

Social Spacetime Topology and Generative Grammars in CW Complexes

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Abstract

This paper develops a CW-complex extension of an event–coupling–primitive 2-complex model of social spacetime. The extension is motivated by structural preservation rather than by representational insufficiency. An enriched 2-complex can encode substantial higher-order information through labels, internal manifolds, strings, amplitudes, operators, auxiliary complexes, or other attached data. Such encodings can therefore remain adequate for many social-scientific and physics-inspired purposes. The present inquiry addresses a different question: whether reduction to a two-dimensional carrier preserves the topology of how higher-order social structures are constituted, related, refined, and recursively generated. The paper treats vertices as events, edges as typed relations or couplings among analytically selected primitives, and faces as primitives relative to a specified research question and resolution. Primitiveness is consequently methodological rather than an assertion of ultimate indivisibility. Higher CW cells are introduced to retain higher incidence, compositional closure, generative genealogy, and multiscale organization as native structural features. A separate generative grammar is then developed to characterize admissible transformations of cellular structure; the CW complex records generated structure, whereas the grammar specifies conditions under which structural generation can occur. This separation permits comparison among informational equivalence, structural equivalence, generative equivalence, and historical equivalence across alternative representations. It also provides a basis for studying higher-order obstructions, refinement processes, scale-dependent topology, and possible fractal or multifractal signatures when suitable metric, measure, or scaling structures are supplied. The framework preserves the 2-complex as a useful lower-dimensional model while identifying conditions under which explicit CW structure yields additional analytical transparency.

Keywords: social spacetime; CW complex; 2-complex; generative grammar; higher-order topology; cellular reduction; multiscale structure; Generative Relational theory

Discussion Paper Note

This paper examines the topology of generated social structure within a Generative Relational approach to social spacetime. Its immediate point of departure is an event–coupling–primitive 2-complex architecture in which vertices represent events, edges represent typed relations or couplings among primitives, and faces represent primitives selected relative to an analytical problem. That architecture remains useful for social-spacetime representation and can be enriched with amplitudes, spins, operators, strings, internal manifolds, and other forms of additional information.

The present inquiry therefore does not treat higher-dimensional CW structure as a remedy for a simple shortage of representational capacity. Its concern is the organization of represented information. A reduction can retain enough data to reconstruct a richer structure while making the incidence, attachment history, higher-order closure, and multiscale organization of that structure external to the topology of the carrier. The paper studies the analytical consequences of retaining those relations as explicit cellular structure.

The CW complex is used as a representation of generated structure. A separate generative grammar is developed to describe admissible structural transformations. This separation permits the paper to ask how a structure was generated, which alternative structures could have been generated under specified conditions, how the grammar itself may vary historically, and which features remain visible or invariant under changes of representation and analytical resolution.

The framework maintains ontological restraint. Cellular, topological, grammatical, geometric, and spacetime language is introduced as formal representation within a research programme. The paper does not infer from the usefulness of these representations that social reality is ultimately constituted by CW complexes or by any particular mathematical object.

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Notices

Status. This working draft develops a formal research programme concerning social-spacetime topology, higher-order cellular organization, and generative grammars. Definitions, formal constructions, section structure, examples, and claims remain subject to revision. The present version contains the completed Introduction and Sections 2–18, including the 2-complex background, the analysis of structural information under dimensional reduction, the CW architecture for social spacetime, the interpretation of higher cells in social composition, the cellular generative grammar, the formal treatment of generative histories and cellular filtrations, the topological invariants of generated social structure, the multiscale topology of generated structure with criteria for fractal and multifractal interpretation, the formal analysis of reduction to enriched 2-complexes, a taxonomy of structural equivalence across representations, a layered dynamics for cellular carriers, cell-associated data, realization laws, and reflexive grammar evolution, an application-level formalization of institutional, normative, social-fact, and knowledge-structure formation, a formal treatment of grammar conflict, partial translation, structural mediation, mediated extension, and reflexive grammar revision, a three-level account separating ontological suspension, relational reality, manifestation reality, and formal observation, an explicit limitation analysis covering cellular nonuniqueness, primitive selection, empirical identifiability, semantic ambiguity, noncellular structure, grammar uncertainty, computational scalability, and the limits of topological explanation, a research programme covering empirical reconstruction, grammar inference, representation benchmarks, multiscale diagnostics, dynamical integration, rewrite foundations, computational infrastructure, and comparative domain studies, and a conclusion consolidating the comparative case for explicit CW structure.

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Statement on the use of language models. The exploratory discussions and preparation of this paper involved OpenAI’s ChatGPT. ChatGPT supported exploratory dialogue, formal reconstruction, argumentative criticism, source planning, and drafting in $\text{\LaTeX}$. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, and bears sole responsibility for the manuscript, including its definitions, formal constructions, arguments, conclusions, and errors. Authorship credit remains with the human author.

1 Introduction

The present section establishes the analytical problem and contribution of a CW-complex extension to a previously developed event–coupling–primitive representation of social spacetime. Its role is to distinguish representational capacity from structural preservation before the paper introduces any higher-dimensional construction. The discussion proceeds conceptually and formally at a programmatic level. It first identifies the social-spacetime object under study, then locates the established 2-complex architecture within the broader framework, specifies the problem created by structurally flattening higher-order organization into enriched two-dimensional data, and states the scope of the proposed CW and grammatical analysis.

1.1 Social-Spacetime Structure as a Generative Object

A social-spacetime model can represent more than a collection of social entities at a given moment. Relations among events, historically accumulated couplings, persistence of relational beings, formation of institutions, stabilization of norms, and changes in the conditions of subsequent formation introduce a generative dimension into social structure. A present configuration can therefore be studied both as an extant structure and as the outcome of a history through which lower-order components were combined, constrained, stabilized, revised, or dissolved.

Generative Relational analysis treats this historical organization as analytically significant. A social object may depend upon relations extending across events and times rather than upon an isolated bearer with intrinsic properties. The structure through which an object becomes constituted can also affect the structures that subsequently become feasible. Institution formation, legal recognition, scientific organization, linguistic stabilization, political ordering, and other social processes may therefore involve histories in which generated structure modifies later conditions of generation.

A formal representation of such processes benefits from distinguishing at least three analytical objects. The first is the set of components treated as primitive for a specified inquiry. The second is the structure generated from those components and their relations. The third is the generative rule system that constrains admissible structural transformations. The present paper develops these objects separately because a generated structure does not, by itself, identify the process or grammar through which it arose, and a grammar does not determine a unique realized history without an initial configuration and a sequence of contingent applications.

The term *primitive* requires particular care. A primitive in the present framework is primitive relative to an analytical question, a chosen resolution, and the interfaces whose internal organization the inquiry elects to resolve. The designation therefore carries no claim of ultimate indivisibility. A corporation, person, legal act, symbolic object, computational system, or other relational being may be treated as a primitive in one model while being represented by an extended internal structure in another. The formalism consequently permits refinement without converting a methodological boundary into an ontological boundary.

