

Processual Justice: A Preliminary Braid-Theoretic Formalization

Wanhong HUANG
huangwanhong@serendip.ngo

Abstract

This paper develops a preliminary formal account of processual justice within Generative Relational Theory, using braid-theoretic trajectory description as one lower-level prototype. The central question is which distinctions among histories belong to the object of justice and which may be normatively compressed. The paper defines justice-relevant history domains; distinguishes terminal-state, endpoint, relational, braid-topological, and normative equivalence; and treats a justice evaluation as inducing a partition of history space. A historical descriptor is normatively sufficient when its equivalence classes do not cross justice-equivalence classes, so the justice quotient provides a theory-relative boundary for principled historical compression. Braid theory is treated conditionally: domain coverage is separated from normative resolution, and identities, roles, harm, benefit, consent, and responsibility enter through explicit decoration rather than being inferred from topology. Reparation is modeled as normative transformation without historical erasure. Set-valued historical reconstruction supports complete, partial, and robust normative identification, while cross-case comparison requires explicit contextual transport. The framework also separates historical uncertainty from variation among admitted normative theories and states reporting rules for representation failure, semantic underdetermination, and theory sensitivity. The proposal is structural rather than a complete substantive theory of justice.

Keywords: processual justice; historical justice; braid theory; normative equivalence; reparation

Discussion Paper Note

This manuscript is the normative continuation of two preceding studies. The first developed braid-theoretic trajectory description as one lower-level formal language within Generative Relational Theory. The second developed a methodology for reconstructing and comparing historical social trajectories under evidential uncertainty. The present paper asks a different question: when, and at what resolution, should the history by which a social state was generated belong to the object of justice?

The paper uses braid theory as a formal prototype rather than as a moral ontology. A braid can preserve some path-sensitive distinctions that an endpoint description removes, but topological difference does not automatically imply normative difference, and topological equivalence does not automatically imply justice equivalence. The central normative task is therefore to specify which historical distinctions a justice theory is warranted in preserving and which it may legitimately quotient out.

The proposal is intentionally broader than procedural justice and narrower than the claim that all justice is historical. It studies conditions under which process or history is constitutive of a justice judgment, together with the formal consequences of making that commitment explicit.

Responsible Use and Rights Reservation

The author encourages good-faith discussion, criticism, independent inquiry, and responsible use of the material in this work. Separately from the licence terms stated on the following page, the author asks users to consider foreseeable harms when adapting or applying the proposed framework. This ethical request leaves legally authorized uses unchanged.

The author retains the rights preserved under CC BY-NC 4.0 and may pursue remedies to which the author is legally entitled for breach of the licence or violation of independently applicable rights. Reuse remains independent from authorial endorsement. Third-party rights require authorization from their respective holders where applicable. Copyright exceptions and limitations remain fully available.

Notices

Status. This manuscript is a revised complete-draft preprint and discussion paper. Sections 1–13 develop the justice-relevant history domain, equivalence and normative-quotient architecture, braid sufficiency and decoration, harm and reparation, historical uncertainty, comparative processual justice, worked examples, interfaces with established justice concepts, scope boundaries, and conclusion. Revision R16 continues the release-preparation stage with a focused source-positioning revision of the manuscript’s interfaces with established justice concepts. The revision strengthens the procedural and reparative interfaces while preserving the formal architecture and the theory-relative scope of the processual proposal.

Licence. Except where otherwise indicated, copyright 2026 Wanhong Huang. This work is made available under the Creative Commons Attribution-NonCommercial 4.0 International License (CC BY-NC 4.0). Subject to its terms, the licence permits sharing and adaptation for noncommercial purposes with appropriate attribution, a link to the licence, and an indication of changes. The licence deed and legal code are available at <https://creativecommons.org/licenses/by-nc/4.0/>.

Statement on the use of language models. OpenAI’s ChatGPT was used for exploratory dialogue, formal reconstruction, source discovery followed by source verification, argumentative criticism, and drafting in $\text{\LaTeX}$. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, verified the cited sources, and bears sole responsibility for the manuscript, including its definitions, formal constructions, arguments, conclusions, and errors. Verification records for cited external sources are maintained in the accompanying source ledger.

1 Introduction

This section establishes the paper’s normative problem and the minimum distinctions needed to formalize it. The central question is whether a justice theory evaluates only a selected social condition or also the history by which that condition was generated. The analysis therefore separates the object of justice, the historical distinctions the theory preserves, and the lower-level descriptors capable of retaining those distinctions. It develops endpoint, historical, braid-topological, and normative equivalence, treats normative quotienting as principled historical compression, and uses braid-theoretic trajectory description as a lower-level prototype rather than as moral equivalence itself.

1.1 Historical Resolution of the Justice Object

A justice theory must evaluate something. Depending on the theory and domain, the object can include holdings, opportunities, rights, institutional rules, social relations, procedures, burdens, benefits, recognition, or other normatively specified features. The present paper focuses on a narrower structural question: whether the evaluative object is exhausted by a social condition at a selected endpoint or whether the history by which that condition was generated can itself contain justice-relevant information.

The question is not new in the broad history of political philosophy. Nozick’s entitlement theory is an explicit historical account of justice in holdings: whether a holding is just depends on the relevant history of acquisition, transfer, and, where necessary, rectification rather than only on whether the resulting distribution instantiates a preferred end-state pattern [7]. The present proposal does not adopt Nozick’s theory of property rights. The relevance of that precedent is narrower: it demonstrates that a justice judgment can be historically indexed and that identical terminal distributions need not receive identical evaluations merely because their endpoints coincide.

The formal problem can be stated independently of any one substantive theory of justice. Let $\mathfrak{D} = (S, M, \alpha, \sim)$ be a GR dynamical instantiation in the sense used by the preceding formal papers [4, 5]. Let $\Gamma(\mathfrak{D})$ be its realized-history space. A justice inquiry ordinarily concerns only some histories in that space, so define the justice-relevant history domain in Equation 1.

$$\Gamma_J \subseteq \Gamma(\mathfrak{D}). \quad (1)$$

Equation 1 is a domain selection, not a claim that every history is an object of justice.

Let Ω_E denote the codomain of an endpoint description relevant to the inquiry. Define the endpoint descriptor in Equation 2.

$$E : \Gamma_J \longrightarrow \Omega_E. \quad (2)$$

The map in Equation 2 can record a terminal allocation, institutional state, relational configuration, or another endpoint object selected by the substantive theory. It need not be a complete physical state.

A justice evaluation is represented here without assuming a numerical justice score. Let Ω_J be an arbitrary set of normative evaluation classes and define the evaluation map in Equation 3.

$$\mathcal{J} : \Gamma_J \longrightarrow \Omega_J. \quad (3)$$

The codomain in Equation 3 can contain binary statuses, partially ordered judgments, qualitative normative categories, or other theory-specific outputs. The map does not presuppose cardinal commensurability among justice judgments.

Equations 2 and 3 permit a precise statement of endpoint sufficiency. Define endpoint equivalence and justice equivalence in Equation 4.

$$\begin{aligned} h_1 \sim_E h_2 &\iff E(h_1) = E(h_2), \\ h_1 \sim_J h_2 &\iff \mathcal{J}(h_1) = \mathcal{J}(h_2). \end{aligned} \quad (4)$$

The endpoint description is sufficient for the justice evaluation exactly when equal endpoints never conceal a justice-relevant historical difference. The condition is stated in Equation 5.

$$\sim_E \subseteq \sim_J. \quad (5)$$

When Equation 5 fails, there exist histories with the same endpoint description but different justice evaluations. The paper calls this situation *endpoint-relative process sensitivity* at the declared endpoint resolution. This is weaker than terminal-state process sensitivity: a coarse endpoint feature can be insufficient even when the complete terminal carrier state remains normatively sufficient. Section 2.4 separates these cases formally. Process sensitivity is therefore relative to the declared descriptive resolution and justice evaluation; it is not a universal claim that every difference of historical path matters morally.

1.2 Normative Position of Processual Justice

The term *processual justice* is used here for an inquiry into the justice-relevance of generated history. This usage should be distinguished from two neighboring traditions.

First, processual justice is not a synonym for procedural justice. Rawls’s discussion of procedural justice distinguishes forms in which a procedure is evaluated in relation to an independently specified just outcome from pure procedural justice, in which a fair procedure helps constitute the justice of the outcome [8]. The present framework asks a different structural question. A procedure is one possible component of history, but the justice-relevant process can also include interactions, transfers, exclusions, injuries, institutional transformations, reparative acts, and persistent relational consequences that extend beyond the specification of one decision procedure. Processual justice therefore concerns the temporal and relational extension of the normative object, not merely whether a decision rule satisfies procedural criteria.

Second, processual justice is not equivalent to Nozickian historical entitlement. Nozick offers a substantive theory of justice in holdings whose historical structure concerns acquisition, transfer, and rectification [7]. The present paper instead develops a meta-normative architecture that can host different substantive theories. Its primary object is the relation among histories that a justice theory regards as normatively equivalent. Property entitlement can instantiate such a structure in one domain, but the formalism is not restricted to property or libertarian premises.

Some inquiries can legitimately satisfy Equation 5. Processual analysis is needed only where the theory cannot justify collapsing the relevant historical distinctions. The methodological commitment is therefore explicit: whether history matters is represented through declared domains, descriptors, and equivalence relations rather than asserted as a universal property of justice.

1.3 Normative Historical Equivalence

The central formal object of the paper is not a scalar justice function but the equivalence relation induced by the evaluation map. Equation 4 defines $\sim_J$ through equality of the declared normative output. The corresponding quotient space is defined in Equation 6.

$$\Gamma_J / \sim_J. \quad (6)$$

Each element of Equation 6 is a class of histories treated as equivalent for the purposes represented by $\mathcal{J}$.

Let the canonical quotient projection be defined in Equation 7.

$$q_J : \Gamma_J \longrightarrow \Gamma_J / \sim_J, \quad q_J(h) := [h]_J. \quad (7)$$

Equation 7 makes explicit that every justice evaluation which identifies more than one history performs some form of historical compression. The relevant normative question is whether that compression is justified.

This leads to the paper’s central conceptual proposal: a justice theory can be analyzed partly through its rule of *principled historical compression*. Such a rule need not preserve microscopic detail. It must preserve the distinctions the theory itself treats as constitutive of justice while permitting other distinctions to be forgotten.

The endpoint descriptor supplies one possible compression. If Equation 5 holds, the justice evaluation factors through the endpoint description on its realized image. If the inclusion fails, endpoint compression has erased at least one distinction required by the justice evaluation. Later sections generalize this observation to braid descriptors and other historical representations.

The normative quotient should not be confused with historical truth. Two histories can be normatively equivalent while remaining factually different. Conversely, two histories can be topologically equivalent while differing in harm, consent, responsibility, or other normatively relevant properties. The paper therefore maintains separate historical, descriptive, and normative equivalence relations throughout.

1.4 Braid-Theoretic Prototype for Historical Distinction

The preceding formal paper develops a braid descriptor as a trajectory-level representation downstream from a generative mechanism and a realized history [5]. The historical-methodology paper then inserts an evidential reconstruction layer and studies when a braid or derived invariant remains identifiable under historical uncertainty [3]. The present paper uses that descriptor for a normative purpose: to make visible the difference between endpoint equivalence and selected path-sensitive equivalence.

Fix a declared fact taxonomy τ . Let $\Gamma_{J,B}^\tau$ be the justice-relevant histories that also satisfy the relational-extraction and braid-admissibility conditions under that taxonomy. Define the domain in Equation 8.

$$\Gamma_{J,B}^\tau \subseteq \Gamma_J. \quad (8)$$

Let the braid descriptor restricted to this domain be given by Equation 9.

$$D_B^\tau : \Gamma_{J,B}^\tau \longrightarrow B_n. \quad (9)$$

The corresponding topological equivalence is defined in Equation 10.

$$h_1 \sim_B^\tau h_2 \iff D_B^\tau(h_1) = D_B^\tau(h_2). \quad (10)$$

A braid descriptor can be normatively sufficient only if equal braid descriptions never collapse a distinction required by the justice evaluation. The preliminary sufficiency condition is stated in Equation 11.

$$\sim_B^\tau \subseteq \sim_J|_{\Gamma_{J,B}^\tau} \quad \text{on } \Gamma_{J,B}^\tau. \quad (11)$$

Equation 11 does not state that braid equivalence is moral equivalence. It states the weaker condition required for justice evaluation to factor through the braid descriptor on the admitted domain.

Three cases are therefore possible. First, the braid descriptor can be too coarse: two histories have the same braid but different justice evaluations. Second, the braid descriptor can be finer than required: two histories have different braids but the same justice evaluation. Third, the braid resolution can match the distinctions required by the justice evaluation on the declared domain. These cases will be formalized later without assuming that one of them holds universally.

Braid theory is useful here because it provides a concrete example in which histories can differ structurally even when selected endpoints agree. It also provides an explicit quotient structure, so the paper can ask whether the distinctions removed by topological equivalence are normatively harmless. The braid formalism is therefore a prototype for process-sensitive justice rather than a universal ontology of just history.

1.5 Research Questions for the Processual Formalism

The manuscript is organized around six research questions.

The first concerns the normative object. Under what conditions is Equation 5 defensible, and under what conditions does a justice evaluation require distinctions among histories sharing the same endpoint?

The second concerns normative equivalence. How should $\sim_J$ be related to endpoint equivalence, historical or relational equivalence, and braid-topological equivalence without identifying any two relations by stipulation?

The third concerns representation. When is a lower-level historical descriptor, especially D_B^τ , sufficient for a justice evaluation in the sense of Equation 11, and when does it preserve too little or more than the normative inquiry requires?

The fourth concerns historical compression. Which quotienting operations legitimately remove normatively irrelevant detail, and what conditions are needed to recover every distinction the justice theory regards as constitutive?

The fifth concerns reparation. How should a reparative continuation be represented when repair improves or restores a normative relation without implying that the earlier injury never occurred? The paper will distinguish normative improvement, acceptability, and historical identity rather than treating mathematical inversion as automatic moral erasure.

The sixth concerns uncertainty and scope. How should processual justice be evaluated when the historical trajectory itself is set-valued under incomplete evidence, and when should the analysis retreat from a braid-level judgment to a coarser or non-topological normative description?

Together, the six questions define a structural contribution: a vocabulary for justice-relevant history, normative equivalence, historical compression, and descriptor adequacy, with braid theory used as one concrete trajectory-level instantiation.

1.6 Scope of the Present Formalization

The formalization is preliminary in scope. Endpoint evaluation remains admissible where endpoint sufficiency can be justified, and processual justice is not identified with procedural justice or historical entitlement. Braid structure is an optional, domain-restricted descriptor; histories with changing component number, unresolved identity, or incompatible relational charts require other representations [5, 3]. The framework also preserves GR’s ontological restraint: histories, braids, quotient classes, and normative descriptors are analytical representations rather than claims about the ultimate ontology of persons, institutions, harms, or justice.

2 Justice-Relevant History Domains

This section defines the formal domain on which processual justice judgments are made and derives the conditions under which an endpoint description is sufficient for those judgments. Its role is foundational: later comparisons among endpoint, historical, braid-topological, and normative equivalence require a clear separation between the histories admitted for evaluation, the information retained by a descriptor, and the normative output itself. The section first recalls the GR finite-history space and treats selection of a justice-relevant subdomain as a substantive commitment. It then defines justice evaluation without assuming a numerical scale, distinguishes complete terminal-state information from coarser endpoint features, introduces general history descriptors, and proves factorization criteria for normative sufficiency. The final subsection separates failure caused by an impoverished endpoint descriptor from process sensitivity that persists even when the full terminal carrier state is retained.

2.1 GR History Spaces and Normative Domain Selection

Let $\mathfrak{D} = (S, M, \alpha, \sim)$ be a GR dynamical instantiation. For an initial state $x_0 \in S$ and a finite mechanism word $w \in M^k$, the companion formal paper defines the realized history as the ordered state sequence generated by the mechanism action [5]. The present paper uses the same history object. A length- k history has the form stated in Equation 12.

$$h = (x_0, x_1, \dots, x_k) \in \Gamma_k(\mathfrak{D}), \quad (12)$$

where each adjacent state pair is related by at least one admissible mechanism action in the underlying instantiation.

The finite-history space is the disjoint union defined in Equation 13.

$$\Gamma(\mathfrak{D}) := \coprod_{k \geq 1} \Gamma_k(\mathfrak{D}). \quad (13)$$

The disjoint union in Equation 13 retains history length as part of the formal identity of a realized history.

The history objects are interpreted within the three-level GR architecture developed explicitly in the historical-methodology paper [3]. Ultimate ontological reality remains suspended. Relational reality contains historically constituted and sufficiently stable objects, roles, institutions, and relations represented by the inquiry. Manifestation reality contains context-dependent records, judgments, utterances, registrations, and other accessible forms through which relational objects become available to reconstruction. In the historical use developed here, a formal history primarily represents the reconstructed relational process, while manifestation-level evidence remains epistemically distinct from that history; manifestation-level objects can themselves enter the history when they are part of the declared research object. Neither membership in the history space nor assignment of a justice value establishes the ultimate ontology of the represented objects.

A justice inquiry need not evaluate every history in $\Gamma(\mathfrak{D})$. The substantive theory and domain determine which histories are admissible objects of the normative inquiry. The justice-relevant domain introduced in Equation 1 is therefore retained as a primitive selection at the present level.

Definition 2.1 (Justice-relevant history domain). A *justice-relevant history domain* is a declared subset $\Gamma_J \subseteq \Gamma(\mathfrak{D})$ whose members are the realized histories to which the justice evaluation of the study is intended to apply.

Definition 2.1 is deliberately domain-relative. Membership in Γ_J can depend on institutional scope, actor class, temporal boundary, legal standing, relational role, or another normatively justified criterion. The formalism does not derive that criterion from the braid structure.

The exclusion set associated with the normative domain is defined in Equation 14.

$$\mathcal{E}_J := \Gamma(\mathfrak{D}) \setminus \Gamma_J. \quad (14)$$

Equation 14 has no built-in moral interpretation. A history can lie in $\mathcal{E}_J$ because it falls outside the question being asked rather than because it is morally irrelevant in every possible inquiry.

Domain selection also precedes the use of a lower-level historical descriptor. Let $D : \Gamma_D \rightarrow \Omega_D$ be any trajectory descriptor with domain $\Gamma_D \subseteq \Gamma(\mathfrak{D})$. The histories on which both the normative evaluation and the descriptor are defined form the common domain in Equation 15.

$$\Gamma_{J,D} := \Gamma_J \cap \Gamma_D. \quad (15)$$

Equation 15 is important for later braid analysis: normative sufficiency of a descriptor can be assessed only on histories for which both sides of the comparison are defined.

2.2 Noncardinal Justice Evaluation

The justice evaluation map $\mathcal{J}$ introduced in Equation 3 is retained without adding numerical structure to its codomain. Package the normative ingredients in the specification defined by Equation 16.

$$\mathfrak{N}_J := (\Gamma_J, \Omega_J, \mathcal{J}), \quad \mathcal{J} : \Gamma_J \longrightarrow \Omega_J. \quad (16)$$

No order, metric, addition, or scalar multiplication is assumed on Ω_J in Equation 16. A substantive justice theory can supply such structure when justified, but the processual architecture does not require it.

The evaluation map induces the normative equivalence relation defined in Equation 17.

$$h_1 \sim_J h_2 \iff \mathcal{J}(h_1) = \mathcal{J}(h_2). \quad (17)$$

Proposition 2.2 (Normative evaluation equivalence). *The relation $\sim_J$ defined in Equation 17 is an equivalence relation on Γ_J .*

Proof. Reflexivity follows from $\mathcal{J}(h) = \mathcal{J}(h)$ for every $h \in \Gamma_J$. Symmetry follows because equality in Ω_J is symmetric. Transitivity follows because $\mathcal{J}(h_1) = \mathcal{J}(h_2)$ and $\mathcal{J}(h_2) = \mathcal{J}(h_3)$ imply $\mathcal{J}(h_1) = \mathcal{J}(h_3)$. $\square$

Proposition 2.2 justifies the quotient $\Gamma_J/\sim_J$ introduced in Equation 6. The quotient records equality of the declared normative outputs; it does not require a cardinal notion of how much more or less just one history is than another.

If a substantive theory supplies a comparative relation on the normative outputs, it can be added separately. Let $\preceq_\Omega$ be an optional binary relation on Ω_J . The corresponding history-level comparison is defined in Equation 18.

$$h_1 \preceq_J h_2 \iff \mathcal{J}(h_1) \preceq_\Omega \mathcal{J}(h_2). \quad (18)$$

Equation 18 shows why equality of normative evaluation and comparative ranking should remain distinct. The processual-equivalence results in this paper concern $\sim_J$; they do not assume that $\preceq_J$ exists or that justice judgments admit a complete ranking.

The canonical quotient map q_J from Equation 7 contains exactly the information needed to recover the normative equivalence class. Define the induced map on justice classes by Equation 19.

$$\overline{\mathcal{J}} : \Gamma_J/\sim_J \longrightarrow \text{im}(\mathcal{J}), \quad \overline{\mathcal{J}}([h]_J) := \mathcal{J}(h). \quad (19)$$

Proposition 2.3 (Canonical factorization of justice evaluation). *The map in Equation 19 is well defined and bijective, and the justice evaluation factors as stated in Equation 20.*

$$\mathcal{J} = \iota_J \circ \overline{\mathcal{J}} \circ q_J, \quad (20)$$

where $\iota_J : \text{im}(\mathcal{J}) \hookrightarrow \Omega_J$ is the inclusion map.

Proof. If $[h_1]_J = [h_2]_J$, then $h_1 \sim_J h_2$, so Equation 17 gives $\mathcal{J}(h_1) = \mathcal{J}(h_2)$. Hence $\overline{\mathcal{J}}$ is well defined. It is surjective onto $\text{im}(\mathcal{J})$ by construction. If $\overline{\mathcal{J}}([h_1]_J) = \overline{\mathcal{J}}([h_2]_J)$, then $\mathcal{J}(h_1) = \mathcal{J}(h_2)$, so $[h_1]_J = [h_2]_J$; therefore it is injective. Equation 20 follows directly from the definitions of q_J , $\overline{\mathcal{J}}$, and ι_J . $\square$

Proposition 2.3 makes precise the sense in which the normative quotient is a lossless representation of the distinctions that the declared evaluation itself preserves, even though it can forget distinctions among histories that receive the same normative output.

2.3 Endpoint Descriptors and History Descriptors

The word *endpoint* can refer either to the complete terminal carrier state or to selected features of that state. These levels should be distinguished before endpoint sufficiency is assessed.

For a finite history $h = (x_0, \dots, x_k) \in \Gamma(\mathfrak{D})$, define the initial-state and terminal-state maps in Equation 21.

$$I, T : \Gamma(\mathfrak{D}) \longrightarrow S, \quad I(h) := x_0, \quad T(h) := x_k. \quad (21)$$

The maps in Equation 21 retain the initial and complete terminal carrier states of every finite history in the GR instantiation. The endpoint analysis uses the restriction of T to Γ_J ; the continuation analysis later uses both boundary maps on the wider history space.

