism* $e : B_n \to \mathbb{Z}$ *satisfying Equation 88.*

$$e(\sigma_i) = 1, \qquad e(\sigma_i^{-1}) = -1 \quad \textit{for } 1 \leq i \leq n - 1. \tag{88}$$

Proof. It is enough to verify that each generating relation in Equation 38 preserves the quantity in Equation 87. The distant-commutation relation contains two positive generators on each side, so both sides have exponent sum 2. The braid relation contains three positive generators on each side, so both sides have exponent sum 3. The inverse relations replace $\sigma_i \sigma_i^{-1}$ or $\sigma_i^{-1} \sigma_i$ by the empty word; both sides then have exponent sum 0. Because $\approx_A$ is generated by these relations together with congruence closure, equivalent words have equal exponent sum. Hence E defines a map $e : B_n \to \mathbb{Z}$.

For words u and v, concatenation gives $E(uv) = E(u) + E(v)$. The descended map therefore satisfies $e(\beta\gamma) = e(\beta) + e(\gamma)$ and is a group homomorphism from B_n to the additive group $\mathbb{Z}$. Surjectivity follows from $e(\sigma_1^k) = k$ for every integer k. This is the standard abelianizing exponent-sum map [2]. □

The permutation and exponent-sum maps preserve complementary information. Their individual noninjectivity can be shown without invoking a classification theorem.

Proposition 7.5 (Coarseness of the elementary invariants)**.** *For* $n \geq 3$*, neither* π_{perm} *nor* e *is injective.*

Proof. For the permutation map, $\pi_{\text{perm}}(\sigma_1^2) = s_1^2 = e$ in S_n, while Proposition 7.4 gives $e(\sigma_1^2) = 2$ and $e(e) = 0$. Hence σ_1^2 is not the identity braid, so π_{perm} is not injective.

For the exponent-sum map, Proposition 7.4 gives equal exponent sums for σ_1 and σ_2. Proposition 7.3 maps them to the distinct adjacent transpositions s_1 and s_2 when $n \geq 3$. Hence $\sigma_1 \neq \sigma_2$ while $e(\sigma_1) = e(\sigma_2) = 1$, so e is not injective. □

Proposition 7.5 illustrates why scalar or finite-valued braid invariants should be interpreted as lower-resolution descriptors rather than substitutes for the full braid class.

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7.3 Relative Completeness of Invariant Families

Several invariants can be used jointly. Let Λ be an index set, and for each $\lambda \in \Lambda$ let $I_\lambda : B_n \to \mathcal{A}_\lambda$ be a derived braid invariant. Define their joint map by Equation 89.

$$I_\Lambda : B_n \longrightarrow \prod_{\lambda\in\Lambda} \mathcal{A}_\lambda, \qquad I_\Lambda(\beta) := \big(I_\lambda(\beta)\big)_{\lambda\in\Lambda}. \tag{89}$$

The product in Equation 89 is the Cartesian product of the invariant codomains. Equality of two joint invariant values therefore means equality of every component invariant.

For two invariants I and J, the equivalence induced by the joint map is the intersection of their individual equivalences. This is stated in Equation 90.

$$\equiv_{(I,J)} = \equiv_I \cap \equiv_J . \tag{90}$$

Equation 90 follows from equality in a Cartesian product: two joint values are equal exactly when their I-components and their J-components are both equal. Adding invariant components can therefore refine a derived description, although it cannot recover distinctions already collapsed by D_B.

Completeness of an invariant should be stated relative to the braids actually realized by the descriptor rather than assumed over all of B_n. Define the realized braid image by Equation 91.

$$\mathcal{B}_D := \mathrm{im}(D_B) \subseteq B_n. \tag{91}$$

Definition 7.6 (Completeness relative to the realized braid image)**.** A derived invariant $I : B_n \to \mathcal{A}$ is *complete relative to* $\mathcal{B}_D$ when its restriction $I|_{\mathcal{B}_D} : \mathcal{B}_D \to \mathcal{A}$ is injective.

The definition gives an exact criterion for preservation of braid-level descriptive resolution.

Proposition 7.7 (Relative completeness criterion)**.** *A derived invariant I is complete relative to $\mathcal{B}_D$ if and only if Equation 92 holds on Γ_B.*

$$\equiv_I = \equiv_B . \tag{92}$$

Proof. Assume first that $I|_{\mathcal{B}_D}$ is injective. Proposition 7.1 already gives $\equiv_B \subseteq \equiv_I$. If $h_1 \equiv_I h_2$, then $I(D_B(h_1)) = I(D_B(h_2))$. Both braid descriptors lie in $\mathcal{B}_D$, so injectivity of the restriction gives $D_B(h_1) = D_B(h_2)$, hence $h_1 \equiv_B h_2$. Thus $\equiv_I \subseteq \equiv_B$, proving equality.

Conversely, assume Equation 92. Let $\beta_1, \beta_2 \in \mathcal{B}_D$ satisfy $I(\beta_1) = I(\beta_2)$. Choose histories $h_1, h_2 \in \Gamma_B$ with $D_B(h_i) = \beta_i$. Equality of invariant values gives $h_1 \equiv_I h_2$. Equation 92 gives $h_1 \equiv_B h_2$, so $\beta_1 = \beta_2$. Hence the restriction of I is injective. □

Definition 7.6 avoids requiring an invariant to classify braid elements that can never occur in the chosen application. It also makes clear that a family of individually incomplete invariants may become complete on a restricted realized image when their joint map is injective there.

For example, the joint elementary map in Equation 93 combines terminal permutation and exponent sum.

$$I_{\text{pe}} : B_n \longrightarrow S_n \times \mathbb{Z}, \qquad I_{\text{pe}}(\beta) := \big(\pi_{\text{perm}}(\beta), e(\beta)\big). \tag{93}$$

Equation 90 implies that the descriptor induced by I_{pe} is at least as fine as either component descriptor. No general completeness claim for I_{pe} is required here.

7.4 Conjugacy-Level Description of Cyclic Re-Cutting

Some trajectory analyses treat the choice of a temporal cut in a recurrent or periodically interpreted history as descriptively irrelevant. Braid conjugacy provides a natural quotient for one important form of such re-cutting.

Define conjugacy on B_n by Equation 94.

$$\beta \sim_{\text{conj}} \beta' \quad \Longleftrightarrow \quad \text{there exists } \alpha \in B_n \text{ such that } \beta' = \alpha^{-1}\beta\alpha. \tag{94}$$

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Conjugacy is an equivalence relation on B_n. Let $\mathrm{Conj}(B_n) := B_n/\!\sim_{\mathrm{conj}}$ denote the set of conjugacy classes, and let the canonical conjugacy descriptor be the map in Equation 95.

$$C_{\mathrm{conj}} : B_n \longrightarrow \mathrm{Conj}(B_n), \qquad C_{\mathrm{conj}}(\beta) := [\beta]_{\mathrm{conj}}. \tag{95}$$

The braid conjugacy problem and the structural significance of conjugacy are classical parts of braid-group theory [2].

A cyclic change in the cut of a factored braid produces conjugate elements. Let $\beta = uv$ with $u, v \in B_n$. Define the cyclicly re-cut braid by $\beta' := vu$. The relation between the two elements is derived in Equation 96.

$$\beta' = vu = u^{-1}(uv)u = u^{-1}\beta u. \tag{96}$$

Equation 96 shows that $C_{\mathrm{conj}}(\beta') = C_{\mathrm{conj}}(\beta)$. Thus a descriptor that factors through conjugacy is insensitive to this change of cut.

The corresponding history-level descriptor is the composite stated in Equation 97.

$$D_{\mathrm{conj}} := C_{\mathrm{conj}} \circ D_B : \Gamma_B \longrightarrow \mathrm{Conj}(B_n). \tag{97}$$

By Proposition 7.1, D_{conj} is weakly coarser than D_B. This coarsening is appropriate only when conjugate braid descriptions are intended to represent the same analytical object. A nonperiodic historical application with a substantively fixed beginning and end may need the full braid element rather than its conjugacy class.

7.5 Markov Equivalence in Closure-Level Description

A braid can be converted into a link by joining corresponding upper and lower endpoints around the outside of the braid cylinder. The resulting object is the *closure* of the braid. Closure provides another trajectory-descriptive quotient because several braid elements, including braids of different strand number, can yield isotopic closed links.

To compare braid indices, define the disjoint union of braid groups by Equation 98.

$$B_* := \coprod_{n \geq 1} B_n. \tag{98}$$

Let $\mathcal{L}$ denote the set of ambient-isotopy classes of oriented links in three-dimensional space. Define the closure map in Equation 99.

$$\mathrm{Cl} : B_* \longrightarrow \mathcal{L}, \qquad \mathrm{Cl}(\beta) := [\widehat{\beta}]_{\mathrm{link}}, \tag{99}$$

where $\widehat{\beta}$ denotes the standard closure of the braid and $[\widehat{\beta}]_{\mathrm{link}}$ its oriented link-isotopy class.

The equivalence relation governing the fibers of Equation 99 is Markov equivalence. Let $\iota_n : B_n \to B_{n+1}$ denote the standard inclusion that sends each generator σ_i of B_n to the generator with the same name in B_{n+1}. Define $\sim_M$ to be the equivalence relation on B_* generated by the two moves in Equation 100.

$$\begin{aligned} &\beta \longmapsto \alpha^{-1}\beta\alpha && \text{within } B_n, \\ &\beta \longmapsto \iota_n(\beta)\sigma_n^{\varepsilon}, \qquad \varepsilon \in \{-1, +1\} && \text{from } B_n \text{ to } B_{n+1}. \end{aligned} \tag{100}$$

The first move is conjugation; the second is stabilization or destabilization. Markov's theorem states that two braids have ambient-isotopic oriented closures exactly when they are related by a finite sequence of these moves [2]. Hence the closure map factors through the Markov quotient and induces a bijection stated in Equation 101.

$$\overline{\mathrm{Cl}} : B_*/\!\sim_M \longrightarrow \mathrm{im}(\mathrm{Cl}) \subseteq \mathcal{L}. \tag{101}$$

The map in Equation 101 is bijective onto the link types represented by braid closures; Alexander's theorem supplies the classical result that every oriented link type has a closed-braid representative, although that stronger coverage result is not required for the present descriptive argument [2].

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Table 5: Selected derived descriptors downstream from the braid element.

Derived descriptor	Preserved feature	Principal information omitted
π_{perm}	Permutation of persistent strand labels	Pure braiding and crossing orientation beyond the endpoint permutation
e	Total signed Artin-generator exponent	Which adjacent strands participate and most nonabelian braid structure
I_{pe}	Joint permutation and exponent-sum data	Braid distinctions unresolved by the pair of elementary invariants
C_{conj}	Conjugacy class of the braid	Choice among conjugate representatives, including cyclic re-cutting of a factored braid
Cl	Oriented link type of the braid closure	Open-braid distinctions removed by conjugation and stabilization-type Markov moves

Any link invariant $K : \mathcal{L} \to \mathcal{A}_K$ can therefore be composed with closure to give a derived braid invariant $K \circ \text{Cl}$. Relative to a fixed B_n, this construction can identify braid elements that are distinct even up to conjugacy because stabilization-mediated Markov equivalence is defined across braid indices. Closure-based quantities should therefore be interpreted as invariants of the closed link representation rather than as complete invariants of the original open braid trajectory.

7.6 Interpretive Scope of Derived Invariants

The constructions above form a hierarchy of progressively coarser descriptive questions. The full braid descriptor asks for the element $D_B(h) \in B_n$. The elementary maps π_{perm} and e ask, respectively, for terminal strand permutation and total signed crossing exponent. The conjugacy descriptor asks for the braid only up to inner relabeling by conjugation, which can remove dependence on a cyclic cut in suitable recurrent interpretations. The closure descriptor asks for the isotopy class of the closed link and therefore forgets still more of the open-braid representation.

Table 5 summarizes the distinctions preserved by these derived descriptors.

More elaborate braid-derived quantities can also be attached after the braid class has been obtained. In dynamical applications, for example, braid and mapping-class methods have been used to estimate topological entropy from observed two-dimensional particle trajectories even when the underlying flow field is poorly known [8]. Such quantities are important examples of the analytical leverage provided by a trajectory-descriptive layer: a topological description can support quantitative downstream analysis without first identifying the full generative rule.

The present paper does not treat any one downstream invariant as universally privileged. Proposition 7.1 and Definition 7.6 instead provide a common criterion for evaluating them. Every derived invariant specifies an additional quotient of braid-level information, and its adequacy depends on the distinctions the analysis intends to preserve. A coarse invariant can be useful when its omitted distinctions are irrelevant to the research question; the same invariant can be inadequate when those distinctions carry substantive meaning.

The next section examines a related benefit of quotient-based description: robustness under admissible perturbation. The central issue there is not merely that a topological invariant takes equal values on equal braid elements, but that an open neighborhood of geometric trajectory perturbations can preserve the same braid class so long as the perturbation does not cross a topology-changing boundary.

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8 Perturbation Robustness of the Braid Descriptor

This section formalizes the robustness property that motivates a topological trajectory description. Its role is to distinguish invariance under admissible deformation from robustness certified by a metric perturbation bound. The objective is to show precisely why braid description can remain stable under substantial geometric variation while also identifying the boundary at which that stability can fail. Section 8.1 derives robustness directly from the homotopy quotient used in the definition of D_B. Section 8.2 then adds a Euclidean perturbation model for strand representatives and derives a sufficient quantitative condition for braid-class preservation. Section 8.3 identifies the collision discriminant as the boundary through which a continuous family must pass before its braid class can change. Section 8.4 relates robustness to descriptive resolution. Section 8.5 states the corresponding uncertainty certificate for reconstructed trajectories.

8.1 Homotopy Robustness of the Descriptor Construction

Corollary 8.3 later extends the same metric-free robustness to every derived braid invariant. The first robustness result itself requires no metric. It follows directly from the homotopy quotient introduced in Equation 32.

Definition 8.1 (Admissible deformation of extracted trajectories)**.** Let $h_0, h_1 \in \Gamma_B$. An *admissible deformation* from $\mathcal{T}_B(h_0)$ to $\mathcal{T}_B(h_1)$ is a continuous map $H : [0,1]\times[0,1] \to \mathcal{U}_n$ satisfying the boundary conditions in Equation 102.

$$\begin{aligned} &H(0,t) = \mathcal{T}_B(h_0)(t), \quad H(1,t) = \mathcal{T}_B(h_1)(t), \\ &H(s,0) = c_0, \qquad\qquad H(s,1) = c_0 \qquad \text{for all } s,t \in [0,1]. \end{aligned} \tag{102}$$

Definition 8.1 is exactly a based homotopy between the two extracted loops. The word *admissible* emphasizes that the deformation remains inside $\mathcal{U}_n$, so the represented components remain pairwise distinct throughout the deformation.

Proposition 8.2 (Homotopy robustness of the braid descriptor)**.** *If an admissible deformation in the sense of Definition 8.1 exists between $\mathcal{T}_B(h_0)$ and $\mathcal{T}_B(h_1)$, then Equation 103 holds.*

$$D_B(h_0) = D_B(h_1). \tag{103}$$

Proof. An admissible deformation is a based homotopy. Equation 32 therefore gives $Q_B(\mathcal{T}_B(h_0)) = Q_B(\mathcal{T}_B(h_1))$. Applying the isomorphism κ_n and using Equation 33 yields Equation 103. □

Proposition 8.2 states the strongest metric-free form of robustness available in the baseline formalism. The descriptor ignores every deformation that remains inside one based-homotopy class. This property is not added after the braid construction; it is the direct consequence of defining the descriptor through the quotient Q_B.

Any derived invariant inherits the same robustness.

Corollary 8.3 (Robustness of downstream braid invariants)**.** *Let $I : B_n \to \mathcal{A}$ be any derived braid invariant and let $D_I = I \circ D_B$ be the corresponding history descriptor. Under the assumptions of Proposition 8.2, Equation 104 holds.*

$$D_I(h_0) = D_I(h_1). \tag{104}$$

Proof. Proposition 8.2 gives equal braid descriptors. Applying I to the common braid element gives Equation 104. □

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8.2 Metric Perturbation Certificates for Geometric Braid Representatives

Homotopy robustness is qualitative: it assumes that an admissible homotopy exists. A concrete trajectory reconstruction may instead come with a geometric error bound. A sufficient metric condition for braid-class preservation can then be derived after adding Euclidean structure at this lower descriptive layer.

Represent a geometric n-strand braid by continuous strand trajectories $z_i : [0,1] \to \mathbb{R}^2$ for $1 \le i \le n$. The strands satisfy $z_i(t) \neq z_j(t)$ for all $i \neq j$ and all t. Their unordered configuration defines the extracted loop through Equation 105.

$$q_z(t) := \{z_1(t), \dots, z_n(t)\} \in \mathcal{U}_n. \tag{105}$$

The labels on the z_i trace persistent strands through time. Because the configuration in Equation 105 is unordered, a strand may end at the initial position of a different label while the configuration itself returns to c_0.

The minimum pairwise separation of the representative $z = (z_1, \dots, z_n)$ is defined in Equation 106.

$$\rho(z) := \min_{\substack{t\in[0,1]\\ 1\le i<j\le n}} \|z_i(t) - z_j(t)\|_2 . \tag{106}$$

The symbol $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^2$. Continuity of the strands and compactness of $[0,1]$ imply that the minimum in Equation 106 exists. Because the representative is collision-free, every pairwise distance is positive. Continuity on the compact domain and finiteness of the strand pairs therefore give $\rho(z) > 0$.

Let $\widetilde{z}_i : [0,1] \to \mathbb{R}^2$ be perturbed strand trajectories with the same labeled endpoints as z_i. Define the perturbation size by Equation 107.

$$\|\widetilde{z} - z\|_\infty := \max_{\substack{t\in[0,1]\\ 1\le i\le n}} \|\widetilde{z}_i(t) - z_i(t)\|_2 . \tag{107}$$

The endpoint condition means $\widetilde{z}_i(0) = z_i(0)$ and $\widetilde{z}_i(1) = z_i(1)$ for every strand i. This condition ensures that the straight-line interpolation used below is relative to the endpoints.

Proposition 8.4 (Separation-margin robustness certificate)**.** *Let z be a collision-free geometric braid representative and let $\widetilde{z}$ be an endpoint-fixed perturbation. If the inequality in Equation 108 holds, then q_z and $q_{\widetilde{z}}$ are based-homotopic in $\mathcal{U}_n$ and therefore define the same braid element.*

$$\|\widetilde{z} - z\|_\infty < \frac{\rho(z)}{2}. \tag{108}$$

Proof. Write $\varepsilon := \|\widetilde{z} - z\|_\infty$. For $s \in [0,1]$, define the straight-line interpolation of the labeled strands by Equation 109.

$$z_i^{(s)}(t) := (1-s)z_i(t) + s\widetilde{z}_i(t). \tag{109}$$

The endpoint condition on $\widetilde{z}_i$ implies $z_i^{(s)}(0) = z_i(0)$ and $z_i^{(s)}(1) = z_i(1)$ for every s.