This question-relative treatment of primitives also affects the interpretation of scale. A coarse model may preserve the interactions relevant to one research question while suppressing internal organization that matters to another. A finer model can resolve the previously suppressed structure, and repeated refinement can reveal patterns that recur or transform across scales. The topology of the resulting family of representations may therefore contain information about structural generation that is difficult to express through a single fixed-resolution model.

1.2 Position of the Event–Coupling–Primitive 2-Complex

The CW extension developed in this paper retains the 2-complex as a central and often sufficient representation of social spacetime. In the established architecture, vertices represent events, edges represent typed relations or couplings among analytically selected primitives, and faces represent the primitives

themselves. Additional physical or social information can be associated with these cells through amplitudes, spins, operators, labels, internal manifolds, strings, or other mathematical structures appropriate to the local model. The resulting 2-complex can therefore carry considerably more information than its cellular dimension alone suggests.

This representational richness is central to the motivation of the present paper. The higher-dimensional CW construction does not arise from a shortage of information capacity in the 2-complex. With sufficiently expressive enrichment, a 2-complex can encode data from structures of arbitrarily high descriptive complexity. It remains especially appropriate for research on event structure, primitive couplings, amplitudes, or dynamics formulated naturally on a two-dimensional carrier.

The CW construction instead treats the 2-complex as a distinguished skeleton within a broader cellular representation. For a CW complex X , the relevant filtration is stated in Equation 1.

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} \subseteq \dots \subseteq X. \quad (1)$$

Within the present interpretation, $X^{(2)}$ retains the event–coupling–primitive social-spacetime architecture. Cells of dimension greater than two are available when the inquiry seeks to preserve higher-order incidence, attachment, or compositional structure explicitly. Their dimensions do not correspond to a fixed sociological hierarchy. An institution, state, market, legal order, scientific field, or other extended relational being may involve cells of several dimensions, and the appropriate decomposition depends upon the analytical question and resolution.

This treatment preserves continuity with the 2-complex programme. Models that gain no analytical advantage from explicit higher cells can remain at the level of $X^{(2)}$. Models concerned with the topology of higher-order constitution can retain additional cells. The choice is therefore methodological and problem-relative rather than a general ranking in which a higher-dimensional model is presumed superior.

1.3 Structural Preservation under Cellular Reduction

The central problem of the paper arises when a higher-order structure is reduced to an enriched 2-complex. Such a reduction may preserve enough information for reconstruction while changing the status of that information within the model. Higher incidence, attachment relations, compositional closure, order of constitution, or recursive refinement can become labels, strings, auxiliary manifolds, or externally stored instructions attached to cells of the reduced carrier. The information may remain available, yet its organizational topology is no longer represented natively by the cellular structure on which the analysis operates.

This distinction motivates a separation between *informational reducibility* and *structural preservation*. Informational reducibility asks whether a representation can encode sufficient data to recover another representation under an agreed decoding procedure. Structural preservation asks whether the relations of incidence, attachment, generation, and multiscale organization that matter to the inquiry remain explicit within the mathematical structure whose topology and transformations are being studied. A representation can satisfy the first condition while offering substantially less transparency under the second.

The difference becomes consequential when the analytical target includes properties defined through higher-order organization. A collection of primitives can exhibit pairwise couplings without acquiring a higher-order compositional closure. Several composite structures can share boundaries, overlap, or participate in a further structure whose properties depend upon the pattern of their incidence. A generated structure can also retain information about a sequence of attachments or refinements that is cumbersome to recover from a flattened encoding. These relations are representable in an enriched 2-complex, but they may cease to be immediate objects of cellular topology.

Multiscale structure strengthens the same problem. If repeated refinement produces recurrent attachment motifs or scale-dependent patterns, reduction to a fixed two-dimensional carrier can store the refinement instructions while obscuring the topology of their recurrence across organizational levels. A CW representation does not by itself establish fractality, because fractal or multifractal claims require

suitable notions of scale, metric, measure, or refinement. It can nevertheless preserve the higher-order cellular architecture on which such scaling questions can be posed without first reconstructing the architecture from auxiliary metadata.

The resulting motivation is therefore narrower and stronger than a claim of greater representational capacity. The paper investigates circumstances in which a higher-dimensional cellular representation makes structural information mathematically native that a reduced 2-complex can only encode indirectly. The relevant gain concerns analytical access to the topology of generation, not the sheer amount of information that can be stored.

1.4 Analytical Contribution, Scope, and Paper Architecture

The paper develops four connected contributions. First, it formulates a CW extension in which the event–coupling–primitive 2-complex remains the social-spacetime skeleton while higher cells record higher-order incidence and compositional structure when such structure is relevant to the research question. The extension preserves the earlier model as a legitimate truncation and therefore treats dimensional enrichment as optional rather than mandatory.

Second, the paper makes primitive selection an explicit component of model construction. Faces represent primitives relative to a research question and analytical resolution. Refinement can replace a face with a more detailed cellular structure when its internal organization becomes relevant. This move allows comparisons across resolutions while avoiding the inference that a chosen primitive is ontologically elementary.

Third, the paper distinguishes generated structure from generative grammar. The CW complex records a realized cellular organization. A separate grammar specifies which cellular transformations are admissible under defined typing, interface, contextual, and historical conditions. This separation supports analysis of structurally similar configurations generated through different histories, different grammars acting on similar configurations, and grammars whose own admissibility conditions change through social development. The formal design of this grammar requires more than a direct transfer of string-production rules because the objects being transformed possess topology, interfaces, embeddings, and potentially higher-dimensional boundaries.

Fourth, the paper develops criteria for comparing CW representations with enriched 2-complex reductions. The comparison distinguishes informational equivalence from structural, generative, historical, and multiscale equivalence. It thereby creates a basis for identifying cases in which dimensional reduction is analytically harmless and cases in which explicit higher-cell structure preserves properties that are central to the inquiry. Homological and homotopical features, obstructions to filling, persistent structures under refinement, and possible scaling signatures become candidate objects of subsequent analysis, subject to the assumptions required by each mathematical construction.

The paper also considers the expressive status of the proposed grammar. The classical Chomsky hierarchy provides a useful comparison for string encodings, while a cellular grammar acts natively on structured objects rather than on linear strings. The analysis therefore separates string-language expressiveness, computational expressiveness, and structural-generative expressiveness. A grammar can be structurally richer than a string grammar without thereby claiming computation beyond recursively enumerable languages. This distinction permits comparison with conventional formal-language classifications while preserving the higher-dimensional character of the modeled objects.

The formalism remains representational. Cellular dimension is not identified with a literal social or physical dimension, higher cells are not assigned universal sociological meanings, and the usefulness of a CW model does not establish CW complexes as the ultimate ontology of social reality. Empirical application will require explicit operationalization of primitives, couplings, higher cells, admissibility conditions, resolution changes, and observable consequences. The paper accordingly treats the proposed architecture as a formal research framework whose adequacy should be assessed relative to particular social-scientific questions.