A substantive theory can evaluate only selected endpoint features. Let $\varepsilon : T(\Gamma_J) \rightarrow \Omega_E$ be a declared endpoint-feature map. The endpoint descriptor of Equation 2 is then derived by Equation 22.

$$E := \varepsilon \circ T. \quad (22)$$

Equation 22 permits E to record a terminal allocation, institutional status, relational pattern, or another normatively selected endpoint object without identifying that object with the complete terminal carrier state.

The two maps induce the equivalence relations defined in Equation 23.

$$\begin{aligned} h_1 \sim_T h_2 &\iff T(h_1) = T(h_2), \\ h_1 \sim_E h_2 &\iff E(h_1) = E(h_2). \end{aligned} \quad (23)$$

Because $E = \varepsilon \circ T$, equal terminal states necessarily have equal endpoint descriptions. The resulting inclusion is stated in Equation 24.

$$\sim_T \subseteq \sim_E. \quad (24)$$

Endpoint descriptors are one member of a broader class of history descriptors. Let $D : \Gamma_{J,D} \rightarrow \Omega_D$ be any descriptor defined on the common domain in Equation 15. Its induced equivalence is defined in Equation 25.

$$h_1 \sim_D h_2 \iff D(h_1) = D(h_2). \quad (25)$$

Definition 2.4 (Normative sufficiency of a history descriptor). A history descriptor D is *normatively sufficient* for $\mathcal{J}$ on $\Gamma_{J,D}$ when the inclusion in Equation 26 holds.

$$\sim_D \subseteq \sim_J|_{\Gamma_{J,D}}. \quad (26)$$

Definition 2.4 requires equal descriptor values never to collapse a distinction preserved by the justice evaluation. It does not require D to have exactly the same resolution as $\mathcal{J}$; a descriptor can retain distinctions that the justice evaluation later disregards.

The sufficiency condition has an equivalent factorization form.

Proposition 2.5 (Factorization criterion for normative descriptor sufficiency). A descriptor $D : \Gamma_{J,D} \rightarrow \Omega_D$ is *normatively sufficient* for $\mathcal{J}$ if and only if there exists a unique map $\mathcal{J}_D : \text{im}(D) \rightarrow \Omega_J$ such that Equation 27 holds.

$$\mathcal{J}|_{\Gamma_{J,D}} = \mathcal{J}_D \circ D. \quad (27)$$

Proof. Assume D is normatively sufficient. For $d \in \text{im}(D)$, choose any $h \in \Gamma_{J,D}$ with $D(h) = d$ and define $\mathcal{J}_D(d) := \mathcal{J}(h)$. If h' is another history with $D(h') = d$, then $h \sim_D h'$. Equation 26 gives $h \sim_J h'$, so $\mathcal{J}(h) = \mathcal{J}(h')$. Thus $\mathcal{J}_D$ is well defined. Equation 27 follows by construction, and uniqueness follows because every element of $\text{im}(D)$ has at least one preimage.

Conversely, assume Equation 27 holds. If $h_1 \sim_D h_2$, then $D(h_1) = D(h_2)$. Applying $\mathcal{J}_D$ gives $\mathcal{J}(h_1) = \mathcal{J}(h_2)$, so $h_1 \sim_J h_2$. Hence Equation 26 holds. $\square$

Proposition 2.5 supplies the formal meaning of a justice evaluation being computable from a lower-level descriptor without returning to the complete history.

The justice quotient can be recovered from a sufficient descriptor as well. Let q_J^D denote the restriction of q_J to $\Gamma_{J,D}$. Normative sufficiency is equivalent to existence of a unique recovery map r_D satisfying Equation 28.

$$r_D : \text{im}(D) \longrightarrow q_J(\Gamma_{J,D}), \quad r_D \circ D = q_J^D. \quad (28)$$

Equation 28 follows from the same fiber argument as Proposition 2.5: equal descriptor values must belong to one justice-equivalence class.

2.4 Conditions for Endpoint Sufficiency

Endpoint sufficiency is the special case of Definition 2.4 obtained by setting $D = E$. The criterion introduced in Equation 5 can therefore be stated in several equivalent forms.

Proposition 2.6 (Endpoint-sufficiency criterion). For the endpoint descriptor $E : \Gamma_J \rightarrow \Omega_E$, the following conditions are equivalent:

1. $\sim_E \subseteq \sim_J$;
2. $\mathcal{J}$ is constant on every fiber $E^{-1}(\{e\})$;
3. there exists a unique map $\mathcal{J}_E : \text{im}(E) \rightarrow \Omega_J$ satisfying Equation 29;
4. there exists a unique map $r_E : \text{im}(E) \rightarrow \Gamma_J / \sim_J$ satisfying Equation 30.

$$\mathcal{J} = \mathcal{J}_E \circ E, \quad (29)$$

$$q_J = r_E \circ E. \quad (30)$$

Proof. Conditions 1 and 3 are equivalent by Proposition 2.5 with $D = E$. Condition 1 is equivalent to Condition 2 because h_1 and h_2 belong to the same fiber of E exactly when $h_1 \sim_E h_2$. Condition 1 implies Condition 4 by the justice-class recovery construction in Equation 28. Conversely, if Equation 30 holds and $E(h_1) = E(h_2)$, then $q_J(h_1) = q_J(h_2)$, so $h_1 \sim_J h_2$. Thus Condition 4 implies Condition 1. $\square$

Proposition 2.6 identifies endpoint sufficiency as a statement about fibers rather than about the philosophical label attached to a theory. A theory can be substantively distributive, relational, corrective, or institutional while still satisfying endpoint sufficiency for a particular question if the selected endpoint object retains every distinction used by its justice evaluation.

When endpoint sufficiency fails, define the endpoint-relative justice-loss relation in Equation 31.

$$\Delta_{J|E} := \{(h_1, h_2) \in \Gamma_J^2 : h_1 \sim_E h_2 \text{ and } h_1 \not\sim_J h_2\}. \quad (31)$$

Equation 31 can also be written as the relation difference $\sim_E \setminus \sim_J$. It records pairs whose distinction is erased by E but retained by $\mathcal{J}$.

Corollary 2.7 (Endpoint insufficiency). *The endpoint descriptor E is normatively insufficient if and only if Equation 32 holds.*

$$\Delta_{J|E} \neq \emptyset. \quad (32)$$

Proof. By Equation 31, the set $\Delta_{J|E}$ is nonempty exactly when some pair lies in $\sim_E$ but not in $\sim_J$. That is precisely failure of the inclusion $\sim_E \subseteq \sim_J$. $\square$

Corollary 2.7 still leaves open why the endpoint descriptor is insufficient. The descriptor can fail because it omits a justice-relevant feature of the terminal state, or because histories with the same complete terminal state receive different justice evaluations. These cases should be distinguished.

Define the terminal-state justice-loss relation in Equation 33.

$$\Delta_{J|T} := \{(h_1, h_2) \in \Gamma_J^2 : h_1 \sim_T h_2 \text{ and } h_1 \not\sim_J h_2\}. \quad (33)$$

Definition 2.8 (Terminal-state sensitivity). *The justice evaluation is terminal-state process-sensitive on Γ_J when Equation 34 holds.*

$$\Delta_{J|T} \neq \emptyset. \quad (34)$$

Definition 2.8 gives a stronger claim than endpoint insufficiency relative to a coarse feature map. It requires two histories with the same complete terminal carrier state to receive different normative evaluations.

The relation between the two tests follows from $\sim_T \subseteq \sim_E$.

Proposition 2.9 (Endpoint-resolution hierarchy). *If $E = \varepsilon \circ T$, then:*

1. endpoint sufficiency of E implies terminal-state sufficiency of T ;
2. terminal-state process sensitivity implies endpoint insufficiency of E ;

3. *terminal-state sufficiency does not, by itself, imply endpoint sufficiency of a coarser E .*

Proof. For item 1, Equation 24 gives $\sim_T \subseteq \sim_E$. If $\sim_E \subseteq \sim_J$, transitivity of set inclusion gives $\sim_T \subseteq \sim_J$.

For item 2, terminal-state process sensitivity gives a pair in $\sim_T \setminus \sim_J$. Since $\sim_T \subseteq \sim_E$, the same pair lies in $\sim_E \setminus \sim_J$, so $\Delta_{J|E} \neq \emptyset$.

For item 3, a coarser endpoint map can identify distinct terminal states. Those states can receive different justice evaluations even when $\mathcal{J}$ is constant on each T -fiber. Hence $\sim_T \subseteq \sim_J$ does not force $\sim_E \subseteq \sim_J$. $\square$

Proposition 2.9 prevents a coarse endpoint measurement from being mistaken for evidence that justice necessarily depends on process. If $\Delta_{J|E} \neq \emptyset$ but $\Delta_{J|T} = \emptyset$, the selected endpoint descriptor is too coarse for the justice evaluation, while the complete terminal carrier state remains sufficient. The stronger processual claim is warranted only when $\Delta_{J|T} \neq \emptyset$, or when another explicitly declared historical descriptor is required to recover the distinctions preserved by $\mathcal{J}$.

The remaining analysis separates domain selection, justice-preserving compression, and the historical resolution required when endpoint information is insufficient.

3 Equivalence Structure of Processual Justice

This section organizes the major equivalence relations needed by the processual account of justice. Its role is to prevent factual identity, endpoint identity, relational similarity, braid-topological equivalence, and normative equivalence from being treated as interchangeable. The section first introduces a common refinement order for equivalence relations. It then locates endpoint, complete-history, relational, and braid equivalence within that order and derives their relations when one descriptor factors through another. The final subsections compare those descriptive relations with justice equivalence and show formally that, absent an additional normative sufficiency condition, no universal inclusion between braid equivalence and justice equivalence can be inferred. Throughout the section, comparisons of equivalence relations are made only after restricting them to a common domain.

3.1 Endpoint Equivalence and the Refinement Order

Let X be a nonempty set and let $\approx$ be an equivalence relation defined on a superset of X . The restriction of $\approx$ to X is defined in Equation 35.

$$\approx|_X := \approx \cap (X \times X). \quad (35)$$

Equation 35 allows two equivalence relations with different original domains to be compared on the histories for which both are defined.

For equivalence relations $\approx_1$ and $\approx_2$ on the same set X , define the refinement order in Equation 36.

$$\approx_1 \preceq \approx_2 \iff \approx_1 \subseteq \approx_2. \quad (36)$$

Under Equation 36, $\approx_1 \preceq \approx_2$ means that $\approx_1$ is at least as fine as $\approx_2$: whenever $\approx_1$ declares two histories equivalent, $\approx_2$ must also declare them equivalent. The finer relation therefore preserves at least all distinctions preserved by the coarser relation.

Proposition 3.1 (Refinement order). *The relation $\preceq$ defined in Equation 36 is a partial order on the set of equivalence relations over X .*

Proof. Reflexivity follows because every relation is a subset of itself. Antisymmetry follows because $\approx_1 \subseteq \approx_2$ and $\approx_2 \subseteq \approx_1$ imply $\approx_1 = \approx_2$. Transitivity follows from transitivity of set inclusion. $\square$

Proposition 3.1 allows later descriptor relations to be compared by information preservation without treating any particular equivalence as normatively privileged.

Section 2.3 defined the complete terminal-state map T and the endpoint-feature descriptor $E = \varepsilon \circ T$. Their induced equivalences satisfy $\sim_T \subseteq \sim_E$ by Equation 24. The refinement notation therefore gives Equation 37.

$$\sim_T \preceq \sim_E . \quad (37)$$

Equation 37 states that a selected endpoint feature can only preserve the complete terminal-state distinctions or forget some of them; it cannot create a distinction between histories whose complete terminal states are identical.

A generic factorization result explains this relation and will be used repeatedly.

Proposition 3.2 (Descriptor factorization and equivalence refinement). *Let $D_1 : X \rightarrow \Omega_1$ and $D_2 : X \rightarrow \Omega_2$ be two descriptors on the same nonempty domain X . Let $\sim_{D_1}$ and $\sim_{D_2}$ be their equality-induced equivalence relations. The following conditions are equivalent:*

1. $\sim_{D_1} \subseteq \sim_{D_2}$;
2. *there exists a unique map $\phi : \text{im}(D_1) \rightarrow \text{im}(D_2)$ satisfying Equation 38.*

$$D_2 = \phi \circ D_1 . \quad (38)$$

Proof. Assume $\sim_{D_1} \subseteq \sim_{D_2}$. For $d \in \text{im}(D_1)$, choose any $x \in X$ with $D_1(x) = d$ and define $\phi(d) := D_2(x)$. If x' also satisfies $D_1(x') = d$, then $x \sim_{D_1} x'$. The assumed inclusion gives $x \sim_{D_2} x'$, so $D_2(x) = D_2(x')$. Hence ϕ is well defined. For every $x \in X$, the construction gives $\phi(D_1(x)) = D_2(x)$, which proves Equation 38. Because every element of $\text{im}(D_1)$ has a preimage in X , the value of any factorization map is forced by D_2 , so ϕ is unique.

Conversely, assume Equation 38. If $x \sim_{D_1} x'$, then $D_1(x) = D_1(x')$. Applying ϕ to both sides gives $D_2(x) = D_2(x')$, so $x \sim_{D_2} x'$. Therefore $\sim_{D_1} \subseteq \sim_{D_2}$. $\square$

Proposition 3.2 shows that equivalence refinement and descriptor factorization are two descriptions of the same information ordering. Equation 22 is the endpoint instance: $E = \varepsilon \circ T$ implies $\sim_T \preceq \sim_E$.

3.2 Historical and Relational Equivalence

The finest equivalence available on a declared history domain is complete historical identity. Define the historical equivalence relation in Equation 39.

$$h_1 \sim_H h_2 \iff h_1 = h_2, \quad h_1, h_2 \in \Gamma_J . \quad (39)$$

The symbol $\sim_H$ does not mean that all empirically indistinguishable histories are identical. It denotes equality of the formal history objects admitted to Γ_J .

Proposition 3.3 (Historical identity is the finest descriptor equivalence). *Let $D : X \rightarrow \Omega_D$ be any descriptor with $X \subseteq \Gamma_J$. Then Equation 40 holds.*

$$\sim_H|_X \subseteq \sim_D . \quad (40)$$

Proof. If $h_1 \sim_H h_2$, then $h_1 = h_2$. A function has the same value on equal arguments, so $D(h_1) = D(h_2)$. Hence $h_1 \sim_D h_2$. $\square$

The historical-methodology paper now makes explicit that relational trajectories are constructed downstream from a declared fact taxonomy and fact regime [3]. To preserve that dependence, fix a fact taxonomy τ and let $\Gamma_{J,R}^\tau \subseteq \Gamma_J$ be the justice-relevant histories for which the declared relational extraction is defined. Let the relational descriptor be Equation 41.

$$D_R^\tau : \Gamma_{J,R}^\tau \longrightarrow \Omega_R^\tau , \quad (41)$$

where Ω_R^τ contains the relational trajectory objects retained by the inquiry. The superscript τ records that a different admissible fact articulation can produce a different research-object construction or relational trajectory.

Relational equivalence is induced by Equation 42.

$$h_1 \sim_R^\tau h_2 \iff D_R^\tau(h_1) = D_R^\tau(h_2). \quad (42)$$

By Proposition 3.3, Equation 43 follows.

$$\sim_H|_{\Gamma_{J,R}^\tau} \preceq \sim_R^\tau. \quad (43)$$

Equation 43 does not imply that relational equivalence is coarser than endpoint equivalence or vice versa. An endpoint descriptor can preserve a terminal feature that D_R^τ does not record, while D_R^τ can preserve path structure that E forgets. An inclusion between $\sim_E$ and $\sim_R^\tau$ therefore requires an additional factorization or sufficiency assumption.

The taxonomy index also prevents a second conflation. For two admissible taxonomies τ_1 and τ_2 , the relations $\sim_R^{\tau_1}$ and $\sim_R^{\tau_2}$ need not be identical even when their domains overlap. Taxonomic robustness, developed in the revised historical-methodology paper, concerns the stability of downstream trajectory conclusions across such alternative articulations [3]. The present paper keeps τ fixed until historical uncertainty and representation dependence are treated explicitly in later sections.

3.3 Braid-Topological Equivalence

Let $\Gamma_{J,B}^\tau \subseteq \Gamma_{J,R}^\tau$ be the justice-relevant histories for which the relational trajectory satisfies the declared braid-admissibility conditions. The braid descriptor introduced in Equation 9 is written here with its fact-regime dependence explicit:

$$D_B^\tau : \Gamma_{J,B}^\tau \longrightarrow B_n. \quad (44)$$

Equation 44 makes the braid descriptor's dependence on the declared fact articulation explicit. The braid quotient acts on the relational trajectory rather than directly on the complete GR history. Accordingly, assume a map π_B^τ exists on the relational-image domain such that Equation 45 holds.

$$D_B^\tau = \pi_B^\tau \circ D_R^\tau|_{\Gamma_{J,B}^\tau}. \quad (45)$$

Equation 45 states the descriptive level relation inherited from Papers I and II: a braid is extracted from a relational trajectory after the research objects and their paths have been constructed.

Define braid-topological equivalence by Equation 46.

$$h_1 \sim_B^\tau h_2 \iff D_B^\tau(h_1) = D_B^\tau(h_2). \quad (46)$$

Proposition 3.4 (Relational refinement of braid equivalence). *Under Equation 45, the inclusion in Equation 47 holds.*

$$\sim_R^\tau|_{\Gamma_{J,B}^\tau} \subseteq \sim_B^\tau. \quad (47)$$

Proof. Equation 45 is a factorization of D_B^τ through D_R^τ . Proposition 3.2 therefore gives Equation 47. $\square$

Combining Proposition 3.3 with Proposition 3.4 gives the descriptive chain in Equation 48.

$$\sim_H|_{\Gamma_{J,B}^\tau} \preceq \sim_R^\tau|_{\Gamma_{J,B}^\tau} \preceq \sim_B^\tau. \quad (48)$$

The successive relations in Equation 48 can be equal in a special application, but equality is not guaranteed. Each descriptive stage can forget distinctions that the previous stage retained.

The endpoint chain and the relational-braid chain are therefore parallel rather than universally nested:

$$\sim_H|_{\Gamma_{J,B}^\tau} \preceq \sim_T|_{\Gamma_{J,B}^\tau} \preceq \sim_E|_{\Gamma_{J,B}^\tau}, \quad \sim_H|_{\Gamma_{J,B}^\tau} \preceq \sim_R^\tau|_{\Gamma_{J,B}^\tau} \preceq \sim_B^\tau. \quad (49)$$

Equation 49 is one of the central structural observations of the paper. Endpoint compression and trajectory compression are two different ways of forgetting historical information. Neither branch is automatically finer than the other.

Table 1: Relation between a descriptive equivalence and the justice equivalence on a common domain.

Case	Relation	Interpretation
Exact normative resolution	$\sim_D = \sim_J$	Descriptor and justice evaluation preserve exactly the same history distinctions.
Surplus descriptive resolution	$\sim_D \subsetneq \sim_J$	Descriptor preserves every justice-relevant distinction and also retains additional distinctions that the evaluation treats as normatively equivalent.
Normative under-resolution	$\sim_J \subsetneq \sim_D$	Descriptor collapses at least one distinction preserved by the justice evaluation and is therefore normatively insufficient.
Mixed resolution	Neither inclusion holds	Descriptor both collapses some justice-relevant distinctions and preserves some distinctions that the evaluation disregards.

3.4 Normative Justice Equivalence

Justice equivalence $\sim_J$ was defined in Equation 17 through equality of the declared justice evaluation. The present subsection asks how a descriptive equivalence $\sim_D$ is situated relative to $\sim_J$ on a common domain $X \subseteq \Gamma_J$.

Definition 2.4 already identifies the key condition, restated on the common domain in Equation 50.

$$\sim_D \subseteq \sim_J|_X. \quad (50)$$

The refinement order makes the interpretation of Equation 50 transparent. A normatively sufficient descriptor is at least as fine as the justice equivalence; it may preserve extra distinctions, but it never collapses a distinction that the declared justice evaluation preserves.

The possible comparable relations are formalized next.

Definition 3.5 (Normative resolution relation). Let $D : X \rightarrow \Omega_D$ be a descriptor on $X \subseteq \Gamma_J$. Its equivalence relation has:

1. *exact normative resolution* when Equation 51 holds;
2. *surplus descriptive resolution* when Equation 52 holds;
3. *normative under-resolution* when Equation 53 holds;
4. *mixed resolution* when neither relation is a subset of the other.

$$\sim_D = \sim_J|_X, \quad (51)$$

$$\sim_D \subsetneq \sim_J|_X, \quad (52)$$

$$\sim_J|_X \subsetneq \sim_D. \quad (53)$$

The four cases in Definition 3.5 are summarized in Table 1.

Table 1 shows that a descriptor need not match the justice quotient exactly in order to be usable. Both exact normative resolution and surplus descriptive resolution satisfy normative sufficiency. Normative under-resolution and mixed resolution do not.

Proposition 3.6 (Normative sufficiency and resolution relation). *A descriptor D is normatively sufficient on X if and only if its relation to $\sim_J$ is either exact normative resolution or surplus descriptive resolution.*

Proof. By Definition 2.4, normative sufficiency is exactly the condition $\sim_D \subseteq \sim_J \mid_X$. If the inclusion is equality, Equation 51 holds. If it is strict, Equation 52 holds. Conversely, either Equation 51 or Equation 52 implies $\sim_D \subseteq \sim_J \mid_X$. $\square$

Proposition 3.6 therefore separates harmless surplus resolution from the two forms of descriptive insufficiency.

For the endpoint descriptor, Equation 5 is therefore the statement that endpoint equivalence has exact or surplus descriptive resolution relative to justice. For the braid descriptor, Equation 11 is the corresponding condition on $\Gamma_{J,B}^\tau$.

No moral claim follows merely from the relative fineness of a descriptive relation. If $\sim_B^\tau \subsetneq \sim_J$, distinct braids can still receive the same justice evaluation. If $\sim_J \subsetneq \sim_B^\tau$, equal braids can conceal a justice-relevant distinction. The direction of inclusion determines informational adequacy, not the substantive reason why a difference matters.

3.5 Inclusion, Incomparability, and Counterexamples

The previous subsections establish the factorization-derived refinements summarized in Equation 54.

$$\sim_T \preceq \sim_E, \quad \sim_R^\tau \mid_{\Gamma_{J,B}^\tau} \preceq \sim_B^\tau. \quad (54)$$

The remaining comparisons require substantive assumptions. In particular, there is no general theorem forcing endpoint equivalence, braid equivalence, and justice equivalence into one linear hierarchy.