It remains to show that no collision occurs during the interpolation. For $i \neq j$, the triangle inequality gives the lower bound in Equation 110.

$$\begin{aligned} \left\| z_i^{(s)}(t) - z_j^{(s)}(t) \right\|_2 &= \left\| z_i(t) - z_j(t) + s\big[(\widetilde{z}_i(t) - z_i(t)) - (\widetilde{z}_j(t) - z_j(t))\big] \right\|_2 \\ &\ge \|z_i(t) - z_j(t)\|_2 - s\,\|\widetilde{z}_i(t) - z_i(t)\|_2 - s\,\|\widetilde{z}_j(t) - z_j(t)\|_2 \\ &\ge \rho(z) - 2\varepsilon. \end{aligned} \tag{110}$$

Equation 108 makes the final expression in Equation 110 strictly positive. Hence all interpolated strand configurations remain collision-free.

The unordered configuration $H(s,t) := \{z_1^{(s)}(t), \dots, z_n^{(s)}(t)\}$ therefore defines a continuous map $H : [0,1]^2 \to \mathcal{U}_n$. Its endpoint conditions are fixed because the strand endpoints are fixed. Thus H is an admissible deformation in the sense of Definition 8.1. Proposition 8.2 gives equal braid descriptors. □

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Proposition 8.4 supplies a computable sufficient condition. It is not claimed to be necessary. Perturbations larger than $\rho(z)/2$ can still remain in the same braid class; the proposition guarantees stability only inside a simple endpoint-fixed sup-norm neighborhood.

The proposition also explains why a topological descriptor can be robust to metric noise without requiring a metric at the highest GR layer. The Euclidean norm enters only after a particular geometric braid representation has been chosen. Metric structure is therefore an optional lower-level instrument used to certify preservation of a previously defined topological class.

8.3 Topology-Changing Role of the Collision Discriminant

The robustness certificate above is controlled by distance from collision. The corresponding topology-changing boundary can be written directly in the ordered configuration ambient space. Define the collision discriminant by Equation 111.

$$\Sigma_n := \left\{(x_1, \ldots, x_n) \in (\mathbb{R}^2)^n : x_i = x_j \text{ for at least one } i \neq j\right\}. \tag{111}$$

The ordered configuration space is the complement $\mathrm{Conf}_n(\mathbb{R}^2) = (\mathbb{R}^2)^n \setminus \Sigma_n$ [4]. The unordered configuration space is obtained by quotienting this collision-free set by the symmetric-group action.

A continuous family of geometric braid representatives can change braid class only if the family ceases to define a based homotopy in the collision-free configuration space. For endpoint-fixed strand families, one concrete way this can occur is passage through Σ_n.

Proposition 8.5 (Collision necessity for class change in an endpoint-fixed strand family)**.** *Let $z_i^{(s)} : [0,1] \to \mathbb{R}^2$ be jointly continuous in $(s,t) \in [0,1]^2$, with labeled endpoints independent of s. Suppose the braid elements represented at $s = 0$ and $s = 1$ are distinct. Then there exist $s^*, t^* \in [0,1]$ and indices $i \neq j$ such that Equation 112 holds.*

$$z_i^{(s^*)}(t^*) = z_j^{(s^*)}(t^*). \tag{112}$$

Proof. Assume, for contradiction, that Equation 112 never occurs. Then the unordered configuration determined by the strands remains in $\mathcal{U}_n$ for every $(s,t) \in [0,1]^2$. Joint continuity and the endpoint condition define an admissible deformation between the $s = 0$ and $s = 1$ extracted loops. Proposition 8.2 then implies that the two braid descriptors are equal, contradicting the assumption that the braid elements are distinct. Hence at least one collision must occur. □

Proposition 8.5 is a statement about continuous endpoint-fixed families of strand representatives. It does not assert that every empirical change in a reconstructed braid is caused by a physical collision. A change in the extraction rule, strand identity assignment, endpoint convention, or data preprocessing can also change the descriptor without corresponding to a continuous family of one fixed extraction datum. The proposition isolates the topology-changing boundary only after those descriptive choices have been held fixed.

The minimum separation $\rho(z)$ in Equation 106 can therefore be interpreted as a local margin from the collision boundary for that geometric representative. As $\rho(z)$ decreases, the simple certified perturbation radius in Proposition 8.4 also decreases. Near-collision trajectories are consequently more sensitive to reconstruction uncertainty than trajectories that remain uniformly separated.

8.4 Resolution Dependence of Robustness

The robustness of braid description is inseparable from its quotient structure. Section 5 showed that the homotopy quotient Q_B removes distinctions among extracted loops in the same based-homotopy class. Proposition 8.2 shows that exactly those removed distinctions become admissible deformations under which the descriptor is invariant.

This relation can be stated abstractly. Let $D : \Gamma_D \to \Omega_D$ be any descriptor and let $\equiv_D$ be its induced equivalence. A perturbation family P is invisible to D whenever every history generated by

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that perturbation remains in the same $\equiv_D$-class. Increasing descriptive resolution means replacing D by a finer descriptor D' with $\equiv_{D'} \subseteq \equiv_D$. Such a refinement can distinguish histories that D treated as equivalent, but every newly distinguished pair is, by definition, a pair against which the refined descriptor is no longer invariant.

For the braid descriptor, the basic tradeoff is therefore between geometric discrimination and homotopy robustness. A descriptor that records exact extracted loops has equivalence $\equiv_T$ and is finer than D_B. It can distinguish two homotopic loops with different geometry. The braid descriptor identifies those loops and is invariant under the corresponding deformation. Equation 113 records this relation.

$$\equiv_T \subseteq \equiv_B \quad \Longrightarrow \quad \text{every } \equiv_B \text{-preserving deformation may contain distinctions visible to } D_T. \tag{113}$$

Equation 113 does not define a numerical tradeoff curve. It identifies the logical source of robustness: a descriptor is stable under precisely those differences that its equivalence relation chooses not to preserve.

This point also applies to the derived invariants of Section 7. Because $\equiv_B \subseteq \equiv_I$ by Proposition 7.1, an invariant descriptor D_I can be robust to distinctions that already change the braid element when those distinct braid elements share the same invariant value. Such robustness comes with additional information loss measured by $\Delta^{\text{desc}}_{I|B}$. Robustness should therefore always be stated together with the descriptive resolution at which it is obtained.

8.5 Certified Braid Stability for Uncertainty Sets

The preceding results can be translated into a simple uncertainty statement for reconstructed trajectories. Let z be a nominal geometric braid representative and let $\varepsilon > 0$ be an error bound. Define the endpoint-fixed uncertainty set in Equation 114.

$$\mathcal{U}_\varepsilon(z) := \left\{ \widetilde{z} : \begin{array}{l} \|\widetilde{z} - z\|_\infty \le \varepsilon, \\ \widetilde{z}_i(0) = z_i(0),\ \widetilde{z}_i(1) = z_i(1) \text{ for every } i \end{array} \right\}. \tag{114}$$

The set in Equation 114 contains every strand reconstruction whose worst-case pointwise deviation from the nominal reconstruction is at most ε and whose endpoints obey the same convention.

Corollary 8.6 (Certified stability under bounded reconstruction error)**.** *If the bound in Equation 115 holds, then every trajectory family $\widetilde{z} \in \mathcal{U}_\varepsilon(z)$ represents the same braid element as z.*

$$\varepsilon < \frac{\rho(z)}{2}. \tag{115}$$

Proof. For every $\widetilde{z} \in \mathcal{U}_\varepsilon(z)$, Equation 114 gives $\|\widetilde{z} - z\|_\infty \le \varepsilon$. Equation 115 therefore implies the strict inequality required by Proposition 8.4, except possibly at the boundary value $\|\widetilde{z} - z\|_\infty = \varepsilon$; the bound on ε remains strictly below $\rho(z)/2$, so the proposition applies. Hence every member of the uncertainty set has the same braid class as z. □

Corollary 8.6 gives a sufficient robustness certificate for imperfect trajectory reconstruction. Its interpretation is limited to the declared extraction model: component identities, endpoint conventions, the number of strands, and the Euclidean embedding are held fixed. Uncertainty in those choices belongs to a different layer and is not covered by the geometric bound.

The broader methodological point is nevertheless important. A braid-theoretic descriptor can remain unchanged over a nontrivial set of geometric reconstructions, and the size of a certified neighborhood can be derived from a separation margin. The next section turns from geometric robustness to historical information. It distinguishes topological path-history, which is retained by the braid descriptor, from mechanism-level collapse-history, which was defined in the earlier GR metamodel and records irreversible loss of distinctions at the generative layer.

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9 Historical Information Layers

This section distinguishes two forms of historical information that occupy different levels of the GR architecture. Its role is to separate the topological path-history retained by a braid descriptor from the mechanism-level collapse-history introduced in the earlier metamodel. The objective is to show that these records can coexist without being identified, and to determine when collapse-history can be recovered from a realized trajectory at all. Section 9.1 derives a cumulative collapse profile from a mechanism word. Section 9.2 defines the corresponding braid path-history. Section 9.3 compares their domains and logical roles. Section 9.4 derives the exact condition under which mechanism-level collapse-history descends to a well-defined function of realized histories. Section 9.5 constructs a joint historical record when both layers are retained.

9.1 Mechanism-Level Collapse History

The earlier metamodel associates each mechanism $m \in M$ with the collapse relation $\Delta^{\text{mech}}(m) \subseteq S \times S$, consisting of pairs that are distinct modulo $\sim$ before the mechanism acts and congruent afterward [6]. The same paper proves that collapse is monotone under subsequent composition: once a distinction has been collapsed, later mechanisms preserving the congruence cannot restore it. The present section applies that result to the prefix products already defined in Equation 13.

Let $w = (m_1, \dots, m_k) \in M^k$ be a mechanism word. Its prefix products $p_j(w)$ describe the cumulative mechanism applied through step j. Define the cumulative collapse relation at step j by Equation 116.

$$C_j(w) := \Delta^{\text{mech}}(p_j(w)) \subseteq S \times S, \qquad 1 \le j \le k. \tag{116}$$

The symbol $C_j(w)$ is derived from the earlier collapse operator and the prefix product. It is not a new primitive.

The relations in Equation 116 form a monotone chain.

Proposition 9.1 (Monotonicity of cumulative collapse along a mechanism word)**.** *For every mechanism word $w = (m_1, \dots, m_k)$ and every $1 \le j < k$, Equation 117 holds.*

$$C_j(w) \subseteq C_{j+1}(w). \tag{117}$$

Proof. Equation 13 gives $p_{j+1}(w) = m_{j+1} p_j(w)$. Let $(x, y) \in C_j(w)$. By Equation 116, $(x, y) \in \Delta^{\text{mech}}(p_j(w))$, so $x \not\sim y$ and $\alpha(p_j(w), x) \sim \alpha(p_j(w), y)$. Because $\sim$ is preserved by every mechanism, applying m_{j+1} to the congruent outputs gives $\alpha(m_{j+1}, \alpha(p_j(w), x)) \sim \alpha(m_{j+1}, \alpha(p_j(w), y))$. The action law identifies these two terms with $\alpha(p_{j+1}(w), x)$ and $\alpha(p_{j+1}(w), y)$, respectively. Hence $(x, y) \in \Delta^{\text{mech}}(p_{j+1}(w)) = C_{j+1}(w)$. □

Proposition 9.1 is the prefix-word form of the earlier monotonicity result. It permits a mechanism sequence to be associated with an ordered record of accumulated distinction loss.

Define the set of length-k monotone collapse chains by Equation 118.

$$\mathscr{C}_k(S, \sim) := \left\{ (R_1, \dots, R_k) : \begin{array}{l} R_j \subseteq S \times S \text{ for } 1 \le j \le k, \\ R_1 \subseteq R_2 \subseteq \cdots \subseteq R_k \end{array} \right\}. \tag{118}$$

The notation $\mathscr{C}_k(S, \sim)$ is local to this paper. The argument $\sim$ records that the relations R_j arise from the congruence-relative collapse operator, even though Equation 118 itself lists only their set-theoretic nesting.

Definition 9.2 (Collapse-history map)**.** For $k \ge 1$, define the *collapse-history map* by Equation 119.

$$H_{\Delta,k} : S \times M^k \longrightarrow \mathscr{C}_k(S, \sim), \qquad H_{\Delta,k}(x_0, w) := \big(C_1(w), \dots, C_k(w)\big). \tag{119}$$

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The initial state x_0 appears in the domain of Equation 119 because the map is defined on the same generative specifications used by Gen_k. Its value depends only on the mechanism word w, since each $C_j(w)$ is defined from the action of the cumulative mechanism on the full carrier. Retaining x_0 in the domain makes the relationship between collapse-history and realized-history fibers explicit in Section 9.4.

Because the chain is monotone, the final relation summarizes whether any collapse occurred along the word. Equation 120 states this equivalence.

$$H_{\Delta,k}(x_0, w) = (\varnothing, \ldots, \varnothing) \quad \Longleftrightarrow \quad C_k(w) = \varnothing. \tag{120}$$

Definition 9.2 makes the collapse chain a function on generative specifications. The forward implication is immediate. For the reverse implication, Proposition 9.1 gives $C_j(w) \subseteq C_k(w)$ for every j, so $C_k(w) = \varnothing$ forces every earlier $C_j(w)$ to be empty.

9.2 Trajectory-Level Path History

Collapse-history belongs to the generative mechanism layer. The braid descriptor records a different historical object: the topological class of a selected realized trajectory. To compare the two layers on a common domain, fix k and define the braid-admissible generative domain by Equation 121.

$$U_{k,B} := \left\{ u \in S \times M^k : \mathrm{Gen}_k(u) \in \Gamma_B \right\}. \tag{121}$$

The notation u in Equation 121 denotes a complete generative specification (x_0, w) of length k.

Definition 9.3 (Braid path-history map)**.** The *braid path-history map* on length-k generative specifications is defined by Equation 122.

$$H_{B,k} : U_{k,B} \longrightarrow B_n, \qquad H_{B,k}(u) := D_B(\mathrm{Gen}_k(u)). \tag{122}$$

Equation 122 is the restriction of the descriptive composite in Equation 27 to the braid descriptor. It records the topological path-history retained after realization, extraction, and homotopy quotienting. Unlike Equation 119, it factors through the realized history by construction: if two generative specifications yield the same history, then their values under $H_{B,k}$ are necessarily equal whenever that history lies in Γ_B.

The two maps therefore answer different historical questions. The map $H_{\Delta,k}$ records which carrier distinctions have become irrecoverably collapsed under the cumulative mechanisms. The map $H_{B,k}$ records which braid class is assigned to the selected relational trajectory generated by those mechanisms. Their codomains, equivalence structures, and dependence on the generative specification are different.

Definition 9.3 formalizes the path-history side of this distinction. The distinction also clarifies the use of the word *history* in the earlier metamodel. The statement that a group of mechanisms has no history there concerns history as irreversible accumulation of lost discrimination: if the mechanism family is a group preserving the congruence, the mechanism-level collapse relation is empty [6]. A braid group element can nevertheless encode a nontrivial path-history because B_n is used here as the codomain of a trajectory descriptor rather than as the mechanism family acting on S.

9.3 Independence of Historical Information Layers

The formal definitions above permit the two records to be compared without identifying them. For $u \in U_{k,B}$, the path-history is $H_{B,k}(u) \in B_n$, while the collapse-history is $H_{\Delta,k}(u) \in \mathscr{C}_k(S, \sim)$. No map between these codomains has been assumed.

The absence of such a map is not merely a missing technical convenience. The two records are generated from different parts of the architecture. The collapse-history uses the mechanism word and the action on the carrier. The path-history uses the realized state sequence, the extraction datum, and the homotopy quotient. Consequently, the axioms developed so far do not imply that either record determines the other.

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Table 6: Two historical information layers distinguished by braid path-history and mechanism-level collapse-history.

	Empty cumulative collapse	Nonempty cumulative collapse
Trivial braid path-history	No recorded topological braiding and no mechanism-level distinction loss	Mechanism-level distinction loss not represented by the selected braid path
Nontrivial braid path-history	Topologically nontrivial relational path without recorded mechanism-level collapse	Topologically nontrivial path together with irreversible mechanism-level distinction loss

Two conditional observations make the separation precise. First, suppose all prefix mechanisms of w are injective modulo $\sim$. Proposition 3 of the earlier metamodel then gives $C_j(w) = \varnothing$ for every j, so Equation 119 is the empty chain. Nothing in this condition forces the braid path-history in Equation 122 to be the identity element of B_n. A braid-admissible extraction can still assign a nontrivial relational loop to the realized history.

Second, suppose a realized history has trivial braid descriptor $D_B(h) = e_{B_n}$, where e_{B_n} denotes the identity element of B_n. This condition says that the extracted loop is null-homotopic. It places no restriction on whether one of the cumulative mechanisms generating h is noninjective modulo $\sim$. A nonempty collapse-history is therefore compatible with a trivial braid path-history whenever the chosen mechanism sequence collapses distinctions not represented by the extracted loop.

These observations can be recorded as a two-axis classification. Table 6 distinguishes path-triviality from mechanism-level collapse. The table states logical possibilities permitted by the formal architecture; whether each cell is realized depends on the concrete dynamical and extraction instantiation.

The lower-right cell is especially important for later applications, but the formal framework does not privilege it. The purpose of the distinction is diagnostic: it prevents a nontrivial braid from being interpreted automatically as irreversible social loss, and it prevents mechanism-level collapse from being inferred from the triviality or nontriviality of the braid alone.

9.4 Descent of Collapse-History to Realized Histories

A braid path-history is a function of the realized history because $H_{B,k} = D_B \circ \mathrm{Gen}_k$. Collapse-history is defined on generative specifications and need not share this property. Generator underdetermination from Section 3.5 therefore creates a precise question: when does $H_{\Delta,k}$ descend through Gen_k to a well-defined function on $\Gamma_k(\mathfrak{D})$?

The necessary and sufficient condition is constancy on the fibers of the generation map.

Proposition 9.4 (Descent criterion for collapse-history)**.** *There exists a map $\overline{H}_{\Delta,k} : \Gamma_k(\mathfrak{D}) \to \mathscr{C}_k(S, \sim)$ satisfying Equation 123 if and only if $H_{\Delta,k}$ is constant on every fiber of* Gen_k.

$$H_{\Delta,k} = \overline{H}_{\Delta,k} \circ \mathrm{Gen}_k \,. \tag{123}$$

Proof. Assume first that a map $\overline{H}_{\Delta,k}$ satisfying Equation 123 exists. If u_1 and u_2 belong to the same fiber of Gen_k, then $\mathrm{Gen}_k(u_1) = \mathrm{Gen}_k(u_2)$. Applying the factorization gives $H_{\Delta,k}(u_1) = \overline{H}_{\Delta,k}(\mathrm{Gen}_k(u_1)) = \overline{H}_{\Delta,k}(\mathrm{Gen}_k(u_2)) = H_{\Delta,k}(u_2)$. Hence $H_{\Delta,k}$ is constant on every generation fiber.