The argument proceeds by first reconstructing the event–coupling–primitive 2-complex as the background model and then isolating the structural information that can become opaque under reduction.

The CW architecture is introduced only after that problem has been stated. Subsequent sections analyze higher-cell composition, formulate the generative grammar, study generative histories and topological invariants, examine multiscale and possible fractal structure, compare explicit CW models with enriched 2-complex encodings, and consider applications and limits. This order is intended to keep the central justification visible throughout the paper: the CW extension earns its role when the topology of structural generation is itself part of the research object.

2 Social Spacetime in the 2-Complex Formalism

This section reconstructs the 2-complex formalism that serves as the background model for the later CW extension. Its objective is to specify the roles of vertices, edges, and faces; clarify the question-relative meaning of a primitive; and state the range of additional information that can already be carried by an enriched 2-complex. The discussion is formal and methodological. It establishes what the existing model can represent before the subsequent analysis examines a different issue: which aspects of higher-order organization can become structurally opaque when such information is compressed into a two-dimensional carrier.

2.1 Event–Coupling–Primitive Architecture

For a research question Q examined at an analytical resolution σ , let $K_{Q,\sigma}$ denote a two-dimensional cellular complex. Its skeleton is given by Equation 2.

$$K_{Q,\sigma}^{(0)} \subseteq K_{Q,\sigma}^{(1)} \subseteq K_{Q,\sigma}^{(2)} = K_{Q,\sigma}. \quad (2)$$

The explicit dependence on Q and σ records that the cellular decomposition belongs to a model constructed for a particular analytical purpose. The notation does not imply that changes in research question or resolution alter the underlying social world; they alter the distinctions made explicit by the representation.

Vertices in $K_{Q,\sigma}^{(0)}$ represent events. An event is treated as a localized occurrence at which the modeled relational configuration has analytical significance. Depending on the application, events may include utterances, exchanges, recognitions, decisions, enactments, appointments, publications, transfers, encounters, or other occurrences specified by the empirical design. Event identity therefore requires an operational criterion in an applied model. The vertex interpretation does not equate persons, institutions, states, norms, or other persistent social beings with points.

Faces in $K_{Q,\sigma}^{(2)}$ represent primitives selected for the inquiry. A primitive is a unit whose internal decomposition is left unresolved at the chosen analytical resolution while its externally relevant relations can be represented through the interfaces and auxiliary data of the model. The term therefore identifies a role within the representation rather than an ultimate element of social reality. A primitive may be internally complex, historically extended, or itself describable by another cellular, manifold, algebraic, or process model.

Edges in $K_{Q,\sigma}^{(1)}$ represent typed relational couplings or interfaces through which incident primitives are connected within the modeled social spacetime. This interpretation uses the incidence structure of the 2-complex: primitive faces meet along edges, while edges terminate or interact at event vertices. An edge need not denote a persistent social relation as a whole. It records the coupling role assigned to the corresponding one-dimensional interface within the chosen decomposition. Causal, communicative, legal, economic, institutional, symbolic, material, or other coupling types can be distinguished when the research question requires them.

The three cellular roles can therefore be stated as a modeling convention: events are localized by vertices, primitive units are represented by faces, and couplings among incident primitives are represented through edges. This convention supplies the base architecture used throughout the paper. Its usefulness depends on whether the selected decomposition permits the empirical or theoretical problem to be expressed without forcing analytically important internal structure into an unresolved primitive.

2.2 Question-Relative Primitives and Analytical Resolution

The status of a face as a primitive requires an explicit relation to the research question. Let $P_{Q,\sigma}$ denote the set of units treated as primitive for Q at resolution σ . A face f represents a member of $P_{Q,\sigma}$ when two modeling conditions are satisfied: the internal decomposition of the represented unit is not required for the target inference at resolution σ , and the interactions relevant to Q can be exposed adequately through the face's interfaces and attached data. These conditions make primitiveness a methodological judgment subject to revision.

A unit may consequently occupy different representational roles across inquiries. A university can be treated as a primitive in a model of interorganizational exchange if the internal departments, appointments, funding channels, and decision procedures are outside the target question. The same university can be resolved into a large internal complex in a study of institutional governance. Likewise, a legal norm, market convention, scientific concept, household, administrative agency, or computational agent can be represented as a primitive at one resolution and decomposed at another.

Refinement can be represented by replacing the unresolved description of a primitive with a structure that exposes distinctions previously suppressed. Let $f \in P_{Q,\sigma}$ be represented by a face at resolution σ , and choose a finer resolution σ' that resolves internal distinctions suppressed at σ . Let $R_{\sigma \rightarrow \sigma'}(f)$ denote the resulting refinement. The refinement relation is stated in Equation 3.

$$R_{\sigma \rightarrow \sigma'}(f) = K_f. \quad (3)$$

Here K_f is a structured representation of the unit previously treated as primitive. Equation 3 is a modeling relation rather than a claim that refinement discovers an ontologically final decomposition. Further refinement can remain possible whenever the inquiry supplies a reason to expose additional structure.

This question-relative treatment also prevents cellular dimension from being read as a fixed hierarchy of social importance. A face selected as primitive can represent an analytically large or small entity; its status follows from resolution, interface adequacy, and the target question. Higher-dimensional cells introduced later in the paper therefore concern explicit higher-order structural organization, while changes in primitive selection concern the resolution at which the base social-spacetime model is constructed. The two operations can interact, but they answer different modeling questions.

2.3 Typed Couplings and Primitive Interfaces

A useful 2-complex requires more than bare incidence. The model must distinguish the types and admissible interpretations of the couplings carried by edges. Let $\mathcal{T}_1$ be a set of coupling types chosen for the inquiry and let the edge-typing map be the function defined in Equation 4.

$$\tau_1 : K_{Q,\sigma}^{(1)} \longrightarrow \mathcal{T}_1. \quad (4)$$

Equation 4 permits edges with similar cellular incidence to play different analytical roles. A legal dependence, material transfer, information channel, normative authorization, market exchange, and interpersonal communication can therefore remain distinguishable even when their local combinatorial arrangement is similar.

Typing can also be extended to faces and vertices when applications require distinctions among primitive classes or event classes. Such typing should remain subordinate to the research question. A model gains little from introducing a large vocabulary of types that never affects interpretation, admissibility, dynamics, or empirical observation. Conversely, a coupling distinction that changes possible interactions or outcomes should remain visible rather than being absorbed into an undifferentiated edge class.

The interface interpretation is especially important for composite primitives. A face can possess a rich internal description while exposing only selected boundary behavior to the surrounding 2-complex. This arrangement resembles an effective description: the model retains the degrees of freedom relevant to interactions across the chosen interface and suppresses internal distinctions whose effects are negligible or represented by aggregated parameters. Such suppression remains revisable when the target question changes.