A finite counterexample makes the point precise. Let the common domain be Equation 55.

$$X := \{h_1, h_2, h_3, h_4\}. \quad (55)$$

Define endpoint, braid, and justice equivalence by the partitions in Equation 56.

$$\begin{aligned} X/\sim_E &= \{\{h_1, h_2\}, \{h_3, h_4\}\}, \\ X/\sim_B &= \{\{h_1, h_3\}, \{h_2, h_4\}\}, \\ X/\sim_J &= \{\{h_1, h_4\}, \{h_2, h_3\}\}. \end{aligned} \quad (56)$$

The partitions in Equation 56 define valid equivalence relations. They also cross-cut one another:

- $h_1 \sim_E h_2$ but $h_1 \not\sim_J h_2$;
- $h_1 \sim_B h_3$ but $h_1 \not\sim_J h_3$;
- $h_1 \sim_J h_4$ but $h_1 \not\sim_E h_4$ and $h_1 \not\sim_B h_4$.

Thus neither $\sim_E$ nor $\sim_B$ is universally comparable with $\sim_J$, and $\sim_E$ and $\sim_B$ need not be comparable with each other.

Proposition 3.7 (No universal ordering of endpoint, braid, and justice equivalence). *Absent additional assumptions connecting the corresponding descriptors or evaluations, none of the inclusions in Equation 57 is logically necessary.*

$$\sim_E \subseteq \sim_B, \quad \sim_B \subseteq \sim_E, \quad \sim_E \subseteq \sim_J, \quad \sim_J \subseteq \sim_E, \quad \sim_B \subseteq \sim_J, \quad \sim_J \subseteq \sim_B. \quad (57)$$

Proof. Equation 56 provides a common-domain construction in which each pair of the three relations is incomparable. For example, $h_1 \sim_E h_2$ while $h_1 \not\sim_B h_2$, so $\sim_E \not\subseteq \sim_B$; and $h_1 \sim_B h_3$ while $h_1 \not\sim_E h_3$, so $\sim_B \not\subseteq \sim_E$. The same construction gives counterexamples to both inclusion directions involving $\sim_J$. Therefore none of the listed inclusions follows from the bare definitions alone. $\square$

Proposition 3.7 shows why braid topology cannot determine processual justice by itself: the same descriptor can be exact, surplus, under-resolved, or mixed relative to different justice evaluations.

The established structure can now be summarized by Equation 58.

$$\begin{aligned} \sim_H &\preceq \sim_T \preceq \sim_E, \\ \sim_H|_{\Gamma_{J,B}^\tau} &\preceq \sim_R|_{\Gamma_{J,B}^\tau} \preceq \sim_B^\tau, \end{aligned} \tag{58}$$

relation to $\sim_J$ requires an explicit normative sufficiency claim.

The first two lines of Equation 58 are descriptive information orderings; the final line requires a normative sufficiency claim. This separation motivates the justice quotient as the boundary of permissible historical compression.

4 Normative Quotients and Principled Historical Compression

This section develops the normative quotient introduced in Sections 1.3 and 2.2 as a formal model of principled historical compression. Its role is to state precisely when a justice theory may forget differences among histories, when a compression destroys a distinction required by the justice evaluation, and what information a sufficient historical descriptor must retain. The section first treats compression maps through their induced equivalence relations and identifies the justice quotient as the coarsest compression that preserves every distinction represented by the declared justice evaluation. It then defines legitimate and illegitimate forgetting relative to that evaluation, derives monotonicity properties for successive compressions, formulates recovery of normatively required distinctions, and concludes by distinguishing exact normative representations from family-relative minimal sufficient descriptors.

4.1 Justice Quotients of History Space

Let $X \subseteq \Gamma_J$ be a nonempty justice-relevant history domain on which a proposed historical representation is defined. A *historical compression map* is a function $C : X \rightarrow \Omega_C$ whose analytical role is to retain selected distinctions among histories while identifying others. The equivalence relation induced by C is defined in Equation 59.

$$h_1 \sim_C h_2 \iff C(h_1) = C(h_2). \tag{59}$$

The term *compression* refers to the possibility that $\sim_C$ contains distinct histories; it does not require a numerical compression ratio.

The canonical quotient associated with C is defined in Equation 60.

$$q_C : X \longrightarrow X/\sim_C, \quad q_C(h) := [h]_C. \tag{60}$$

The original map C and the quotient map q_C carry the same equality-level information. Define the quotient-image correspondence in Equation 61.

$$\overline{C} : X/\sim_C \longrightarrow \text{im}(C), \quad \overline{C}([h]_C) := C(h). \tag{61}$$

Proposition 4.1 (Compression quotient representation). *The map $\overline{C}$ in Equation 61 is well defined and bijective, and C factors as Equation 62.*

$$C = \iota_C \circ \overline{C} \circ q_C, \tag{62}$$

where $\iota_C : \text{im}(C) \hookrightarrow \Omega_C$ is the inclusion map.

Proof. If $[h_1]_C = [h_2]_C$, then $h_1 \sim_C h_2$, so Equation 59 gives $C(h_1) = C(h_2)$. Thus $\overline{C}$ is well defined. It is surjective onto $\text{im}(C)$ by construction. If $\overline{C}([h_1]_C) = \overline{C}([h_2]_C)$, then $C(h_1) = C(h_2)$, so $h_1 \sim_C h_2$ and $[h_1]_C = [h_2]_C$. Hence $\overline{C}$ is injective. Equation 62 follows directly from the definitions. $\square$

Proposition 4.1 permits the analysis to compare compression maps through their induced equivalence relations without treating codomain labels as substantively important.

The justice quotient on X is obtained by restricting $\sim_J$ to X . Define the restricted justice quotient map in Equation 63.

$$q_J^X : X \longrightarrow X / (\sim_J|_X), \quad q_J^X(h) := [h]_{J,X}. \quad (63)$$

The codomain in Equation 63 contains exactly the distinctions that the declared justice evaluation preserves on X .

The relation $\sim_J|_X$ is therefore a natural compression boundary for the declared evaluation. A compression relation strictly finer than $\sim_J|_X$ retains some distinctions that the evaluation treats as normatively equivalent. A compression relation equal to $\sim_J|_X$ retains exactly the normative distinctions. A coarser relation can identify histories whose justice evaluations differ.

4.2 Normative Boundaries of Historical Forgetting

The language of legitimate forgetting is always relative to a declared justice evaluation. It does not assert that a discarded historical distinction is morally irrelevant under every theory or for every institutional purpose.

Definition 4.2 (Justice-preserving historical compression). A historical compression map $C : X \rightarrow \Omega_C$ is *justice-preserving relative to $\mathcal{J}$* when Equation 64 holds.

$$\sim_C \subseteq \sim_J|_X. \quad (64)$$

A distinction forgotten by a justice-preserving compression is called *legitimately forgotten relative to $\mathcal{J}$* .

Definition 4.2 is the compression-language counterpart of normative sufficiency in Definition 2.4. A compression can therefore retain more historical information than the justice evaluation requires while remaining justice-preserving.

Define the justice-loss relation of C in Equation 65.

$$\Delta_{J|C} := \sim_C \setminus \sim_J|_X. \quad (65)$$

The relation $\Delta_{J|C}$ contains the pairs of histories identified by C whose justice evaluations remain distinct.

Proposition 4.3 (Justice preservation and loss). A compression C is *justice-preserving if and only if Equation 66 holds*.

$$\Delta_{J|C} = \emptyset. \quad (66)$$

Proof. By Equation 65, the loss relation is empty exactly when every pair in $\sim_C$ also belongs to $\sim_J|_X$. This is Equation 64. $\square$

Proposition 4.3 makes illegitimate forgetting formally visible without requiring a cardinal measure of normative loss.

Successive compression has a monotonicity property. Let $C_1 : X \rightarrow \Omega_1$ and $C_2 : X \rightarrow \Omega_2$ be compression maps such that C_2 factors through C_1 . Assume Equation 67.

$$C_2 = \psi \circ C_1 \quad (67)$$

for some map $\psi : \text{im}(C_1) \rightarrow \Omega_2$. Proposition 3.2 then gives Equation 68.

$$\sim_{C_1} \subseteq \sim_{C_2}. \quad (68)$$

Proposition 4.4 (Refinement preservation under legitimate compression). Under Equation 67, if C_2 is *justice-preserving*, then C_1 is *justice-preserving*.

Proof. Justice preservation of C_2 gives $\sim_{C_2} \subseteq \sim_J \mid_X$. Equation 68 gives $\sim_{C_1} \subseteq \sim_{C_2}$. Transitivity of set inclusion yields $\sim_{C_1} \subseteq \sim_J \mid_X$, so C_1 is justice-preserving. $\square$

Proposition 4.4 has an asymmetric converse. A fine compression can be justice-preserving while a coarser successor ceases to be justice-preserving. The justice quotient identifies the limit of such coarsening.

Corollary 4.5 (Compression boundary at exact normative resolution). *Suppose $C_\star : X \rightarrow \Omega_\star$ has exact normative resolution: $\sim_{C_\star} = \sim_J \mid_X$. Let $C_+ : X \rightarrow \Omega_+$ factor through $C_\star$ and satisfy Equation 69.*

$$\sim_{C_\star} \subsetneq \sim_{C_+}. \quad (69)$$

Then C_+ is not justice-preserving.

Proof. Equation 69 provides a pair (h_1, h_2) in $\sim_{C_+}$ that is not in $\sim_{C_\star}$. Because $\sim_{C_\star} = \sim_J \mid_X$, the pair is not justice-equivalent. Hence it belongs to $\Delta_{J|C_+}$, which is nonempty. Proposition 4.3 then shows that C_+ is not justice-preserving. $\square$

Corollary 4.5 formalizes the intuitive limit of principled historical compression: once a representation has reached exactly the justice quotient, every strict additional coarsening erases at least one distinction represented by the declared justice evaluation.

4.3 Recovery of Normatively Required Distinctions

A justice-preserving compression need not permit reconstruction of the complete history. It must permit reconstruction of the history's justice-equivalence class. This weaker recovery target is defined next.

Definition 4.6 (Normative recovery map). Let $C : X \rightarrow \Omega_C$ be a historical compression. A *normative recovery map* for C is a map r_C satisfying Equation 70.

$$r_C : \text{im}(C) \longrightarrow X / (\sim_J \mid_X), \quad r_C \circ C = q_J^X. \quad (70)$$

Definition 4.6 asks only for recovery of the justice class. It places no requirement on reconstruction of fact-level, relational, geometric, or other history details that the justice evaluation does not preserve.

Proposition 4.7 (Recovery criterion for justice-preserving compression). *A historical compression C is justice-preserving if and only if there exists a unique normative recovery map r_C satisfying Equation 70.*

Proof. Assume C is justice-preserving. For $c \in \text{im}(C)$, choose any $h \in X$ with $C(h) = c$ and define $r_C(c) := q_J^X(h)$. If h' is another history with $C(h') = c$, then $h \sim_C h'$. Justice preservation gives $h \sim_J h'$, so $q_J^X(h) = q_J^X(h')$. Thus r_C is well defined. Equation 70 holds by construction, and uniqueness follows because every element of $\text{im}(C)$ has a preimage in X .

Conversely, suppose a map r_C satisfies Equation 70. If $C(h_1) = C(h_2)$, then applying r_C gives $q_J^X(h_1) = q_J^X(h_2)$. Hence $h_1 \sim_J h_2$, so $\sim_C \subseteq \sim_J \mid_X$. Therefore C is justice-preserving. $\square$

Proposition 4.7 expresses normative sufficiency as a recoverability condition. Every distinction required by $\mathcal{J}$ must remain distinguishable after compression.

The same condition can be stated pairwise. Define the set of justice-required distinctions in Equation 71.

$$\mathcal{D}_J(X) := \{(h_1, h_2) \in X^2 : h_1 \not\sim_J h_2\}. \quad (71)$$

Proposition 4.8 (Pairwise preservation criterion). *A compression C is justice-preserving if and only if Equation 72 holds.*

$$(h_1, h_2) \in \mathcal{D}_J(X) \implies C(h_1) \neq C(h_2). \quad (72)$$

Proof. Equation 72 is the contrapositive form of $C(h_1) = C(h_2) \implies h_1 \sim_J h_2$, which is exactly $\sim_C \subseteq \sim_J \mid_X$. $\square$

Proposition 4.8 clarifies the meaning of historical recovery in this paper. A descriptor can lose large amounts of historical detail and still be normatively sufficient when every distinction in $\mathcal{D}_J(X)$ remains recoverable from its output.

Exact normative resolution strengthens recovery. If $\sim_C = \sim_J|_X$, then the recovery map r_C is bijective. If $\sim_C \subsetneq \sim_J|_X$, then C retains surplus descriptive distinctions and r_C is necessarily noninjective on $\text{im}(C)$.

Proposition 4.9 (Exact resolution and bijective normative recovery). *For a justice-preserving compression C , the normative recovery map r_C is bijective if and only if Equation 73 holds.*

$$\sim_C = \sim_J|_X. \quad (73)$$

Proof. Assume Equation 73. Surjectivity of r_C follows because every justice class in the restricted quotient has a representative in X . If $r_C(C(h_1)) = r_C(C(h_2))$, then $h_1 \sim_J h_2$. Equation 73 gives $h_1 \sim_C h_2$, so $C(h_1) = C(h_2)$. Thus r_C is injective.

Conversely, assume r_C is bijective. Justice preservation already gives $\sim_C \subseteq \sim_J|_X$. If $h_1 \sim_J h_2$, then $q_J^X(h_1) = q_J^X(h_2)$. Equation 70 gives $r_C(C(h_1)) = r_C(C(h_2))$. Injectivity of r_C implies $C(h_1) = C(h_2)$, so $h_1 \sim_C h_2$. Hence $\sim_J|_X \subseteq \sim_C$, which proves equality. $\square$

4.4 Minimal Sufficient Historical Descriptors

The justice quotient is the coarsest possible justice-preserving compression at the level of equivalence relations. This statement concerns normative information, not empirical accessibility. A concrete historical descriptor can be unavailable, unstable, difficult to construct, or restricted to a smaller domain even when the abstract justice quotient is well defined.

The coarsest property is stated formally next.

Proposition 4.10 (Justice quotient as the greatest legitimate compression relation). *Let $\mathcal{E}_J(X)$ be the set of equivalence relations on X that satisfy Equation 74.*

$$\mathcal{E}_J(X) := \{ \approx : \approx \text{ is an equivalence relation on } X, \quad \approx \subseteq \sim_J|_X \}. \quad (74)$$

Then $\sim_J|_X$ is the greatest element of $\mathcal{E}_J(X)$ under set inclusion.

Proof. The relation $\sim_J|_X$ belongs to $\mathcal{E}_J(X)$ because it is an equivalence relation and is a subset of itself. For every $\approx \in \mathcal{E}_J(X)$, Equation 74 gives $\approx \subseteq \sim_J|_X$. Hence no member of $\mathcal{E}_J(X)$ is strictly coarser while remaining justice-preserving. $\square$

Proposition 4.10 makes the justice quotient the formal endpoint of principled historical compression relative to $\mathcal{J}$. The result does not imply that one empirical descriptor realizes that endpoint.

Let $\mathcal{D}_X$ be a declared family of historical descriptors on the common domain X . Define the justice-preserving subfamily in Equation 75.

$$\mathcal{D}_X^J := \{ D \in \mathcal{D}_X : \sim_D \subseteq \sim_J|_X \}. \quad (75)$$

Definition 4.11 (Family-minimal sufficient descriptor). A descriptor $D_\star \in \mathcal{D}_X^J$ is *family-minimal sufficient* when Equation 76 holds.

$$\forall D \in \mathcal{D}_X^J, \quad \sim_D \subseteq \sim_{D_\star}. \quad (76)$$

Definition 4.11 selects a coarsest justice-preserving descriptor within the declared family. Such a descriptor need not exist, and when it exists it need not have exact normative resolution.

The relation to the justice quotient becomes sharper when the family contains an exact descriptor.

Proposition 4.12 (Exact descriptor in a family). *Suppose $\mathcal{D}_X^J$ contains a descriptor D_E with Equation 77.*

$$\sim_{D_E} = \sim_J|_X. \quad (77)$$

If $D_\star$ is family-minimal sufficient, then $D_\star$ also satisfies Equation 77.

Proof. Because $D_\star$ is family-minimal sufficient, Equation 76 applied to D_E gives $\sim_{D_E} \subseteq \sim_{D_\star}$. Equation 77 therefore gives $\sim_J \upharpoonright_X \subseteq \sim_{D_\star}$. Justice preservation of $D_\star$ gives the reverse inclusion. Hence $\sim_{D_\star} = \sim_J \upharpoonright_X$. $\square$

Proposition 4.12 shows that an exact justice-level descriptor, when available inside the declared family, fixes the coarsest sufficient resolution attainable within that family.

If the family contains no exact descriptor, a family-minimal sufficient descriptor can retain unavoidable surplus resolution. If the family contains several incomparable sufficient descriptors and no coarsest member, no family-minimal sufficient descriptor exists. These possibilities are analytically important because an abstract normative quotient and an available descriptive language need not coincide.

A descriptor with exact normative resolution represents the justice quotient up to relabeling.

Corollary 4.13 (Uniqueness of exact normative representation up to bijection). *If $D : X \rightarrow \Omega_D$ satisfies $\sim_D = \sim_J \upharpoonright_X$, then there exists a unique bijection b_D satisfying Equation 78.*

$$b_D : \text{im}(D) \longrightarrow X / (\sim_J \upharpoonright_X), \quad b_D \circ D = q_J^X. \quad (78)$$

Proof. The unique normative recovery map exists by Proposition 4.7. It is bijective by Proposition 4.9. Set $b_D := r_D$. $\square$

Corollary 4.13 shows that exact descriptive realization of the justice quotient is unique only at the level of preserved distinctions. Different codomains, labels, geometric models, or symbolic systems can encode the same normative partition.

Principled historical compression is therefore theory-relative: the justice quotient is the coarsest abstract representation preserving the distinctions encoded by the declared evaluation. Concrete descriptors may remain finer, but any strict coarsening beyond that boundary is normatively lossy.

5 Braid-Theoretic Representation of Justice-Relevant Histories

This section specializes the historical-compression framework of Section 4 to braid-theoretic trajectory description. Its role is to determine when a braid descriptor is merely defined, when it is sufficiently expressive for a declared justice evaluation, when it preserves more topological distinction than justice requires, and when it collapses a distinction that justice preserves. The section keeps two questions separate throughout: *domain adequacy*, concerning which justice-relevant histories admit the braid representation at all, and *resolution adequacy*, concerning whether equal braid descriptions are normatively interchangeable on that admitted domain. Section 5.1 defines the relevant braid domain and its omission set. Section 5.2 places the braid descriptor inside the justice quotient architecture. Section 5.3 proves equivalent criteria for braid sufficiency and distinguishes local sufficiency from global coverage. Section 5.4 derives the quotient induced when topology retains more distinctions than justice requires. Section 5.5 identifies topological aliasing and derives the condition that any later normative decoration must satisfy to repair it. The braid formalism remains downstream from the fact-conditioned relational trajectory architecture of Papers I and II rather than being treated as a generator or an ontology of justice [5, 3].

5.1 Braid Domains for Justice Evaluation

Fix a fact taxonomy τ and a justice-relevant history domain Γ_J . Section 3.3 introduced the subset $\Gamma_{J,B}^\tau$ of histories for which the declared relational extraction and braid construction are both defined. The inclusion map is written in Equation 79.

$$\iota_{J,B}^\tau : \Gamma_{J,B}^\tau \hookrightarrow \Gamma_J. \quad (79)$$

Equation 79 records domain inclusion only. Membership in $\Gamma_{J,B}^\tau$ does not mean that the history is just, morally admissible, or normatively preferred. It means only that the history is both justice-relevant and representable by the declared braid construction.

The justice-relevant histories omitted by the braid representation are collected in Equation 80.

$$\mathcal{O}_B^\tau := \Gamma_J \setminus \Gamma_{J,B}^\tau. \quad (80)$$

The set $\mathcal{O}_B^\tau$ can be nonempty because of changing component number, insufficiently persistent research-object identity, failure of collision exclusion, incompatible endpoint conventions, missing relational positions, or another declared braid-domain condition. Paper II treats several of these failures at the historical-reconstruction level [3].

Definition 5.1 (Complete justice-domain braid coverage). The declared braid representation has *complete justice-domain coverage* when Equation 81 holds.

$$\Gamma_{J,B}^\tau = \Gamma_J. \quad (81)$$

Definition 5.1 asks only whether every justice-relevant history lies inside the braid domain. It does not ask whether the braid preserves all distinctions required by justice.

The distinction between domain and resolution can be made explicit by restricting the justice evaluation to the braid domain. Define Equation 82.

$$\mathcal{J}_B^\tau := \mathcal{J}|_{\Gamma_{J,B}^\tau} : \Gamma_{J,B}^\tau \longrightarrow \Omega_J. \quad (82)$$

All braid-sufficiency claims in the next subsections are initially claims about $\mathcal{J}_B^\tau$. They become claims about the full justice domain only when Definition 5.1 is also satisfied.

Proposition 5.2 (Independence of braid coverage and normative sufficiency). *Complete justice-domain braid coverage and normative sufficiency of the braid descriptor are logically independent conditions.*

Proof. For coverage without sufficiency, let every history in Γ_J be braid-admissible but let two histories h_1, h_2 have the same braid and different justice evaluations. Then Equation 81 holds while braid sufficiency fails.

For sufficiency without coverage, let $\Gamma_{J,B}^\tau$ be a proper nonempty subset of Γ_J and suppose the braid descriptor is injective on $\Gamma_{J,B}^\tau$. Then equal braid values imply equal histories and therefore equal justice evaluations on the admitted domain, while $\mathcal{O}_B^\tau$ is nonempty. $\square$

Proposition 5.2 prevents a common overstatement. A perfectly sufficient braid descriptor on its own domain can still be globally incomplete for justice because some justice-relevant histories have no braid representation. Conversely, universal braid-domain coverage would not by itself show that braid equivalence is normatively adequate.

5.2 Braid Descriptors inside the Normative Architecture

On $\Gamma_{J,B}^\tau$, the taxonomy-indexed braid descriptor is the map from Equation 44, restated in Equation 83 for the present section.

$$D_B^\tau : \Gamma_{J,B}^\tau \longrightarrow B_n. \quad (83)$$

Only the realized image $\text{im}(D_B^\tau) \subseteq B_n$ is required for the normative analysis.

The induced braid equivalence is Equation 84.

$$h_1 \sim_B^\tau h_2 \iff D_B^\tau(h_1) = D_B^\tau(h_2). \quad (84)$$

The corresponding braid quotient map is Equation 85.

$$q_B^\tau : \Gamma_{J,B}^\tau \longrightarrow \Gamma_{J,B}^\tau / \sim_B^\tau, \quad q_B^\tau(h) := [h]_{B,\tau}. \quad (85)$$

By Proposition 4.1, the quotient in Equation 85 and the realized braid image carry the same equality-level information.