Conversely, assume $H_{\Delta,k}$ is constant on every fiber of Gen_k. For a history $h \in \Gamma_k(\mathfrak{D})$, choose any $u \in S \times M^k$ with $\mathrm{Gen}_k(u) = h$; such a u exists because Gen_k is surjective. Define $\overline{H}_{\Delta,k}(h) := H_{\Delta,k}(u)$. If u' is another generative specification of the same history, then u and u' lie in the same generation fiber, so the assumed constancy gives $H_{\Delta,k}(u) = H_{\Delta,k}(u')$. The definition is therefore well defined. Substitution yields Equation 123. □

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Proposition 9.4 identifies an epistemic limitation of trajectory-level analysis. A realized state sequence can determine its mechanism-level collapse-history only when all generative specifications compatible with that sequence carry the same collapse profile. If one realized history can arise from two mechanism words with different cumulative collapse chains, the realized history alone does not identify which collapse-history occurred.

The ambiguity can be represented explicitly. For $h \in \Gamma_k(\mathfrak{D})$, define the set of compatible collapse histories by Equation 124.

$$\mathfrak{H}_{\Delta,k}(h) := \{H_{\Delta,k}(u) : u \in \mathcal{G}_k(h)\}, \tag{124}$$

where $\mathcal{G}_k(h) = \mathrm{Gen}_k^{-1}(\{h\})$ is the generation fiber defined in Section 3.5. Equation 124 is a singleton exactly when collapse-history is locally identifiable for that realized history.

Corollary 9.5 (Local identifiability of collapse-history)**.** *For a fixed history $h \in \Gamma_k(\mathfrak{D})$, all generative specifications compatible with h have the same collapse-history if and only if Equation 125 holds.*

$$|\mathfrak{H}_{\Delta,k}(h)| = 1. \tag{125}$$

Proof. Equation 124 is the image of the generation fiber under $H_{\Delta,k}$. Its cardinality is one exactly when that image contains one collapse chain, which is equivalent to constancy of $H_{\Delta,k}$ on the fiber $\mathcal{G}_k(h)$. □

The same result applies a fortiori to braid data. Because a braid descriptor is generally coarser than the full realized history, equality of braid path-history cannot resolve a collapse-history ambiguity that is already present within a single realized-history fiber.

9.5 Combined Historical Records

When the generative mechanism sequence is available, the two historical layers can be retained jointly without forcing either to reduce to the other. Define the joint historical record on $U_{k,B}$ by Equation 126.

$$H_{\text{joint},k} : U_{k,B} \longrightarrow B_n \times \mathscr{C}_k(S,\sim), \qquad H_{\text{joint},k}(u) := (H_{B,k}(u), H_{\Delta,k}(u)). \tag{126}$$

The product in Equation 126 is Cartesian. Its two coordinates retain topological path-history and cumulative collapse-history as separate components.

Let $\equiv^G_{B,k}$ denote equality under the path-history map on generative specifications, and let $\equiv^G_{\Delta,k}$ denote equality under the collapse-history map. These relations are defined in Equation 127.

$$\begin{aligned} u_1 \equiv^G_{B,k} u_2 \quad &\Longleftrightarrow \quad H_{B,k}(u_1) = H_{B,k}(u_2), \\ u_1 \equiv^G_{\Delta,k} u_2 \quad &\Longleftrightarrow \quad H_{\Delta,k}(u_1) = H_{\Delta,k}(u_2). \end{aligned} \tag{127}$$

The superscript G indicates that these equivalences are defined on generative specifications rather than directly on realized histories.

The equivalence induced by the joint record is the intersection of the two component equivalences.

Proposition 9.6 (Equivalence induced by the joint historical record)**.** *For $u_1, u_2 \in U_{k,B}$, Equation 128 holds.*

$$H_{\text{joint},k}(u_1) = H_{\text{joint},k}(u_2) \quad \Longleftrightarrow \quad \left(u_1 \equiv^G_{B,k} u_2 \text{ and } u_1 \equiv^G_{\Delta,k} u_2\right). \tag{128}$$

Proof. Equality in the Cartesian product $B_n \times \mathscr{C}_k(S,\sim)$ is componentwise. Equation 126 therefore gives equality of joint records exactly when both component equalities hold. Equation 127 converts those component equalities into the two displayed equivalences. □

Proposition 9.6 shows that joint historical equality requires simultaneous agreement at both the path-history and collapse-history levels. If the descent criterion in Proposition 9.4 holds, the joint record can itself be defined on realized histories. In that case, define $\overline{H}_{\text{joint},k}$ by Equation 129.

$$\overline{H}_{\text{joint},k}(h) := \left(D_B(h), \overline{H}_{\Delta,k}(h)\right) \quad \text{for } h \in \Gamma_k(\mathfrak{D}) \cap \Gamma_B. \tag{129}$$

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When collapse-history does not descend, Equation 129 is unavailable as a single-valued history descriptor. The braid path-history remains defined, while the collapse component must be represented by the set-valued ambiguity in Equation 124 or by additional information about the generating mechanisms.

The distinction between path-history and collapse-history therefore survives even when both are retained jointly. A braid element records the topological class of a selected generated trajectory. A collapse chain records cumulative loss of distinctions under the mechanisms that generated the system's evolution. Their coexistence supplies a richer historical record, while their formal separation prevents one kind of history from being inferred from the other without an explicit bridge. The next section uses this separation to characterize the expressive profile of braid-theoretic trajectory description and to identify the level, domain, and resolution restrictions that prevent it from serving as a generalized language for dynamics description.

10 Expressive Characteristics of Braid-Theoretic Description

This section characterizes the braid formalism as a lower-level descriptive language before the manuscript introduces a general criterion for dynamical-language generality. Its role is diagnostic. The objective is to state positively what the braid descriptor represents, to identify the formal sources of its selectivity, and to separate three different restrictions that can otherwise be conflated: its downstream position relative to generative mechanisms, its restricted domain of braid-admissible histories, and its quotient-based descriptive resolution. Section 10.1 formalizes the descriptive level through the fibers of the generation and braid-observation maps. Section 10.2 derives domain restriction from the admissibility conditions of the braid extraction datum. Section 10.3 isolates the topological selectivity of the descriptor and proves invariance under endpoint-preserving reparameterization. Section 10.4 states resolution restriction in terms of descriptive collapse. Section 10.5 summarizes the positive compositional and invariant-bearing capacities of the language. Section 10.6 relates the resulting profile to combinations of the four GR meta-primitives.

10.1 Generator Visibility at the Descriptive Level

The braid descriptor occupies a downstream position in the architecture developed in Sections 3–9. For a fixed history length k, the relevant generative domain is the braid-admissible set $U_{k,B}$ defined in Equation 121. The realized history is produced by the restriction of Gen_k to this set, and the corresponding braid-level observation is the path-history map $H_{B,k}$ from Equation 122.

To compare what the realized-history layer and braid layer can distinguish among generative specifications, define two relations on $U_{k,B}$. The generation-fiber relation is defined in Equation 130.

$$u_1 \equiv_{\mathrm{Gen},k} u_2 \quad \Longleftrightarrow \quad \mathrm{Gen}_k(u_1) = \mathrm{Gen}_k(u_2). \tag{130}$$

The braid-observation relation on generative specifications is defined in Equation 131.

$$u_1 \equiv^{G}_{B,k} u_2 \quad \Longleftrightarrow \quad H_{B,k}(u_1) = H_{B,k}(u_2). \tag{131}$$

The superscript G in Equation 131 indicates that the relation is defined on generative specifications rather than on histories. This notation is consistent with the path-history relation introduced in Equation 127.

Because $H_{B,k} = D_B \circ \mathrm{Gen}_k$ on $U_{k,B}$, every distinction already lost by the realization map is necessarily lost at the braid level.

Proposition 10.1 (Inheritance of generator underdetermination)**.** *For every $k \geq 1$, the inclusion in Equation 132 holds on $U_{k,B}$.*

$$\equiv_{\mathrm{Gen},k} \subseteq \equiv^{G}_{B,k} . \tag{132}$$

Proof. If $u_1 \equiv_{\mathrm{Gen},k} u_2$, Equation 130 gives $\mathrm{Gen}_k(u_1) = \mathrm{Gen}_k(u_2)$. Applying the function D_B to the common realized history yields $D_B(\mathrm{Gen}_k(u_1)) = D_B(\mathrm{Gen}_k(u_2))$. Equation 122 identifies these terms with $H_{B,k}(u_1)$ and $H_{B,k}(u_2)$, respectively. Equation 131 then gives $u_1 \equiv^{G}_{B,k} u_2$. □

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Proposition 10.1 formalizes the level restriction of braid description. A braid descriptor can refine neither a distinction that never appears in the realized history nor a distinction that the realization map has already collapsed. This statement does not imply that the generative mechanism is unknowable by every method. It states only that a descriptor whose input is the realized history cannot recover generator distinctions absent from that input without auxiliary information.

The compatible generative specifications for one braid value can be written explicitly. For $b \in \mathrm{im}(H_{B,k})$, define the generator-compatibility set in Equation 133.

$$\mathcal{U}_{B,k}(b) := H_{B,k}^{-1}(\{b\}) \subseteq U_{k,B}. \tag{133}$$

The fiber decomposition from Equation 29 specializes to Equation 134.

$$\mathcal{U}_{B,k}(b) = \bigcup_{h \in D_B^{-1}(\{b\}) \cap \Gamma_k(\mathfrak{D})} \mathcal{G}_k(h). \tag{134}$$

Equation 134 separates two sources of generator ambiguity. Several realized histories can share one braid value because D_B is noninjective, and several generative specifications can share one realized history because Gen_k is noninjective. A unique generator can be inferred from the braid description only when the relevant union reduces to a singleton.

10.2 Domain Restriction in Braid Admissibility

The braid descriptor is not defined on the entire history collection $\Gamma(\mathfrak{D})$ unless the chosen extraction datum makes every history braid-admissible. Its actual domain is the subset Γ_B fixed in Definition 4.1. The histories omitted by the descriptor are collected in Equation 135.

$$\mathcal{E}_B := \Gamma(\mathfrak{D}) \setminus \Gamma_B. \tag{135}$$

The symbol $\mathcal{E}_B$ denotes domain exclusion and carries no evaluative meaning. A history can belong to $\mathcal{E}_B$ because the chosen braid formalism is inappropriate for that history even when the history is perfectly meaningful at the higher GR level.

The standard extraction datum of Section 4.1 imposes several linked requirements. It fixes an integer n, traces n persistent components, requires their extracted configuration to remain in the collision-free space $\mathcal{U}_n$, and uses a based-loop endpoint convention. These requirements arise from the combination of the constraint and change primitives: constraint selects the admissible configuration domain, while change is represented as continuous motion of the selected persistent components inside that domain.

The resulting exclusions can be stated directly.

Proposition 10.2 (Sufficient conditions for exclusion from the standard braid domain)**.** *Fix a braid-admissible extraction datum $\mathcal{E}_B = (n, c_0, \Gamma_B, \mathcal{T}_B)$ of Definition 4.1. A realized history cannot belong to Γ_B under this datum if the intended extraction requires any one of the following conditions:*

1. *the number of represented components changes from the fixed value n during the descriptive interval;*
2. *two represented components coincide at some extracted time, so the configuration leaves $\mathcal{U}_n$;*
3. *persistent component identity cannot be traced well enough to define the selected strands continuously;*
4. *the chosen endpoint convention cannot produce a loop based at c_0.*

Proof. Each condition contradicts one of the requirements built into Definition 4.1. Condition 1 contradicts the fixed configuration space $\mathcal{U}_n$. Condition 2 contradicts membership in the collision-free configuration space. Condition 3 prevents the construction of the continuous strand trajectories required for an element of the loop space in Equation 30. Condition 4 contradicts the base-loop requirement $q(0) = q(1) = c_0$. Hence the corresponding history cannot be assigned an image under the extraction map specified by the fixed datum. □

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Proposition 10.2 concerns the standard B_n construction used in this paper. It does not imply that branching, fusion, birth, death, open endpoints, or changing identities are intrinsically indescribable. Such processes require a different lower-level formalism, a modified extraction datum, or an extension such as a groupoid, category, cobordism-like structure, or another process representation. The higher GR meta-language remains agnostic about which extension is appropriate.

The domain restriction is therefore relative to the declared descriptor. A suitably narrow target class can satisfy $\mathcal{E}_B = \varnothing$, even though the same braid formalism is partial relative to a broader class of dynamics. This relativity will be made explicit in the next section, which develops a preliminary criterion for generalized dynamical languages.

10.3 Reparameterization Invariance of Topological Description

The braid descriptor preserves topological path information at a particular resolution. It does not preserve the full geometry or timing of the extracted trajectory. This selectivity comes from the homotopy quotient Q_B rather than from the algebraic discreteness of B_n itself. The input to Q_B is the continuous loop space $\Omega(\mathcal{U}_n, c_0)$, and the output records only the based-homotopy class.

One simple consequence is invariance under endpoint-preserving reparameterization. Let q be a based loop in the loop space defined by Equation 30. Let $\varphi : [0,1] \to [0,1]$ be a continuous strictly increasing bijection satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. The reparameterized loop is $q \circ \varphi$.

Proposition 10.3 (Invariance under monotone reparameterization)**.** *For every based loop q and every endpoint-preserving increasing homeomorphism φ as above, Equation 136 holds.*

$$Q_B(q \circ \varphi) = Q_B(q). \tag{136}$$

Consequently, replacing an extracted loop by such a reparameterization leaves the braid descriptor unchanged.

Proof. Define the interpolation of parameters by $\theta_s(t) := (1-s)t + s\varphi(t)$ for $(s,t) \in [0,1]^2$. Because both t and $\varphi(t)$ lie in $[0,1]$, so does $\theta_s(t)$. Define $H(s,t) := q(\theta_s(t))$. Continuity of q and θ_s makes H continuous. At $s = 0$, $H(0,t) = q(t)$, and at $s = 1$, $H(1,t) = q(\varphi(t))$. The endpoint conditions give $H(s,0) = q(0) = c_0$ and $H(s,1) = q(1) = c_0$ for every s. Hence q and $q \circ \varphi$ are based-homotopic, and Equation 32 gives Equation 136. Equation 33 then gives the same braid element after either loop is used in the extraction stage. □

Proposition 10.3 makes one aspect of descriptive selectivity explicit. The braid class records the order and topological intertwining of the represented trajectories while ignoring changes in traversal rate generated by monotone reparameterization. Exact duration, speed, acceleration, and other metric-temporal quantities therefore require additional structure if they are analytically relevant.

A similar qualification applies to internal state. The baseline extraction map $\mathcal{T}_B$ takes values only in the configuration space of the selected components. Internal variables, semantic content, causal labels, probabilities, or institutional attributes affect D_B only when the extraction datum is explicitly enriched to encode them. The absence of such information in the baseline braid descriptor is therefore a consequence of representational choice, not a claim that those dimensions lack dynamical importance.

This point also explains why the absence of a metric is not the principal reason braid theory is non-general. The earlier GR metamodel deliberately avoids a metric at the highest level [6], and Section 8 showed that a metric can be introduced later when a concrete geometric robustness certificate is useful. The deeper expressive restriction is that the braid descriptor retains a topological quotient of selected trajectories rather than the full set of distinctions available in the generated history.

10.4 Resolution-Induced Descriptive Collapse

The resolution restriction of braid description is measured exactly by the descriptive collapse relation derived in Section 4.5. Assume the baseline relation $\approx_\Gamma$ is braid-compatible. Then Proposition 6.2 gives

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$\Delta_B^{\text{desc}} = \equiv_B \setminus \approx_\Gamma$. The braid descriptor is therefore faithful at the declared baseline resolution precisely when no excess equivalence is introduced.

Definition 10.4 (Baseline-faithful braid description)**.** The braid descriptor is *baseline-faithful* on Γ_B when the equality in Equation 137 holds.

$$\equiv_B = \approx_\Gamma . \tag{137}$$

The definition has several equivalent forms already derived in earlier sections.

Proposition 10.5 (Equivalent formulations of baseline faithfulness)**.** *Assume $\approx_\Gamma$ is braid-compatible. The following conditions are equivalent:*

1. *the braid descriptor is baseline-faithful in the sense of Definition 10.4;*
2. *$\Delta_B^{\text{desc}} = \varnothing$;*
3. *the induced map $\overline{D}_B : \Gamma_B/\approx_\Gamma \to B_n$ is injective;*
4. *the canonical quotient map $q_{\Gamma B} : \Gamma_B/\approx_\Gamma \to \Gamma_B/\equiv_B$ is injective.*

Proof. The equivalence of conditions 1 and 2 follows from Equation 72 together with the inclusion $\approx_\Gamma \subseteq \equiv_B$. The equivalence of conditions 2 and 3 is Proposition 4.10. The equivalence of conditions 2 and 4 is Corollary 6.3. Hence all four conditions are equivalent. □

Proposition 10.5 shows that resolution restriction is not the same as domain restriction. A braid descriptor can be defined on every history in a target class and still be too coarse relative to the distinctions that class intends to preserve. Conversely, a descriptor can be faithful on every history it admits while remaining undefined on histories outside Γ_B.

The same logic extends to downstream invariants. Even when D_B is baseline-faithful, an invariant I can introduce additional collapse $\Delta_{I|B}^{\text{desc}}$. Section 7 therefore treats each invariant as a further descriptive choice rather than as a cost-free summary of the braid.

10.5 Compositional Capacity with Derived Invariants

The preceding restrictions coexist with several positive expressive characteristics. First, the braid descriptor is compositional whenever the extraction map is composition-compatible. Proposition 4.6 gives the law in Equation 138.

$$D_B(h_2 \star h_1) = D_B(h_2) D_B(h_1). \tag{138}$$

Equation 138 allows extended trajectory histories to be represented from shorter descriptive segments while retaining the noncommutative order of braid multiplication.

Second, the Artin relations provide local equivalences among composed crossing descriptions. The braid relation and distant-commutation relation in Equation 38 identify distinct words only when the defining algebra declares them equivalent. This gives the language a structured notion of sameness beyond raw sequence identity.

Third, the descriptor supports derived invariants. For any map $I : B_n \to \mathcal{A}$, Equation 80 gives a downstream descriptor $D_I = I \circ D_B$. The braid description can therefore function as an intermediate structural representation from which permutation data, exponent sums, conjugacy classes, closure-based link types, complexity-sensitive quantities, or other appropriate derived constructions can be obtained. This extensibility is one reason braid representation can be useful even when the underlying generative rule remains unidentified.

Fourth, the homotopy quotient gives deformation robustness. Proposition 8.2 shows that admissibly homotopic extracted trajectories have the same braid descriptor, while Proposition 8.4 gives a lower-level metric certificate under a geometric instantiation. The positive expressive profile is therefore not reducible to information loss: quotienting can remove distinctions precisely so that structural features remain stable under transformations declared irrelevant.

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Table 7: Primitive combinations governing the expressive profile of the braid-theoretic descriptor.