The edge therefore carries a relational role that differs from the internal content of either incident face. An edge may mediate compatibility, exchange, constraint, transmission, or coupling between primitives whose internal structures are represented differently. The same primitive can participate in several edge types, and the same edge type can connect primitives belonging to different face types when the model's admissibility conditions permit such incidence.

2.4 Amplitudes, Representation Labels, and Internal Face Data

The 2-complex architecture can carry information substantially richer than its bare cellular structure. The present framework allows amplitudes, representation labels such as spins where a spin-foam-like construction is intended, operator-valued data, local state spaces, internal manifolds, path or string data, probability assignments, empirical attributes, and other structures whose analytical role is explicitly defined. The admissible enrichment depends on the application and need not be uniform across all faces or edges.

For bookkeeping, let $D(c)$ denote the additional data associated with a cell c of the 2-complex. The structured 2-complex used in an application can then be represented by Equation 5.

$$\mathcal{K}_{Q,\sigma} = (K_{Q,\sigma}, \tau, D), \quad (5)$$

where τ collects the typing information required by the model and D collects defined auxiliary structures. Equation 5 deliberately leaves the content of D open. A specific model must state which data are present, which cells carry them, how they transform, and what empirical or theoretical interpretation they receive.

A face treated as primitive can therefore contain an internal manifold or another structured state space without ceasing to be a primitive at the level of $K_{Q,\sigma}$. The internal structure is represented as data associated with the face because the global model does not currently resolve that structure through its own cellular incidence. Similarly, string-like information can encode histories, internal configurations, boundary states, or compositional instructions. Operators acting on such data can represent transformations while the carrier remains a 2-complex.

This flexibility explains why the later CW extension cannot be justified by raw information capacity. A sufficiently enriched 2-complex can encode a wide range of higher-dimensional or internally complex structures. The analytical question concerns the status of that encoded information. Data stored inside $D(f)$ can describe a higher-order incidence pattern without making that pattern part of the incidence structure of $K_{Q,\sigma}$ itself. The subsequent analysis of structural reduction develops the consequences of this distinction.

2.5 Relational Beings as Distributed 2-Complex Structures

Persistent social beings require a representation distinct from the assignment of one social object to one cell dimension. A person, institution, state, currency, scientific field, legal order, or other history-bearing relational being can be represented through a distributed structure involving many events, couplings, and primitives. Depending on the model, such a being may correspond to a subcomplex, a temporally indexed family of subcomplexes, a stability class over changing configurations, or an observable defined on portions of the complex.

This distributed treatment preserves historical constitution. A university, for example, can persist through changes in personnel, buildings, regulations, and internal organization because its modeled identity may depend on a structured continuity among many events and couplings rather than on one immutable cell. Likewise, a person represented socially through interactions, records, roles, commitments, and embodied events need not be collapsed into a vertex merely because vertices represent events in the underlying complex.

The distinction between cells and relational beings also protects the model from a fixed sociological reading of dimension. Cellular dimension indicates a structural role in the representation. Social identity, persistence, institutional unity, and historical continuity are properties that can emerge from or be defined

over distributed configurations. Their adequate representation may require additional state variables, equivalence criteria, temporal indexing, or later higher-cell structure, depending on the question being investigated.

Within the three-level discipline used by Generative Relational analysis, such constructions belong to the formal representation of relational reality rather than to an assertion that social beings are literally cellular complexes in an ultimate ontological sense. The model tracks historically constituted and causally or generatively relevant structure while leaving ultimate ontology suspended. This restraint remains in force when the paper later introduces higher CW cells.

2.6 Representational Scope of the Enriched 2-Complex

The enriched 2-complex is sufficient for many questions concerning social spacetime. It can represent events and their incidence, typed couplings among selected primitives, complex internal states associated with primitives, amplitudes or transition weights, operator dynamics, and histories encoded in auxiliary data. Where the research objective concerns these objects directly, an explicit higher-dimensional cellular extension may add complexity without improving the analysis.

The boundary of the 2-complex approach arises from a different consideration. Suppose an inquiry treats the organization of several primitive configurations as an analytical object in its own right, and suppose the incidence, attachment, filling, refinement, or higher-order topology of those configurations affects the target explanation. The required information can still be encoded in D or in auxiliary constructions. The resulting representation, however, places the higher-order organization in metadata relative to the carrier on which cellular topology is computed.

This observation yields the background premise for the remainder of the paper. The 2-complex is retained as a strong and often sufficient social-spacetime model. The subsequent CW construction is motivated only where the topology of organization above the selected primitive level is itself part of the research object. The subsequent analysis therefore examines what can change analytically when higher-order generative structure is reduced to, or reconstructed from, an enriched 2-complex.

3 Structural Information under Dimensional Reduction

This section isolates the formal problem that motivates the later CW construction. Its objective is to distinguish three questions that are easily conflated when an enriched 2-complex can encode higher-order information: whether the information can be stored, whether a richer structure can be reconstructed from that storage, and whether the relevant organization remains part of the topology of the carrier on which analysis is performed. The discussion introduces a reduction formalism, gives a concrete example in which identical 2-skeleta support topologically different higher-dimensional extensions, and identifies the classes of structural information for which explicit higher cells can matter. The section remains agnostic about the grammar that generates those cells; a later section on cellular generative grammar will treat generation separately from the generated CW structure.

3.1 Dimensional Reduction as an Encoding Problem

Let X denote a candidate higher-dimensional cellular representation associated with a research question $\mathcal{Q}$ and analytical resolution σ . The 2-skeleton $X^{(2)}$ retains the event–coupling–primitive architecture developed in Section 2. A reduction to an enriched 2-complex can preserve the 2-skeleton as the carrier while transferring information carried by cells of dimension greater than two into auxiliary data.

For the present section, let $\mathfrak{R}_2$ denote such a reduction procedure. Its output is defined in Equation 6.

$$\mathfrak{R}_2(X) = (X^{(2)}, \tau_2, D_X), \quad (6)$$

where τ_2 contains the typing retained on the 2-skeleton and D_X contains whatever auxiliary information the reduction procedure chooses to store about higher cells, attaching maps, internal structures, histories,

or other features of X . The notation permits D_X to be arbitrarily rich. Equation 6 therefore does not presuppose that dimensional reduction is lossy in an information-theoretic sense.

A reduction is *exactly reconstructive* on a class $\mathcal{C}$ of models when there exists a decoding procedure Δ_2 such that Equation 7 holds for each $X \in \mathcal{C}$.

$$\Delta_2(\mathfrak{R}_2(X)) \cong X. \quad (7)$$

The equivalence symbol in Equation 7 is intentionally left to be specified by the application. It may denote cellular isomorphism, homeomorphism, homotopy equivalence, equivalence with respect to a selected family of observables, or another relation justified by the research design. An exact reduction can therefore retain enough information to recover the source representation even though the carrier of the reduced model remains two-dimensional.