The justice quotient restricted to the braid domain is written in Equation 86.

$$q_J^{B,\tau} : \Gamma_{J,B}^\tau \longrightarrow \Gamma_{J,B}^\tau / \left(\sim_J|_{\Gamma_{J,B}^\tau} \right), \quad q_J^{B,\tau}(h) := [h]_{J,B,\tau}. \quad (86)$$

Equations 85 and 86 make the normative comparison precise: the first partitions histories by topological trajectory description, while the second partitions them by the declared justice evaluation.

Define the braid-relative justice-loss relation in Equation 87.

$$\Delta_{J|B}^\tau := \sim_B^\tau \setminus \sim_J|_{\Gamma_{J,B}^\tau}. \quad (87)$$

A pair in $\Delta_{J|B}^\tau$ has equal braid description but different justice evaluation.

Define the converse surplus-distinction relation in Equation 88.

$$\Sigma_{B|J}^\tau := \sim_J|_{\Gamma_{J,B}^\tau} \setminus \sim_B^\tau. \quad (88)$$

A pair in $\Sigma_{B|J}^\tau$ receives the same justice evaluation while retaining distinct braid descriptions.

The two relations in Equations 87 and 88 separate two very different mismatches between topology and justice. The first is a normative failure of braid resolution; the second is harmless surplus resolution when no justice-loss pair remains.

5.3 Braid Sufficiency for Justice Evaluation

Definition 5.3 (Local braid sufficiency). The braid descriptor D_B^τ is *locally sufficient for justice* on $\Gamma_{J,B}^\tau$ when Equation 89 holds.

$$\sim_B^\tau \subseteq \sim_J|_{\Gamma_{J,B}^\tau}. \quad (89)$$

Definition 5.3 is the braid-specific instance of Definition 2.4. It says that braid equality is strong enough to guarantee normative equality on the admitted domain.

The factorization and recovery consequences can now be stated together.

Proposition 5.4 (Equivalent criteria for local braid sufficiency). *The following conditions are equivalent:*

1. Equation 89 holds;
2. the braid-relative justice-loss relation is empty;
3. there exists a unique map $\mathcal{J}_B^\sharp : \text{im}(D_B^\tau) \rightarrow \Omega_J$ satisfying Equation 90;
4. there exists a unique map $r_B^\tau : \text{im}(D_B^\tau) \rightarrow \Gamma_{J,B}^\tau / (\sim_J|_{\Gamma_{J,B}^\tau})$ satisfying Equation 91.

$$\mathcal{J}_B^\tau = \mathcal{J}_B^\sharp \circ D_B^\tau, \quad (90)$$

$$q_J^{B,\tau} = r_B^\tau \circ D_B^\tau. \quad (91)$$

Proof. The equivalence of Conditions 1 and 2 follows directly from Equation 87. The equivalence of Conditions 1 and 3 follows from Proposition 2.5 applied to $D = D_B^\tau$ and domain $\Gamma_{J,B}^\tau$. The equivalence of Conditions 1 and 4 follows from the justice-class recovery criterion in Equation 28, again restricted to the braid domain. $\square$

Proposition 5.4 provides the exact sense in which justice can be evaluated from braid information alone: the justice evaluation or, equivalently, the justice-equivalence class must be recoverable from the realized braid value without returning to the complete history.

Local sufficiency should be distinguished from a stronger global claim.

Definition 5.5 (Global braid adequacy for a justice evaluation). The declared braid representation is *globally adequate* for $\mathcal{J}$ on Γ_J when it has complete justice-domain coverage and is locally sufficient, as stated in Equation 92.

$$\Gamma_{J,B}^\tau = \Gamma_J, \quad \sim_B^\tau \subseteq \sim_J. \quad (92)$$

Definition 5.5 is intentionally demanding. A justice theory can make legitimate local use of braid structure even when global adequacy fails. The scope claim must then remain restricted to $\Gamma_{J,B}^\tau$.

Corollary 5.6 (Exact braid resolution). *Assume local braid sufficiency. The braid descriptor has exact normative resolution on $\Gamma_{J,B}^\tau$ if and only if the recovery map r_B^τ from Equation 91 is bijective.*

Proof. This is the braid-specific instance of the exact-resolution recovery result established for justice-preserving compressions in Section 4.3. Exact normative resolution is $\sim_B^\tau = \sim_J \upharpoonright_{\Gamma_{J,B}^\tau}$, which is equivalent to one-to-one correspondence between realized braid classes and justice classes. $\square$

Corollary 5.6 separates two successful uses of topology. Exact braid resolution means the braid image and justice quotient preserve the same distinctions. Local sufficiency without exactness means the braid remains useful but retains more structure than the justice evaluation itself requires.

5.4 Surplus Topological Resolution

Assume throughout this subsection that D_B^τ is locally sufficient. Surplus topological resolution is the strict inclusion stated in Equation 93.

$$\sim_B^\tau \subsetneq \sim_J \upharpoonright_{\Gamma_{J,B}^\tau}. \quad (93)$$

Equation 93 means that at least two histories have different braid values while the justice evaluation treats them as equivalent. Equivalently, $\Delta_{J|B}^\tau$ is empty and $\Sigma_{B|J}^\tau$ is nonempty.

Because local sufficiency supplies the map $\mathcal{J}_B^\sharp$ from Equation 90, the justice evaluation induces an equivalence relation directly on the realized braid image. Define Equation 94.

$$\beta_1 \approx_J^{B,\tau} \beta_2 \iff \mathcal{J}_B^\sharp(\beta_1) = \mathcal{J}_B^\sharp(\beta_2), \quad \beta_1, \beta_2 \in \text{im}(D_B^\tau). \quad (94)$$

The quotient of the realized braid image by Equation 94 is the amount of braid-level information that remains after surplus topological distinctions are normatively forgotten.

Define the braid-image normative quotient map in Equation 95.

$$q_{J \leftarrow B}^\tau : \text{im}(D_B^\tau) \longrightarrow \text{im}(D_B^\tau) / \approx_J^{B,\tau}, \quad q_{J \leftarrow B}^\tau(\beta) := [\beta]_{J \leftarrow B, \tau}. \quad (95)$$

Proposition 5.7 (Justice quotient recovered from a sufficient braid image). *If D_B^τ is locally sufficient, there exists a unique bijection Θ_B^τ satisfying Equation 96.*

$$\Theta_B^\tau : \text{im}(D_B^\tau) / \approx_J^{B,\tau} \longrightarrow \Gamma_{J,B}^\tau / \left(\sim_J \upharpoonright_{\Gamma_{J,B}^\tau} \right), \quad \Theta_B^\tau \circ q_{J \leftarrow B}^\tau \circ D_B^\tau = q_J^{B,\tau}. \quad (96)$$

Proof. For a braid-image class $[\beta]_{J \leftarrow B, \tau}$, choose a history h with $D_B^\tau(h) = \beta$. Define the image of the braid class to be the justice class $[h]_{J,B,\tau}$. If another history h' has braid value β' in the same $\approx_J^{B,\tau}$ -class, Equation 94 gives $\mathcal{J}_B^\sharp(\beta) = \mathcal{J}_B^\sharp(\beta')$. Equation 90 then gives $\mathcal{J}(h) = \mathcal{J}(h')$, so the justice class is independent of the representative and the map is well defined.

Surjectivity follows because every justice class on the braid domain has a history representative and therefore a braid value. If two braid-image classes map to the same justice class, their representatives have equal justice evaluations and hence are related by $\approx_J^{B,\tau}$, so the braid-image classes are equal. Thus the map is injective. The commuting identity in Equation 96 follows from the construction, and uniqueness follows because the images of histories generate the realized braid-image quotient. $\square$

Proposition 5.7 shows that surplus topological resolution is not a defect. A sufficient braid representation can first preserve more trajectory distinction than justice requires and then be normatively quotiented to exactly the justice partition. Distinct braid classes in one $\approx_J^{B,\tau}$ -class remain historically or topologically different even though the declared justice evaluation treats them as equivalent.

The surplus relation also clarifies interpretive restraint. If $\Sigma_{B|J}^\tau$ is nonempty, braid difference alone cannot be interpreted as justice difference. A substantive argument is still required to explain why a particular topological distinction should or should not matter normatively.

5.5 Descriptor Refinement for Braid-Level Aliasing

Topological insufficiency begins when Equation 97 holds.

$$\Delta_{J|B}^\tau \neq \emptyset. \quad (97)$$

By Proposition 5.4, Equation 97 is equivalent to failure of local braid sufficiency. There is then no well-defined function from the realized braid value alone to the justice evaluation or justice class.

The failure can have two forms already distinguished in Section 3.4. If $\sim_J|_{\Gamma_{J,B}^\tau} \subsetneq \sim_B^\tau$, the braid is normatively under-resolved. If neither relation contains the other, the resolution is mixed: the braid both collapses some justice-relevant distinctions and retains some distinctions that justice disregards.

A useful diagnostic is the set of justice-relevant pairs that the braid aliases. Define Equation 98.

$$\mathcal{A}_{B \rightarrow J}^\tau := \{(h_1, h_2) \in (\Gamma_{J,B}^\tau)^2 : D_B^\tau(h_1) = D_B^\tau(h_2), \mathcal{J}(h_1) \neq \mathcal{J}(h_2)\}. \quad (98)$$

Equation 98 contains the same ordered pairs as $\Delta_{J|B}^\tau$, but its functional form emphasizes the practical problem: histories aliased by topology still require different normative treatment.

One response is to refine the braid descriptor with additional historical or normative information. Let $N : X \rightarrow \Omega_N$ be an auxiliary descriptor on a nonempty domain $X \subseteq \Gamma_{J,B}^\tau$. Define the joint descriptor in Equation 99.

$$D_{B \oplus N}^\tau : X \longrightarrow B_n \times \Omega_N, \quad D_{B \oplus N}^\tau(h) := (D_B^\tau(h), N(h)). \quad (99)$$

Let $\sim_N$ and $\sim_{B \oplus N}^\tau$ be the equivalence relations induced by N and Equation 99, respectively.

Proposition 5.8 (Joint-descriptor refinement). *The equivalence relation induced by Equation 99 is the intersection stated in Equation 100.*

$$\sim_{B \oplus N}^\tau = \sim_B^\tau|_X \cap \sim_N. \quad (100)$$

Proof. Two histories have equal joint descriptor values exactly when both coordinates are equal. Equality of the first coordinate is $\sim_B^\tau$ on X ; equality of the second is $\sim_N$. Thus both conditions hold exactly on the intersection in Equation 100. $\square$

Proposition 5.8 shows that adding information can only refine the braid equivalence on the common domain. The exact repair condition is stated next.

Proposition 5.9 (Repair criterion for braid insufficiency). *The joint descriptor $D_{B \oplus N}^\tau$ is normatively sufficient on X if and only if Equation 101 holds.*

$$(\sim_B^\tau|_X \cap \sim_N) \subseteq \sim_J|_X. \quad (101)$$

Equivalently, every braid-aliased justice-distinct pair in X must be separated by N .

Proof. By Proposition 5.8, the equivalence relation of the joint descriptor is the left-hand side of Equation 101. Definition 2.4 then gives the first equivalence.

For the pairwise statement, suppose a pair in X is braid-equivalent but not justice-equivalent. If it were also N -equivalent, the pair would belong to the left-hand side of Equation 101 but not its right-hand side, violating the inclusion. Conversely, if N separates every such braid-aliased justice-distinct pair, every pair remaining in the intersection is justice-equivalent, so the inclusion holds. $\square$

Proposition 5.9 fixes the repair task: an added descriptor must separate every justice-distinct pair aliased by topology. If that cannot be justified, the analysis should restrict the domain or use a different historical descriptor rather than infer the missing distinction from braid structure.

6 Normative Decoration of Trajectory Structure

Section 5 established that raw braid topology can fail to preserve distinctions required by a justice evaluation and derived the exact condition under which an auxiliary descriptor repairs that insufficiency. The present section specifies one important class of such auxiliary descriptors. Its role is to represent historically situated identities, roles, harm, benefit, consent, and responsibility without treating any of those categories as properties contained intrinsically in a braid. The section first derives a finite site structure from a justice-relevant history and uses it to separate persistent research-object identity from temporally variable role. It then introduces typed normative label maps without assuming cardinal scales or universal semantics. These components are packaged into a normative decoration descriptor and combined with the braid descriptor. The final subsection proves that any decoration determined entirely by braid topology is informationally redundant and therefore cannot repair a normatively insufficient braid representation.

6.1 Historical Identities and Roles

Fix a nonempty domain $X \subseteq \Gamma_{J,B}^\tau$ on which the braid descriptor D_B^τ is defined. Let $h = (x_0, \dots, x_k) \in X$. The history length determines the state-index set and transition-index set in Equation 102.

$$T_h := \{0, 1, \dots, k\}, \quad J_h := \{1, 2, \dots, k\}. \quad (102)$$

The index $j \in J_h$ refers to the transition from x_{j-1} to x_j .

The fact-conditioned research-object construction of Paper II permits a history to be analyzed through a finite set of justice-relevant object labels. Let $\mathcal{A}_h$ be the finite set of analytical labels selected for history h . Let $\mathcal{O}_\tau(h)$ be the set of research objects available for h under fact taxonomy τ . The identity registry in Equation 103 connects the analytical labels to those research objects.

$$\iota_h : \mathcal{A}_h \longrightarrow \mathcal{O}_\tau(h). \quad (103)$$

When distinct labels are intended to denote distinct tracked objects, ι_h is required to be injective.

The identity registry should be separated from the roles occupied by those objects. Define the object-time site set in Equation 104.

$$S_h^{\text{obj}} := \mathcal{A}_h \times T_h. \quad (104)$$

Let Ω_{role} be the declared set of role labels admitted by the inquiry. The role map is defined in Equation 105.

$$r_h : S_h^{\text{obj}} \longrightarrow \Omega_{\text{role}}. \quad (105)$$

The value $r_h(a, t)$ records the role assigned to tracked object a at historical index t .

Equations 103 and 105 encode different information. The registry asks which research object a strand-like analytical label denotes. The role map asks what historically situated role that object occupies. A continuing object can therefore undergo role change while retaining its declared identity.

Define the role-change set in Equation 106.

$$C_{\text{role}}(h) := \{(a, j) \in \mathcal{A}_h \times J_h : r_h(a, j-1) \neq r_h(a, j)\}. \quad (106)$$

Equation 106 records changes in role without representing them as changes of research-object identity.

Definition 6.1 (Identity-role context). The *identity-role context* of h is the structured object defined in Equation 107.

$$C_{\text{IR}}(h) := (\mathcal{A}_h, \iota_h, r_h, C_{\text{role}}(h)). \quad (107)$$

Definition 6.1 does not itself evaluate justice. It preserves contextual distinctions that can later matter to a justice evaluation. Two geometrically identical trajectories can represent different historical objects or different role histories, and a normative theory can treat those differences as relevant or irrelevant according to its own criteria.

6.2 Harm, Benefit, Consent, and Responsibility Labels

The next step introduces labels that can carry normatively relevant information while remaining distinct from the final justice evaluation. The construction is typed because harm, benefit, consent, and responsibility need not have the same semantic domain.

The transition-participation site set is derived from the object labels and transition indices in Equation 108.

$$S_h^{\text{tr}} := \mathcal{A}_h \times J_h. \quad (108)$$

A site $(a, j) \in S_h^{\text{tr}}$ means that the inquiry is evaluating the relation of tracked object a to transition j . The formalism does not assume that every object is an active participant in every transition; a label codomain can contain an inapplicable value where the domain semantics require one.

Let Ω_{harm} , Ω_{ben} , Ω_{cons} , and Ω_{resp} be the declared codomains for harm, benefit, consent, and responsibility labels, respectively. No order, metric, additive operation, or common scale is assumed on these sets. Define the four label maps in Equation 109.

$$\begin{aligned} H_h : S_h^{\text{tr}} &\longrightarrow \Omega_{\text{harm}}, & B_h : S_h^{\text{tr}} &\longrightarrow \Omega_{\text{ben}}, \\ C_h : S_h^{\text{tr}} &\longrightarrow \Omega_{\text{cons}}, & R_h : S_h^{\text{tr}} &\longrightarrow \Omega_{\text{resp}}. \end{aligned} \quad (109)$$

The four coordinates in Equation 109 are intentionally independent. For example, a transition can be harmful to an object without the formalism inferring absence of consent, or can benefit an object without the formalism inferring absence of responsibility.

A single transition can also have different labels for different tracked objects. For $a, b \in \mathcal{A}_h$, Equation 110 is permitted.

$$H_h(a, j) \neq H_h(b, j). \quad (110)$$

The same actor-relative possibility applies to benefit, consent, and responsibility.

The label maps need not be binary. A substantive theory can choose codomains containing qualitative categories, partially ordered statuses, intervals, structured legal states, or other domain-appropriate objects. The only requirement at the present level is that the codomain and interpretation of each label family be declared before the label is used in a justice evaluation.

To distinguish normative labeling from unsupported certainty, let $\perp_{\text{harm}} \in \Omega_{\text{harm}}$, $\perp_{\text{ben}} \in \Omega_{\text{ben}}$, $\perp_{\text{cons}} \in \Omega_{\text{cons}}$, and $\perp_{\text{resp}} \in \Omega_{\text{resp}}$ be optional indeterminacy values when the chosen label scheme permits them. The condition in Equation 111 illustrates their use.

$$C_h(a, j) = \perp_{\text{cons}} \quad (111)$$

means that the declared reconstruction does not determine the consent label at that site. The symbol $\perp_{\text{cons}}$ does not mean refusal; it means unresolved status in that label scheme. A later section will treat uncertainty across multiple compatible histories rather than within one label value.

Definition 6.2 (Normative label structure). The *normative label structure* of h is the tuple in Equation 112.

$$L_N(h) := (H_h, B_h, C_h, R_h). \quad (112)$$

The four label families in Definition 6.2 are examples of normatively relevant information rather than an exhaustive taxonomy. A particular theory can omit a family, refine it, or add further categories such as obligation, vulnerability, authorization, domination, desert, or reparative status. The formal requirement is that such additions enter through an explicit descriptor rather than being read directly from braid topology.

6.3 Decorated Trajectories and Justice Evaluation

The identity-role context and normative label structure can now be packaged into a single auxiliary descriptor. Let Ω_N be the set of all decoration structures admitted by the declared decoration scheme. Define the normative decoration descriptor in Equation 113.

$$N : X \longrightarrow \Omega_N, \quad N(h) := (C_{\text{IR}}(h), L_N(h)). \quad (113)$$

Equation 113 does not assert that all justice-relevant information must be contained in the listed coordinates. It defines the decoration used by the present prototype.

The decorated braid descriptor is the joint descriptor in Equation 114.

$$D_{\text{dec}}^\tau : X \longrightarrow B_n \times \Omega_N, \quad D_{\text{dec}}^\tau(h) := (D_B^\tau(h), N(h)). \quad (114)$$

Equation 114 is the specialization of $D_{B \oplus N}^\tau$ from Equation 99 to the decoration defined in Equation 113.

Let $\sim_N$ and $\sim_{\text{dec}}^\tau$ denote the equivalence relations induced by N and D_{dec}^τ . Proposition 5.8 immediately gives Equation 115.

$$\sim_{\text{dec}}^\tau = \sim_B^\tau|_X \cap \sim_N. \quad (115)$$

The braid-relative justice loss that remains after decoration is defined in Equation 116.

$$\Delta_{J|B,N}^\tau := (\sim_B^\tau|_X \cap \sim_N) \setminus \sim_J|_X. \quad (116)$$

Equation 116 contains exactly those history pairs that remain aliased by both topology and the decoration while still receiving different justice evaluations.

Proposition 6.3 (Decoration adequacy criterion). *The decorated descriptor D_{dec}^τ is normatively sufficient on X if and only if Equation 117 holds.*

$$\Delta_{J|B,N}^\tau = \emptyset. \quad (117)$$

Proof. By Equation 115, the equivalence relation induced by the decorated descriptor is $\sim_B^\tau|_X \cap \sim_N$. Definition 2.4 requires this relation to be contained in $\sim_J|_X$. By Equation 116, that inclusion holds exactly when the relation difference is empty. $\square$

Proposition 6.3 states the formal purpose of normative decoration: the decoration must eliminate every unresolved justice distinction inside a braid fiber. It need not eliminate distinctions between histories that the justice evaluation itself treats as equivalent.

When Equation 117 holds, Proposition 2.5 gives a unique map $\mathcal{J}_{\text{dec}}^\tau$ on the realized decorated image satisfying Equation 118.

$$\mathcal{J}|_X = \mathcal{J}_{\text{dec}}^\tau \circ D_{\text{dec}}^\tau. \quad (118)$$

Equation 118 does not mean that the braid coordinate and decoration coordinate contribute independently or additively. The factorization can depend on their joint configuration.

The decoration can itself be decomposed into components. Define the four label descriptors in Equation 119.

$$N_{\text{harm}}(h) := H_h, \quad N_{\text{ben}}(h) := B_h, \quad N_{\text{cons}}(h) := C_h, \quad N_{\text{resp}}(h) := R_h. \quad (119)$$

Together with the identity-role descriptor $N_{\text{IR}}(h) := C_{\text{IR}}(h)$, the full decoration in Equation 113 is their joint product descriptor.

Let $\mathcal{K}_N := \{\text{IR, harm, ben, cons, resp}\}$ index these five components. For any $K \subseteq \mathcal{K}_N$, let N_K denote the joint descriptor formed from the components indexed by K . Define the braid-relative sufficiency family in Equation 120.

$$\mathcal{S}_N := \{K \subseteq \mathcal{K}_N : (\sim_B^\tau|_X \cap \sim_{N_K}) \subseteq \sim_J|_X\}. \quad (120)$$

Equation 120 supports a component audit: a label family is necessary relative to the chosen component scheme when removing it from an otherwise sufficient set causes the inclusion to fail.

The audit should not be interpreted as proving that the retained labels are metaphysically fundamental. It only identifies which declared components are required to preserve the justice distinctions represented by $\mathcal{J}$ on X .

6.4 Separation between Topology and Normative Meaning

The previous subsection shows how normative decoration can refine braid equivalence. Definition 6.4 isolates the limiting case in which the added decoration is already a function of the braid value. The present subsection identifies a strict limitation: a decoration that is already determined by the braid value cannot add the information needed to repair a braid-level normative failure.

Definition 6.4 (Topology-determined decoration). A decoration $N : X \rightarrow \Omega_N$ is *topology-determined* on X when there exists a map ν defined on $D_B^\tau(X)$ such that Equation 121 holds.

$$N = \nu \circ D_B^\tau|_X. \quad (121)$$

Proposition 6.5 (Topological redundancy). *If N is topology-determined on X , then Equation 122 holds.*

$$\sim_{\text{dec}}^\tau = \sim_B^\tau|_X. \quad (122)$$

Proof. If two histories are braid-equivalent on X , Equation 121 forces them to have equal N values. Thus $\sim_B^\tau|_X \subseteq \sim_N$. Equation 115 then gives $\sim_{\text{dec}}^\tau = \sim_B^\tau|_X \cap \sim_N = \sim_B^\tau|_X$. $\square$

The preceding result has an immediate consequence.