Primitive combination	Formal realization	Expressive consequence
Constraint + Change	Selection of Γ_B and continuous motion in the fixed collision-free configuration space $\mathcal{U}_n$	Produces domain restriction when relevant dynamics involve changing component number, collisions, branching, fusion, nontraceable identity, or incompatible endpoints
Change + Structure	Elementary crossing vocabulary Σ_n, history concatenation, braid multiplication, and the Artin relations	Gives ordered compositional description but privileges dynamics that admit a braid-compatible relational trajectory representation
Structure + Difference	Quotient by based homotopy and the induced relation $\equiv_B$	Produces resolution restriction by identifying geometrically distinct trajectories inside one topological class
Constraint + Difference	Admissible deformation class together with the quotient Q_B	Makes robustness relative to the declared constraints: changing the admissible deformation class changes which differences are ignored
All four	Composite $D_B = \kappa_n \circ Q_B \circ \mathcal{T}_B$ together with its domain, composition, and induced equivalence	Yields a coherent language for the topological organization of selected generated relational trajectories rather than a complete language for arbitrary dynamical generation

These properties can be summarized as a combination of compositionality, noncommutative order sensitivity, equivalence-class structure, invariant support, and deformation robustness. None requires that the braid descriptor encode the law generating the trajectory.

10.6 Restriction Sources in Primitive Combinations

The expressive characteristics above can be traced to combinations of the four GR meta-primitives rather than assigned to isolated symbols. Table 7 summarizes the main interactions.

The first row of Table 7 explains domain restriction. The second explains why the language is particularly suited to persistent relational trajectories whose ordered interactions can be represented by braid composition. The third explains resolution restriction and its associated robustness. The fourth makes explicit that robustness is constraint-relative: a difference is invisible only relative to the deformation class that the descriptor permits.

The remaining restriction is the descriptive level itself. It is not produced by one braid relation. It follows from the placement of the entire braid construction downstream from Gen_k. Proposition 10.1 states the resulting information ordering on generative specifications. Together, the three principal restrictions of the present braid formalism can therefore be stated as follows:

1. *level restriction*: the braid descriptor is a function of realized histories and inherits any generator underdetermination already present in the realization map;

2. *domain restriction*: the descriptor is defined on Γ_B, which can be a proper subset of the full history space;

3. *resolution restriction*: within Γ_B, the descriptor can identify histories that remain distinct under the declared baseline equivalence.

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These restrictions are logically distinct. A chosen target class can remove the domain restriction by ensuring that every relevant history is braid-admissible. A sufficiently coarse baseline equivalence can remove the resolution restriction by making the braid descriptor faithful relative to that baseline. Neither adjustment changes the downstream logical position of the descriptor relative to generation.

The distinction among the three restrictions provides the bridge to the next section. A claim that a formal language is "general" is underspecified unless it states the target class of dynamical systems, the descriptive resolution to be preserved, the required compositional structure, and whether representation of the generative rule is part of the claim. The next section therefore develops a preliminary formal criterion for generalized dynamical languages and uses the braid formalism as its first test case.

11 Preliminary Generality Criteria for Dynamical Languages

This section proposes a preliminary criterion for judging when a lower-level formal language is sufficiently general for a declared class of dynamics. Its role is broader than the braid construction itself. The preceding sections have shown that a useful trajectory descriptor can be compositional, invariant-bearing, and robust while remaining restricted in level, domain, and resolution. A criterion for generality must therefore state which systems are covered, which distinctions must remain recoverable, and which compositional and primitive roles the language is required to realize. The objective is to formulate that comparison without reintroducing a universal state space, metric, order, or arithmetic into the GR meta-language.

The construction proceeds in seven stages. Section 11.1 defines the target class and the baseline equivalences relative to which generality is assessed. Section 11.2 defines a trajectory-descriptive language schema using the four GR meta-primitives. Section 11.3 formalizes coverage. Section 11.4 separates preservation of required distinctions from optional matching of the baseline resolution. Section 11.5 states compositional and primitive adequacy. Section 11.6 combines these conditions into a relative notion of trajectory-generality. Section 11.7 then adds representation of the generative layer and distinguishes trajectory-generality from dynamics-generality. The criteria are a revisable formal proposal rather than a claim that every useful dynamical language must be organized in exactly this way.

11.1 Relative Generality Specification

Generality is assessed relative to a declared target. Let $\mathfrak{C}$ be a nonempty class of dynamical systems of the form $\mathfrak{D} = (S, M, \alpha, \sim)$ introduced in Section 3. For each $\mathfrak{D} \in \mathfrak{C}$, let $\Gamma(\mathfrak{D})$ denote the corresponding realized-history space defined in Equation 17.

For each system $\mathfrak{D} \in \mathfrak{C}$, choose an equivalence relation $\approx_{\mathfrak{D}}$ on $\Gamma(\mathfrak{D})$. The relation $\approx_{\mathfrak{D}}$ states which distinctions the analysis requires the language to preserve. Two histories in different $\approx_{\mathfrak{D}}$-classes must remain distinguishable at the declared level of analysis. A language may preserve additional distinctions inside one baseline class; such additional resolution is not, by itself, a failure of generality.

Let the four meta-primitives be collected in the finite set defined by Equation 139.

$$\mathsf{P}_4 := \{\mathsf{C}, \mathsf{Ch}, \mathsf{S}, \mathsf{D}\}, \tag{139}$$

where C denotes constraint, Ch change, S structure, and D difference. The symbol P_4 labels the four primitives fixed by the earlier metamodel and introduces no algebraic operation among them.

Definition 11.1 (Relative generality specification)**.** A *relative generality specification* is a triple defined in Equation 140.

$$\mathfrak{R} := \left(\mathfrak{C}, \{\approx_{\mathfrak{D}}\}_{\mathfrak{D}\in\mathfrak{C}}, \mathsf{P}_{\text{req}}\right), \qquad \mathsf{P}_{\text{req}} \subseteq \mathsf{P}_4. \tag{140}$$

The class $\mathfrak{C}$ declares the systems covered by the claim, the family $\{\approx_{\mathfrak{D}}\}$ declares the minimum distinctions that must remain recoverable, and P_{req} declares which GR meta-primitives the language must explicitly realize.

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For the present paper, the strongest default choice is $\mathsf{P}_{\text{req}} = \mathsf{P}_4$. Allowing a proper subset in Definition 11.1 keeps the criterion usable for intentionally partial descriptive languages.

Definition 11.1 makes generality explicitly relative. Narrowing the target class changes the required coverage, while refining the baseline equivalence increases the number of distinctions that a language must preserve. Coarsening the baseline permits a language to retain additional distinctions without losing adequacy.

11.2 A Four-Primitive Trajectory-Descriptive Language Schema

A language intended to describe trajectories across a class of systems should make its descriptive domain, descriptive objects, composition rule, and induced distinctions explicit. The present criterion represents those roles through a system-indexed schema.

Definition 11.2 (Trajectory-descriptive language schema)**.** A *trajectory-descriptive language schema* $\mathcal{L}$ on $\mathfrak{C}$ assigns to each $\mathfrak{D} \in \mathfrak{C}$ a quadruple defined in Equation 141.

$$\mathcal{L}[\mathfrak{D}] := \left(\Gamma_{\mathcal{L}}(\mathfrak{D}), \Omega_{\mathcal{L}}(\mathfrak{D}), D_{\mathcal{L}}^{\mathfrak{D}}, \odot_{\mathcal{L}}^{\mathfrak{D}}\right), \tag{141}$$

where:

1. $\Gamma_{\mathcal{L}}(\mathfrak{D}) \subseteq \Gamma(\mathfrak{D})$ is the domain of histories admitted by the language;
2. $\Omega_{\mathcal{L}}(\mathfrak{D})$ is the set of descriptive objects used for that system;
3. $D_{\mathcal{L}}^{\mathfrak{D}} : \Gamma_{\mathcal{L}}(\mathfrak{D}) \to \Omega_{\mathcal{L}}(\mathfrak{D})$ is the trajectory-description map;
4. $\odot_{\mathcal{L}}^{\mathfrak{D}}$ is a partial binary operation on $\Omega_{\mathcal{L}}(\mathfrak{D})$ used to represent composition of descriptions when the corresponding histories are composable.

The quadruple in Equation 141 supplies one descriptive realization of the four GR primitive roles. Constraint appears in the admissible domain $\Gamma_{\mathcal{L}}(\mathfrak{D})$. Change appears through the history-sensitive description $D_{\mathcal{L}}^{\mathfrak{D}}(h)$ assigned to an admitted realized process. Structure appears through $\odot_{\mathcal{L}}^{\mathfrak{D}}$. Difference is derived from the distinctions preserved or collapsed by the description map relative to the declared baseline.

For each $\mathfrak{D}$, define the equivalence induced by the descriptor through Equation 142.

$$h_1 \equiv_{\mathcal{L}}^{\mathfrak{D}} h_2 \quad \Longleftrightarrow \quad D_{\mathcal{L}}^{\mathfrak{D}}(h_1) = D_{\mathcal{L}}^{\mathfrak{D}}(h_2), \qquad h_1, h_2 \in \Gamma_{\mathcal{L}}(\mathfrak{D}). \tag{142}$$

Equation 142 is the direct generalization of Equation 20. It makes descriptive difference a relation derived from the map rather than a separately imposed metric.

Definition 11.2 does not require a common codomain for all systems in $\mathfrak{C}$. Generality here concerns a common formal role and construction scheme rather than placement of heterogeneous histories into one universal representational space.

11.3 Domain Coverage Defects

The first dimension of generality is whether the language is defined throughout the declared target class. For each system, define the domain-exclusion set by Equation 143.

$$\mathcal{E}_{\mathcal{L}}^{\text{dom}}(\mathfrak{D}) := \Gamma(\mathfrak{D}) \setminus \Gamma_{\mathcal{L}}(\mathfrak{D}). \tag{143}$$

Definition 11.3 (Coverage adequacy)**.** The language schema $\mathcal{L}$ has *coverage adequacy* on $\mathfrak{C}$ when Equation 144 holds for every $\mathfrak{D} \in \mathfrak{C}$.

$$\mathcal{E}_{\mathcal{L}}^{\text{dom}}(\mathfrak{D}) = \varnothing. \tag{144}$$

Equation 144 is equivalent to $\Gamma_{\mathcal{L}}(\mathfrak{D}) = \Gamma(\mathfrak{D})$. The defect-set formulation remains useful because it identifies which histories lie outside the language when coverage is partial.

Coverage adequacy is relative to $\mathfrak{C}$. A language can be total on a structurally restricted class even when it is undefined on histories outside that class.

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11.4 Baseline Faithfulness and Resolution Matching

Coverage alone is insufficient. A total descriptor can still identify two histories that the declared baseline requires it to keep separate. The converse situation, in which the descriptor distinguishes histories that the baseline treats as equivalent, is different: it represents surplus resolution rather than loss of required information.

Fix $\mathfrak{D} \in \mathfrak{C}$ and restrict $\approx_{\mathfrak{D}}$ to $\Gamma_{\mathcal{L}}(\mathfrak{D})$. The language is *baseline-faithful* when Equation 145 holds.

$$h_1 \equiv_{\mathcal{L}}^{\mathfrak{D}} h_2 \quad \Longrightarrow \quad h_1 \approx_{\mathfrak{D}} h_2. \tag{145}$$

Equation 145 means that equal descriptions never collapse a distinction that the baseline requires the analysis to preserve.

Define the corresponding under-discrimination relation by Equation 146.

$$\Delta_{\mathcal{L}}^{\text{under}}(\mathfrak{D}) := \left\{ (h_1, h_2) : h_1 \not\approx_{\mathfrak{D}} h_2 \text{ and } h_1 \equiv_{\mathcal{L}}^{\mathfrak{D}} h_2 \right\}. \tag{146}$$

Equation 146 generalizes the braid-descriptive collapse relation of Definition 4.9.

Baseline faithfulness has an equivalent recovery formulation. Let $\pi_{\approx}^{\mathfrak{D}} : \Gamma_{\mathcal{L}}(\mathfrak{D}) \to \Gamma_{\mathcal{L}}(\mathfrak{D})/\!\approx_{\mathfrak{D}}$ denote the canonical quotient projection.

Proposition 11.4 (Baseline-class recovery criterion)**.** *The descriptor $D_{\mathcal{L}}^{\mathfrak{D}}$ is baseline-faithful if and only if there exists a unique map $r_{\mathcal{L}}^{\mathfrak{D}} : \mathrm{im}(D_{\mathcal{L}}^{\mathfrak{D}}) \to \Gamma_{\mathcal{L}}(\mathfrak{D})/\!\approx_{\mathfrak{D}}$ satisfying Equation 147.*

$$r_{\mathcal{L}}^{\mathfrak{D}} \circ D_{\mathcal{L}}^{\mathfrak{D}} = \pi_{\approx}^{\mathfrak{D}}. \tag{147}$$

Proof. Assume baseline faithfulness. For $\omega \in \mathrm{im}(D_{\mathcal{L}}^{\mathfrak{D}})$, choose h with $D_{\mathcal{L}}^{\mathfrak{D}}(h) = \omega$ and define $r_{\mathcal{L}}^{\mathfrak{D}}(\omega) := [h]_{\approx}$. If h' is another history with the same descriptive value, then $h \equiv_{\mathcal{L}}^{\mathfrak{D}} h'$. Baseline faithfulness gives $h \approx_{\mathfrak{D}} h'$, so the definition is independent of the representative. Equation 147 follows. Uniqueness follows because every element of the image has the form $D_{\mathcal{L}}^{\mathfrak{D}}(h)$.

Conversely, suppose a map satisfying Equation 147 exists. If $h_1 \equiv_{\mathcal{L}}^{\mathfrak{D}} h_2$, then the two descriptor values are equal. Applying $r_{\mathcal{L}}^{\mathfrak{D}}$ and using Equation 147 gives $[h_1]_{\approx} = [h_2]_{\approx}$, hence $h_1 \approx_{\mathfrak{D}} h_2$. □

Proposition 11.4 clarifies the adequacy requirement: a language can retain more detail than the baseline and remain adequate so long as the baseline class is recoverable from the description.

For analyses that require the descriptor to operate at exactly the declared baseline resolution, impose the additional *baseline-invariance* condition in Equation 148.

$$h_1 \approx_{\mathfrak{D}} h_2 \quad \Longrightarrow \quad h_1 \equiv_{\mathcal{L}}^{\mathfrak{D}} h_2. \tag{148}$$

Failure of Equation 148 indicates surplus resolution. Define the surplus-resolution relation by Equation 149.

$$\Xi_{\mathcal{L}}^{\text{fine}}(\mathfrak{D}) := \left\{ (h_1, h_2) : h_1 \approx_{\mathfrak{D}} h_2 \text{ and } h_1 \not\equiv_{\mathcal{L}}^{\mathfrak{D}} h_2 \right\}. \tag{149}$$

The symbol $\Xi_{\mathcal{L}}^{\text{fine}}$ records that the language is finer than the baseline on the listed pairs. It is not treated as an adequacy defect unless exact resolution matching is itself part of the research requirement.

Definition 11.5 (Resolution matching)**.** A descriptor is *resolution-matched* to the baseline on its domain when it is both baseline-faithful and baseline-invariant.

Proposition 11.6 (Resolution-matching criterion)**.** *For a fixed system $\mathfrak{D}$, the following statements are equivalent:*

1. *the descriptor is resolution-matched to the baseline;*
2. *the induced and baseline relations agree on the language domain, as stated in Equation 150;*

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3. *both* $\Delta_{\mathcal{L}}^{\text{under}}(\mathfrak{D})$ *and* $\Xi_{\mathcal{L}}^{\text{fine}}(\mathfrak{D})$ *are empty.*

$$\equiv_{\mathcal{L}}^{\mathfrak{D}} = \approx_{\mathfrak{D}} \big|_{\Gamma_{\mathcal{L}}(\mathfrak{D})}. \tag{150}$$

Proof. Baseline faithfulness and baseline invariance give the two inclusions needed for Equation 150. Equality of the two relations makes both defect sets empty by their definitions. Conversely, emptiness of $\Delta_{\mathcal{L}}^{\text{under}}$ gives baseline faithfulness, while emptiness of $\Xi_{\mathcal{L}}^{\text{fine}}$ gives baseline invariance. □

The distinction between baseline faithfulness and resolution matching is important for generality. A fine-grained language can remain general for a coarser baseline if the coarser class is recoverable from its outputs. Exact matching is a stronger optional requirement.

11.5 Structural Adequacy

The third dimension of generality concerns structure. Let $\text{Comp}_{\mathcal{L}}(\mathfrak{D})$ denote the set of ordered pairs (h_2, h_1) such that $h_1, h_2 \in \Gamma_{\mathcal{L}}(\mathfrak{D})$, the histories are composable in the sense of Section 3.3, and $h_2 \star h_1 \in \Gamma_{\mathcal{L}}(\mathfrak{D})$. Define the compositional defect set by Equation 151.

$$\mathcal{E}_{\mathcal{L}}^{\text{comp}}(\mathfrak{D}) := \left\{ (h_2, h_1) \in \text{Comp}_{\mathcal{L}}(\mathfrak{D}) : \begin{array}{l} D_{\mathcal{L}}^{\mathfrak{D}}(h_2 \star h_1) \\ \neq D_{\mathcal{L}}^{\mathfrak{D}}(h_2) \odot_{\mathcal{L}}^{\mathfrak{D}} D_{\mathcal{L}}^{\mathfrak{D}}(h_1) \end{array} \right\}. \tag{151}$$

Definition 11.7 (Compositional adequacy)**.** The language schema $\mathcal{L}$ has *compositional adequacy* on $\mathfrak{C}$ when Equation 152 holds for every $\mathfrak{D} \in \mathfrak{C}$.

$$\mathcal{E}_{\mathcal{L}}^{\text{comp}}(\mathfrak{D}) = \varnothing. \tag{152}$$

The four-primitives perspective adds a second structural requirement: the language should state where each required primitive is realized. Define the primitive realization profile of $\mathcal{L}[\mathfrak{D}]$ by Equation 153.

$$\mathcal{P}_{\mathcal{L}}(\mathfrak{D}) := \left(\Gamma_{\mathcal{L}}(\mathfrak{D}), D_{\mathcal{L}}^{\mathfrak{D}}, \odot_{\mathcal{L}}^{\mathfrak{D}}, \equiv_{\mathcal{L}}^{\mathfrak{D}} \right). \tag{153}$$

The entries of Equation 153 correspond respectively to the descriptive roles of constraint, change, structure, and difference.

Definition 11.8 (Primitive adequacy)**.** The language schema $\mathcal{L}$ has *primitive adequacy* relative to $\mathfrak{R}$ when, for every $\mathfrak{D} \in \mathfrak{C}$, each primitive in P_{req} has an explicit realization in $\mathcal{P}_{\mathcal{L}}(\mathfrak{D})$ and the realization satisfies the corresponding declared constraints of the schema.

Definition 11.8 is structural rather than numerical. It asks whether the language makes its commitments visible and whether those commitments can be checked against the required primitive roles.

11.6 Trajectory-Generality Relative to a Declared Class

The preceding conditions can now be combined into a preliminary notion of generalized trajectory description.

Definition 11.9 (Trajectory-generality)**.** A trajectory-descriptive language schema $\mathcal{L}$ is called *trajectory-general* relative to $\mathfrak{R}$ when the following conditions hold for every $\mathfrak{D} \in \mathfrak{C}$:

1. coverage adequacy in the sense of Definition 11.3;
2. baseline faithfulness in the sense of Equation 145;
3. compositional adequacy in the sense of Definition 11.7;
4. primitive adequacy in the sense of Definition 11.8.