This formulation clarifies the problem addressed by the paper. The existence of Δ_2 settles a question of recoverability. It does not imply that properties of X have become properties of the carrier $X^{(2)}$. A higher-dimensional attaching map stored inside D_X can be decoded later, while homology, incidence, or cellular boundary calculations performed directly on the 2-skeleton continue to operate on $X^{(2)}$. The distinction concerns where structural information resides in the model and which operations expose it without an additional reconstruction step.

3.2 Recoverable Information and Carrier-Native Structure

The distinction between encoded and native structure can be stated relative to an analytical quantity. Let I be an invariant, structural statistic, or other well-defined quantity associated with models in $\mathcal{C}$. The reduction $\mathfrak{R}_2$ makes I *recoverable* when there exists a procedure $\hat{I}$ satisfying Equation 8.

$$I(X) = \hat{I}(\mathfrak{R}_2(X)) \quad \text{for all } X \in \mathcal{C}. \quad (8)$$

Recoverability permits $\hat{I}$ to inspect and decode D_X . It is therefore compatible with an enriched 2-complex carrying complete higher-dimensional information.

A stronger condition is useful when the topology of the carrier is itself the analytical object. Call I *carrier-native under reduction* when its value is determined by the retained 2-skeleton and the structures declared native to that carrier, without reconstructing higher cells from auxiliary data. If U_2 denotes the operation that forgets the auxiliary data and returns the carrier, then carrier-native preservation requires a suitable quantity I_2 satisfying Equation 9.

$$I(X) = I_2(U_2(\mathfrak{R}_2(X))) = I_2(X^{(2)}). \quad (9)$$

The two conditions answer different questions. Equation 8 asks whether the reduced representation contains enough information to recover I . Equation 9 asks whether the same quantity remains visible as a property of the cellular carrier after reduction. A model can satisfy the first condition and fail the second.

Definition 3.1 (Structural transparency relative to an inquiry). Let $\mathcal{I}_Q$ be the family of structural quantities used to answer research question Q . A representation is structurally transparent relative to Q to the extent that the members of $\mathcal{I}_Q$ are computable or characterizable from the structures treated as native in that representation, with no reconstruction of suppressed organization required.

Definition 3.1 makes transparency problem-relative. A reduction that obscures higher-cell incidence can remain fully adequate when $\mathcal{I}_Q$ concerns only event incidence, primitive couplings, or face-level amplitudes. The same reduction becomes analytically inconvenient when $\mathcal{I}_Q$ contains quantities defined by higher attaching maps, filling relations, or multiscale cellular organization. Structural transparency therefore supplies a criterion for representation choice without assigning intrinsic superiority to either dimensional form.

3.3 Identical 2-Skeleta and Distinct Higher-Dimensional Topology

A simple cellular construction demonstrates that the 2-skeleton alone does not determine the topology of a higher-dimensional extension. Consider the common 2-skeleton $K = S^2$. Construct two spaces by attaching a single 3-cell e^3 along different maps from its boundary sphere. Let $\alpha_{\text{id}} : S^2 \rightarrow S^2$ be the identity map and let $\alpha_0 : S^2 \rightarrow S^2$ be a constant map. The two extensions are defined in Equation 10.

$$X_{\text{id}} = S^2 \cup_{\alpha_{\text{id}}} e^3, \quad X_0 = S^2 \cup_{\alpha_0} e^3. \quad (10)$$

Both spaces have the same 2-skeleton. Their higher-dimensional topologies differ because the attaching maps perform different structural work. Attaching the 3-cell by the identity fills the sphere and produces a contractible space homeomorphic to a 3-ball. Attaching the same cell at a single point leaves the original 2-sphere and contributes a 3-sphere summand up to the corresponding quotient description, giving a space homotopy equivalent to $S^2 \vee S^3$.

The homological distinction is stated in Equation 11.

$$\tilde{H}_n(X_{\text{id}}; \mathbb{Z}) = 0 \quad \text{for all } n, \quad \tilde{H}_n(X_0; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n = 2, 3, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Equation 11 establishes the formal point needed here: identical 2-skeleta can participate in higher-dimensional extensions with different topological invariants. The difference lies in the attaching structure, not in the common carrier S^2 .

An enriched reduction can record whether the suppressed attaching map was α_{id} or α_0 . If D_X stores that information and Δ_2 reconstructs the corresponding 3-cell, the reduction can be exactly reconstructive in the sense of Equation 7. The carrier topology nevertheless remains the topology of S^2 in both reduced models. Recovering the difference in Equation 11 requires consultation of the encoded higher structure or reconstruction of the original extension.

Claim 3.2 (Carrier underdetermination by the 2-skeleton). Within a class of CW extensions admitting cells of dimension greater than two, the 2-skeleton does not in general determine the topology of the full extension. Consequently, a reduction that retains the 2-skeleton as carrier can preserve higher-dimensional topology only through additional data or restrictions on the admissible extension class.

Claim 3.2 does not establish that an enriched 2-complex is inadequate. It identifies the location of the additional burden: the reduction scheme must encode enough attachment information to distinguish extensions that the 2-skeleton itself cannot distinguish. When those attachment relations are central to the research question, retaining them as cells and attaching maps can make the target structure directly available to topological analysis.

3.4 Higher Incidence, Filling, and Compositional Closure

The previous example concerns one 3-cell, yet the same issue extends to large systems of higher cells. A cellular model contains more than a list of cell attributes. It specifies which cells occur in the boundaries or closures of other cells, how attaching maps identify boundary points with existing structure, and which closed configurations receive higher-dimensional fillers. These relations create an incidence architecture across dimensions.

For a regular CW complex, one convenient representation of this architecture is the cellular incidence relation. If c and d are cells, write $c \preceq d$ when c lies in the closure of d . The resulting partially ordered structure records which lower-dimensional cells participate in the boundary organization of higher-dimensional ones. The present paper does not require regularity for every later construction, but regular CW complexes provide a transparent case in which incidence can be handled combinatorially without ambiguity about cell closures.

Suppose a collection of primitive faces and their couplings forms a closed 2-dimensional configuration z . The existence of z does not by itself determine whether the configuration is filled by a 3-cell. If a 3-cell c has cellular boundary z , then Equation 12 holds in the relevant cellular chain representation.

$$\partial_3 c = z, \quad \partial_2 z = 0. \quad (12)$$

The second relation in Equation 12 follows from the chain condition $\partial_2 \partial_3 = 0$. Its analytical interpretation should remain model-specific. The formalism permits an inquiry to distinguish a closed primitive configuration that receives a higher-order filler from a combinatorially similar configuration that remains unfilled. Whether that distinction corresponds to institutional constitution, normative closure, coordinated organization, or another social process depends on the typing and empirical interpretation supplied by the application.

Flattening this distinction into a 2-complex can retain it as a label indicating that a particular 2-cycle is filled. The label is informationally serviceable. Explicit higher cells add a different affordance: filling relations become part of the boundary operators and incidence structure used by the topology itself. Questions about which cycles survive, which are killed by attachments, which structures share higher-dimensional boundaries, or how fillings interact can then be posed directly on the cellular object.