Corollary 6.6 (Topology-determined decoration cannot repair braid insufficiency). *Suppose D_B^τ is normatively insufficient on X . If N is topology-determined on X , then D_{dec}^τ is also normatively insufficient on X .*

Proof. Braid insufficiency means $\sim_B^\tau|_X \not\subseteq \sim_J|_X$. By Proposition 6.5, the decorated equivalence is the same relation. Therefore it also fails the normative-sufficiency inclusion. $\square$

Corollary 6.6 shows that repair requires information varying inside at least one braid fiber. A pairwise formulation follows from the aliasing set in Equation 98.

Proposition 6.7 (Necessary within-fiber variation). *Assume $\mathcal{A}_{J|B}^\tau \cap (X \times X) \neq \emptyset$. If N repairs braid insufficiency on X , then Equation 123 holds for every aliasing pair (h_1, h_2) in that set.*

$$N(h_1) \neq N(h_2). \quad (123)$$

Proof. Every pair in $\mathcal{A}_{J|B}^\tau$ is braid-equivalent and justice-distinct. Proposition 5.9 states that a successful auxiliary descriptor must separate every such pair. Therefore Equation 123 is necessary. $\square$

Proposition 6.7 gives the minimum informational condition for successful repair. Braid generators and derived topological invariants do not intrinsically encode harm, consent, responsibility, entitlement, or reparative status; such meanings require a justified semantic or normative mapping. The decoration labels are likewise theory-relative representations within GR's ontologically restrained architecture. Topology supplies path distinctions, decoration supplies additional historical or normative information, and the justice evaluation determines which distinctions the joint representation must preserve.

7 Historical Persistence of Harm and Reparative Transformation

Section 6 introduced harm, benefit, consent, responsibility, identity, and role as explicit historical decorations rather than meanings contained intrinsically in braid topology. The present section studies how an injurious history can be followed by a reparative continuation without treating the continuation as deletion of the earlier history. Its role is to distinguish four objects that are often conflated in informal accounts of repair: historical extension, normative acceptability, transition between justice-equivalence classes, and algebraic inversion of a trajectory descriptor. The section first derives a continuation operation and a non-cardinal notion of injurious history from the harm labels already defined. It then distinguishes acceptability from historical identity, represents reparation as a possible transformation of normative history class, and proves that braid-group inversion is neither by itself a sufficient nor a necessary representation of repair.

7.1 Injurious Histories and Reparative Continuations

Fix a nonempty domain $X \subseteq \Gamma_{J,B}^\tau$ and let $h = (x_0, \dots, x_k) \in X$. A continuation must begin at the terminal state of the history it extends. Let $c = (y_0, \dots, y_m) \in \Gamma(\mathfrak{D})$ satisfy $I(c) = T(h)$, using the boundary-state maps from Equation 21. The concatenated history is defined in Equation 124.

$$h \star c := (x_0, \dots, x_k, y_1, \dots, y_m). \quad (124)$$

Equation 124 removes the duplicated boundary state and preserves the complete earlier prefix. Boundary matching makes the sequence-level concatenation well formed, but it does not by itself establish that the concatenated sequence is an admissible realized history of $\mathfrak{D}$. The continuation set below therefore requires the extended history itself to lie in the declared justice domain, and hence in the realized-history domain on which that justice domain is defined.

The set of justice-relevant continuations of h is defined in Equation 125.

$$\text{Cont}_J(h) := \{c \in \Gamma(\mathfrak{D}) : I(c) = T(h) \text{ and } h \star c \in \Gamma_J\}. \quad (125)$$

The history-extension relation is then defined in Equation 126.

$$h \sqsubseteq_J h' \iff \exists c \in \text{Cont}_J(h) \text{ such that } h' = h \star c. \quad (126)$$

The relation in Equation 126 records historical inclusion by continuation rather than normative comparison.

Section 6.2 defined the harm-label map $H_h : S_h^{\text{tr}} \rightarrow \Omega_{\text{harm}}$ without assuming a numerical scale on Ω_{harm} . A substantive theory can nevertheless designate which harm labels count as injurious for the question under study. Let $\Omega_{\text{harm}}^- \subseteq \Omega_{\text{harm}}$ be that declared set of injury-bearing labels. Define the injury-site set in Equation 127.

$$\text{Inj}(h) := \{(a, j) \in S_h^{\text{tr}} : H_h(a, j) \in \Omega_{\text{harm}}^-\}. \quad (127)$$

Definition 7.1 (Injurious history). A history h is *injurious relative to the declared harm scheme* when Equation 128 holds.

$$\text{Inj}(h) \neq \emptyset. \quad (128)$$

Definition 7.1 is deliberately relative to the declared harm codomain and injury subset. It does not infer injury from crossing number, endpoint inequality, or another topological feature.

Historical extension alone does not guarantee that a later evaluation uses the same labels for the earlier part of the history. To formalize historical non-erasure without excluding genuine reinterpretation, the relevant stability condition is stated explicitly.

Let $h' = h \star c$ with $c \in \text{Cont}_J(h)$. A prefix-compatible object continuation includes an injective label map $j_{h,h'}^A : \mathcal{A}_h \hookrightarrow \mathcal{A}_{h'}$ whose interpretation is that the earlier tracked object remains identifiable in the extended history. This label map induces the transition-site embedding defined in Equation 129.

$$j_{h,h'}^{\text{tr}} : S_h^{\text{tr}} \longrightarrow S_{h'}^{\text{tr}}, \quad j_{h,h'}^{\text{tr}}(a, j) := (j_{h,h'}^A(a), j). \quad (129)$$

Call the continuation *prefix compatible for the harm encoding* when such an object continuation is declared and Equation 130 holds for every earlier transition site.

$$H_{h'}(j_{h,h'}^{\text{tr}}(a, j)) = H_h(a, j), \quad (a, j) \in S_h^{\text{tr}}. \quad (130)$$

Equation 130 means only that appending later events does not, by itself, relabel an already encoded earlier harm site after the earlier site has been embedded into the extended history. A later evidential revision or conceptual reclassification can instead be represented by changing the reconstruction or decoration scheme, which is distinct from simple continuation.

Proposition 7.2 (Historical persistence of encoded injury). *Let h be injurious and let $h' = h \star c$ be a prefix-compatible continuation satisfying Equation 130. Then Equation 131 holds.*

$$j_{h,h'}^{\text{tr}}(\text{Inj}(h)) \subseteq \text{Inj}(h'). \quad (131)$$

In particular, h' remains a history containing an encoded injury site.

Proof. Choose an arbitrary earlier injury site $(a, j) \in \text{Inj}(h)$. Equation 127 places its harm label in Ω_{harm}^- . Equation 130 preserves that label at the embedded site $j_{h,h'}^{\text{tr}}(a, j)$. The embedded site therefore belongs to $\text{Inj}(h')$. Because the choice was arbitrary, the image inclusion follows. $\square$

Reparation is a distinct normative assessment of the continuation rather than the negation of Equation 131. Define the continuation-pair domain in Equation 132.

$$\mathfrak{C}_J := \{(h, c) : h \in \Gamma_J, c \in \text{Cont}_J(h)\}. \quad (132)$$

Let Ω_P be a declared codomain for reparation assessment and let $\Omega_P^{\text{rep}} \subseteq \Omega_P$ be the subset designated as reparative. A reparation-assessment map is defined in Equation 133.

$$P : \mathfrak{C}_J \longrightarrow \Omega_P. \quad (133)$$

Definition 7.3 (Reparative continuation). A continuation $c \in \text{Cont}_J(h)$ is *reparative relative to the declared reparation scheme* when Equation 134 holds.

$$P(h, c) \in \Omega_P^{\text{rep}}. \quad (134)$$

The definition permits a reparative continuation to coexist with persistent historical injury. Reparation can alter later relations, statuses, obligations, benefits, responsibilities, or justice evaluations without making the earlier transition cease to belong to the history. This abstraction is intentionally broader than any single doctrine of reparations. Walker’s account of moral repair treats acknowledgment of wrongdoing, acceptance of responsibility, and reparative action as analytically connected while also emphasizing the reconstruction of damaged moral relations. Work on reparations in transitional settings likewise treats reparation as a family of practices with multiple possible aims rather than as literal restoration of an untouched past; de Greiff, for example, analyzes recognition, civic trust, and social solidarity as relevant goals of large-scale reparations programs. International human-rights guidance similarly distinguishes restitution, compensation, rehabilitation, satisfaction, and guarantees of non-repetition as different forms of reparation. These neighboring literatures motivate keeping the map P open to several substantively specified forms of repair, while leaving the present formalism neutral about which combination a particular theory should require [11, 2, 10].

7.2 Normative Acceptability across Distinct Histories

The justice evaluation $\mathcal{J} : \Gamma_J \rightarrow \Omega_J$ does not require a total ranking. Normative acceptability can therefore be introduced without cardinalizing justice. Let $\Omega_J^{\text{acc}} \subseteq \Omega_J$ be the set of justice outputs designated as acceptable for the question under study. Define the acceptable-history set in Equation 135.

$$\Gamma_J^{\text{acc}} := \mathcal{J}^{-1}(\Omega_J^{\text{acc}}). \quad (135)$$

A history can therefore be acceptable without being assigned a numerical justice score or being ranked against every alternative.

For $h \in \Gamma_J$ and $c \in \text{Cont}_J(h)$, call the continuation *acceptability restoring* when Equation 136 holds.

$$h \notin \Gamma_J^{\text{acc}}, \quad h \star c \in \Gamma_J^{\text{acc}}. \quad (136)$$

Equation 136 is independent of Definition 7.3: a substantive theory can require a reparative act even when the prior history was already inside an acceptable class, or can regard some transition into acceptability as nonreparative because the change occurred for another reason.

Historical identity, endpoint restoration, justice-class convergence, and acceptability should be kept distinct. Let $g \in \Gamma_J$ be a comparison history. For an extended history $h' = h \star c$, define endpoint convergence to g by Equation 137.

$$E(h') = E(g). \quad (137)$$

Define justice convergence to g by Equation 138.

$$\mathcal{J}(h') = \mathcal{J}(g). \quad (138)$$

Historical identity would require the stronger condition in Equation 139.

$$h' = g. \quad (139)$$

Equations 137–139 are not equivalent conditions.

Proposition 7.4 (Justice convergence does not require historical identity). *Suppose $h', g \in \Gamma_J$ satisfy Equation 138. Then $h' \sim_J g$. Historical identity $h' = g$ follows only if $\sim_J$ coincides with $\sim_H$ on the pair $\{h', g\}$.*

Proof. Equation 138 and the definition of $\sim_J$ in Equation 17 give $h' \sim_J g$. The implication from justice equivalence to historical identity would require $\sim_J \subseteq \sim_H$ on the relevant domain. Because $\sim_H$ is complete historical identity and justice equivalence can be strictly coarser, the implication is not available without the stated additional condition. $\square$

The distinction becomes especially important when an injurious history is repaired.

Proposition 7.5 (Acceptability with historical non-erasure). *Let h be injurious, let $c \in \text{Cont}_J(h)$ be prefix compatible for the harm encoding, and suppose $h' = h \star c \in \Gamma_J^{\text{acc}}$. Then h' is acceptable while Equation 140 holds.*

$$\text{Inj}(h') \neq \emptyset. \quad (140)$$

Proof. Acceptability follows from $h' \in \Gamma_J^{\text{acc}}$. Because h is injurious and the continuation is prefix compatible, Proposition 7.2 gives a nonempty embedded image of $\text{Inj}(h)$ inside $\text{Inj}(h')$. The image is nonempty because $j_{h,h'}^{\text{tr}}$ is injective and h is injurious, so $\text{Inj}(h')$ is nonempty. $\square$

Proposition 7.5 formalizes one meaning of historical non-erasure: later acceptability does not imply that the earlier injury is absent from the represented history. This statement does not deny that a substantive theory can regard successful reparation as normatively transformative. It states only that transformation of normative status is different from deletion of historical content.

7.3 Reparative Transformation of Normative History Class

The justice quotient provides a natural representation of normative transformation that does not require a numerical improvement scale. Let $\mathcal{Q}_J := \Gamma_J / \sim_J$ and let $q_J : \Gamma_J \rightarrow \mathcal{Q}_J$ be the quotient map from Section 2.2. For any continuation pair $(h, c) \in \mathfrak{C}_J$, define the normative class transition in Equation 141.

$$\Theta_J(h, c) := (q_J(h), q_J(h \star c)) \in \mathcal{Q}_J \times \mathcal{Q}_J. \quad (141)$$

Equation 141 records whether the continuation changes the justice-equivalence class of the complete history.

Definition 7.6 (Class-transforming continuation). A continuation $c \in \text{Cont}_J(h)$ is *class transforming* for $\mathcal{J}$ when Equation 142 holds.

$$q_J(h) \neq q_J(h \star c). \quad (142)$$

Definition 7.6 separates class transformation from reparative status. The two are logically distinct. A reparative continuation can leave the declared justice output unchanged when $\mathcal{J}$ is too coarse to register the repair, while a class-changing continuation need not be reparative unless the reparation assessment identifies it as such.

The acceptable subset of the quotient is defined in Equation 143.

$$\mathcal{Q}_J^{\text{acc}} := q_J(\Gamma_J^{\text{acc}}). \quad (143)$$

Equation 143 is well defined because histories in one justice-equivalence class have the same $\mathcal{J}$ value and therefore either all belong to Γ_J^{acc} or all lie outside it.

Definition 7.7 (Acceptable-class transition). A continuation $c \in \text{Cont}_J(h)$ is an *acceptable-class transition* when Equation 144 holds.

$$q_J(h) \notin \mathcal{Q}_J^{\text{acc}}, \quad q_J(h \star c) \in \mathcal{Q}_J^{\text{acc}}. \quad (144)$$

Definition 7.7 is the quotient-level equivalent of Equation 136. It represents repair as movement between normative history classes rather than erasure of the earlier prefix.

A repaired history can also converge to the justice class of a comparison history without becoming historically identical to it. Define the target-class condition in Equation 145.

$$q_J(h \star c) = q_J(g). \quad (145)$$

Proposition 7.8 (Normative convergence under retained history). *Suppose Equation 145 holds. Then $h \star c \sim_J g$. If, in addition, $h \star c \neq g$, then the two histories remain distinct under complete historical identity $\sim_H$. If h is injurious and c is prefix compatible for the harm encoding, then $\text{Inj}(h \star c) \neq \emptyset$.*

Proof. Equality of quotient classes in Equation 145 is equivalent to justice equivalence by definition of the quotient, which gives $h \star c \sim_J g$. Under the manuscript's definition of $\sim_H$ as complete historical identity, the additional condition $h \star c \neq g$ entails historical non-equivalence. If h is injurious and c is prefix compatible, Proposition 7.2 preserves a nonempty embedded image of the earlier injury-site set in the extended history. $\square$

Proposition 7.8 separates three claims that a coarse formulation can otherwise blur: convergence in justice class, identity of complete histories, and persistence of encoded injury. A substantive theory may therefore place a repaired history and a comparison history in the same normative class while a finer descriptive representation continues to preserve their different historical contents. This is the quotient-theoretic sense in which reparative transformation can coexist with historical non-erasure.

The reparation assessment can now be combined with the class transition. Define the reparative transition set in Equation 146.

$$\mathfrak{R}_P := \{(h, c) \in \mathfrak{C}_J : P(h, c) \in \Omega_P^{\text{rep}}\}. \quad (146)$$

The image $\Theta_J(\mathfrak{R}_P)$ then records which justice-class transitions are actually realized by continuations that the declared reparation scheme regards as reparative. The framework therefore permits substantive theories to distinguish repair that changes justice class, repair that restores acceptability, repair that reaches a target class, and repair that is recognized as significant even when the current justice descriptor is too coarse to register a class change.

7.4 Reparative Interpretation of Braid Inversion

Braid groups contain inverses, and the concatenation of a braid with its inverse gives the identity element. This algebraic fact can provide a useful geometric analogy for undoing a trajectory, but it does not by itself represent moral reparation.

Restrict attention to histories and continuations for which braid composition is defined under one compatible strand registry, endpoint convention, and fact taxonomy. Let h and c be such that $h \star c \in \Gamma_{J,B}^\tau$. The braid descriptor is composition-compatible when Equation 147 holds.

$$D_B^\tau(h \star c) = D_B^\tau(h)D_B^\tau(c). \quad (147)$$

Equation 147 is the continuation form of the compositional braid architecture developed in the companion formal paper.

Definition 7.9 (Inverse-braid continuation). A braid-admissible continuation c of h is an *inverse-braid continuation* when Equation 148 holds.

$$D_B^\tau(c) = D_B^\tau(h)^{-1}. \quad (148)$$

Under Equation 147 and Definition 7.9, the extended braid is the identity element $e \in B_n$, as stated in Equation 149.

$$D_B^\tau(h \star c) = D_B^\tau(h)D_B^\tau(h)^{-1} = e. \quad (149)$$

This conclusion is topological. It does not state that the extended history is normatively equivalent to a history in which the earlier events never occurred.

Proposition 7.10 (Topological cancellation does not erase encoded injury). *Let h be injurious and let c be a prefix-compatible inverse-braid continuation satisfying Equation 147. Then both conditions in Equation 150 hold.*

$$D_B^\tau(h \star c) = e, \quad \text{Inj}(h \star c) \neq \emptyset. \quad (150)$$

Proof. The first condition follows from Equation 149. The second follows from Proposition 7.2, because prefix compatibility preserves every earlier injury site. $\square$

Proposition 7.10 shows directly that the identity braid can contain both uninjured and injury-bearing histories in the same braid fiber. Conversely, Definition 7.3 imposes no braid-inverse requirement on a reparative continuation.

Proposition 7.11 (Independence of braid inversion and reparative status). *Absent an additional axiom relating P to D_B^τ , neither implication in Equation 151 follows from the present definitions.*

$$\begin{aligned} D_B^\tau(c) = D_B^\tau(h)^{-1} &\not\Rightarrow P(h, c) \in \Omega_P^{\text{rep}}, \\ P(h, c) \in \Omega_P^{\text{rep}} &\not\Rightarrow D_B^\tau(c) = D_B^\tau(h)^{-1}. \end{aligned} \quad (151)$$

Proof. The reparation-assessment map P and the braid descriptor D_B^τ are independent declared maps in the present architecture. For the first implication, one can assign a continuation the inverse braid value while assigning $P(h, c)$ outside Ω_P^{rep} ; none of the definitions is violated. For the second, one can assign $P(h, c)$ inside Ω_P^{rep} while choosing any composable continuation braid different from the inverse. An implication in either direction therefore requires an additional substantive axiom connecting reparation to topology. $\square$

Proposition 7.11 limits the inversion analogy to the selected topological descriptor. Historical extension preserves the earlier prefix, prefix-compatible decoration preserves encoded injury sites, and reparation can still alter the justice class or restore acceptability. The plurality of reparative forms recognized in moral-repair, transitional-justice, and international remedial frameworks further cautions against identifying repair with one algebraic operation alone [11, 2, 10]. Any coincidence between braid inversion and reparative status therefore represents an additional modeling commitment whose substantive justification must be supplied independently of the group law.

8 Justice Evaluation under Historical Uncertainty

The preceding sections treated justice evaluation on histories already admitted to the justice domain. Historical inquiry rarely supplies one uniquely reconstructed history. The present section therefore propagates reconstruction uncertainty into processual justice rather than selecting one preferred history prematurely. Its role is to distinguish identification of a complete justice evaluation from identification of a coarser normative feature and from robustness across alternative reconstruction regimes. Section 8.1 derives the justice-relevant reconstruction set from the set-valued historical methodology of Paper II [3]. Section 8.2 defines normative identification inside one regime and robustness across a declared class of regimes. Section 8.3 formalizes partial normative identifiability through derived features of the justice evaluation. Section 8.4 then derives monotonicity and sensitivity results under evidential strengthening or weakening.

8.1 Set-Valued Historical Reconstruction

Fix an evidential record Y and a historical reconstruction regime η whose fact taxonomy, evidential constraints, auxiliary assumptions, identity rules, and representational commitments determine a compatible history set $\mathcal{R}_\eta(Y) \subseteq \Gamma(\mathfrak{D})$. The present paper evaluates only histories that also belong to the justice-relevant domain Γ_J . Define the justice-relevant reconstruction set in Equation 152.

$$\mathcal{R}_J(Y; \eta) := \mathcal{R}_\eta(Y) \cap \Gamma_J. \quad (152)$$

Equation 152 preserves the distinction between historical compatibility and normative scope. A history can be compatible with the record while lying outside the justice question currently being asked.

The justice evaluation $\mathcal{J} : \Gamma_J \rightarrow \Omega_J$ maps every history in $\mathcal{R}_J(Y; \eta)$ to a normative output. The evidence-compatible justice-value set is defined in Equation 153.

$$\mathcal{V}_J(Y; \eta) := \{\mathcal{J}(h) : h \in \mathcal{R}_J(Y; \eta)\}. \quad (153)$$

Equation 153 is the normative image of the reconstructed history set. It records uncertainty in the justice evaluation induced by uncertainty about the underlying history.

The corresponding set of justice-equivalence classes is defined in Equation 154.

$$\mathcal{Q}_J(Y; \eta) := \{q_J(h) : h \in \mathcal{R}_J(Y; \eta)\} \subseteq \Gamma_J / \sim_J. \quad (154)$$

Proposition 2.3 implies that $\mathcal{V}_J(Y; \eta)$ and $\mathcal{Q}_J(Y; \eta)$ carry the same partition-level information, because $\overline{\mathcal{J}}$ is bijective on the justice quotient.

Definition 8.1 (Normative identifiability under one reconstruction regime). The justice evaluation is *identified from Y under η* when the two conditions in Equation 155 hold.

$$\mathcal{R}_J(Y; \eta) \neq \emptyset, \quad |\mathcal{V}_J(Y; \eta)| = 1. \quad (155)$$

The first condition in Equation 155 prevents an empty reconstruction from producing vacuous agreement. The second permits several histories to remain historically nonidentified so long as they all belong to one justice-equivalence class.

Let $\omega \in \Omega_J$ and define the justice fiber in Equation 156.

$$\mathcal{F}_J(\omega) := \mathcal{J}^{-1}(\{\omega\}). \quad (156)$$

Proposition 8.2 (Fiber criterion for normative identifiability). Assume $\mathcal{R}_J(Y; \eta) \neq \emptyset$. The justice evaluation is identified from Y with value ω under η if and only if Equation 157 holds.

$$\mathcal{R}_J(Y; \eta) \subseteq \mathcal{F}_J(\omega). \quad (157)$$

Proof. If Equation 157 holds, every compatible justice-relevant history has justice value ω , so Equation 153 gives $\mathcal{V}_J(Y; \eta) = \{\omega\}$. Conversely, if $\mathcal{V}_J(Y; \eta) = \{\omega\}$, then every $h \in \mathcal{R}_J(Y; \eta)$ satisfies $\mathcal{J}(h) = \omega$ and therefore lies in $\mathcal{F}_J(\omega)$. $\square$

Proposition 8.2 shows that normative identification does not require historical identification. Several compatible histories can remain open while the evidence supports one justice evaluation because all of those histories lie in the same justice fiber.