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The zero-defect part of Definition 11.9 is stated in Equation 154.

$$\begin{aligned} \mathcal{E}^{\mathrm{dom}}_{\mathcal{L}}(\mathfrak{D}) &= \varnothing, \\ \Delta^{\mathrm{under}}_{\mathcal{L}}(\mathfrak{D}) &= \varnothing, \qquad \text{for every } \mathfrak{D} \in \mathfrak{C}, \\ \mathcal{E}^{\mathrm{comp}}_{\mathcal{L}}(\mathfrak{D}) &= \varnothing \end{aligned} \tag{154}$$

together with primitive adequacy.

A trajectory-general language may retain distinctions inside one baseline class. When exact agreement with the declared resolution is additionally required, Definition 11.5 gives the stronger notion.

Definition 11.10 (Resolution-matched generality)**.** A language is *resolution-matched trajectory-general* relative to $\mathfrak{R}$ when it is trajectory-general and, for each $\mathfrak{D} \in \mathfrak{C}$, its induced equivalence is resolution-matched to $\approx_{\mathfrak{D}}$.

Thus $\Xi^{\mathrm{fine}}_{\mathcal{L}}$ need not be empty for trajectory-generality, while it must be empty for resolution-matched trajectory-generality.

The relative character of Definition 11.9 remains essential. Narrowing the target class can remove coverage defects, while coarsening the baseline can make distinction faithfulness easier to satisfy. Generality claims should therefore report their target class and baseline family rather than use the adjective "general" without qualification.

11.7 Generative Adequacy for Dynamics-Generality

Trajectory-generality concerns the generated-history layer. A stronger claim that a language is general for dynamics additionally requires representation of the generative side of the system.

For each $\mathfrak{D} = (S, M, \alpha, \sim)$ and each positive integer k, let $U_k(\mathfrak{D}) := S \times M^k$ be the set of generative specifications used by the generation map in Equation 18. Choose an equivalence relation $\approx^{\mathfrak{D}}_{G,k}$ on $U_k(\mathfrak{D})$ that states which generator-side distinctions the analysis requires the language to preserve.

Definition 11.11 (Generative representation)**.** A *generative representation* for $\mathcal{L}$ on $\mathfrak{D}$ at length k consists of:

1. a set $\Omega^{\mathfrak{D}}_{G,k}$ of generative descriptions;
2. a map $R^{\mathfrak{D}}_{G,k} : U_k(\mathfrak{D}) \to \Omega^{\mathfrak{D}}_{G,k}$;
3. a semantic realization map $\mathrm{Sem}^{\mathfrak{D}}_k : \Omega^{\mathfrak{D}}_{G,k} \to \Omega_{\mathcal{L}}(\mathfrak{D})$ satisfying Equation 155.

$$D^{\mathfrak{D}}_{\mathcal{L}} \circ \mathrm{Gen}_k = \mathrm{Sem}^{\mathfrak{D}}_k \circ R^{\mathfrak{D}}_{G,k}, \tag{155}$$

on every generative specification whose realized history lies in $\Gamma_{\mathcal{L}}(\mathfrak{D})$.

Define the induced generator equivalence by Equation 156.

$$u_1 \equiv^{\mathfrak{D},k}_{G,\mathcal{L}} u_2 \quad \Longleftrightarrow \quad R^{\mathfrak{D}}_{G,k}(u_1) = R^{\mathfrak{D}}_{G,k}(u_2). \tag{156}$$

A generative representation is *generator-faithful* at the declared resolution when Equation 157 holds.

$$\equiv^{\mathfrak{D},k}_{G,\mathcal{L}} \subseteq \approx^{\mathfrak{D}}_{G,k} . \tag{157}$$

Equation 157 permits the generator representation to retain more distinctions than the baseline requires. Exact resolution matching at the generator level is the stronger equality in Equation 158.

$$\equiv^{\mathfrak{D},k}_{G,\mathcal{L}} = \approx^{\mathfrak{D}}_{G,k} . \tag{158}$$

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Definition 11.12 (Dynamics-generality)**.** Let a relative trajectory-generality specification $\mathfrak{R}$ be extended by a family of generator equivalences $\{\approx^{\mathfrak{D}}_{G,k}\}$. A language schema $\mathcal{L}$ is *dynamics-general relative to this extended specification* when:

1. $\mathcal{L}$ is trajectory-general relative to $\mathfrak{R}$;
2. for every $\mathfrak{D} \in \mathfrak{C}$ and every relevant k, a generative representation in the sense of Definition 11.11 exists;
3. the generative representation satisfies generator faithfulness in Equation 157.

A stronger resolution-matched form can additionally require Equation 158 together with Definition 11.10.

Proposition 11.13 (Relation between the two generality levels)**.** *Dynamics-generality implies trajectory-generality for the underlying trajectory specification. The converse does not follow from the trajectory-level conditions alone.*

Proof. The first statement follows directly from item 1 of Definition 11.12. The second follows because Definition 11.9 introduces no generator-side codomain or map and imposes no semantic compatibility condition of the form in Equation 155. □

11.8 Scope of the Preliminary Criterion

The criteria above are preliminary. They capture forms of adequacy that can be stated using the current GR meta-language: coverage, preservation of required distinctions, composition, explicit primitive realization, and an optional generator-side representation. Several stronger notions of generality remain outside the present construction.

First, Definition 11.2 requires a common formal role across systems but does not formalize naturality under transformations between systems. A stronger cross-system language theory would specify morphisms between dynamical systems and require descriptive assignments to commute with those morphisms.

Second, the criterion is extensional with respect to history equivalence. It asks whether required distinctions remain recoverable but does not rank correct representations by computational complexity, interpretability, empirical estimability, or explanatory usefulness.

Third, dynamics-generality requires a representation of generative specifications and semantic compatibility with trajectory descriptions. It does not require that the represented generative structure be inferable from observed trajectories. Identifiability is an epistemic property of an inverse problem and should remain separate from representability.

The preliminary criterion is therefore diagnostic. It turns an undifferentiated question about whether a language is "general" into explicit questions about target class, minimum preserved distinctions, domain coverage, compositionality, primitive realization, optional resolution matching, and generator representation. The next section applies these criteria to the braid-theoretic formalism developed in the preceding sections.

12 Generality Profile of the Braid Formalism

This section applies the criteria of Section 11 to the braid-theoretic trajectory descriptor constructed in Sections 4–10. Its role is diagnostic rather than absolute. The objective is to identify exactly when the braid formalism satisfies trajectory-generality, when it satisfies the stronger resolution-matched form, and which failures arise from domain restriction, loss of required distinctions, composition, or generator resolution. Section 12.1 embeds the braid construction into the general language schema. Section 12.2 derives its domain, baseline-faithfulness, resolution-surplus, and composition profile. Section 12.3 states sufficient conditions for relative trajectory-generality and identifies the native braid baseline as a resolution-matched special case. Section 12.4 states the corresponding failure conditions for

broader domains and finer required resolutions. Section 12.5 evaluates the generator layer. Section 12.6 consolidates the resulting profile.

12.1 Braid-Theoretic Instantiation of the Language Schema

Let $\mathfrak{C}_B$ be a class of GR dynamical systems such that each $\mathfrak{D} \in \mathfrak{C}_B$ is equipped with a braid-admissible extraction datum in the sense of Definition 4.1. Write that datum as $\mathcal{E}_B^{\mathfrak{D}} = (n_{\mathfrak{D}}, c_{0,\mathfrak{D}}, \Gamma_B(\mathfrak{D}), \mathcal{T}_B^{\mathfrak{D}})$. The braid index may vary across systems.

For each $\mathfrak{D} \in \mathfrak{C}_B$, define the braid language schema by Equation 159.

$$\mathcal{L}_B[\mathfrak{D}] := \left(\Gamma_B(\mathfrak{D}), B_{n_{\mathfrak{D}}}, D_B^{\mathfrak{D}}, \cdot\right), \tag{159}$$

where $D_B^{\mathfrak{D}}$ is the descriptor derived from $\mathcal{E}_B^{\mathfrak{D}}$ by Equation 33, and $\cdot$ denotes braid multiplication on the relevant group.

Equation 159 is an instance of Definition 11.2. Constraint is represented by the admitted history domain and the extraction assumptions, change by the history-sensitive braid description, structure by braid multiplication and the Artin relations, and difference by the induced braid equivalence and the distinctions it preserves or collapses.

12.2 Coverage, Faithfulness, Resolution Surplus, and Composition

Fix a relative generality specification $\mathfrak{R}_B = (\mathfrak{C}_B, \{\approx_{\mathfrak{D}}\}, \mathsf{P}_{\text{req}})$. The domain defect is given by Equation 160.

$$\mathcal{E}_B^{\text{dom}}(\mathfrak{D}) := \Gamma(\mathfrak{D}) \setminus \Gamma_B(\mathfrak{D}). \tag{160}$$

The braid under-discrimination relation, which records loss of required baseline distinctions, is defined in Equation 161.

$$\Delta_B^{\text{under}}(\mathfrak{D}) := \left\{(h_1, h_2) \in \Gamma_B(\mathfrak{D})^2 : h_1 \not\approx_{\mathfrak{D}} h_2 \text{ and } h_1 \equiv_B^{\mathfrak{D}} h_2\right\}. \tag{161}$$

The complementary surplus-resolution relation is defined in Equation 162.

$$\Xi_B^{\text{fine}}(\mathfrak{D}) := \left\{(h_1, h_2) \in \Gamma_B(\mathfrak{D})^2 : h_1 \approx_{\mathfrak{D}} h_2 \text{ and } h_1 \not\equiv_B^{\mathfrak{D}} h_2\right\}. \tag{162}$$

A nonempty set in Equation 162 means that braid description retains more distinctions than the baseline requires. It prevents exact resolution matching but does not by itself violate trajectory-generality.

For systems whose extraction datum has a declared composition rule, define the compositional defect by Equation 163.

$$\mathcal{E}_B^{\text{comp}}(\mathfrak{D}) := \left\{(h_2, h_1) \in \text{Comp}_B(\mathfrak{D}) : D_B^{\mathfrak{D}}(h_2 \star h_1) \neq D_B^{\mathfrak{D}}(h_2) D_B^{\mathfrak{D}}(h_1)\right\}. \tag{163}$$

The defect conditions can now be stated without treating surplus resolution as information loss.

Proposition 12.1 (Braid adequacy conditions)**.** *For a fixed $\mathfrak{D} \in \mathfrak{C}_B$, the following statements hold.*

1. *Coverage adequacy holds if and only if $\Gamma_B(\mathfrak{D}) = \Gamma(\mathfrak{D})$.*
2. *Baseline faithfulness holds exactly when the braid equivalence is contained in the baseline equivalence on the admitted domain. Equivalently, the relation $\Delta_B^{\text{under}}(\mathfrak{D})$ is empty.*
3. *Baseline invariance holds exactly when the baseline equivalence is contained in the braid equivalence. Equivalently, the relation $\Xi_B^{\text{fine}}(\mathfrak{D})$ is empty.*
4. *The braid descriptor is resolution-matched to the baseline if and only if $\equiv_B^{\mathfrak{D}} = \approx_{\mathfrak{D}}$.*

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5. *If $\mathcal{T}_B^{\mathfrak{D}}$ is composition-compatible, then $\mathcal{E}_B^{\mathrm{comp}}(\mathfrak{D}) = \varnothing$.*

Proof. Item 1 follows from Equation 160.

Item 2 is a direct restatement of baseline faithfulness in Equation 145. The failure pairs are exactly those collected by Equation 161.

Item 3 is the braid instance of baseline invariance in Equation 148. The failure pairs are exactly those collected by Equation 162.

Item 4 follows because equality of equivalence relations is equivalent to the two inclusions in items 2 and 3.

Item 5 follows from Proposition 4.6. □

The earlier braid-compatibility condition of Definition 4.7 is exactly item 3 of Proposition 12.1. It is therefore a condition for baseline invariance and resolution matching, not a prerequisite for baseline faithfulness.

Primitive adequacy remains explicit on the admitted domain.

Proposition 12.2 (Primitive adequacy of the braid schema)**.** *Let $\mathsf{P}_{\mathrm{req}} \subseteq \mathsf{P}_4$. For every $\mathfrak{D} \in \mathfrak{C}_B$, the braid schema of Equation 159 explicitly realizes every primitive in $\mathsf{P}_{\mathrm{req}}$ through the constructions of Section 4.*

Proof. Constraint is represented by $\Gamma_B(\mathfrak{D})$ and the admissibility conditions of the extraction datum. Change is represented by the extracted trajectory and its braid description. Structure is represented by loop concatenation, braid multiplication, and the Artin relations. Difference is represented by the induced braid equivalence and, relative to a declared baseline, by the under-discrimination relation and the optional surplus-resolution relation. Thus every required primitive has an explicit descriptive realization. □

12.3 Relative Trajectory-Generality and the Native Braid Resolution

The revised criterion permits a braid descriptor to be trajectory-general even when it is finer than the declared baseline. What matters is that it cover the target histories, preserve every distinction required by the baseline, preserve composition, and realize the required primitives.

Proposition 12.3 (Relative trajectory-generality of braid description)**.** *Let $\mathfrak{C}_B^{\mathrm{full}}$ be a target class such that, for every $\mathfrak{D} \in \mathfrak{C}_B^{\mathrm{full}}$:*

1. *$\Gamma_B(\mathfrak{D}) = \Gamma(\mathfrak{D})$;*
2. *$\equiv_B^{\mathfrak{D}} \subseteq \approx_{\mathfrak{D}}$;*
3. *$\mathcal{T}_B^{\mathfrak{D}}$ is composition-compatible;*
4. *$\mathsf{P}_{\mathrm{req}} \subseteq \mathsf{P}_4$ is realized by the braid schema.*

Then $\mathcal{L}_B$ is trajectory-general relative to the resulting specification.

Proof. Item 1 gives coverage adequacy. Item 2 gives baseline faithfulness by Proposition 12.1, item 2. Item 3 gives compositional adequacy by item 5 of the same proposition. Item 4 gives primitive adequacy by Proposition 12.2. Definition 11.9 therefore applies. □

The native braid baseline remains an important special case. Define $\approx_{\mathfrak{D}}^{B} := \equiv_B^{\mathfrak{D}}$. Then both baseline faithfulness and baseline invariance hold automatically.

Corollary 12.4 (Resolution-matched trajectory-generality at native braid resolution)**.** *Let $\mathfrak{C}_B^{\mathrm{full}}$ satisfy coverage and composition compatibility as in Proposition 12.3, and set $\approx_{\mathfrak{D}} = \equiv_B^{\mathfrak{D}}$ for every target system. With $\mathsf{P}_{\mathrm{req}} = \mathsf{P}_4$, the braid schema is resolution-matched trajectory-general.*

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Proof. Proposition 12.3 gives trajectory-generality. Equality of the baseline and braid equivalence gives resolution matching by Proposition 12.1, item 4. □

The corollary is intentionally relative. It states that braid theory is fully adequate at its native topological resolution on a class whose histories all admit the chosen extraction and composition structure. It does not extend that adequacy to arbitrary histories or finer required distinctions.

12.4 Failure Conditions for Broader Classes and Finer Required Resolutions

Under the revised criterion, three kinds of defect can defeat trajectory-generality: missing coverage, loss of required baseline distinctions, and compositional failure. Missing primitive realization can also defeat the criterion when a required primitive has not been supplied.

Proposition 12.5 (Failure of trajectory-generality from braid defects)**.** *Let $\mathfrak{R}_B$ be a relative generality specification for the braid language. If there exists $\mathfrak{D} \in \mathfrak{C}_B$ for which at least one of the conditions in Equation 164 holds, then $\mathcal{L}_B$ is not trajectory-general relative to $\mathfrak{R}_B$.*

$$\begin{aligned} \mathcal{E}_B^{\mathrm{dom}}(\mathfrak{D}) &\neq \varnothing, \\ \Delta_B^{\mathrm{under}}(\mathfrak{D}) &\neq \varnothing, \\ \mathcal{E}_B^{\mathrm{comp}}(\mathfrak{D}) &\neq \varnothing. \end{aligned} \tag{164}$$

Proof. A nonempty domain defect violates coverage adequacy. A nonempty under-discrimination relation violates baseline faithfulness. A nonempty compositional defect violates compositional adequacy. Each condition therefore contradicts one of the requirements in Definition 11.9. □

A nonempty Ξ_B^{fine} does not appear in Equation 164. It means that braid description retains distinctions beyond the baseline. Such surplus resolution prevents resolution-matched trajectory-generality but remains compatible with trajectory-generality when the baseline class is recoverable from the braid description.

This distinction changes the interpretation of a coarser baseline. For example, if the baseline identifies all histories with the same extracted endpoints, a braid descriptor can be strictly finer and still baseline-faithful because equal braid descriptions imply equal extracted endpoints. The extra braid distinctions are surplus resolution rather than an adequacy failure.

The remaining failure modes retain the interpretations developed earlier. Domain defects arise when histories fall outside the braid-admissible class, such as changing-component processes under a fixed-strand datum. Under-discrimination arises when the baseline preserves distinctions that braid equivalence removes. Compositional defects arise when the extraction rule does not preserve history composition.

12.5 Generator-Resolution Profile of Braid-Derived Representation

The stronger criterion of dynamics-generality requires a representation of generative specifications. The path-history map $H_{B,k} = D_B \circ \mathrm{Gen}_k$ provides a minimal braid-derived generator representation on the braid-admissible generative domain $U_{k,B}$. Define Equation 165.

$$R^G_{B,k} := H_{B,k} : U_{k,B} \longrightarrow B_n. \tag{165}$$

With the identity semantic map on B_n, Equation 166 holds.

$$D_B \circ \mathrm{Gen}_k = \mathrm{id}_{B_n} \circ R^G_{B,k}. \tag{166}$$

The induced generator equivalence is the braid-observation relation $\equiv^G_{B,k}$ introduced in Equation 131. The revised faithfulness condition permits the braid-derived representation to retain more generator-side distinctions than the declared baseline but forbids it from collapsing distinctions the baseline requires.

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Table 8: Generality profile of the braid-theoretic trajectory descriptor relative to the preliminary criteria of Section 11.

Criterion	Braid-theoretic status	Formal condition
Coverage	Conditional	Adequate exactly when $\Gamma_B(\mathfrak{D}) = \Gamma(\mathfrak{D})$ for every target system
Baseline faithfulness	Conditional	Adequate exactly when $\equiv_B^{\mathfrak{D}} \subseteq \approx_{\mathfrak{D}}$
Resolution matching	Optional stronger condition	Holds exactly when $\equiv_B^{\mathfrak{D}} = \approx_{\mathfrak{D}}$
Compositional adequacy	Strong under an explicit assumption	Adequate when $\mathcal{T}_B^{\mathfrak{D}}$ is composition-compatible
Primitive adequacy	Explicitly realized on the admitted domain	Constraint, change, structure, and difference are instantiated by Section 4
Trajectory-generality	Relative rather than universal	Holds when coverage, baseline faithfulness, composition, and primitive adequacy hold
Generator faithfulness	Limited without enrichment	The minimal representation $R_{B,k}^G = D_B \circ \mathrm{Gen}_k$ is faithful when $\equiv_{B,k}^G \subseteq \approx_{G,k}^{\mathfrak{D}}$
Dynamics-generality	Requires generator-side adequacy in addition to trajectory-generality	Fails at any generator resolution that preserves distinctions collapsed by the braid-derived generator representation

Proposition 12.6 (Faithfulness condition for braid-derived generator representation)**.** *Let $\approx_{G,k}^{\mathfrak{D}}$ be the declared generator equivalence on $U_{k,B}$. The braid-derived representation $R_{B,k}^G$ is generator-faithful if and only if Equation 167 holds.*

$$\equiv_{B,k}^G \subseteq \approx_{G,k}^{\mathfrak{D}} . \tag{167}$$

It is resolution-matched at the generator level if and only if equality holds.