This distinction is especially relevant to collective composition. Pairwise couplings among primitives can be represented entirely within the 2-skeleton. A higher-order composite can require information about a whole boundary configuration that is not reducible to the existence of its individual pairwise interfaces. Explicit higher cells permit the model to record the constitution of such composites through attachment. The later analysis of higher cells in social composition will develop this interpretation; the present section establishes only the representational problem that motivates it.

3.5 Generative Genealogy beyond Static Topology

The topology of a completed CW complex does not by itself determine the historical order in which socially interpreted structures were generated. Two construction histories can yield isomorphic or homotopy-equivalent terminal complexes. Generative genealogy therefore requires additional structure beyond the final CW object, such as a filtration indexed by social or analytical time, a sequence of rewrite events, or another record of formation.

Let a generative history be represented provisionally by the sequence in Equation 13.

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{T-1}} X_T, \quad (13)$$

where each ϕ_t records a structural transformation whose admissibility will be specified later by the generative grammar. Equation 13 is deliberately broader than the skeletal filtration by cellular dimension. The index t represents an ordering relevant to generation, whereas cell dimension records structural level in a cellular decomposition. These orderings need not coincide.

A reduction can encode the entire sequence in Equation 13 within auxiliary data. Exact reconstruction may then remain possible. The resulting history, however, becomes external to the reduced carrier unless the reduction introduces a native structure that represents the sequence and its transformation interfaces. This observation parallels the distinction between a final network and the process that generated it: identical terminal carriers can be associated with different histories, and those histories can matter when later admissibility depends on path-dependent institutional, legal, normative, or informational conditions.

The present paper therefore separates three objects that will be formalized progressively: the selected primitive architecture, the generated CW structure, and the generative history governed by a grammar. A single CW complex represents the first two only at a chosen stage. The later analysis of generative histories will add an explicit history object. The current section's point is narrower: encoding a history as face data or strings can preserve its informational content while leaving its order and structural transformations outside the carrier topology being analyzed.

3.6 Multiscale Organization and Refinement Structure

Question-relative primitiveness makes changes of resolution part of the modeling problem. Section 2.2 represented refinement of one primitive by a structured model. A multiscale inquiry may require many such refinements and relations among them. Let Σ be an ordered family of analytical resolutions. A multiscale representation can then involve a family of complexes $\{X_\sigma\}_{\sigma \in \Sigma}$ together with refinement or coarse-graining maps between selected pairs of resolutions.

For a sequence of increasingly resolved models, the basic structure is written in Equation 14.

$$X_{\sigma_0} \xrightarrow{\rho_{01}} X_{\sigma_1} \xrightarrow{\rho_{12}} X_{\sigma_2} \xrightarrow{\rho_{23}} \dots \quad (14)$$

The maps ρ_{ij} must be defined according to the application; they need not be inclusions, and different refinement procedures can preserve different observables. Equation 14 therefore describes a family of representations together with inter-resolution structure rather than one canonical hierarchy supplied automatically by CW theory.

This qualification matters for fractal and multifractal analysis. A high-dimensional CW complex is not thereby fractal, and a sequence such as Equation 14 does not by itself establish a scaling law. Fractal interpretation requires a defensible notion of scale and usually additional metric, measure, counting, or refinement data. The potential advantage of retaining explicit higher-order and inter-resolution structure is methodological: recurrence of attachment motifs, changes in incidence statistics, persistence of holes, or scaling of cell counts can be studied across the family without first unpacking every organizational level from metadata attached to a fixed 2-complex.

A flattened representation can still encode the entire family $\{X_\sigma\}$, including every refinement map, inside auxiliary data. The distinction remains the same as in previous subsections. Information can be recoverable while multiscale organization is absent from the native carrier. This paper therefore treats possible fractal signatures as properties of a suitably enriched system of scales and transformations, not as consequences of cellular dimension alone.

3.7 Analytical Criteria for Retaining Higher Structure

The preceding analysis supports a conditional criterion for extending the 2-complex. Explicit higher-dimensional cellular structure is useful when the research question depends on relations that are naturally defined through higher attachments, incidence, fillings, or inter-level organization and when keeping those relations native materially simplifies or clarifies the analysis. An enriched 2-complex remains preferable when the target quantities are already native to the 2-skeleton or when higher-order information functions only as auxiliary state data.

The choice can be organized around four tests. First, the *recoverability test* asks whether the reduced representation retains enough information for the target quantities to be reconstructed. Second, the *native-structure test* asks whether those quantities are properties of the reduced carrier or require decoding of auxiliary data. Third, the *transformation test* asks whether the operations central to the inquiry act naturally on the reduced representation or require repeated reconstruction of suppressed higher structure. Fourth, the *multiscale test* asks whether relations across resolutions or organizational levels are themselves analytical objects.

These tests do not yield a universal preference for CW complexes. A 2-complex with rich face and edge data can satisfy an inquiry efficiently and preserve all information that matters. The case for a CW extension becomes stronger when the same inquiry repeatedly reconstructs higher incidence, attachment, or refinement relations in order to compute its central quantities. At that point, the allegedly auxiliary information is performing structural work that can be represented directly in the cellular architecture.

The problem established in this section can therefore be summarized as a distinction among storage, reconstruction, and structural visibility. Dimensional reduction can be informationally exact while changing which relations are native to the model. The next section introduces a CW architecture designed to retain selected higher-order relations as explicit cellular structure while preserving the event-coupling-primitive 2-skeleton as its base social-spacetime layer.

4 CW Architecture for Social Spacetime

Section 3 identified a representational problem: a reduction to an enriched 2-complex can retain enough information to reconstruct higher-order organization while relocating that organization from the topology of the carrier into auxiliary data. The present section introduces the cellular architecture used to retain selected higher-order relations as native structure. Its objective is constructive rather than generative. It specifies the relation between the established event–coupling–primitive 2-complex and a CW extension, defines the attaching data carried by higher cells, distinguishes cellular dimension from social or ontological rank, and states a conservative criterion for how far the extension should proceed. The rules that make particular attachments socially admissible are intentionally deferred to Section 6; the CW complex introduced here records a generated structure, while the later grammar will govern its generation.

4.1 The Social-Spacetime 2-Skeleton

Fix a research question Q and analytical resolution σ . Let $K_{Q,\sigma}$ be the enriched 2-complex introduced in Section 2, with vertices interpreted as events, edges as typed couplings or interfaces among selected primitives, and faces as the primitives treated as unresolved at the chosen resolution. The present construction retains this object as the base social-spacetime layer. A CW extension therefore begins with a cellular identification between $K_{Q,\sigma}$ and the 2-skeleton of a larger complex.