8.2 Robust Normative Evaluation across Compatible Histories

Definition 8.1 concerns uncertainty inside one declared reconstruction regime. Robustness asks whether the same normative conclusion survives alternative regimes that the inquiry regards as admissible.

Let $\mathfrak{U}_J$ be a nonempty family of reconstruction regimes for the same justice question. Its members can differ in evidential interpretation, fact taxonomy, auxiliary assumptions, identity reconstruction, relational representation, or another declared component, provided the justice evaluation $\mathcal{J}$ and the normative codomain Ω_J remain comparable across the family.

Define the aggregate justice-value set in Equation 158.

$$\mathcal{V}_J(Y; \mathfrak{U}_J) := \bigcup_{\eta \in \mathfrak{U}_J} \mathcal{V}_J(Y; \eta). \quad (158)$$

The regimes with no justice-relevant reconstruction are collected in Equation 159.

$$\mathcal{E}_J^\emptyset(Y; \mathfrak{U}_J) := \{\eta \in \mathfrak{U}_J : \mathcal{R}_J(Y; \eta) = \emptyset\}. \quad (159)$$

Definition 8.3 (Robust normative identifiability). A justice value $\omega \in \Omega_J$ is *robustly identified from Y over $\mathfrak{U}_J$* when Equation 160 holds.

$$\mathcal{E}_J^\emptyset(Y; \mathfrak{U}_J) = \emptyset, \quad \mathcal{V}_J(Y; \mathfrak{U}_J) = \{\omega\}. \quad (160)$$

Definition 8.3 requires every declared regime to produce at least one justice-relevant history and all such histories to receive the same justice value. An empty regime is therefore not treated as evidence of agreement.

The aggregate reconstruction set in Equation 161 provides an equivalent formulation.

$$\mathcal{R}_J(Y; \mathfrak{U}_J) := \bigcup_{\eta \in \mathfrak{U}_J} \mathcal{R}_J(Y; \eta). \quad (161)$$

Proposition 8.4 (Aggregate-fiber criterion for robust normative identification). Assume $\mathcal{E}_J^\emptyset(Y; \mathfrak{U}_J) = \emptyset$. A justice value ω is *robustly identified over $\mathfrak{U}_J$ if and only if Equation 162 holds*.

$$\mathcal{R}_J(Y; \mathfrak{U}_J) \subseteq \mathcal{F}_J(\omega). \quad (162)$$

Proof. By Equations 158 and 161, the aggregate justice-value set is the image of the aggregate reconstruction set under $\mathcal{J}$. The same fiber argument as Proposition 8.2 therefore applies. The nonempty-regime condition is retained separately by Definition 8.3. $\square$

Proposition 8.4 shows that robustness across reconstruction regimes is equivalent to placing the entire aggregate history set inside one justice fiber, once empty-regime failures are excluded.

The distinction between identification and robustness is important. A singleton $\mathcal{V}_J(Y; \eta_0)$ under a reference regime η_0 does not imply that alternative plausible reconstructions preserve the same conclusion. Conversely, robust identification can coexist with substantial historical disagreement when all admissible histories across all declared regimes remain in one justice fiber.

8.3 Partial Normative Identifiability

The complete justice evaluation can remain nonidentified while a coarser normative feature is stable. This situation is analogous to partial topological identifiability in Paper II but occurs at the normative level.

Let $\Phi : \Omega_J \rightarrow \Omega_\Phi$ be a declared derived normative feature. Examples can include acceptability status, a reparative-status class, or another coarser property of the justice output. The evidence-compatible feature set is defined in Equation 163.

$$\mathcal{V}_\Phi(Y; \eta) := \{\Phi(\omega) : \omega \in \mathcal{V}_J(Y; \eta)\}. \quad (163)$$

Definition 8.5 (Partial normative identifiability). The reconstruction has *partial normative identifiability through* Φ when Equation 164 holds.

$$\mathcal{R}_J(Y; \eta) \neq \emptyset, \quad |\mathcal{V}_J(Y; \eta)| > 1, \quad |\mathcal{V}_\Phi(Y; \eta)| = 1. \quad (164)$$

Definition 8.5 means that the evidence does not identify one complete justice output but does identify the selected coarser feature. The warranted report is therefore the feature value rather than one arbitrarily selected element of $\mathcal{V}_J(Y; \eta)$.

Section 7.2 introduced the designated acceptable subset $\Omega_J^{\text{acc}} \subseteq \Omega_J$. Define the acceptability indicator in Equation 165.

$$\Phi_{\text{acc}} : \Omega_J \longrightarrow \{0, 1\}, \quad \Phi_{\text{acc}}(\omega) := \begin{cases} 1, & \omega \in \Omega_J^{\text{acc}}, \\ 0, & \omega \notin \Omega_J^{\text{acc}}. \end{cases} \quad (165)$$

Equation 165 allows acceptability to be identified even when the finer justice class is not.

Proposition 8.6 (Stable acceptability under nonidentified justice value). *Suppose $\mathcal{R}_J(Y; \eta) \neq \emptyset$ and $|\mathcal{V}_J(Y; \eta)| > 1$. Acceptability is identified under η if and only if one of the two conditions in Equation 166 holds.*

$$\mathcal{V}_J(Y; \eta) \subseteq \Omega_J^{\text{acc}}, \quad \text{or} \quad \mathcal{V}_J(Y; \eta) \cap \Omega_J^{\text{acc}} = \emptyset. \quad (166)$$

Proof. The acceptability feature is identified exactly when $\mathcal{V}_{\Phi_{\text{acc}}}(Y; \eta)$ is a singleton. By Equation 165, that singleton is $\{1\}$ precisely when every compatible justice value lies in Ω_J^{acc} , and it is $\{0\}$ precisely when none does. $\square$

Proposition 8.6 therefore permits a stable acceptability judgment even when the finer justice value remains unresolved.

The same construction applies to several derived features simultaneously. Let $\{\Phi_\ell\}_{\ell=1}^m$ be a finite family of feature maps. Define their joint map in Equation 167.

$$\Phi_{\text{joint}} := (\Phi_1, \dots, \Phi_m) : \Omega_J \longrightarrow \prod_{\ell=1}^m \Omega_{\Phi_\ell}. \quad (167)$$

The joint feature can refine a partial normative conclusion without pretending that the complete justice value has been identified.

8.4 Evidential Sensitivity of Processual Judgments

The set-valued architecture also makes the effect of evidential strengthening or weakening explicit. Consider two reconstruction regimes η_1 and η_2 that use the same justice domain, fact taxonomy, auxiliary assumptions, and interpretation rules except that η_2 contains evidential constraints at least as strong as those in η_1 . Represent this relation by the reconstruction inclusion in Equation 168.

$$\mathcal{R}_J(Y; \eta_2) \subseteq \mathcal{R}_J(Y; \eta_1). \quad (168)$$

Equation 168 is the operative meaning of evidential strengthening in the present subsection.

Table 2: Reporting states for justice evaluation under historical uncertainty.

Reconstruction status	Formal condition	Warranted report
No justice-relevant reconstruction	$\mathcal{R}_J(Y; \eta) = \emptyset$	The declared reconstruction regime does not support a justice evaluation at the stated domain.
Complete normative identification	$ \mathcal{V}_J(Y; \eta) = 1$ with nonempty reconstruction	Report the identified justice value or corresponding justice class.
Partial normative identification	$ \mathcal{V}_J(Y; \eta) > 1$ and $ \mathcal{V}_\Phi(Y; \eta) = 1$	Report the identified feature and retain uncertainty about the finer justice value.
Normative nonidentification	$ \mathcal{V}_\Phi(Y; \eta) > 1$ at the reporting resolution	Report the compatible feature or justice-value set rather than one selected value.
Regime-sensitive normative identification	One reference regime identifies a value but $\mathcal{V}_J(Y; \mathfrak{U}_J)$ is not a singleton	Qualify the conclusion as reconstruction-sensitive and identify the variation included in the robustness class.
Robust normative identification	Equation 160 holds	Report the value as robust over the declared regime family.

Proposition 8.7 (Monotonicity of normative possibility under evidential strengthening). *If Equation 168 holds, then Equation 169 follows.*

$$\mathcal{V}_J(Y; \eta_2) \subseteq \mathcal{V}_J(Y; \eta_1). \quad (169)$$

For every derived feature $\Phi : \Omega_J \rightarrow \Omega_\Phi$, Equation 170 also holds.

$$\mathcal{V}_\Phi(Y; \eta_2) \subseteq \mathcal{V}_\Phi(Y; \eta_1). \quad (170)$$

Proof. Equation 169 follows by applying the function $\mathcal{J}$ to both sides of the reconstruction inclusion in Equation 168. Equation 170 then follows by applying Φ to the two justice-value sets. $\square$

Proposition 8.7 states that stronger evidence can eliminate previously compatible normative possibilities but cannot introduce a justice value that was absent from the weaker reconstruction, provided the remaining regime components are held fixed. The proposition does not apply when new evidence also changes the fact taxonomy, interpretation rule, or justice domain.

Evidential weakening can therefore destroy identification. Suppose η_0 is a reference regime and η^{-j} is the regime obtained by removing one record constraint or record item j while holding the remaining commitments fixed. Define the justice-materiality condition in Equation 171.

$$\mathcal{V}_J(Y; \eta^{-j}) \neq \mathcal{V}_J(Y; \eta_0). \quad (171)$$

When Equation 171 holds, item j is material to the complete justice-value set under the declared reconstruction scheme.

A stronger identification-specific condition is given in Equation 172.

$$|\mathcal{V}_J(Y; \eta_0)| = 1, \quad |\mathcal{V}_J(Y; \eta^{-j})| > 1. \quad (172)$$

Equation 172 means that one record item is material to identification of the complete justice value. The analogous feature-level test replaces $\mathcal{V}_J$ with $\mathcal{V}_\Phi$.

The major uncertainty states are summarized in Table 2.

Table 2 extends maximally warranted reporting to the normative layer: uncertainty is propagated rather than resolved by selecting one admissible history, and identified coarser features may be reported when the complete justice value is not. Historical reconstruction, complete normative identification, feature identification, and robustness across regimes remain distinct conditions.

9 Comparative Processual Justice

This section extends the processual formalism from one historical case to comparisons across cases. Its role is to distinguish comparison of complete justice evaluations from comparison of selected normative features and from recurrence of joint process-normative motifs. The objective is to make cross-case transport explicit so that case-specific roles, fact taxonomies, normative labels, and justice codomains are not silently treated as interchangeable. Section 9.1 defines common comparative descriptors and their uncertainty sets. Section 9.2 combines process and normative descriptors into recurrent justice-relevant motifs. Section 9.3 derives a cross-case equivalence on the disjoint union of case domains and states the conditions under which the transport preserves case-specific justice distinctions. Section 9.4 then formalizes the information loss introduced by comparative compression and gives reporting rules for full, feature-level, motif-level, and non-comparable cases. The construction parallels the comparative trajectory methodology of Paper II while adding an explicit normative transport layer [3].

9.1 Cross-Case Normative Descriptors

Let $\mathcal{C}$ be a finite nonempty set of cases. For each $c \in \mathcal{C}$, let the case-specific justice specification be Equation 173.

$$\mathfrak{N}_{J,c} := (\Gamma_{J,c}, \Omega_{J,c}, \mathcal{J}_c), \quad \mathcal{J}_c : \Gamma_{J,c} \longrightarrow \Omega_{J,c}. \quad (173)$$

Equation 173 allows different cases to use different justice codomains when the substantive normative objects are not literally identical. A study using one common justice theory can recover the special case $\Omega_{J,c} = \Omega_J$ and $\mathcal{J}_c = \mathcal{J}|_{\Gamma_{J,c}}$.

Cross-case comparison requires a common comparative codomain. Let Ω_{cmp} be such a codomain and let the case-specific normative transport map be Equation 174.

$$\psi_c : \Omega_{J,c} \longrightarrow \Omega_{\text{cmp}}. \quad (174)$$

The map ψ_c records which case-specific normative distinctions are retained for the comparative question. Its justification is substantive rather than topological: a comparison can require explicit correspondences among fact categories, research objects, roles, harm or consent labels, institutional meanings, or other contextual features.

The transported comparative descriptor for case c is defined in Equation 175.

$$K_c : \Gamma_{J,c} \longrightarrow \Omega_{\text{cmp}}, \quad K_c := \psi_c \circ \mathcal{J}_c. \quad (175)$$

Equation 175 makes the comparative resolution explicit. Two cases can therefore be compared through a shared feature of their justice evaluations even when their complete justice codomains differ.

Historical uncertainty should propagate through the transport. For case c , let Y_c be its evidential record and let η_c be its declared reconstruction regime. Define the comparative value set in Equation 176.

$$\mathcal{V}_{\text{cmp}}(c) := \{K_c(h) : h \in \mathcal{R}_{J,c}(Y_c; \eta_c)\} \subseteq \Omega_{\text{cmp}}. \quad (176)$$

The notation $\mathcal{R}_{J,c}$ is the case-specific version of the justice-relevant reconstruction set from Equation 152.

Definition 9.1 (Cross-case comparative identifiability). The transported normative value is *identified* for case c when Equation 177 holds.

$$\mathcal{R}_{J,c}(Y_c; \eta_c) \neq \emptyset, \quad |\mathcal{V}_{\text{cmp}}(c)| = 1. \quad (177)$$

Definition 9.1 does not require the complete justice value $\mathcal{J}_c(h)$ to be identified. The common comparative feature can be stable while finer case-specific normative distinctions remain unresolved.

For two cases $c, d \in \mathcal{C}$, define the comparative overlap set in Equation 178.

$$\mathcal{V}_{\text{cmp}}^\cap(c, d) := \mathcal{V}_{\text{cmp}}(c) \cap \mathcal{V}_{\text{cmp}}(d). \quad (178)$$

When both case sets are singletons, a nonempty intersection means that the identified transported values agree. When one or both sets contain several values, Equation 178 supports only the weaker claim that the cases share at least one compatible comparative value.

9.2 Recurrent Justice-Relevant Motifs

Comparison of justice values alone can omit the process structure that motivates the present framework. A justice-relevant motif therefore combines a declared process descriptor with a transported normative descriptor.

For each case c , let $D_{\text{traj},c}$ be a process descriptor defined on a comparison domain $X_c \subseteq \Gamma_{J,c}$, with common codomain Ω_{traj} . The descriptor can be braid-theoretic, decoration-sensitive, or another justified trajectory descriptor. Define Equation 179.

$$D_{\text{traj},c} : X_c \longrightarrow \Omega_{\text{traj}}. \quad (179)$$

The common codomain in Equation 179 means that the cases have already been placed under a declared process-comparison protocol of the kind developed in Paper II [3].

Restrict the transported normative descriptor K_c to X_c and define the joint justice-relevant motif descriptor in Equation 180.

$$M_c : X_c \longrightarrow \Omega_{\text{traj}} \times \Omega_{\text{cmp}}, \quad M_c(h) := (D_{\text{traj},c}(h), K_c(h)). \quad (180)$$

Equation 180 retains one declared process pattern together with one declared comparative normative value. Neither coordinate is assumed to determine the other.

The evidence-compatible motif set for case c is defined in Equation 181.

$$\mathcal{M}(c) := \{M_c(h) : h \in \mathcal{R}_{J,c}(Y_c; \eta_c) \cap X_c\}. \quad (181)$$

Definition 9.2 (Identified justice-relevant motif). A motif $m \in \Omega_{\text{traj}} \times \Omega_{\text{cmp}}$ is *identified for case c* when the reconstruction intersected with X_c is nonempty and Equation 182 holds.

$$\mathcal{M}(c) = \{m\}. \quad (182)$$

Definition 9.2 requires both process and transported normative coordinates to be stable on the compatible comparison-domain histories of the case.

Define the identification support of motif m in Equation 183.

$$\text{ISupp}_J(m) := \{c \in \mathcal{C} : \mathcal{M}(c) = \{m\}\}. \quad (183)$$

Definition 9.3 (Recurrent justice-relevant motif). An identified motif m is *recurrent across the case set* when Equation 184 holds.

$$|\text{ISupp}_J(m)| \geq 2. \quad (184)$$

Definition 9.3 is descriptive. Recurrence of m does not establish a common generator, one universal moral mechanism, or identity of the complete justice evaluations. It establishes only repeated identification of the declared joint process-normative profile.

Robust recurrence can be defined by replacing each reference reconstruction regime η_c with a declared regime family $\mathfrak{U}_{J,c}$ and requiring every regime in that family to yield the same motif. Define the regime-aggregated motif set in Equation 185.

$$\mathcal{M}(c; \mathfrak{U}_{J,c}) := \bigcup_{\eta \in \mathfrak{U}_{J,c}} \{M_c(h) : h \in \mathcal{R}_{J,c}(Y_c; \eta) \cap X_c\}. \quad (185)$$

A motif is robustly identified for case c when every regime in $\mathfrak{U}_{J,c}$ has a nonempty comparison-domain reconstruction and $\mathcal{M}(c; \mathfrak{U}_{J,c}) = \{m\}$. The robust support set is then defined in Equation 186.

$$\text{RISupp}_J(m) := \{c \in \mathcal{C} : m \text{ is robustly identified for } c\}. \quad (186)$$

Equation 186 distinguishes recurrence at one selected reconstruction from recurrence that survives the declared uncertainty analysis.

9.3 Comparative Equivalence and Contextual Transport

The transported descriptors K_c permit a cross-case equivalence to be defined without pretending that histories from different cases inhabit one original justice domain.

Let the comparison domain of case c be $X_c \subseteq \Gamma_{J,c}$. Form the disjoint union in Equation 187.

$$X_{\sqcup} := \coprod_{c \in \mathcal{C}} X_c. \quad (187)$$

Define the piecewise comparative descriptor in Equation 188.

$$K_{\sqcup} : X_{\sqcup} \longrightarrow \Omega_{\text{cmp}}, \quad K_{\sqcup}|_{X_c} := K_c|_{X_c}. \quad (188)$$

The equality-induced relation on $X_{\sqcup}$ is Equation 189.

$$h_c \sim_{\text{cmp}} h_d \iff K_c(h_c) = K_d(h_d). \quad (189)$$

Proposition 9.4 (Cross-case comparative equivalence). *The relation $\sim_{\text{cmp}}$ defined in Equation 189 is an equivalence relation on $X_{\sqcup}$.*

Proof. Equation 189 is equality of the values of the function $K_{\sqcup}$ defined in Equation 188. Equality-induced relations are reflexive, symmetric, and transitive. $\square$

Proposition 9.4 provides a formally coherent comparative partition. Its substantive interpretation remains dependent on the transport maps ψ_c . The existence of one common codomain does not by itself establish that the case-specific meanings have been aligned adequately.

Within a single case, the transported descriptor factors through the case-specific justice evaluation. Let $\sim_{J,c}$ be equality under $\mathcal{J}_c$ and let $\sim_{K,c}$ be equality under K_c . Proposition 3.2 gives Equation 190.

$$\sim_{J,c}|_{X_c} \subseteq \sim_{K,c}. \quad (190)$$

Equation 190 states that comparative transport can preserve or merge case-specific justice classes; it cannot split one justice class when $K_c = \psi_c \circ \mathcal{J}_c$.

Definition 9.5 (Justice-faithful contextual transport). The transport ψ_c is *justice-faithful* on X_c when its restriction to $\mathcal{J}_c(X_c)$ is injective.

Definition 9.5 therefore treats injectivity on the realized case-specific justice image as the criterion for preserving all represented within-case justice distinctions.

Proposition 9.6 (Equivalence preservation under faithful transport). *The transport ψ_c is justice-faithful on X_c if and only if Equation 191 holds.*

$$\sim_{J,c}|_{X_c} = \sim_{K,c}. \quad (191)$$

Proof. If ψ_c is injective on $\mathcal{J}_c(X_c)$, then $K_c(h_1) = K_c(h_2)$ implies $\mathcal{J}_c(h_1) = \mathcal{J}_c(h_2)$. Thus $\sim_{K,c} \subseteq \sim_{J,c}|_{X_c}$. Equation 190 gives the reverse inclusion.

Conversely, assume Equation 191. Let $\omega_1, \omega_2 \in \mathcal{J}_c(X_c)$ satisfy $\psi_c(\omega_1) = \psi_c(\omega_2)$. Choose $h_1, h_2 \in X_c$ with $\mathcal{J}_c(h_i) = \omega_i$. Then $K_c(h_1) = K_c(h_2)$, so $h_1 \sim_{K,c} h_2$. Equation 191 gives $h_1 \sim_{J,c} h_2$, and hence $\omega_1 = \omega_2$. Therefore the restricted transport is injective. $\square$

Justice-faithful transport preserves all case-specific justice distinctions represented in X_c . It still does not establish that two equal transported values from different cases have identical historical reasons, identical fact configurations, or identical institutional meaning. Those stronger claims require further case-specific argument.

9.4 Normative Resolution of Cross-Case Compression

A common comparative codomain is often intentionally coarser than the complete justice codomains of the cases. Such compression can be legitimate for a limited comparative question while being insufficient for claims of complete normative equivalence.

Define the transport-induced justice-loss relation for case c in Equation 192.

$$\Delta_{\text{cmp}|J}^c := \sim_{K,c} \setminus \sim_{J,c}|_{X_c}. \quad (192)$$

Equation 192 contains pairs of case histories that the comparative descriptor merges even though the case-specific justice evaluation distinguishes them.

Proposition 9.7 (Comparative transport loss). *The transport ψ_c is justice-faithful on X_c if and only if Equation 193 holds.*

$$\Delta_{\text{cmp}|J}^c = \emptyset. \quad (193)$$

Proof. By Equation 190, $\sim_{J,c}|_{X_c} \subseteq \sim_{K,c}$. The relation difference in Equation 192 is empty exactly when the reverse inclusion also holds. By Proposition 9.6, equality of the two equivalences is equivalent to justice-faithfulness. $\square$

Proposition 9.7 identifies losslessness of the comparative transport with preservation of the case-specific justice partition.

When $\Delta_{\text{cmp}|J}^c \neq \emptyset$, the common descriptor is a deliberate compression of case-specific justice information. A comparative equality at the transported resolution is stated in Equation 194.

$$K_c(h_c) = K_d(h_d). \quad (194)$$

Equation 194 supports equality only at the declared comparative resolution. It does not support equality of $\mathcal{J}_c(h_c)$ and $\mathcal{J}_d(h_d)$, because those values can inhabit different codomains and can retain distinctions erased by ψ_c or ψ_d .

The same caution applies to recurrent motifs. Equality of $M_c(h_c)$ and $M_d(h_d)$ means that the declared process coordinate and transported normative coordinate agree. It does not establish: a shared historical generator, identity of the complete normative evaluation, identity of the reasons supporting that evaluation, or identity of the reparative response appropriate to the two cases.

The major comparison states are summarized in Table 3.

Table 3 makes the comparative compression boundary explicit. Case-specific justice evaluation, transported comparison, and recurrent process-normative motifs are distinct objects. Equality at a common resolution warrants complete normative identity only under a faithful transport.