Proof. By Equation 165, equality of generator representations is exactly the relation $\equiv_{B,k}^G$. Generator faithfulness is therefore Equation 157 specialized to the braid-derived representation, which gives Equation 167. Resolution matching is the stronger equality in Equation 158. □

Corollary 12.7 (Need for generative enrichment at finer generator resolution)**.** *If $\approx_{G,k}^{\mathfrak{D}}$ is a proper subset of $\equiv_{B,k}^G$, then the braid-derived representation cannot satisfy generator faithfulness. Dynamics-generality at that finer generator resolution therefore requires additional generator-side structure beyond the braid descriptor.*

Proof. Proper inclusion gives two generative specifications that have the same braid-derived representation while remaining distinct under the declared generator baseline. Equation 167 therefore fails. □

The result identifies the logical limit of braid-derived generator labels. They are adequate only when every generator-side distinction required by the analysis is recoverable from the generated braid. A finer generator question requires an enriched representation of the mechanism layer.

12.6 Consolidated Generality Profile

Table 8 summarizes the revised profile.

The profile supports three conclusions.

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First, braid theory can be trajectory-general relative to a carefully declared class and minimum descriptive resolution. Proposition 12.3 states the sufficient conditions, while Corollary 12.4 gives a resolution-matched native-braid special case.

Second, the standard braid formalism fails trajectory-generality when the target class contains histories outside the braid domain, when the required baseline distinctions are finer than braid equivalence, or when the extraction rule fails to preserve composition. A descriptor that is finer than the baseline can remain trajectory-general.

Third, braid description alone does not provide dynamics-generality at a generator resolution that preserves distinctions invisible in the generated braid. Composing the braid descriptor with the generation map labels mechanism-bearing specifications by their resulting braid, but that operation adds no generator distinction absent from the trajectory.

The resulting characterization is positive and bounded. Braid theory is a compositional, topologically robust, invariant-bearing language for describing a class of intertwined relational trajectories. Its adequacy is relative to the histories admitted, the minimum distinctions required, and the level of analysis. The next section develops formal examples that instantiate these success and failure modes.

13 Formal Examples

This section instantiates the preceding constructions in a small set of deliberately elementary examples. Its role is diagnostic: each example isolates one distinction that the general formalism is designed to preserve. The objective is not to model a substantive empirical domain, but to verify that the definitions and propositions of Sections 3–12 behave as intended in explicit cases. Section 13.1 exhibits identical extracted endpoints with distinct braid classes. Section 13.2 constructs geometrically distinct trajectories with one braid description. Section 13.3 isolates information loss at the extraction stage. Section 13.4 gives two generative mechanisms that produce the same realized history while carrying different mechanism-level collapse histories. Section 13.5 gives a changing-component process outside the standard braid domain. Section 13.6 combines these examples to illustrate the relative character of trajectory-generality.

13.1 Common Extracted Endpoints with Distinct Braid Classes

The first example makes explicit why endpoint information is strictly coarser than braid information. Take $n = 2$ and choose the base configuration $c_0 = \{(-1, 0), (1, 0)\} \in \mathcal{U}_2$.

Define the trivial geometric representative $z^{(0)}$ by Equation 168.

$$z_1^{(0)}(t) := (-1, 0), \qquad z_2^{(0)}(t) := (1, 0), \qquad 0 \le t \le 1. \tag{168}$$

Its unordered configuration $q_0(t) := \{z_1^{(0)}(t), z_2^{(0)}(t)\}$ is the constant based loop at c_0. Hence its class in $\pi_1(\mathcal{U}_2, c_0)$ is the identity, and the corresponding braid element is $e \in B_2$.

Define a second geometric representative $z^{(1)}$ by Equation 169.

$$\begin{aligned} z_1^{(1)}(t) &:= \big(-\cos(\pi t), \sin(\pi t)\big), \\ z_2^{(1)}(t) &:= \big(\cos(\pi t), -\sin(\pi t)\big). \end{aligned} \tag{169}$$

At $t = 0$, Equation 169 gives the labeled positions $(-1, 0)$ and $(1, 0)$. At $t = 1$, the two labeled positions have exchanged: $z_1^{(1)}(1) = (1, 0)$ and $z_2^{(1)}(1) = (-1, 0)$. Because the configuration is unordered, the initial and terminal configurations are both c_0.

The two strands never collide. Their difference is given by Equation 170.

$$z_1^{(1)}(t) - z_2^{(1)}(t) = \big(-2\cos(\pi t), 2\sin(\pi t)\big), \qquad \Big\| z_1^{(1)}(t) - z_2^{(1)}(t) \Big\|_2 = 2. \tag{170}$$

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Equation 170 therefore defines a loop $q_1(t) := \{z_1^{(1)}(t), z_2^{(1)}(t)\}$ in $\mathcal{U}_2$. Under the fixed isomorphism $\kappa_2 : \pi_1(\mathcal{U}_2, c_0) \to B_2$, choose the conventional orientation so that $Q_B(q_1)$ corresponds to the generator σ_1.

Choose braid-admissible histories $h_0, h_1 \in \Gamma_B$ whose extracted loops satisfy $\mathcal{T}_B(h_j) = q_j$ for $j \in \{0, 1\}$. The two histories therefore satisfy Equation 171.

$$D_\partial(h_0) = D_\partial(h_1) = (c_0, c_0), \qquad D_B(h_0) = e, \qquad D_B(h_1) = \sigma_1. \tag{171}$$

Equation 171 applies the descriptors to the histories h_0 and h_1 themselves. The endpoint descriptor identifies the two extracted trajectories, while the braid descriptor distinguishes them. This gives an explicit instance of the strict inclusion $\equiv_B \subsetneq \equiv_\partial$ from Proposition 6.5.

13.2 Distinct Geometric Trajectories with One Braid Description

The second example isolates homotopy-stage descriptive collapse. Start from the exchange trajectory in Equation 169. For a parameter $a \in [0, \frac{1}{4}]$, define a radially deformed family by Equation 172.

$$\begin{aligned} r_a(t) &:= 1 + a\sin^2(\pi t), \\ z_{1,a}(t) &:= r_a(t)\big(-\cos(\pi t), \sin(\pi t)\big), \\ z_{2,a}(t) &:= -z_{1,a}(t). \end{aligned} \tag{172}$$

The factor $r_a(t)$ equals 1 at $t = 0$ and $t = 1$, so every member of the family has the same labeled endpoints as Equation 169. The two strands remain separated because Equation 173 holds.

$$\|z_{1,a}(t) - z_{2,a}(t)\|_2 = 2r_a(t) \geq 2. \tag{173}$$

Let $q_a(t) := \{z_{1,a}(t), z_{2,a}(t)\}$. For two parameters $a_0, a_1 \in [0, \frac{1}{4}]$, define the interpolation $a(s) := (1-s)a_0 + sa_1$ and set $H(s,t) := q_{a(s)}(t)$. Equation 173 guarantees that $H(s,t) \in \mathcal{U}_2$ for every $(s,t) \in [0,1]^2$. Because all members of the family share the same endpoints, H is an admissible deformation in the sense of Definition 8.1. Choose histories $h_a \in \Gamma_B$ such that $\mathcal{T}_B(h_a) = q_a$ for the parameter values considered.

Proposition 8.2 therefore yields Equation 174.

$$D_B(h_{a_0}) = D_B(h_{a_1}) = \sigma_1. \tag{174}$$

For $a_0 \neq a_1$, the parameterized loops are not equal because their radii differ whenever $\sin(\pi t) \neq 0$. Hence $q_{a_0} \neq q_{a_1}$ while Equation 174 holds.

If exact extracted-loop equality is used as the baseline relation for this example, the pair belongs to the homotopy-stage collapse relation $\Delta^{\text{desc}}_{B,\text{hom}}$ defined in Equation 58. The example therefore exhibits the mechanism by which geometric variation is intentionally removed while topological trajectory structure is retained.

13.3 Extraction-Stage Collapse of an Internal Distinction

The third example separates extraction-stage loss from the homotopy quotient. Let the carrier of a concrete instantiation be $S := \mathcal{U}_2 \times \{0, 1\}$. The second factor is an internal binary marker that the extraction will ignore. Let $M = \{m\}$ with composition $mm = m$, and define the action by Equation 175.

$$\alpha\big(m, (c, r)\big) := (c, r) \qquad \text{for } (c, r) \in S. \tag{175}$$

The one-element semigroup in Equation 175 acts trivially on both coordinates. Take the two initial states $x^{(0)} := (c_0, 0)$ and $x^{(1)} := (c_0, 1)$. Equation 18 then produces the two length-one histories stated in Equation 176.

$$\begin{aligned} h^{(0)} &:= \text{Gen}_1(x^{(0)}, (m)) = \big((c_0, 0), (c_0, 0)\big), \\ h^{(1)} &:= \text{Gen}_1(x^{(1)}, (m)) = \big((c_0, 1), (c_0, 1)\big). \end{aligned} \tag{176}$$

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The histories are distinct because their internal markers differ.

Define the extraction map to discard the marker and return the constant loop at the first coordinate, as stated in Equation 177.

$$\mathcal{T}_B(h^{(0)})(t) := c_0, \qquad \mathcal{T}_B(h^{(1)})(t) := c_0 \qquad \text{for } 0 \le t \le 1. \tag{177}$$

Hence $\mathcal{T}_B(h^{(0)}) = \mathcal{T}_B(h^{(1)})$. If exact history identity is used as the baseline relation, then $h^{(0)} \not\approx_\Gamma h^{(1)}$. Equation 57 therefore gives Equation 178.

$$\big(h^{(0)}, h^{(1)}\big) \in \Delta^{\text{desc}}_{B,\text{ext}}. \tag{178}$$

No homotopy quotient is required to produce the loss in Equation 178; the two histories have already become identical after $\mathcal{T}_B$ acts.

The example shows why descriptive collapse should be decomposed by stage. A braid descriptor can omit information because the chosen trajectory projection excludes an internal variable even before topology is considered.

13.4 Common Realized History with Distinct Collapse-Histories

The fourth example isolates generator underdetermination and the independence of path-history from collapse-history. Let the carrier be $S := \mathbb{N}_0$, the nonnegative integers, and let the carrier congruence $\sim$ be equality. Define two transformations on S by Equation 179.

$$m(x) := 1, \qquad n(x) := x + 1 \qquad \text{for every } x \in \mathbb{N}_0. \tag{179}$$

The transformation m is constant and therefore noninjective. The transformation n is the unilateral shift and is injective but not surjective. Let M be the semigroup of transformations generated by m and n under composition, and let $\alpha(g, x) := g(x)$ be its natural action. This choice avoids requiring an identity or an inverse in the mechanism semigroup.

Consider the two length-one generative specifications in Equation 180.

$$u_m := (0, (m)), \qquad u_n := (0, (n)). \tag{180}$$

Equation 179 gives $m(0) = n(0) = 1$. Therefore the realization map sends both specifications to the same history, as stated in Equation 181.

$$\text{Gen}_1(u_m) = \text{Gen}_1(u_n) = (0, 1). \tag{181}$$

Thus $u_m \equiv_{\text{Gen},1} u_n$ by Equation 130. Every trajectory descriptor acting only on the realized history must assign the same descriptive value to the two specifications.

Their mechanism-level collapse relations are nevertheless different. Because the congruence is equality, the definition of Δ^{mech} recalled in Section 9.1 reduces to ordinary failure of injectivity. The constant map m collapses every distinct pair of carrier states; for example, $(0, 2) \in \Delta^{\text{mech}}(m)$. The shift n is injective, so $\Delta^{\text{mech}}(n) = \varnothing$. These conclusions are stated together in Equation 182.

$$(0, 2) \in \Delta^{\text{mech}}(m), \qquad \Delta^{\text{mech}}(n) = \varnothing. \tag{182}$$

For $k = 1$, Definition 9.2 therefore gives distinct collapse histories, as recorded in Equation 183.

$$H_{\Delta,1}(u_m) \neq H_{\Delta,1}(u_n). \tag{183}$$

Equations 181 and 183 exhibit the non-descent condition of Proposition 9.4. The same realized history has at least two compatible collapse histories. Consequently, $\mathfrak{H}_{\Delta,1}((0, 1))$ contains at least two elements, and Corollary 9.5 confirms that mechanism-level collapse-history is not locally identifiable from the realized history alone.

If a braid extraction is assigned to the common history $(0, 1)$, the two generative specifications also have the same path-history because Definition 9.3 gives $H_{B,1} = D_B \circ \text{Gen}_1$. The example therefore realizes one of the logically independent cases described in Section 9.3: identical path-history can coexist with different collapse-histories.

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13.5 Fixed-Strand Coverage of Changing Component Number

The fifth example illustrates a domain defect rather than a resolution defect. Consider an intended relational process with a time-dependent number of represented components defined by Equation 184.

$$N(t) := \begin{cases} 2, & 0 \le t \le \frac{1}{2}, \\ 3, & \frac{1}{2} < t \le 1. \end{cases} \tag{184}$$

The transition at $t = \frac{1}{2}$ can be interpreted formally as one represented component branching into two descendants. The precise semantics of the branching are irrelevant to the present point; Equation 184 states only that a faithful trajectory representation requires two persistent traces before the transition and three after it.

Suppose, for contradiction, that this process were represented by one standard braid extraction datum $\mathcal{E}_B = (n, c_0, \Gamma_B, \mathcal{T}_B)$ of Definition 4.1. Every value of $\mathcal{T}_B(h)$ would then belong to $\mathcal{U}_n$ and would therefore contain exactly n configuration points at every time. Faithfulness to the intended component count would require both equations in Equation 185.

$$n = 2 \quad \text{for } t < \tfrac{1}{2}, \qquad n = 3 \quad \text{for } t > \tfrac{1}{2}. \tag{185}$$

No fixed integer n satisfies both conditions. Hence the process cannot belong to Γ_B under a standard fixed-n braid datum that aims to preserve the branching component distinction.

The example is a direct instance of Proposition 10.2, item 1. It produces a nonempty domain defect whenever the target class includes the branching history, as recorded in Equation 186.

$$h_{\mathrm{br}} \in \mathcal{E}_B^{\mathrm{dom}}(\mathfrak{D}). \tag{186}$$

Nothing in Equation 186 implies that branching is indescribable in GR. It states that the fixed-strand braid language is the wrong lower-level descriptor for this history unless it is extended or replaced.

13.6 Relative Generality under Specification Changes

The preceding examples can be combined to show why generality is relative to both target class and descriptive resolution.

Define a narrow class $\mathfrak{C}_{\mathrm{ex}}$ whose histories are all represented by collision-free two-strand based loops in $\mathcal{U}_2$, whose extraction map is the identity on those loops, and whose composition is loop concatenation. Let the baseline equivalence for each member of this class be based-homotopy equivalence, which is exactly the native braid equivalence by Proposition 4.4.

For this class, every target history is braid-admissible, so the domain defect is empty. The baseline is the native braid baseline, so under-discrimination and surplus resolution are both empty. Composition compatibility is built into the class definition. The four primitives are explicitly realized by Section 4. The assumptions of Proposition 12.4 are therefore satisfied, giving Equation 187.

$$\mathcal{L}_B \text{ is trajectory-general relative to } (\mathfrak{C}_{\mathrm{ex}}, \{\equiv_B\}, \mathsf{P}_4)\,. \tag{187}$$

Now enlarge the target class by adjoining the branching process of Section 13.5. Denote the enlarged class by $\mathfrak{C}_{\mathrm{ex}}^{+}$. Equation 186 supplies a nonempty domain defect for at least one member of the enlarged class. Proposition 12.5 therefore yields Equation 188.

$$\mathcal{L}_B \text{ is not trajectory-general relative to } \mathfrak{C}_{\mathrm{ex}}^{+} \text{ under the same fixed-strand braid schema.} \tag{188}$$

A second failure can occur without enlarging the target class. Keep $\mathfrak{C}_{\mathrm{ex}}$ but refine the declared baseline from based-homotopy equivalence to exact extracted-loop equality. The family in Section 13.2 contains distinct loops $q_{a_0} \neq q_{a_1}$ with equal braid descriptors. Under the finer baseline, that pair belongs to $\Delta_B^{\mathrm{under}}$. Proposition 12.5 therefore gives Equation 189.

$$\mathcal{L}_B \text{ is not trajectory-general relative to the exact-loop baseline on } \mathfrak{C}_{\mathrm{ex}}. \tag{189}$$

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Table 9: Formal examples and the distinctions they instantiate.

Example	Formal phenomenon	Main result
Constant loop versus one exchange	Endpoint equivalence with distinct braid classes	Endpoint description is strictly coarser than braid description
Radially deformed exchanges	Homotopy-stage descriptive collapse	Distinct geometric trajectories can share one braid class
Internal binary marker	Extraction-stage descriptive collapse	Information can be lost before the homotopy quotient acts
Constant mechanism versus shift mechanism	Generator underdetermination with different collapse-history	One realized path-history need not determine the mechanism-level history of lost distinctions
Two-to-three component branching	Domain defect	Fixed-n braid extraction does not cover changing-component processes
Narrow versus enlarged target class	Relative trajectory-generality	Generality changes with target class and declared descriptive resolution

Equations 187–189 provide a concrete demonstration of the relative generality criterion. The same formal braid language can satisfy trajectory-generality for a restricted class at a topological resolution, fail after a branching history is added to the class, and also fail on the original class when the required descriptive resolution is refined beyond braid equivalence.

Table 9 consolidates the examples and the formal distinctions they instantiate.

The examples confirm the internal logic of the formalism without assigning substantive social, physical, or normative interpretations to the braid symbols. They also identify the distinct reasons why two histories may become descriptively equal: trajectory extraction can remove variables, homotopy can remove geometric detail, and later invariants can remove further braid-level distinctions. Conversely, domain failure is not an information quotient at all; it is the absence of a description for a target history. The next section places these distinctions back into the wider GR representational architecture and states how the higher meta-language, generative models, realized histories, and multiple descriptive languages can coexist without being collapsed into one formal level.

14 GR Representational Architecture

This section places the braid construction within a broader GR representational architecture. Its role is to consolidate the logical levels developed throughout the manuscript without turning one lower-level descriptor into the ontology of the whole system. The objective is to formalize how the highest meta-language, a concrete generative instantiation, realized histories, trajectory-descriptive languages, and downstream invariants can coexist as distinct but connected layers. Section 14.1 defines the layered architecture. Section 14.2 introduces families of trajectory descriptors on a common history domain and derives their joint resolution. Section 14.3 distinguishes refinement, coarsening, and incomparability among descriptive languages. Section 14.4 states the conditions under which information can be recovered across layers. Section 14.5 formulates the resulting representational pluralism and identifies the specific place occupied by braid-theoretic description.