Definition 4.1 (Social CW extension). A *social CW extension* of $K_{Q,\sigma}$ is a pair $(X_{Q,\sigma}, \iota_{Q,\sigma})$ in which $X_{Q,\sigma}$ is a CW complex and $\iota_{Q,\sigma}$ is a cellular isomorphism satisfying Equation 15.

$$\iota_{Q,\sigma} : K_{Q,\sigma} \xrightarrow{\cong} X_{Q,\sigma}^{(2)}. \quad (15)$$

Definition 4.1 makes the relation to the earlier framework explicit. The CW construction does not replace the 2-complex. It fixes the 2-complex, up to the specified cellular identification, as the 2-skeleton on which additional cells may be attached. Any amplitude, representation label, operator, internal state space, or other data already assigned to $K_{Q,\sigma}$ can therefore remain part of the base model. The extension adds explicit higher cellular organization only where the research question requires it.

Figure 1 summarizes the architecture. The horizontal sequence is a skeletal filtration. Each stage retains the previously constructed cells and adds a family of cells of one higher dimension. The figure represents cellular construction order, not chronological social time; historical generation will be treated separately in Section 7.

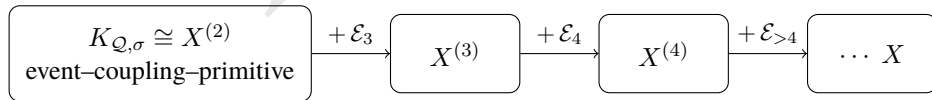


Figure 1: CW extension of the established social-spacetime 2-complex. The 2-skeleton is retained as the event–coupling–primitive layer, while the successive cell families $\mathcal{E}_n$ make selected higher attachment and incidence relations native to the cellular carrier. The filtration is structural and should not be identified with chronological history.

4.2 Inductive Attachment beyond the Primitive Layer

The standard CW construction forms an n -skeleton by attaching n -dimensional disks to the preceding skeleton through maps from their boundary spheres [1]. For the present model, let Λ_n index the n -cells introduced at dimension $n \geq 3$. For each $\lambda \in \Lambda_n$, let D_λ^n be a closed n -disk with boundary S_λ^{n-1} and let the attaching map be the map defined in Equation 16.

$$\alpha_{n,\lambda} : S_\lambda^{n-1} \longrightarrow X_{Q,\sigma}^{(n-1)}. \quad (16)$$

The next skeleton is obtained by the quotient construction in Equation 17.

$$X_{Q,\sigma}^{(n)} = \left(X_{Q,\sigma}^{(n-1)} \sqcup \prod_{\lambda \in \Lambda_n} D_\lambda^n \right) / \sim_\alpha, \quad (17)$$

where each boundary point $x \in S_\lambda^{n-1}$ is identified with $\alpha_{n,\lambda}(x)$. Equation 17 supplies the structural operation required by the paper: a higher cell is present in the carrier only when its disk and attaching map are included in the CW construction. Encoding the same attaching map inside face metadata can remain informationally equivalent, but it places the map outside the cellular incidence on which the native topology is computed.

The attaching map carries more information than the set-theoretic image of the boundary. Define the attaching trace of an n -cell e_λ^n by Equation 18.

$$\text{Tr}(e_\lambda^n) = \alpha_{n,\lambda}(S_\lambda^{n-1}) \subseteq X_{Q,\sigma}^{(n-1)}. \quad (18)$$

Equation 18 records where the boundary lands, while the full map $\alpha_{n,\lambda}$ records how it lands. Distinct attaching maps can have the same or similar images while differing by winding, identification, orientation, degree, or other homotopically relevant behavior. The CW architecture therefore treats the attaching map, rather than only the list of lower-dimensional cells touched by the attachment, as part of the structural record.

The construction in Equation 17 can stop at any finite dimension justified by the inquiry or continue through an unbounded sequence of finite-dimensional stages. Nothing in the social interpretation requires an a priori maximal dimension. Equally, nothing requires dimensions above two when the target quantities remain structurally transparent on the 2-skeleton.

4.3 Higher Incidence as Carrier-Native Structure

The principal analytical gain of Definition 4.1 is the relocation of selected higher relations from auxiliary descriptions into the cellular carrier. Once an n -cell is attached, the complex records its relationship to lower skeleta through the characteristic and attaching maps. Higher incidence can then enter homological, homotopical, combinatorial, or other topological constructions without a preliminary decoding operation of the kind represented by Δ_2 in Section 3.1.

For general CW complexes, the attaching map need not embed its boundary sphere, and the closure of a cell need not behave like a combinatorially transparent closed ball with an embedded boundary. It is therefore useful to distinguish the general construction from an optional regular specialization. Let $\chi_{n,\lambda} : D_\lambda^n \rightarrow X$ denote the characteristic map of a cell. A social CW extension is called *regular* in the present paper when each $\chi_{n,\lambda}$ restricts to a homeomorphism from D_λ^n onto the closure of e_λ^n . Under this additional assumption, cell closures and their boundaries admit a particularly transparent incidence interpretation.

Regularity is a modeling choice rather than a default substantive claim. A nonregular attaching map may be useful when the represented structure contains identifications, repeated boundary traversal, quotient-like organization, or another configuration for which an embedded boundary is too restrictive. Conversely, applications centered on explicit compositional parts, shared boundaries, or combinatorial incidence may benefit from a regular model because the cell structure is easier to inspect and compare. The chosen regularity class should therefore follow the research question and the intended structural quantities.

This distinction also disciplines later social interpretations. A higher cell should not automatically be described as “filling a collection of faces” unless the attaching structure actually supports that description. In a regular complex the language of a boundary assembled from lower cells can often be literal. In a general CW complex the boundary sphere can map into the lower skeleton with identifications, so the appropriate object is the attaching map itself. Section 5 will develop social interpretations only after this formal distinction has been retained.

4.4 Question-Relative Primitives and Cellular Refinement

The CW extension inherits the methodological status of primitiveness established in Section 2.2. Let $\mathcal{E}_2(K_{\mathcal{Q},\sigma})$ denote the set of open 2-cells in the base 2-complex. For the inquiry $(\mathcal{Q}, \sigma)$, the primitive family is defined in Equation 19.

$$\mathcal{P}_{\mathcal{Q},\sigma} = \mathcal{E}_2(K_{\mathcal{Q},\sigma}). \quad (19)$$

Equation 19 is a statement about the chosen cellularization. It does not imply that the represented objects lack internal structure. A primitive face can carry internal data, including a manifold, state space, string-like history, operator algebra, or another representation, while the global model continues to treat the face as one unresolved unit at resolution σ .

When a research question requires internal structure to become part of the carrier, the appropriate operation is a change of model or resolution rather than a reinterpretation of the existing face as ontologically elementary. A finer representation $K_{\mathcal{Q},\sigma'}$ may replace a coarse primitive by a subcomplex or a more detailed family of cells while preserving a specified external interface. A coarse-graining map can then be introduced when the application supplies a defensible relation between the two resolutions. Such a map is additional model structure; CW theory alone does not provide a canonical coarse-graining between arbitrary cellular decompositions.