Table 3: Reporting levels for comparative processual justice.

Comparative state	Formal basis	Warranted report
Justice-faithful comparison	Each relevant transport is injective on the realized case-specific justice image	The common descriptor preserves the represented justice distinctions within each compared case; cross-case semantic equivalence remains limited to the declared transport.
Feature-level comparison	Comparative values are identified but at least one transport is noninjective	Report agreement or difference in the selected normative feature without claiming equality of complete justice evaluation.
Justice-relevant motif recurrence	The same identified pair of process and transported normative values occurs across at least two cases	Report recurrence of the declared joint motif without inferring a shared causal generator or universal normative mechanism.
Uncertainty overlap	$\mathcal{V}_{\text{cmp}}^\cap(c, d) \neq \emptyset$ while one or both case sets are nonsingleton	Report compatible overlap rather than equality of identified case values.
Contextual non-comparability	No substantively justified common transport or process-comparison protocol is available	Retain separate case analyses or compare at a weaker descriptor level that can be justified.

10 Formal Worked Examples

This section gives five finite worked examples that isolate the main distinctions developed in Sections 2–9. Its role is diagnostic rather than empirical: each example is deliberately small so that endpoint equivalence, braid equivalence, normative decoration, reparative continuation, and historical uncertainty can be inspected without introducing case-specific substantive assumptions. The examples do not claim that the displayed justice values follow from braid topology. Instead, the justice maps and label schemes are declared explicitly and then used to test the formal criteria developed earlier. Section 10.1 exhibits terminal-state process sensitivity. Section 10.2 exhibits surplus topological resolution. Section 10.3 shows how decoration can repair braid insufficiency. Section 10.4 illustrates reparation without historical erasure. Section 10.5 shows robust normative identification despite unresolved history and topology.

10.1 Equal Endpoints with Unequal Justice Evaluations

Let the justice-relevant domain contain the two histories in Equation 195.

$$X_1 := \{h_A, h_B\} \subseteq \Gamma_J. \quad (195)$$

Assume each history has two transitions and is represented by Equation 196.

$$h_A := (x_0, x_A, x_*), \quad h_B := (y_0, y_B, x_*), \quad (196)$$

where the terminal carrier state is the same state x_* . The terminal-state descriptor therefore satisfies Equation 197.

$$T(h_A) = T(h_B) = x_*. \quad (197)$$

Hence $h_A \sim_T h_B$.

Declare two distinct justice values $\omega_A, \omega_B \in \Omega_J$ and define the justice evaluation on X_1 by Equation 198.

$$\mathcal{J}(h_A) := \omega_A, \quad \mathcal{J}(h_B) := \omega_B, \quad \omega_A \neq \omega_B. \quad (198)$$

Equation 198 places the pair outside justice equivalence: $h_A \not\sim_J h_B$. Combining this fact with Equation 197 gives Equation 199.

$$(h_A, h_B) \in \Delta_{J|T}. \quad (199)$$

By Definition 2.8, Equation 199 exhibits terminal-state process sensitivity.

If an endpoint feature descriptor is defined by $E = \varepsilon \circ T$, then Equation 24 gives $h_A \sim_E h_B$ as well. The endpoint-relative loss relation therefore also contains the pair, as stated in Equation 200.

$$(h_A, h_B) \in \Delta_{J|E}. \quad (200)$$

The example does not specify why $\omega_A \neq \omega_B$. A substantive justice theory could base the difference on consent, responsibility, coercion, prior entitlement, role, harm, or another historically relevant distinction. The formal point is narrower: complete endpoint equality is compatible with normative inequality.

10.2 Distinct Braids with Equal Normative Evaluation

Let $X_2 := \{h_1, h_2\}$ be a two-history subset of a common braid-admissible domain with braid index $n = 2$. The group B_2 is generated by σ_1 . Declare the braid values in Equation 201.

$$D_B^\tau(h_1) := \sigma_1, \quad D_B^\tau(h_2) := \sigma_1^2. \quad (201)$$

The two braid values are distinct, so Equation 202 holds.

$$h_1 \not\sim_B^\tau h_2. \quad (202)$$

Let $\omega_* \in \Omega_J$ and define the justice evaluation by Equation 203.

$$\mathcal{J}(h_1) = \mathcal{J}(h_2) = \omega_*. \quad (203)$$

Equation 203 gives $h_1 \sim_J h_2$. Thus the pair belongs to the surplus-distinction relation from Equation 88, as shown in Equation 204.

$$(h_1, h_2) \in \Sigma_{B|J}^\tau. \quad (204)$$

Because X_2 contains only two histories and their braid values are distinct, the restriction of braid equivalence to X_2 is the identity relation on X_2 . The justice equivalence is the universal equivalence on X_2 . Therefore Equation 205 holds.

$$\sim_B^\tau|_{X_2} \subsetneq \sim_J|_{X_2}. \quad (205)$$

By Definition 3.5, the braid descriptor has surplus descriptive resolution relative to justice on X_2 . It is normatively sufficient even though it distinguishes histories that the declared justice evaluation treats as equivalent. This example shows why topological difference should not be converted automatically into moral difference.

10.3 Equal Braids with Distinct Normative Decoration

Let $X_3 := \{h_C, h_N\}$ be a two-history braid-admissible domain with one tracked object a and one transition. Assume the two histories have the same braid value as stated in Equation 206.

$$D_B^\tau(h_C) = D_B^\tau(h_N) = \sigma_1. \quad (206)$$

The subscripts C and N are merely labels for the two synthetic histories.

Use the consent and harm label codomains in Equation 207.

$$\Omega_{\text{cons}} := \{c^+, c^-\}, \quad \Omega_{\text{harm}} := \{h^0, h^-\}, \quad (207)$$

where the symbols are interpreted in this example as consent present, consent absent, no declared injury, and declared injury, respectively. No ordering or cardinal magnitude is attached to the four symbols.

Let the only transition-participation site be $(a, 1)$. Define the relevant decoration coordinates by Equation 208.

$$\begin{aligned} C_{h_C}(a, 1) &:= c^+, & H_{h_C}(a, 1) &:= h^0, \\ C_{h_N}(a, 1) &:= c^-, & H_{h_N}(a, 1) &:= h^-. \end{aligned} \quad (208)$$

Assume all remaining decoration components are equal between the histories. Equation 208 then implies Equation 209.

$$N(h_C) \neq N(h_N). \quad (209)$$

Declare distinct justice values as Equation 210.

$$\mathcal{J}(h_C) := \omega_C, \quad \mathcal{J}(h_N) := \omega_N, \quad \omega_C \neq \omega_N. \quad (210)$$

Equations 206 and 210 show that raw braid topology is normatively insufficient on X_3 :

$$(h_C, h_N) \in \Delta_{J|B}^\tau. \quad (211)$$

Equation 211 identifies the pair as a concrete braid-aliasing failure.

The decorated descriptor, however, assigns the values in Equation 212.

$$\begin{aligned} D_{\text{dec}}^\tau(h_C) &:= (\sigma_1, N(h_C)), \\ D_{\text{dec}}^\tau(h_N) &:= (\sigma_1, N(h_N)). \end{aligned} \quad (212)$$

By Equation 209, the two decorated values are unequal. Since X_3 contains only the two histories, its decorated equivalence is the identity relation and is therefore contained in justice equivalence. Equivalently, the residual gap is empty:

$$\Delta_{J|B,N}^\tau = \emptyset. \quad (213)$$

Equation 213 shows that the declared decoration removes the residual justice loss in this finite domain. Proposition 6.3 then gives normative sufficiency of the decorated descriptor on X_3 . The example isolates the function of decoration: it contributes information that varies inside one braid fiber and thereby separates a justice-distinct pair that topology alone aliases.

10.4 Harm Followed by Repair

Let an initial two-strand history h_I have one transition and let c_R be a one-transition continuation. The continued history is defined by Equation 214.

$$h_R := h_I \star c_R. \quad (214)$$

Assume the tracked object set contains a and that the initial transition is injurious under the declared harm scheme:

$$(a, 1) \in \text{Inj}(h_I). \quad (215)$$

Equation 215 fixes the injurious prefix before any reparative continuation is appended. Assume the decoration is prefix-compatible in the sense used by the historical non-erasure result of Section 7. Then Equation 216 follows.

$$(a, 1) \in \text{Inj}(h_R). \quad (216)$$

Thus the earlier injury remains part of the continued history.

Let the reparation codomain contain a designated reparative value $p^{\text{rep}} \in \Omega_P^{\text{rep}}$ and declare Equation 217.

$$P(h_I, c_R) := p^{\text{rep}}. \quad (217)$$

Let $\omega_I \notin \Omega_J^{\text{acc}}$ and $\omega_R \in \Omega_J^{\text{acc}}$ be two justice values and define Equation 218.

$$\mathcal{J}(h_I) := \omega_I, \quad \mathcal{J}(h_R) := \omega_R. \quad (218)$$

Equation 218 therefore describes an acceptability-restoring normative transition while Equation 216 retains the injurious prefix.

To exhibit the limitation of braid inversion explicitly, suppose the braid values of the initial history and continuation satisfy Equation 219.

$$D_B^\tau(h_I) := \sigma_1, \quad D_B^\tau(c_R) := \sigma_1^{-1}. \quad (219)$$

Under the compositionality condition of Equation 147, Equation 220 follows.

$$D_B^\tau(h_R) = \sigma_1 \sigma_1^{-1} = e. \quad (220)$$

Equations 216 and 220 coexist. The identity braid therefore does not erase the injurious history. In this synthetic example the continuation is also declared reparative by Equation 217, but the reparative status comes from P , not from algebraic inversion.

10.5 Historical Uncertainty with Stable Justice Evaluation

Let one evidential record Y be analyzed under two admissible reconstruction regimes η_1 and η_2 . Assume the justice-relevant reconstruction sets are Equation 221.

$$\mathcal{R}_J(Y; \eta_1) := \{h_1, h_2\}, \quad \mathcal{R}_J(Y; \eta_2) := \{h_2, h_3\}. \quad (221)$$

The histories are not identified because each regime contains more than one compatible history and the aggregate set contains three histories.

Assume their braid values are distinct as stated in Equation 222.

$$D_B^\tau(h_1) := e, \quad D_B^\tau(h_2) := \sigma_1, \quad D_B^\tau(h_3) := \sigma_1^2. \quad (222)$$

Thus neither history nor braid value is identified from the record under the displayed regimes.

Let the justice evaluation nevertheless satisfy Equation 223.

$$\mathcal{J}(h_1) = \mathcal{J}(h_2) = \mathcal{J}(h_3) = \omega_*. \quad (223)$$

Applying Equation 153 to the two regimes gives Equation 224.

$$\mathcal{V}_J(Y; \eta_1) = \mathcal{V}_J(Y; \eta_2) = \{\omega_*\}. \quad (224)$$

Let $\mathfrak{U}_J := \{\eta_1, \eta_2\}$. The aggregate reconstruction set is Equation 225.

$$\mathcal{R}_J(Y; \mathfrak{U}_J) = \{h_1, h_2, h_3\}. \quad (225)$$

By Equation 223, this aggregate set lies entirely in one justice fiber:

$$\mathcal{R}_J(Y; \mathfrak{U}_J) \subseteq \mathcal{F}_J(\omega_*). \quad (226)$$

Equation 226 is the aggregate-fiber condition required for robust normative identification. No regime in $\mathfrak{U}_J$ has an empty justice-relevant reconstruction. Proposition 8.4 therefore yields robust normative identification of ω_* .

This example shows that processual justice need not wait for complete historical or topological identification. The evidence can leave several histories and several braid classes open while still supporting one robust justice evaluation, provided all admissible reconstructions fall in the same normative fiber.

The five examples are summarized in Table 4.

The examples show that endpoint structure, braid structure, decoration, reparation, and evidential uncertainty answer different questions and should remain separate from the justice evaluation itself.

Table 4: Formal distinctions isolated by the worked examples.

Example	Formal pattern	Main conclusion
Equal endpoints, unequal justice	$h_A \sim_T h_B$ and $h_A \not\sim_J h_B$	Complete terminal-state equality can be normatively insufficient.
Distinct braids, equal justice	$h_1 \not\sim_B h_2$ and $h_1 \sim_J h_2$	Topology can retain surplus distinctions without becoming normatively inadequate.
Equal braids, distinct decoration	$h_C \sim_B h_N$, $N(h_C) \neq N(h_N)$, and $h_C \not\sim_J h_N$	Additional within-fiber information can repair braid insufficiency.
Harm followed by repair	Reparative continuation, retained injury, and identity final braid	Reparation can transform normative status without deleting the injurious prefix.
Historical uncertainty, stable justice	Several histories and braid values, one justice fiber across regimes	Normative identification can be robust despite unresolved historical and topological structure.

11 Interfaces with Established Justice Concepts

The processual formalism is intended to clarify the historical resolution at which a justice evaluation operates. Its role is therefore comparative and diagnostic rather than substitutive: established theories supply substantive criteria, while the present framework asks which distinctions among histories those criteria require a representation to preserve. The discussion below considers procedural justice, historical entitlement, distributive and relational evaluation, and corrective or reparative concerns before stating the narrower second-order contribution of processual justice. The expression *processual justice* is used here in this stipulated structural sense. Contemporary philosophical mappings of justice already distinguish procedural and substantive, corrective and distributive, and relational and non-relational dimensions, so the interfaces considered here should remain plural rather than being collapsed into a single taxonomy [6].

11.1 Procedural Justice

Procedural justice evaluates the normative properties of procedures and the relation between a procedure, participation in it, and the result it produces. Rawls distinguishes perfect, imperfect, and pure procedural justice, thereby showing that the justice of a procedure can stand in different relations to an independently specified criterion for a just result [8]. Legal-philosophical treatments also emphasize that procedural justice raises questions of legitimate decision, participation, and accuracy that cannot be reduced to the distributive value of the outcome alone [9]. These traditions provide an important neighboring domain for the present analysis because a procedure can itself be part of the history that a justice theory evaluates.

Processual justice extends the object of inquiry beyond the boundaries of a single decision procedure while remaining substantively open about which historical features matter. A justice-relevant history can include interactions preceding the procedure, role assignments, consent conditions, harms and benefits generated during implementation, institutional transformations, and reparative continuations after a decision has been made. Conversely, a substantive theory may legitimately treat a well-specified procedure as sufficient for its own evaluation. The distinction is therefore one of analytical object and resolution rather than a ranking of procedural and processual approaches.

Let $D_{\text{proc}} : X \rightarrow \Omega_{\text{proc}}$ be a descriptor of the procedurally relevant information retained by a substantive theory on a history domain $X \subseteq \Gamma_J$. The procedural descriptor is normatively sufficient exactly when Equation 227 holds.

$$\sim_{D_{\text{proc}}} \subseteq \sim_J|_X. \quad (227)$$

Equation 227 does not define procedural justice. It states an information condition: histories identified by the procedural descriptor must also be identified by the declared justice evaluation. Failure of the inclusion means that some distinction required by the justice theory lies outside the selected procedural description.

The distinction between procedural and processual analysis is therefore visible at the level of equivalence relations. Two histories can instantiate the same described procedure while differing in an earlier relation of domination, an identity or role assignment, a consent condition, an injury, or a later reparative response that the declared justice evaluation treats as relevant. A different theory can regard the procedure as normatively decisive and make D_{proc} sufficient even though additional historical differences remain. The processual framework accommodates both possibilities by locating the disagreement in the declared normative quotient.

11.2 Historical Entitlement

Nozick's entitlement theory provides a prominent modern example of a justice theory whose evaluation of holdings is historical rather than determined solely by an end-state or patterned distribution. Its principles concern acquisition, transfer, and rectification, so the route by which a holding arose can matter to its justice [7]. Historical entitlement therefore supplies a clear instance in which endpoint identity need not imply normative identity. The present framework abstracts this structural feature without adopting the substantive property theory in which Nozick develops it.

Let $D_{\text{ent}} : X \rightarrow \Omega_{\text{ent}}$ represent the historical information that a declared entitlement theory uses on X . The relation between entitlement history and the justice evaluation can be tested by Equation 228.

$$\sim_{D_{\text{ent}}} \subseteq \sim_J|_X. \quad (228)$$

If Equation 228 holds, the declared entitlement history is sufficient for the justice evaluation on that domain. Failure of the inclusion indicates that the represented theory requires additional information beyond the selected entitlement descriptor.

The scope difference remains substantial. Processual justice does not endorse particular principles of acquisition, transfer, property, or rectification, and historical justice need not be exhausted by entitlement history. A justice-relevant history can contain recognition, domination, consent, institutional position, responsibility, injury, or reparation that a holdings-based descriptor does not encode. Braid topology is one descriptive level further removed: preservation of path structure cannot substitute for semantically typed entitlement history unless an explicit sufficiency argument establishes that the topological descriptor preserves every distinction required by the evaluation.

11.3 Distributive and Relational Evaluation

Distributive theories evaluate how benefits, burdens, opportunities, resources, or other goods are allocated according to a substantive criterion. Rawls's theory is a central modern example of institutional distributive evaluation, although its scope and principles cannot be reduced to a bare allocation vector [8, 6]. Relational egalitarian approaches emphasize another dimension: the character of social relations, including whether persons stand to one another as equals rather than through oppression, domination, subordination, or status hierarchy [1, 6].

The processual framework treats distributive and relational information as potentially distinct descriptors whose historical sufficiency must be assessed rather than presumed. Let $D_{\text{dist}} : X \rightarrow \Omega_{\text{dist}}$ retain the distributive information selected by an inquiry and let $D_{\text{rel}} : X \rightarrow \Omega_{\text{rel}}$ retain the relational information selected by the same inquiry. Their joint descriptor is defined in Equation 229.

$$D_{\text{DR}}(h) := (D_{\text{dist}}(h), D_{\text{rel}}(h)). \quad (229)$$

By the joint-descriptor logic developed earlier in the paper, its induced equivalence is given by Equation 230.

$$\sim_{D_{\text{DR}}} = \sim_{D_{\text{dist}}} \cap \sim_{D_{\text{rel}}}. \quad (230)$$

Equations 229 and 230 allow several substantive architectures without deciding in advance which axis is fundamental. A theory can be distributively sufficient, relationally sufficient, jointly sufficient, or insufficient under both declared descriptors. Processual justice is consequently neither a synonym for relational egalitarianism nor a classification of distributive theories as endpoint-only.

The relevant issue is the temporal resolution internal to the theory itself. A distributive object can be history-sensitive when legitimate holdings depend on acquisition or transfer. A relational evaluation can, in some cases, be made from a sufficiently rich relational state without recovering the path that produced it. Labels such as *distributive* and *relational* therefore do not determine process sensitivity. The determining question is whether the descriptor used by the theory preserves the historical distinctions preserved by its justice evaluation.

11.4 Corrective and Reparative Dimensions

Corrective justice is commonly distinguished from distributive justice through its concern with wrongful transactions, responsibility, and remedial relations between parties, although theories disagree about the exact basis and content of correction [6]. Philosophical work on moral repair broadens the neighboring terrain by examining how wrongdoing damages moral relations and how responses such as acknowledgment, validation, trust repair, forgiveness, and making amends can contribute to their reconstruction [11]. These literatures are especially relevant to the present paper because a remedial act can transform the normative status of a continuing history while leaving the occurrence of the earlier wrong historically intact.

Let $P(h, c)$ be the reparative assessment introduced in Section 7 for a continuation c of history h . A corrective theory can supply a more specific descriptor $D_{\text{corr}}(h, c)$ containing the facts it regards as relevant to correction, such as the wrong, parties, responsibility relation, loss, duty, remedy, or institutional status. The present framework requires a claim of corrective sufficiency to satisfy the same information condition used elsewhere. Define Equation 231 on an admitted continuation domain Z .

$$\sim_{D_{\text{corr}}} \subseteq \sim_P|_Z, \quad (231)$$

where $\sim_P$ is equality under the declared reparative assessment. Equation 231 asks whether the corrective descriptor preserves every distinction that the reparative evaluation itself preserves.

The historical non-erasure result developed earlier can now be located more precisely. A later remedy may change the normative class of the complete history while the earlier wrong remains one of its constitutive events. Restoration of an endpoint, compensation, apology, institutional correction, acknowledgment, or another remedial act therefore need not make the repaired history equivalent to a history in which the wrong never occurred. This formal distinction does not determine what successful repair requires. It preserves conceptual room for substantive accounts that assign continuing significance to responsibility, recognition, trust, acknowledgment, or other consequences of wrongdoing.

The braid-inversion example from Section 7.4 reinforces the same separation at the descriptor level. Algebraic cancellation in a trajectory representation is neither necessary nor sufficient for correction. A remedial continuation must be evaluated through the historical and normative information required by the corrective or reparative theory, even where the underlying braid product simplifies to the identity.

11.5 Meta-Normative Object Specification in Processual Justice

The common structure across the preceding interfaces is a second-order problem of object specification. The paper uses *meta-normative* in this limited structural sense: the formalism concerns which distinctions a declared justice evaluation requires its object to retain. It does not advance a metaethical claim about the semantics, truth conditions, or ontology of moral judgment.

For a justice evaluation $\mathcal{J}$ on domain $X \subseteq \Gamma_J$, Section 4 identified the justice-required distinction relation. Restricted to X , it is written in Equation 232.

$$\mathcal{D}_J(X) := \{(h_1, h_2) \in X^2 : h_1 \not\sim_J h_2\}. \quad (232)$$

A substantive theory of justice determines which pairs enter $\mathcal{D}_J(X)$. The processual formalism then asks which historical descriptors preserve those pairs and which compressions erase them.

Let $\mathcal{D}_X$ be a declared family of candidate descriptors on X . The theory-relative sufficient-descriptor family is defined in Equation 233.

$$\mathcal{S}_J(X) := \{D \in \mathcal{D}_X : \sim_D \subseteq \sim_J|_X\}. \quad (233)$$

Equation 233 provides a formal criterion for representational adequacy relative to the declared theory and domain. A descriptor belongs to $\mathcal{S}_J(X)$ when its compression preserves every distinction represented by $\mathcal{D}_J(X)$.

Procedural, entitlement-based, distributive, relational, corrective, reparative, and other justice theories can consequently differ along two dimensions at once: their substantive evaluations and the historical resolution at which those evaluations are defined. Some admit coarse endpoint compression. Others require acquisition histories, role relations, injury and responsibility, consent, acknowledgment, or reparative continuation. Some require joint descriptors assembled from several such domains. The processual architecture makes these informational commitments explicit without ranking the substantive theories by the quantity of history they retain.

Once a theory declares a justice evaluation, however, a representation of its justice object incurs a theory-relative information obligation: it must preserve the distinctions that the evaluation treats as normatively different. This obligation is the narrower contribution of processual justice in the present paper. It organizes the principled retention and compression of history across substantive theories while leaving the substantive grounds of justice open to those theories themselves.