14.1 Layered GR Representational Architecture

The preceding construction has introduced several formal objects at different logical levels. They can be organized without adding a new ontological claim.

At the highest level, the GR meta-language supplies the four primitives collected in P_4 by Equation 139. These primitives state what a dynamical formalism must be able to express at the chosen level

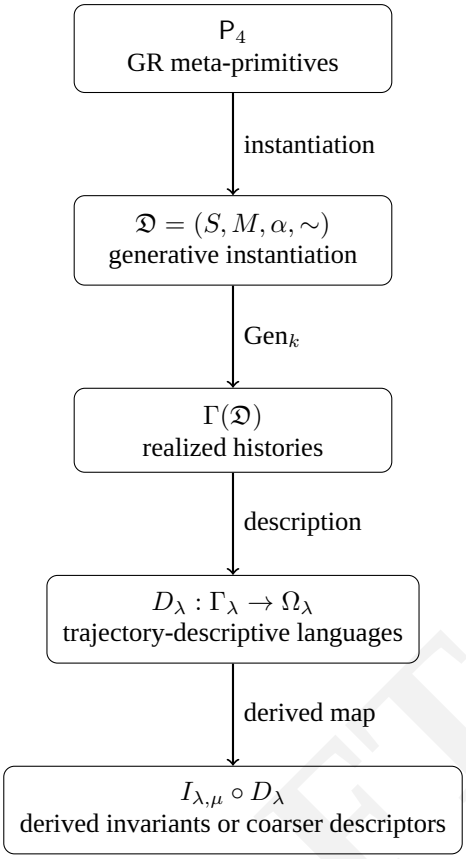

Figure 3: Layered GR representational architecture. Each downward step introduces additional commitments or a lower-resolution representation. The arrows do not imply that the lower layer uniquely determines the layer above it.

of abstraction: constraint, change, structure, and difference. They do not yet select a carrier, mechanism family, topology, metric, or braid structure.

A concrete dynamical instantiation is then given by $\mathfrak{D} = (S, M, \alpha, \sim)$. From $\mathfrak{D}$, the mechanism action generates histories in $\Gamma(\mathfrak{D})$ through the maps Gen_k of Equation 18. A trajectory-descriptive language acts downstream on selected histories. The braid descriptor $D_B : \Gamma_B \to B_n$ is one such language. Derived invariants such as $I \circ D_B$ occupy a further downstream level.

The layered architecture is summarized by Figure 3.

Figure 3 records dependency rather than equivalence. The four-primitives layer constrains what a lower formalism should make explicit. The generative instantiation determines which histories can be realized. A trajectory descriptor then selects and organizes features of those histories. A derived invariant can compress that description further.

The architecture therefore distinguishes two kinds of downward operation. The transition from $\mathfrak{D}$ to $\Gamma(\mathfrak{D})$ is generative: mechanisms acting on carrier states produce histories. The transition from $\Gamma(\mathfrak{D})$ to Ω_λ is descriptive: histories already generated are mapped into a chosen representational space. The transition from Ω_λ to a derived invariant codomain is a further descriptive coarsening or transformation. These operations should not be conflated merely because each can be written as a map.

To state the architecture compactly for one fixed system, define a *representational stack* by Equation 190.

$$\mathscr{R}(\mathfrak{D}) := (\mathsf{P}_4,\, \mathfrak{D},\, \Gamma(\mathfrak{D}),\, \{D_\lambda\}_{\lambda\in\Lambda})\,, \tag{190}$$

where Λ is an index set labeling the trajectory-descriptive languages currently used for the system. Equation 190 is a bookkeeping device. It does not imply that the family $\{D_\lambda\}_{\lambda\in\Lambda}$ is exhaustive or that a privileged descriptor exists.

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14.2 Families of Trajectory-Descriptive Languages

One realized history can support several descriptions. A braid descriptor can record topological intertwining; another language might retain exact geometry, symbolic event order, network structure, or a domain-specific semantic annotation. The architecture should therefore permit several descriptors without forcing their codomains into one common metric space.

Let Λ be a nonempty index set. For each $\lambda \in \Lambda$, let $D_\lambda : \Gamma_\lambda \to \Omega_\lambda$ be a trajectory-descriptive map on the same underlying system $\mathfrak{D}$. The domains Γ_λ may differ.

The histories on which every descriptor in the family is simultaneously defined form the common domain in Equation 191.

$$\Gamma_\Lambda := \bigcap_{\lambda\in\Lambda} \Gamma_\lambda. \tag{191}$$

If Equation 191 is empty, no joint comparison over all descriptors is available without restricting the family or changing the domain.

Assume henceforth that Γ_Λ is nonempty. Define the joint descriptor by Equation 192.

$$D_\Lambda : \Gamma_\Lambda \longrightarrow \prod_{\lambda\in\Lambda} \Omega_\lambda, \qquad D_\Lambda(h) := \big(D_\lambda(h)\big)_{\lambda\in\Lambda}. \tag{192}$$

The Cartesian product in Equation 192 is used only to collect already-defined descriptions. It does not impose a metric, order, or arithmetic across their codomains.

Let $\equiv_\lambda$ denote the equivalence induced by D_λ, and let $\equiv_\Lambda$ denote the equivalence induced by D_Λ. The joint equivalence is the intersection of the component equivalences.

Proposition 14.1 (Equivalence induced by a descriptor family)**.** *On the common domain Γ_Λ, Equation 193 holds.*

$$\equiv_\Lambda = \bigcap_{\lambda\in\Lambda} \equiv_\lambda . \tag{193}$$

Proof. Take $h_1, h_2 \in \Gamma_\Lambda$. By definition of the equivalence induced by a descriptor, $h_1 \equiv_\Lambda h_2$ holds exactly when $D_\Lambda(h_1) = D_\Lambda(h_2)$. Equation 192 makes equality of the product values equivalent to $D_\lambda(h_1) = D_\lambda(h_2)$ for every $\lambda \in \Lambda$. Each component equality is equivalent to $h_1 \equiv_\lambda h_2$. Hence the pair belongs to every relation $\equiv_\lambda$, which is exactly membership in the intersection on the right-hand side of Equation 193. □

Proposition 14.1 implies that a joint descriptor is at least as fine as each of its components. For every $\lambda \in \Lambda$, Equation 194 holds.

$$\equiv_\Lambda \subseteq \equiv_\lambda . \tag{194}$$

Equation 194 follows because an intersection of relations is contained in each relation being intersected.

The joint construction does not recover information already lost by every component descriptor. If a pair of histories lies in $\equiv_\lambda$ for every $\lambda \in \Lambda$, then Equation 193 makes that pair indistinguishable even under the joint descriptor. Multiple descriptive languages can therefore complement one another only when at least one language preserves the relevant distinction.

14.3 Descriptive-Resolution Relations

The descriptive-resolution order of Definition 3.6 provides a precise way to compare two languages on a common domain. Let $D_a : \Gamma_* \to \Omega_a$ and $D_b : \Gamma_* \to \Omega_b$ be two descriptors defined on the same history set.

The relation $D_a \preceq D_b$ means $\equiv_{D_a} \subseteq \equiv_{D_b}$. Proposition 3.7 shows that this relation holds exactly when D_b factors through the image of D_a. Thus one descriptor is a coarsening of another when its value can be computed from the finer descriptor without returning to the original history.

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The factorization can be restated in architectural form. If $D_a \preceq D_b$, there exists a unique map $\varphi_{ba} : \mathrm{im}(D_a) \to \mathrm{im}(D_b)$ such that Equation 195.

$$D_b = \iota_b \circ \varphi_{ba} \circ D_a^{\mathrm{im}}, \tag{195}$$

with notation as in Proposition 3.7. Equation 195 identifies a dependency relation among descriptors.

Two descriptors need not be comparable in this order.

Definition 14.2 (Descriptive incomparability)**.** Descriptors D_a and D_b on a common domain are *descriptively incomparable* when Equation 196 holds.

$$\equiv_{D_a} \not\subseteq \equiv_{D_b}, \qquad \equiv_{D_b} \not\subseteq \equiv_{D_a} . \tag{196}$$

Definition 14.2 means that each descriptor distinguishes at least one pair that the other identifies. Incomparability is therefore different from saying that one language is more detailed overall. The two languages preserve partially different distinctions.

The joint descriptor provides a common refinement even when the components are incomparable.

Proposition 14.3 (Joint refinement of two descriptors)**.** *For descriptors D_a and D_b on a common domain, let $D_{a,b}$ be their joint descriptor. Then Equation 197 holds.*

$$\equiv_{D_{a,b}} = \equiv_{D_a} \cap \equiv_{D_b}, \qquad D_{a,b} \preceq D_a, \qquad D_{a,b} \preceq D_b. \tag{197}$$

Proof. The equality follows from Proposition 14.1 for a two-element index set. The two resolution relations follow because the intersection is contained in each component equivalence relation. □

Proposition 14.3 gives a formal version of methodological complementarity. A geometric descriptor and a braid descriptor, for example, need not be ordered by descriptive resolution if each preserves distinctions the other omits. Their joint use can retain both sets of distinctions on the common domain.

The proposition should not be interpreted as requiring every analysis to maximize descriptive resolution. A finer joint descriptor can sacrifice robustness, simplicity, interpretability, or empirical estimability. The choice of descriptor family remains relative to the research question and the baseline distinctions that the analysis intends to preserve.

14.4 Cross-Level Recovery Conditions

The architecture in Figure 3 is directed downward. Recovery in the opposite direction requires additional conditions. The preceding sections have already derived such conditions for specific maps; they can now be stated in a common form.

Let $f : X \to Y$ be any map occurring between adjacent representational layers. Exact recovery of $x \in X$ from the value $f(x)$ is possible on a subset $X_0 \subseteq X$ only if the restricted map $f|_{X_0}$ is injective. If the restriction is injective, it has an inverse on its image.

Proposition 14.4 (Local recovery criterion for a representational layer)**.** *Let $f : X \to Y$ be a map and let $X_0 \subseteq X$. There exists a map $R : f(X_0) \to X_0$ satisfying Equation 198 if and only if $f|_{X_0}$ is injective.*

$$R(f(x)) = x \qquad \textit{for every } x \in X_0. \tag{198}$$

Proof. Assume first that such a map R exists. If $f(x_1) = f(x_2)$ for $x_1, x_2 \in X_0$, then applying R gives $x_1 = R(f(x_1)) = R(f(x_2)) = x_2$. Hence the restriction is injective.

Conversely, if $f|_{X_0}$ is injective, every $y \in f(X_0)$ has exactly one preimage in X_0. Define $R(y)$ to be that unique preimage. Equation 198 follows. □

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Proposition 14.4 unifies several earlier results. Exact extracted-loop recovery from a braid element requires injectivity of $Q_B^{\mathcal{T}}$ by Proposition 5.4. Exact history recovery additionally requires injectivity of $\mathcal{T}_B$ by Corollary 5.5. Recovery of a unique generative specification from a realized history requires a singleton generation fiber $\mathcal{G}_k(h)$, equivalently local injectivity of Gen_k on the relevant subset. Recovery from an invariant value requires injectivity of the invariant on the realized braid image, as formalized by Proposition 7.7.

The architecture therefore supports a general non-reduction principle.

Remark 14.5 (Representational non-reduction)**.** A lower representational layer should not be identified with an upstream layer unless the corresponding downward map is injective at the declared resolution. Equality of two braid descriptions does not imply equality of full histories when D_B is noninjective. Equality of two realized histories does not imply equality of generative specifications when Gen_k is non-injective. Equality of two invariant values does not imply equality of braid elements when the invariant is incomplete on the realized image.

Remark 14.5 is methodological rather than ontological. It states when one representation can or cannot recover another; it does not assert that the represented objects are ultimate ontological layers of reality.

14.5 Representational Pluralism within GR

The layered architecture permits several lower-level languages to coexist because the highest GR meta-language does not require one privileged representational geometry. A concrete analysis can therefore choose a descriptor according to the distinctions relevant to its research question.

Let $\mathcal{D}(\mathfrak{D})$ denote the collection of trajectory-descriptive maps currently admitted for a system $\mathfrak{D}$. The architecture imposes three minimal requirements on the use of a member $D_\lambda \in \mathcal{D}(\mathfrak{D})$.

First, the domain Γ_λ and its admissibility conditions should be explicit. This identifies the constraint commitments of the descriptor.

Second, the induced equivalence $\equiv_\lambda$ should be explicit or derivable. This identifies the descriptive resolution and makes visible which distinctions are preserved or collapsed.

Third, if the descriptor is used compositionally, its structural operation and compatibility with history composition should be stated. This identifies the structure commitments of the descriptor.

These requirements are summarized in Equation 199.

$$\mathcal{A}(D_\lambda) := (\Gamma_\lambda,\, \equiv_\lambda,\, \odot_\lambda)\,, \tag{199}$$

where $\odot_\lambda$ may be omitted when the descriptor is intentionally non-compositional. The symbol $\mathcal{A}(D_\lambda)$ is introduced only as a record of explicit commitments; it is not a new adequacy score.

The braid descriptor occupies one position in this plural architecture. Its admissible domain is Γ_B; its induced equivalence is $\equiv_B$; and its compositional operation is braid multiplication when the extraction is composition-compatible. Its distinctive strengths are topological path sensitivity, compositionality, support for derived invariants, and robustness under admissible deformation. Its limitations are equally explicit: standard braid description requires a suitable fixed-strand extraction, quotients out geometric distinctions inside a homotopy class, and does not by itself represent a generator at a resolution finer than the observed braid.

Other descriptors can occupy different positions without being reduced to braid theory. A metric trajectory descriptor may preserve geometric distances that braid description removes. A symbolic event language may preserve semantic event types that a bare braid does not encode. A network descriptor may represent changing adjacency structures even when no useful braid extraction exists. The GR architecture allows these languages to be compared through their domains, equivalence relations, composition laws, and primitive realizations rather than assuming that one representation must subsume the others.

This pluralism also clarifies the role of joint description. When two descriptors preserve complementary distinctions and share a nonempty common domain, Proposition 14.3 constructs a joint descriptor

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that is finer than either component. When their domains do not overlap, no such joint description follows automatically. When a research question requires distinctions absent from every available descriptor, a new language or a richer instantiation is required.

The representational architecture can therefore be summarized by the dependency statement in Equation 200.

$$\mathsf{P}_4 \;\rightsquigarrow\; \mathfrak{D} \xrightarrow{\text{Gen}} \Gamma(\mathfrak{D}) \xrightarrow{\{D_\lambda\}} \{\Omega_\lambda\} \xrightarrow{\{I_{\lambda,\mu}\}} \{\mathcal{A}_{\lambda,\mu}\}. \tag{200}$$

The symbol $\rightsquigarrow$ in Equation 200 denotes instantiation of the meta-language rather than a function between sets. The ordinary arrows denote maps whose domains and codomains are fixed by a concrete construction. The family notation emphasizes that several trajectory descriptors and several downstream invariants can coexist.

Equation 200 locates the contribution of the present paper within the wider GR architecture. Braid theory supplies one trajectory-descriptive branch. Its value lies in the distinctions it preserves, the quotient structure it makes explicit, and the robust topological information derived from realized relational trajectories. The next section collects the remaining scope boundaries and open problems before the manuscript closes with its main conclusions.

15 Scope Boundaries of the Present Formalism

This section identifies the main boundaries that remain after the braid-theoretic trajectory-descriptive formalism has been constructed and evaluated. Its role is to separate limitations that can be addressed by a direct extension of the present architecture from limitations that require genuinely new representational structure. The objective is not to complete those extensions inside the current manuscript, but to state them precisely enough that later work can inherit a well-defined problem. Section 15.1 treats endpoint variation through a fundamental-groupoid extension. Section 15.2 formalizes the obstruction created by changing component number. Section 15.3 shows how semantic or internal-state information can be added through joint description without being identified with braid topology. Section 15.4 isolates the problem of evolving extraction rules and evolving baseline equivalences. Section 15.5 formulates trajectory reconstruction from incomplete observations as a set-valued inverse problem. Section 15.6 states the unresolved cross-system compatibility problem.

15.1 Endpoint Variation through a Fundamental-Groupoid Extension

The baseline braid construction uses based loops in $\mathcal{U}_n = \mathrm{UConf}_n(\mathbb{R}^2)$ and therefore assigns each admissible trajectory to the automorphism group of one selected configuration c_0. Histories whose extracted trajectories begin and end at different unordered configurations require a broader codomain.

Let $\mathcal{P}(\mathcal{U}_n)$ denote the set of continuous paths $q : [0,1] \to \mathcal{U}_n$. For $q \in \mathcal{P}(\mathcal{U}_n)$, define its source and target by $s(q) := q(0)$ and $t(q) := q(1)$. Two paths with the same source and target are equivalent when they are homotopic relative to their endpoints.

The *fundamental groupoid* of $\mathcal{U}_n$ is the structure defined in Equation 201.

$$\Pi_1(\mathcal{U}_n) := \left\{ \begin{array}{l} \text{objects: } c \in \mathcal{U}_n, \\ \text{morphisms } c \to c' : \text{ endpoint-preserving homotopy classes of paths } q \\ \text{with } q(0) = c \text{ and } q(1) = c' \end{array} \right\}. \tag{201}$$

Composition is induced by path concatenation whenever the target of the first path equals the source of the second. Every morphism is invertible because reversing a path gives an inverse up to endpoint-preserving homotopy.

A trajectory descriptor with variable extracted endpoints can therefore be defined without forcing all paths to close at one base configuration. Let $\Gamma_\Pi \subseteq \Gamma(\mathfrak{D})$ be a history domain and let $\mathcal{T}_\Pi : \Gamma_\Pi \to \mathcal{P}(\mathcal{U}_n)$ be an extraction map. Define the groupoid-valued descriptor in Equation 202.

$$D_\Pi : \Gamma_\Pi \longrightarrow \mathrm{Mor}(\Pi_1(\mathcal{U}_n)), \qquad D_\Pi(h) := [\mathcal{T}_\Pi(h)]_\partial, \tag{202}$$

where $\mathrm{Mor}(\Pi_1(\mathcal{U}_n))$ denotes the collection of all morphisms of the fundamental groupoid.

The based-loop braid descriptor is recovered as the automorphism sector of Equation 202. The automorphism group at c_0 is stated in Equation 203.

$$\mathrm{Aut}_{\Pi_1(\mathcal{U}_n)}(c_0) = \pi_1(\mathcal{U}_n, c_0) \cong B_n. \tag{203}$$

Equation 203 follows directly from the definitions: an automorphism of c_0 in the groupoid is an endpoint-preserving homotopy class of loops based at c_0, which is an element of the fundamental group.

The based-loop formalism developed here is therefore one sector of a broader endpoint-sensitive description. The groupoid extension addresses endpoint variation while preserving the same topological quotient principle. It does not address changing component number, branching, or the generative-rule problem considered below.

15.2 Fixed-Cardinality Configuration Sectors

The standard braid domain fixes the number of represented components. A natural first attempt to relax that assumption is to collect all configuration spaces into one disjoint union. Define the configuration-sector space in Equation 204.

$$\mathcal{U}_* := \coprod_{n \geq 1} \mathcal{U}_n. \tag{204}$$

Each sector $\mathcal{U}_n$ in Equation 204 contains configurations with exactly n points. The coproduct topology makes every sector open and closed in $\mathcal{U}_*$.

The disjoint union alone does not solve the changing-cardinality problem.

Proposition 15.1 (Continuous trajectories cannot change configuration sector)**.** *Let $q : [0,1] \to \mathcal{U}_*$ be continuous. Then there exists one integer $n \geq 1$ such that Equation 205 holds.*

$$q([0,1]) \subseteq \mathcal{U}_n. \tag{205}$$

Proof. The interval $[0,1]$ is connected, so its continuous image $q([0,1])$ is connected. Each $\mathcal{U}_n$ is open and closed in the disjoint union $\mathcal{U}_*$. If the image met two distinct sectors $\mathcal{U}_n$ and $\mathcal{U}_m$ with $n \neq m$, the intersections of the image with those sectors would give a separation of $q([0,1])$ into nonempty disjoint open subsets. This contradicts connectedness. Hence the image lies in one sector. □

Proposition 15.1 shows that simply replacing $\mathcal{U}_n$ with $\mathcal{U}_*$ does not permit births, deaths, branching, or fusion to appear as ordinary continuous trajectories. Such processes require additional inter-sector morphisms that are not paths inside the disjoint union.

A future extension can therefore be described abstractly by a category $\mathcal{C}_*$ whose objects include configurations in the sectors $\mathcal{U}_n$ and whose morphisms contain both within-sector path classes and additional inter-sector processes. The required structural condition is stated schematically in Equation 206.

$$\mathrm{Hom}_{\mathcal{C}_*}(c_n, c_m) \neq \varnothing \quad \text{may be permitted for } n \neq m, \tag{206}$$

where $c_n \in \mathcal{U}_n$ and $c_m \in \mathcal{U}_m$. Equation 206 is a requirement on a future extension rather than a completed construction. The present paper does not specify which inter-sector morphisms should represent splitting, fusion, entry, exit, or emergence.

The result sharpens the domain limitation diagnosed in Section 13.5: variable component number is not repaired by taking a larger disjoint union of ordinary braid configuration spaces. New morphism types are required.

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15.3 Joint Enrichment of Trajectory Description

A bare braid descriptor records topological trajectory structure and does not, by itself, encode the semantic type of an interaction, the internal state of a component, or a normative interpretation. Section 14.2 already provides a general way to enrich such a description without altering braid equivalence itself.

Let $D_S : \Gamma_S \to \Omega_S$ be a second descriptor whose codomain records semantic or internal-state information relevant to an application. No particular structure on Ω_S is assumed. On the common domain $\Gamma_{B,S} := \Gamma_B \cap \Gamma_S$, define the joint descriptor in Equation 207.

$$D_{B,S} : \Gamma_{B,S} \longrightarrow B_n \times \Omega_S, \qquad D_{B,S}(h) := (D_B(h), D_S(h)) \,. \tag{207}$$

By Proposition 14.1, the equivalence induced by $D_{B,S}$ is the intersection stated in Equation 208.

$$\equiv_{B,S} = \equiv_B \cap \equiv_S \,. \tag{208}$$

Equation 208 gives a disciplined route to decorated trajectory description. If two histories have the same braid topology but differ in a semantic variable preserved by D_S, the joint descriptor can distinguish them. Conversely, semantic enrichment does not retroactively change what the braid component means; it adds a second descriptive channel.

This construction is especially important for later applications to justice, trust, governance, or law. Those domains can require distinctions such as role, consent, authorization, power, responsibility, or institutional status. The present paper does not define those domain-specific descriptors. Equation 207 only states how such information can be combined with braid topology once an appropriate domain language has been independently justified.

15.4 Evolving Descriptive Regimes

The current formalism fixes both the extraction rule $\mathcal{T}_B$ and the baseline equivalence $\approx_\Gamma$ during an analysis. This assumption becomes restrictive when the system's own historical development changes what can be observed, what counts as the same component, or which distinctions the analysis treats as relevant.

Let R be an index set of descriptive regimes. For each $r \in R$, suppose there is a descriptor $D_r : \Gamma_r \to \Omega_r$ and a baseline equivalence $\approx_r$ on Γ_r. The family is stated in Equation 209.

$$\mathfrak{E} := \{(\Gamma_r, D_r, \approx_r) : r \in R\} \,. \tag{209}$$

Equation 209 permits the descriptive rule itself to vary. It does not yet say how descriptions produced under different regimes should be compared.

A comparison between two regimes r and r' requires additional structure. On a shared set of histories $\Gamma_r \cap \Gamma_{r'}$, one possible requirement is a comparison map $\chi_{r'r} : \Omega_r \to \Omega_{r'}$ satisfying Equation 210.

$$D_{r'}(h) = \chi_{r'r}(D_r(h)) \qquad \text{for every } h \text{ in the declared comparison domain.} \tag{210}$$

When a map satisfying Equation 210 exists, the later regime description factors through the earlier one on that comparison domain. When no such map exists, the two regimes can preserve incomparable distinctions and should not be treated as different encodings of one fixed descriptive object.

The same issue applies to the baseline equivalences. A change from $\approx_r$ to $\approx_{r'}$ can alter which pairs count as descriptive collapse even when the underlying map D_r remains fixed. The evolving-congruence problem identified in the earlier GR metamodel therefore has a descriptive analogue here: the rules governing what counts as a preserved distinction may themselves be history-dependent.

The present manuscript keeps $\mathcal{T}_B$ and $\approx_\Gamma$ fixed so that descriptive collapse and robustness are well defined. A full theory of evolving descriptive regimes requires transition rules such as Equation 210 or another explicitly justified comparison structure.

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15.5 Set-Valued Reconstruction of Incomplete Observations

The formalism developed so far takes the extracted trajectory as determined once a history and extraction rule are given. Empirical reconstruction can be less determinate. Historical archives, partial observations, identity uncertainty, or missing temporal order can support several trajectories consistent with the available evidence.

Let Y denote an observation object and let $\mathcal{R}_B(Y) \subseteq \Omega(\mathcal{U}_n, c_0)$ be the set of braid-admissible loops consistent with that observation under a declared reconstruction procedure. The set of compatible braid elements is defined in Equation 211.

$$\mathcal{B}(Y) := \{\kappa_n(Q_B(q)) : q \in \mathcal{R}_B(Y)\} \subseteq B_n. \tag{211}$$

Equation 211 is set-valued and introduces no probability model.

Definition 15.2 (Braid identifiability from an observation)**.** The braid class is *identifiable from* Y under the declared reconstruction procedure when Equation 212 holds.

$$|\mathcal{B}(Y)| = 1. \tag{212}$$

Definition 15.2 permits exact topological identification even when the exact geometric trajectory remains uncertain. If $\mathcal{R}_B(Y)$ contains several geometrically different loops but all lie in the same based-homotopy class, Equation 212 still holds.

Derived invariants can remain identifiable even when the full braid class is not. For $I : B_n \to \mathcal{A}$, define the compatible invariant set in Equation 213.

$$\mathcal{A}_I(Y) := \{I(\beta) : \beta \in \mathcal{B}(Y)\}. \tag{213}$$

The invariant is identifiable from Y when $|\mathcal{A}_I(Y)| = 1$. Because I can be noninjective, this condition can hold even when $|\mathcal{B}(Y)| > 1$.

The set-valued construction clarifies a methodological advantage of topological description. Uncertainty in the exact trajectory does not automatically imply uncertainty in every topological characteristic. At the same time, identifiability in Equation 212 depends on the declared reconstruction set $\mathcal{R}_B(Y)$. A social-scientific methodology for constructing that set belongs to a later application paper rather than to the present formal manuscript.

15.6 Cross-System Naturality

The preliminary generality criterion of Section 11 is system-indexed. It requires the same formal roles across a class of systems but does not define morphisms between those systems. A stronger language theory would ask whether descriptions behave coherently when a dynamical system is transformed, restricted, aggregated, or embedded into another system.

Suppose future work defines a category $\mathbf{Dyn}_{\mathrm{GR}}$ whose objects are GR dynamical systems and whose morphisms $F : \mathfrak{D} \to \mathfrak{D}'$ induce history maps $\Gamma(F) : \Gamma(\mathfrak{D}) \to \Gamma(\mathfrak{D}')$. Suppose also that a descriptive language assigns codomain maps $\Omega(F) : \Omega_{\mathcal{L}}(\mathfrak{D}) \to \Omega_{\mathcal{L}}(\mathfrak{D}')$. The natural compatibility condition would be the commutativity relation in Equation 214.

$$D_{\mathcal{L}}^{\mathfrak{D}'} \circ \Gamma(F) = \Omega(F) \circ D_{\mathcal{L}}^{\mathfrak{D}}. \tag{214}$$

Equation 214 states that transforming a history and then describing it should agree with describing the original history and then transforming the description.

The current paper does not define $\mathbf{Dyn}_{\mathrm{GR}}$, $\Gamma(F)$, or $\Omega(F)$. Equation 214 is therefore an open structural criterion rather than a theorem of the present formalism. Its inclusion identifies a precise next problem: system-relative trajectory-generality can be strengthened into a cross-system language theory only after the admissible transformations between systems have themselves been formalized.

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Table 10: Scope boundaries of the present braid-theoretic trajectory formalism.

Boundary	Present limitation	Formal direction
Variable extracted endpoints	Baseline descriptor uses based loops at one configuration	Replace the single fundamental group by a fundamental-groupoid descriptor
Changing component number	Continuous trajectories in the disjoint union of configuration sectors remain in one fixed-n sector	Introduce new inter-sector morphisms beyond ordinary configuration-space paths
Semantic or internal-state information	Bare braid topology does not encode domain semantics	Combine braid description with independently justified semantic descriptors through joint description
Evolving descriptive rules	$\mathcal{T}_B$ and $\approx_\Gamma$ are fixed during an analysis	Introduce indexed descriptive regimes and explicit comparison maps between them
Incomplete observation	Baseline extraction is deterministic once a history is given	Use set-valued reconstruction and test identifiability of braid classes or downstream invariants
Cross-system comparison	Generality criterion is system-indexed and lacks system morphisms	Define admissible GR system transformations and impose a naturality condition such as Equation 214
Generative explanation	Braid description remains downstream from the mechanism layer	Enrich the generator-side representation when distinctions among mechanisms sharing one braid must be preserved

15.7 Boundary Summary for Subsequent Work

The boundaries identified in this section fall into different formal classes. Table 10 summarizes the current status.

The table separates extensions of the current descriptor from problems requiring a new language layer. Endpoint variation can be handled within the same path-homotopy logic by moving from a fundamental group to a fundamental groupoid. Semantic enrichment can be handled by joint description. By contrast, changing component number requires morphisms that ordinary paths in configuration space cannot provide, and cross-system naturality requires a theory of system transformations not yet present in the highest-layer metamodel.

The scope boundaries also delimit the claim of the present paper. The manuscript has constructed and analyzed one braid-theoretic trajectory-descriptive language in GR, established its information flow, equivalence structure, invariant support, robustness, historical layers, expressive restrictions, and relative generality profile, and located it inside a plural representational architecture. It has not supplied a universal language for all dynamics, a complete generator theory, a semantic theory for particular application domains, or an empirical reconstruction methodology. These remaining problems define the interfaces through which later work on historical social dynamics, justice, trust, governance, law, and other domains can extend the present construction without changing its logical level retrospectively.

16 Conclusion

This section consolidates the formal results of the paper and states the scope of the resulting contribution. The manuscript began from a distinction among a generative rule, a realized history, and a subsequent description of selected structure in that history. Its objective was to determine whether braid theory can be

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derived as one trajectory-descriptive formalism within GR, to identify the information such a formalism preserves and removes, and to state more precisely what would be required for a lower-level language to count as generalized relative to a declared class of dynamics. The argument supports a bounded conclusion: braid-theoretic trajectory description provides a coherent and useful formal language for the topological organization of a restricted class of intertwined relational histories, while its generality depends on the level, domain, and descriptive resolution at which the claim is made.

The construction retains the hierarchy of the earlier GR metamodel. Constraint, change, structure, and difference remain the highest abstract primitives. Realized histories are derived from the mechanism action. A descriptor is then defined on selected histories. The braid descriptor follows from trajectory extraction, homotopy quotienting, and algebraic identification with the Artin braid group. The separation keeps braid generators distinct from generative mechanisms and allows a group-valued descriptor to coexist with irreversible mechanism-level dynamics.

The resulting information flow is explicit. Trajectory extraction can remove features of a full realized history before topology enters. The homotopy quotient can then identify geometrically distinct extracted trajectories that belong to the same topological class. The algebraic braid-group isomorphism preserves the resolution of that class and introduces no additional quotient. Section 5 formalized this sequence and decomposed braid-descriptive collapse into extraction-stage and homotopy-stage components. The later invariant analysis showed that downstream quantities introduce further coarsening only when they identify braid elements that the complete braid descriptor still distinguishes.

The equivalence analysis makes the descriptive resolution similarly explicit. Exact history identity, baseline history equivalence, extracted-loop equality, homotopy equivalence, braid equivalence, and endpoint equivalence are distinct relations. Their implications depend on the map used at each stage and on the baseline distinctions declared relevant by the analysis. In particular, endpoint agreement does not determine braid equivalence, while braid equivalence does not determine exact geometric trajectory identity. The braid descriptor therefore occupies an intermediate resolution: it retains topological path information that endpoint descriptions erase and intentionally removes geometric variation within an admissible homotopy class.

This quotient structure is also the source of robustness. The braid descriptor is invariant under admissible based homotopy by construction. After a Euclidean geometric realization is supplied, Section 8 derived a sufficient stability certificate from the minimum pairwise strand separation. The resulting robustness does not require a metric at the highest GR layer. The metric appears only as an optional lower-level instrument for certifying preservation of a topological class. The analysis also identified the complementary limitation: robustness weakens near topology-changing boundaries, and a continuous fixed-extraction family that changes braid class must leave the collision-free configuration space.

A second major distinction concerns historical information. The paper separates topological path-history from mechanism-level collapse-history. The former is recorded by the braid descriptor; the latter records distinctions irreversibly collapsed by the generative mechanism sequence. These layers can vary independently. Two generative specifications can produce the same realized history and hence the same braid path-history while carrying different collapse histories. Conversely, a nontrivial braid history can occur without mechanism-level collapse. This separation resolves the apparent tension between a reversible braid group and an earlier GR account in which irreversible history depends on noninvertible mechanisms.

The paper then used the braid construction to derive a preliminary criterion for generalized dynamical languages. Generality was made relative to a target class of systems, a declared baseline equivalence on histories, and a declared set of primitive requirements. Within that specification, trajectory-generality requires coverage adequacy, preservation of every distinction required by the baseline, compositional adequacy, and primitive adequacy. Exact equality between the language-induced equivalence and the baseline is treated as the stronger condition of resolution matching. Dynamics-generality is stronger: it additionally requires a generator-side representation that preserves the declared distinctions among generative specifications and remains semantically compatible with the trajectory description.

Applying this criterion to braid theory yields a mixed but precise profile. Braid description can be

trajectory-general for a braid-complete target class whenever braid equivalence preserves every distinction required by the baseline. Native braid equivalence gives a resolution-matched special case. The same language fails trajectory-generality when the target class includes histories outside its fixed-strand domain, when the baseline preserves distinctions removed by braid equivalence, or when extraction fails to preserve composition. A braid descriptor that is finer than the baseline can remain trajectory-general. At the generative level, a braid trace can label mechanism-bearing specifications only at a resolution that already treats all specifications producing the same braid as equivalent. Any finer generator resolution requires additional generator-side structure.

The formal examples in Section 13 show that these restrictions are not merely verbal qualifications. Common endpoints can coexist with different braid classes. Geometrically distinct trajectories can share one braid description. Internal distinctions can disappear at the extraction stage before any topological quotient is applied. Different mechanisms can generate the same realized history while carrying different collapse histories. A changing-component process can lie outside every fixed-n standard braid domain. The same braid language can therefore satisfy the trajectory-generality criterion for one declared class and fail it after either the target class or the required descriptive resolution is changed.

The wider GR architecture developed in Section 14 places these results within a plural representational framework. A realized history can support several trajectory-descriptive languages, and those languages need not be linearly ordered by informativeness. Some can be refinements or coarsenings of one another, while others preserve incomparable distinctions. Joint description can recover complementary information when the component descriptors have a common domain, although it cannot reconstruct distinctions lost by every component. The architecture therefore treats braid theory as one branch among potentially several lower-level descriptions rather than as a privileged representation of relational dynamics.

The scope boundaries derived in Section 15 indicate several directions in which the formalism can be extended without obscuring its present claims. Variable extracted endpoints can be represented by a fundamental groupoid rather than a single based fundamental group. Changing component number requires genuinely new inter-sector morphisms and cannot be obtained merely by taking a disjoint union of fixed-n configuration spaces. Semantic and internal-state information can be added through independently justified joint descriptors. Evolving extraction rules and evolving baseline equivalences require an explicit theory of descriptive regimes. Incomplete observations naturally lead to set-valued reconstruction and identifiability questions. Cross-system generality ultimately requires a formal account of admissible transformations between GR dynamical systems and an associated naturality condition.

These extensions also clarify what the current paper does not claim. The braid descriptor is not presented as an ontology of relational reality, a universal language for dynamics, a complete generator theory, or a substitute for domain semantics. Its mathematical usefulness follows from a narrower property: it gives a compositional algebraic representation of topological trajectory classes and supports robust derived invariants once an appropriate relational trajectory has been extracted. That narrower role is compatible with substantial methodological value precisely because some research problems concern the structural organization of realized histories even when the complete generative rule is unknown or underdetermined.

The paper therefore contributes two connected formal proposals. The first is a braid-based construction for describing generated trajectories downstream from the four GR meta-primitives. The second is a preliminary framework for judging the relative generality of lower-level dynamical languages through explicit criteria for coverage, preserved distinctions, composition, primitive realization, and generator representation. Braid theory serves as the first developed example and as a stress test of that framework.

The next stage of the research programme can now move in two directions without conflating them. One direction is formal: richer trajectory languages can be developed for endpoint variation, changing component number, semantic decoration, evolving descriptive regimes, and cross-system transformation. The other direction is applicative: historical social dynamics, processual justice, interactive justice, trust formation, governance, and law can each supply domain-specific extraction rules and baseline distinctions whose adequacy must be argued independently. The present manuscript provides the formal

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interface between those future applications and the higher GR meta-language.

The resulting position is deliberately modest. Braid-theoretic trajectory description does not explain every dynamical process and does not recover every hidden mechanism. It establishes a precise way to describe a class of generated relational trajectories, to identify which distinctions that description preserves, to quantify where information is lost through quotienting, to exploit topological robustness, and to state the conditions under which stronger claims of generality would be warranted. That level of formal clarity is sufficient to make later applications testable against the commitments of the language they use rather than relying on braid imagery as metaphor alone.

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