12 Scope Boundaries of the Processual Formalism

This section states the conditions under which the processual formalism should be used and the conditions under which its conclusions must be weakened, reformulated, or withheld. Its role is methodological rather than defensive: the preceding sections deliberately separate history, descriptor, normative decoration, justice evaluation, reconstruction uncertainty, and cross-case transport because failure at any one layer does not imply failure at every other layer. The section first treats justice-relevant histories for which standard braid structure is unavailable. It then separates historical uncertainty from normative indeterminacy about the justice evaluation itself. The third subsection states semantic and ontological restraints on interpreting formal objects. The fourth examines dependence on descriptor choice and representation architecture. The final subsection consolidates these limits into a reporting discipline for processual claims.

12.1 Descriptor Coverage of Justice-Relevant History Domains

The processual formalism is broader than its braid-theoretic prototype. A justice-relevant history need not admit a fixed-strand braid representation. Section 5.1 defined the braid-admissible justice domain $\Gamma_{J,B}^\tau \subseteq \Gamma_J$ and the omitted-domain set $\mathcal{O}_B^\tau$. The present subsection treats omission as a representational limitation rather than as absence of justice-relevant history.

Let $\mathcal{D}_J$ be a declared family of history descriptors. For each $D \in \mathcal{D}_J$, let $\Gamma_D \subseteq \Gamma_J$ be its domain. The descriptor-coverage set of a history $h \in \Gamma_J$ is defined in Equation 234.

$$\mathfrak{D}_J(h) := \{D \in \mathcal{D}_J : h \in \Gamma_D\}. \quad (234)$$

Equation 234 can be empty even when h is a legitimate member of the justice domain. The formalism therefore distinguishes normative admissibility from descriptor availability.

Definition 12.1 (Descriptor-covered justice domain). The *descriptor-covered justice domain* generated by $\mathcal{D}_J$ is the set in Equation 235.

$$\Gamma_{J,\mathfrak{D}} := \bigcup_{D \in \mathcal{D}_J} \Gamma_D. \quad (235)$$

The uncovered domain is correspondingly defined in Equation 236.

$$\mathcal{O}_{\mathfrak{D}} := \Gamma_J \setminus \Gamma_{J,\mathfrak{D}}. \quad (236)$$

Definition 12.1 makes descriptor coverage a property of the declared representational family rather than of justice relevance itself. A nonempty $\mathcal{O}_{\mathfrak{D}}$ means only that the current descriptor family is incomplete for the justice domain.

Several historically common situations can produce $h \notin \Gamma_{J,B}^\tau$: the number of tracked objects changes, persistent identity cannot be reconstructed, the relevant process is not representable by a planar noncollision trajectory, endpoint conventions cannot be justified, or the fact taxonomy changes so that one fixed research-object registry ceases to be adequate. Paper II treats such changes as possible fact-regime or ontology changes rather than ordinary braid collisions [3].

The appropriate response is representation retreat rather than forced closure. Let D_1 be unavailable on h and let D_2 be a coarser or structurally different descriptor defined on h . The justice question can continue at D_2 when D_2 satisfies the normative-sufficiency criterion on the relevant domain. Thus braid inadmissibility does not imply processual irrelevance.

Proposition 12.2 (Representation retreat under descriptor failure). *Let $X \subseteq \Gamma_J$ be nonempty. Suppose D_1 is undefined on some members of X , while $D_2 : X \rightarrow \Omega_2$ is defined on all of X and satisfies Equation 237.*

$$\sim_{D_2} \subseteq \sim_J|_X. \quad (237)$$

Then D_2 is a normatively sufficient descriptor on X regardless of the domain failure of D_1 .

Proof. Equation 237 is exactly Definition 2.4 applied to D_2 on X . The availability or nonavailability of another descriptor D_1 does not enter that condition. $\square$

Proposition 12.2 formalizes a general reporting principle: the strongest unavailable representation should not determine whether the justice question itself is reportable. The appropriate level is the strongest descriptor whose domain and resolution are both warranted.

12.2 Normative Indeterminacy

Section 8 treated uncertainty about which history occurred while holding the justice evaluation $\mathcal{J}$ fixed. A different problem arises when the normative theory itself does not determine one unique evaluation map.

Let $X \subseteq \Gamma_J$ be a nonempty domain. Define a nonempty family of admissible justice evaluations in Equation 238.

$$\mathfrak{J}_X := \{\mathcal{J}_\lambda : X \rightarrow \Omega_{J,\lambda} \mid \lambda \in \Lambda_J\}. \quad (238)$$

The index λ can represent competing normative theories, unresolved interpretive parameters, alternative legal-normative standards, or another explicitly admitted source of normative variation.

Because the codomains $\Omega_{J,\lambda}$ need not be identical, a cross-theory comparison requires a common feature rather than direct equality of complete outputs. Let Ω_Φ be a common feature codomain and let $\Phi_\lambda : \Omega_{J,\lambda} \rightarrow \Omega_\Phi$ be a declared theory-relative feature map. For history $h \in X$, define the theory-compatible feature set in Equation 239.

$$\mathcal{V}_\Phi^{\text{th}}(h) := \{\Phi_\lambda(\mathcal{J}_\lambda(h)) : \lambda \in \Lambda_J\}. \quad (239)$$

Definition 12.3 (Theory-robust normative feature). A comparative normative feature is *theory-robust* for h when Equation 240 holds.

$$|\mathcal{V}_\Phi^{\text{th}}(h)| = 1. \quad (240)$$

Definition 12.3 should not be confused with robust normative identification from Section 8.2. The latter varies the historical reconstruction while keeping the justice evaluation fixed. The present definition keeps the history fixed while varying the admitted normative evaluation.

Both uncertainties can occur simultaneously. For evidential record Y and reconstruction regime family $\mathfrak{U}_J$, restrict each reconstructed history set to the common normative-theory domain X using Equation 241.

$$\mathcal{R}_{J,X}(Y; \eta) := \mathcal{R}_J(Y; \eta) \cap X. \quad (241)$$

The restriction in Equation 241 ensures that every evaluation $\mathcal{J}_\lambda$ is applied only where it is defined. Define the combined history-theory feature set in Equation 242.

$$\mathcal{V}_\Phi^{\text{joint}}(Y; \mathfrak{U}_J, \mathfrak{J}_X) := \bigcup_{\eta \in \mathfrak{U}_J} \bigcup_{\lambda \in \Lambda_J} \{\Phi_\lambda(\mathcal{J}_\lambda(h)) : h \in \mathcal{R}_{J,X}(Y; \eta)\}. \quad (242)$$

Definition 12.4 (Joint historical-normative robustness). The feature value $\phi \in \Omega_\Phi$ is *jointly robust* over $(\mathfrak{U}_J, \mathfrak{J}_X)$ when every reconstruction regime has a nonempty theory-domain reconstruction set and Equation 243 holds. The nonemptiness requirement is $\mathcal{R}_{J,X}(Y; \eta) \neq \emptyset$ for every $\eta \in \mathfrak{U}_J$.

$$\mathcal{V}_\Phi^{\text{joint}}(Y; \mathfrak{U}_J, \mathfrak{J}_X) = \{\phi\}. \quad (243)$$

Definition 12.4 therefore requires stability across both declared reconstruction variation and declared normative-theory variation.

The distinction matters for interpretive restraint. A single identified history can remain normatively indeterminate when several admitted justice theories classify it differently. Conversely, several histories can remain historically unresolved while a selected normative feature remains stable under every admitted theory.

The paper does not attempt to solve normative disagreement by topological representation. Topology can help preserve or distinguish histories. It cannot select the morally correct justice evaluation from a family of competing substantive theories.

12.3 Semantic-Ontological Restraint

The formal objects used in this paper have declared analytical roles. They should not be promoted into stronger semantic or ontological claims without additional argument.

The three-level GR architecture distinguishes ultimate ontological reality, relational reality, and manifestation reality. The present formalism operates primarily on representations of relational and manifestation-level histories while suspending claims about ultimate ontology. A strand, braid generator, fact token, role label, harm label, or justice class is therefore a representational object whose interpretation depends on the declared research construction.

Let $D : X \rightarrow \Omega_D$ be a formal descriptor and let $\mathcal{S}$ be a set of substantive semantic claims. A semantic interpretation relation is defined in Equation 244.

$$\text{Sem}_D \subseteq \Omega_D \times \mathcal{S}. \quad (244)$$

The relation $(d, s) \in \text{Sem}_D$ means that substantive claim s is admitted as an interpretation of descriptor value d under the declared domain semantics.

Equation 244 is intentionally a relation rather than a required function. One descriptor value can support several substantively distinct readings, and one substantive claim can be compatible with several formal values.

Definition 12.5 (Descriptor-determined meaning). A semantic claim $s \in \mathcal{S}$ is *descriptor-determined* on X when its truth-status assignment is constant on every $\sim_D$ equivalence class represented in X .

Definition 12.5 is the semantic analogue of the non-reification criterion used in Paper II. If histories sharing one braid value differ in whether they instantiate harm, consent, responsibility, domination, legal validity, or another substantive property, that property is not determined by the braid.

The normative-decoration results of Section 6 make the same point formally. A topology-determined decoration cannot repair topological normative insufficiency. The needed information must vary inside at least one braid fiber.

Ontological restraint also applies to the fact regime. A legal fact, social fact, institutional fact, historical fact, or another domain category can be represented inside the GR architecture without being declared an ultimate ontological kind. Likewise, representation of a research object through a fact configuration does not establish that the object is reducible to that configuration.

The scope rule is therefore: formal adequacy establishes adequacy of a declared representation relative to a declared question. It does not by itself establish ontological identity, semantic uniqueness, causal completeness, or moral truth.

12.4 Descriptor-Dependent Representation Choice

A processual justice conclusion can depend on the descriptor chosen to represent history. This dependence should be stated explicitly rather than treated as a defect that can always be eliminated.

Let $\mathcal{D}_X$ be a family of descriptors defined on the same nonempty domain $X \subseteq \Gamma_J$. For $D \in \mathcal{D}_X$, define the descriptor-induced justice-loss set in Equation 245.

$$\Delta_{J|D} := \sim_D \setminus \sim_J|_X. \quad (245)$$

The sufficient-descriptor family from Section 11.5 can equivalently be written as Equation 246.

$$\mathcal{S}_J(X) = \{D \in \mathcal{D}_X : \Delta_{J|D} = \emptyset\}. \quad (246)$$

The existence of several members of $\mathcal{S}_J(X)$ means that justice-relevant history can admit several adequate representations. Those representations need not be equally fine, equally interpretable, equally robust, or equally convenient.

To compare representational resolution, reuse the descriptor refinement order: for $D_1, D_2 \in \mathcal{D}_X$, define Equation 247.

$$D_1 \preceq_X D_2 \iff \sim_{D_1} \subseteq \sim_{D_2}. \quad (247)$$

Under Equation 247, D_1 is at least as fine as D_2 .

A maximally compressed sufficient representation inside the declared family is any maximal element of $\mathcal{S}_J(X)$ under $\preceq_X$, equivalently any sufficient descriptor that cannot be strictly coarsened within $\mathcal{D}_X$ while preserving sufficiency. This family-relative concept is the same compression principle developed in Section 4.4, stated here in the language of representation choice.

No rule requires selection of the coarsest sufficient descriptor in every application. A finer descriptor can be preferable because it is more transparent, supports later comparison, preserves legally or historically salient distinctions beyond the current justice evaluation, or remains stable under future revisions of the normative theory.

Representation choice should therefore consider at least four dimensions: domain coverage, normative sufficiency, robustness under reconstruction and taxonomic variation, and semantic interpretability. These dimensions can trade off. A descriptor with excellent topological robustness can have narrow domain coverage; a broad descriptor can be normatively under-resolved; a very fine descriptor can be sufficient but difficult to compare across cases.

The formalism accordingly treats representation choice as part of the declared methodological regime rather than as a hidden preprocessing step.

12.5 Reporting Rules for Processual Claims

The previous limitations can be consolidated into a reporting profile. Let η be a declared historical reconstruction regime, let D be the selected historical descriptor, let $\mathfrak{U}_J$ be an optional robustness family, and let $\mathfrak{J}_X$ be an optional family of admitted justice evaluations on a declared common domain X . Let $T_3 := \{0, 1, \text{NA}\}$. The processual-justice applicability profile is defined in Equation 248.

$$\mathbf{p}_J := (p_{\text{dom}}, p_{\text{rec}}, p_{\text{desc}}, p_{\text{suff}}, p_{\text{id}}, p_{\text{rob}}, p_{\text{sem}}) \in \{0, 1\}^5 \times T_3^2. \quad (248)$$

Each component refers to a distinct methodological condition:

1. $p_{\text{dom}} = 1$ when the justice domain relevant to the claim is explicitly declared;
2. $p_{\text{rec}} = 1$ when the compatible justice-relevant reconstruction is nonempty;
3. $p_{\text{desc}} = 1$ when the selected descriptor is defined on every history needed for the claim;
4. $p_{\text{suff}} = 1$ when the descriptor preserves every justice distinction required at the claimed normative resolution;
5. $p_{\text{id}} = 1$ when the claimed justice value or normative feature is identified under the reference reconstruction regime;
6. $p_{\text{rob}} = 1$ when the claimed value has been shown robust over the declared historical and, where applicable, normative variation; and
7. $p_{\text{sem}} = 1$ when the substantive interpretation being reported is justified by the declared semantic mapping rather than inferred from formal structure alone.

The first five coordinates in Equation 248 are binary. The robustness and semantic coordinates use T_3 so that an unassessed condition can be reported as NA rather than conflated with failure. The profile remains a compact reporting device rather than a complete theory of evidential strength.

The maximally warranted reporting rule is stated in Definition 12.6.

Definition 12.6 (Maximally warranted processual report). A processual-justice report is *maximally warranted* when it states the most informative normative conclusion that is:

1. defined on the reconstructed justice-relevant domain;
2. represented by a descriptor available on that domain;
3. identified at the claimed normative resolution;
4. no stronger than the established descriptor-sufficiency result;
5. qualified by the robustness actually assessed; and
6. limited to substantive meanings supported by the declared semantic interpretation.

Definition 12.6 permits principled retreat. If a full braid is unavailable, a non-braid history descriptor can still support a justice conclusion. If the complete justice value is not identified, a coarser normative feature can be reported. If a reference regime identifies a value but the robustness family does not, the report should state regime sensitivity. If no justified semantic transport exists across cases, comparison should remain at a weaker level or be withheld.

Table 5 summarizes the principal failure modes.

Table 5 treats these limits as part of the method itself. The framework can state which distinctions a declared justice evaluation requires, which descriptors preserve them, which reconstructions support them, and which conclusions survive declared uncertainty; it does not by formal structure settle social ontology, choose the correct substantive theory, or convert topological regularity into moral truth.

Table 5: Scope boundaries and corresponding reporting responses.

Limitation	Formal symptom	Reporting response
Descriptor-domain failure	Some justice-relevant histories lie outside the selected descriptor domain	Retreat to another justified descriptor or restrict the claim explicitly to the covered domain.
Normative under-resolution	$\Delta_{J D} \neq \emptyset$	Refine the descriptor or avoid claiming that the descriptor alone determines the justice evaluation.
Historical nonidentification	$ \mathcal{R}_J(Y; \eta) > 1$	Propagate the compatible histories into the justice-value set rather than selecting one history silently.
Normative nonidentification	$ \mathcal{V}_J(Y; \eta) > 1$	Report a compatible value set or a coarser identified feature.
Theory sensitivity	$ \mathcal{V}_{\Phi}^{\text{th}}(h) > 1$	State that the normative conclusion depends on the admitted theory or standard.
Robustness failure	The aggregate value set over the declared regime family is nonsingleton	Report sensitivity to the varied assumptions, evidence, taxonomy, or normative standard.
Semantic underdetermination	Formal equivalence classes support several incompatible substantive interpretations	Report the formal result separately and withhold the unsupported semantic claim.
Cross-case transport failure	No justified common normative or process transport exists	Retain separate case analyses or compare only at a weaker justified resolution.

13 Conclusion

This paper has developed a preliminary formal account of processual justice in which a justice evaluation can depend on the history by which a social condition is generated rather than only on a selected endpoint. The proposal is structural and theory-relative. It does not supply one substantive moral principle; it asks which distinctions among histories a declared justice evaluation preserves, which distinctions it compresses, and whether a lower-level representation retains the information required by that evaluation.

The first contribution is a formal separation of historical and normative resolutions. A justice inquiry begins with a declared history domain Γ_J and an evaluation $\mathcal{J} : \Gamma_J \rightarrow \Omega_J$. Endpoint evaluation remains admissible whenever endpoint equivalence is contained in justice equivalence. Stronger process sensitivity arises only when the declared endpoint representation, or even the complete terminal carrier state, collapses distinctions that the justice evaluation preserves. Historical identity, terminal-state equivalence, endpoint equivalence, relational equivalence, braid equivalence, and justice equivalence are therefore distinct relations whose ordering must be derived from factorization or added normative assumptions rather than imposed by terminology.

The second contribution is the general sufficiency principle developed across Sections 3 and 4. A historical descriptor is normatively sufficient when its equivalence classes remain inside justice-equivalence classes. Exact normative resolution is one possibility, but a descriptor may retain surplus historical distinctions without becoming defective. The justice quotient $\Gamma_J / \sim_J$ therefore acts as a theory-relative compression boundary: information can be discarded within one justice class, while any strict coarsening beyond that boundary loses a distinction represented by the evaluation. Concrete descriptor families need not contain an exact realization of this quotient, and several different representations can preserve the same normative partition.

The third contribution is a conditional placement of braid theory inside that architecture. Braid-theoretic trajectory description is useful only on histories satisfying its domain requirements, and domain coverage is independent of normative resolution. Distinct braids can belong to one justice class, producing harmless surplus topological resolution. Equal braids can also receive different justice evaluations, producing braid-level insufficiency. When that occurs, successful refinement must add information that varies inside the affected braid fibers. Historical identity, role, harm, benefit, consent, and responsibility were therefore represented through explicit decoration rather than inferred from generators, crossing signs, exponent sums, conjugacy classes, or other topological features.

The fourth contribution concerns reparation and historical non-erasure. A reparative continuation can transform the normative class of a history while preserving the earlier injurious prefix and its encoded injury sites. Endpoint restoration, later acceptability, or even algebraic cancellation to the identity braid does not imply that the earlier wrong disappears from the represented history. Braid inversion and reparative status remain independent unless a substantive theory explicitly connects them. This distinction allows repair to change normative history without rewriting the historical path that made repair necessary.

The fifth contribution is epistemic. Historical reconstruction was treated as set-valued, so compatible histories induce compatible justice-value sets rather than one silently selected narrative. Complete normative identification occurs when all compatible justice-relevant histories lie in one justice fiber; coarser features can remain identified when the complete justice value is not. Robustness requires the conclusion to survive the declared family of reconstruction regimes. The scope analysis further separates historical uncertainty from normative-theory indeterminacy: a fixed history can receive different evaluations under different admitted theories, and a feature is jointly robust only when it survives both sources of variation.

The sixth contribution concerns comparison. Case-specific justice evaluations can have different codomains, so cross-case analysis requires explicit transport into a common comparative codomain. A transport is justice-faithful only when it preserves the represented case-specific justice partition. Coarser transports warrant feature-level comparison but not complete normative identity. Likewise, recurrent joint process-normative motifs record repeated structure at the declared comparative resolution; they do not establish a shared generator, universal moral mechanism, or identical reasons for judgment.

The complete methodological architecture is summarized in Equation 249.

$$\begin{aligned} \text{historical reconstruction} &\longrightarrow \text{historical descriptor} \longrightarrow \text{normative evaluation,} \\ \text{descriptor adequacy} &\neq \text{historical identification} \neq \text{normative truth.} \end{aligned} \tag{249}$$

Equation 249 expresses the separation maintained throughout the paper. A descriptor can be adequate without uniquely recovering the past; a normative conclusion can remain stable across several histories; and neither representational adequacy nor robustness selects the correct substantive theory of justice.

This theory-relative structure also clarifies the relation to established justice concepts. Procedural, entitlement-based, distributive, relational, corrective, and reparative theories can require different historical resolutions. Some questions can be endpoint-sufficient; others require acquisition histories, role relations, injury and responsibility, consent, or reparative continuation. The processual formalism supplies a common language for stating those information requirements without reducing the theories to one another.

The same restraint applies to ontology and semantics. Within the GR three-level architecture, the ultimate ontology of persons, institutions, facts, harms, rights, and justice remains suspended. Fact configurations, strands, braids, decorations, and justice classes are representational objects whose validity depends on declared domains and interpretation rules. Formal precision therefore does not convert topological regularity into moral truth or make one fact taxonomy ontologically final.

The practical implication is the maximally warranted reporting rule developed in Section 12.5. A processual-justice claim should be no stronger than the declared justice domain, historical reconstruction, descriptor coverage, normative sufficiency, identification result, robustness analysis, and semantic interpretation jointly support. Where a braid representation fails, another descriptor may remain adequate. Where the complete justice value is unresolved, a coarser feature may still be identified. Where

comparison lacks a justified transport, separate case analysis is preferable to forced equivalence.

The paper's principal claim is therefore limited but consequential: justice theories retain some historical distinctions and discard others, and those acts of retention and compression can be made explicit. Processual justice, in the preliminary sense developed here, is the disciplined analysis of which historical distinctions belong to a normative evaluation, how representations preserve or erase them, how uncertainty propagates through them, and how later repair can transform a normative history without erasing its past.

References

- [1] Elizabeth S. Anderson. What is the point of equality? *Ethics*, 109(2):287–337, 1999.
- [2] Pablo de Greiff. Justice and reparations. In Pablo de Greiff, editor, *The Handbook of Reparations*, pages 451–477. Oxford University Press, Oxford, 2006.
- [3] Wanhong Huang. Braid-theoretic trajectory description of historical social dynamics: A preliminary methodological framework. Preprint manuscript supplied by the author, 2026.
- [4] Wanhong Huang. A metamodel for ontologically heterogeneous social dynamics, part i: The highest abstract layer. Working manuscript supplied by the author, 2026.
- [5] Wanhong Huang. A preliminary formalization of braid-theoretic trajectory description in generative relational theory. Preprint manuscript supplied by the author, 2026.
- [6] David Miller. Justice. In Edward N. Zalta and Uri Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2026 edition, 2026.
- [7] Robert Nozick. *Anarchy, State, and Utopia*. Basic Books, New York, 1974.
- [8] John Rawls. *A Theory of Justice*. Belknap Press of Harvard University Press, Cambridge, MA, revised edition, 1999.
- [9] Lawrence B. Solum. Procedural justice. *Southern California Law Review*, 78:181–321, 2004.
- [10] United Nations General Assembly. Basic principles and guidelines on the right to a remedy and reparation for victims of gross violations of international human rights law and serious violations of international humanitarian law. Resolution 60/147, UN Doc. A/RES/60/147, 2005. Adopted 16 December 2005.
- [11] Margaret Urban Walker. *Moral Repair: Reconstructing Moral Relations after Wrongdoing*. Cambridge University Press, Cambridge and New York, 2006.