The same qualification applies to the higher extension. Refining the primitive layer can alter which higher attachments remain appropriate, because a former boundary map into one coarse face may need to be lifted, replaced, or redistributed over a finer cellular structure. Consequently, a family of models indexed by resolution should record explicit inter-resolution maps when comparisons across σ are part of the inquiry. Section 9 will develop this issue as a multiscale problem. The present architecture requires only that the dependence on $\mathcal{Q}$ and σ remain visible from the beginning.

This treatment prevents a fixed mapping from social categories to cell dimensions. A person, institution, state, legal order, scientific field, or market can be represented as a face at one resolution, as a distributed subcomplex at another, or through observables on a larger cellular configuration. The model's primitive choice is justified by analytical adequacy, interface behavior, and resolution, not by a claim that a named social entity naturally belongs to one universal dimension.

4.5 Cell Types and Organizational Dimension

Cell dimension alone does not determine social meaning. The CW architecture therefore supplements dimension with explicit cell typing when the application requires distinctions among higher structural roles. Let $\mathcal{E}_n(X_{\mathcal{Q},\sigma})$ be the family of open n -cells and let $\mathcal{T}_n$ be a model-specific type set. The typing map is defined in Equation 20.

$$\tau_n : \mathcal{E}_n(X_{\mathcal{Q},\sigma}) \longrightarrow \mathcal{T}_n, \quad n \geq 3. \quad (20)$$

A type can distinguish different forms of composition, institutional relation, normative closure, legal integration, representational dependence, or another structural role specified by a concrete application. The formalism does not require those interpretations in the abstract theory; Equation 20 only provides a place for them when they matter.

The distinction between dimension and type is essential. Two 3-cells can have different social interpretations while sharing the same cellular dimension. Conversely, structures serving a similar substantive role can occupy different dimensions under different primitive selections or cellular decompositions. Cellular dimension should therefore be read as the position of a cell within the chosen attachment architecture. It indicates the dimensional level at which the cell is added relative to the existing skeleton. It does not, by itself, measure social importance, causal strength, institutional scale, complexity, or ontological depth.

This interpretation also separates CW dimension from physical spacetime dimension. The event-coupling-primitive 2-skeleton can remain the physics-inspired social-spacetime carrier used in earlier work. Cells of dimension greater than two add explicit organization of structures represented on that

carrier. Their dimension records cellular attachment order within the representation; it need not be interpreted as an additional physical coordinate or as evidence that social spacetime literally possesses the corresponding geometric dimension.

Optional cell-associated data can be retained at higher dimensions as well. If $D_n(e)$ denotes model-specific data carried by an n -cell, the application must state what those data represent and how they interact with the attaching structure. Higher cells therefore do not eliminate the enriched-data approach. The CW extension changes which relations are represented through native cellular attachment, while labels, amplitudes, operators, and internal state spaces remain available whenever they serve a defined analytical role.

4.6 Generated Structure and Generative Law

The construction so far specifies an architecture for represented structures, not a theory of which structures can arise. Given a completed or partially completed $X_{Q,\sigma}$, its cells and attaching maps record a particular structural configuration. They do not by themselves explain why one attaching map occurred, why another candidate attachment was excluded, which transformations were historically available, or how an existing structure changes the admissibility of later structures.

This separation can be expressed by distinguishing a realized cellular state from the rule system later used to generate such states. Let Γ denote, provisionally, the generative grammar to be developed in Section 6. The present section assigns no internal form to Γ . For bookkeeping, the structural carrier and generative rule system can be kept as distinct coordinates of the modeling object defined in Equation 21.

$$\mathfrak{M}_{Q,\sigma} = (X_{Q,\sigma}, \Gamma). \quad (21)$$

In Equation 21, the first coordinate records generated cellular structure and the second denotes the rule system governing admissible structural generation. Keeping the coordinates distinct prevents two different formal objects from being conflated. The existence of an n -cell records that the modeled structure contains the corresponding attachment. A grammar must additionally specify the conditions under which that attachment, deletion, refinement, replacement, identification, or other structural transformation is admissible. The grammar may depend on types, interfaces, context, history, and the existing CW state; those dependencies require a separate formal treatment.

The distinction also leaves open the possibility that the same terminal CW complex can be produced by different histories or grammars. Structural equivalence of generated outputs need not imply generative equivalence. Section 7 will make the history object explicit, and Section 11 will compare several notions of equivalence across representations.

4.7 Minimal Explicit Extension Relative to an Inquiry

The framework gains little from adding cellular dimensions that perform no analytical work. The representational criterion developed in Section 3.7 can therefore be expressed as a minimal-extension principle. Let $\mathcal{I}_Q$ denote the structural quantities required to answer Q , as in Definition 3.1. For a candidate CW model X , define the inquiry-relative explicit dimension by Equation 22 whenever the minimum exists.

$$d_{\mathcal{I}}(X) = \min \left\{ N \geq 2 : \text{each } I \in \mathcal{I}_Q \text{ is structurally transparent on } X^{(N)} \right\}. \quad (22)$$

Equation 22 does not define an intrinsic dimension of society. It defines the smallest skeleton of a specified representation on which the inquiry's target structural quantities are native in the sense established in Section 3.2. If the required quantities are already native to $X^{(2)}$, then $d_{\mathcal{I}}(X) = 2$ and the enriched 2-complex remains the appropriate representation. If a target quantity depends essentially on explicit 3-cell attachment, the minimum can rise to three, and similarly for higher stages.

Remark 4.2 (Inquiry-relative dimensionality). The value in Equation 22 depends on the chosen representation, primitive decomposition, target quantities, and meaning of structural transparency. It should

therefore be interpreted as a methodological property of a modeling problem, not as a universal scalar characteristic of a social system.

The minimal-extension principle preserves continuity with the earlier 2-complex work. Higher CW structure is introduced when it exposes relations that would otherwise require recurrent reconstruction from auxiliary data, and it is omitted when the 2-skeleton already supports the analysis. The resulting architecture is therefore layered but nonmaximalist: the event–coupling–primitive 2-complex remains the base model, while explicit higher cells are added only for structurally relevant organization.

The present section has specified the carrier on which the remainder of the paper operates. A social CW extension retains the established 2-skeleton, adds higher cells through explicit attaching maps, keeps primitive selection indexed by question and resolution, separates cell dimension from substantive social type, and distinguishes generated topology from the law of generation. Section 5 next examines what kinds of social composition can be represented by higher cells and which interpretive constraints are required before cellular attachment can be read as constitutive social organization.

5 Higher Cells in Social Composition

Section 4 introduced higher CW cells as carrier-native attachments above the event–coupling–primitive 2-skeleton. The present section specifies the conditions under which such attachments can be interpreted as social composition. Its objective is semantic and structural rather than generative: it distinguishes an assembly of lower-dimensional structures from a represented composite, defines the lower-dimensional support through which a higher cell is constituted, separates topological filling from substantive social closure, and describes overlap among composites. The discussion also identifies configurations that can be treated as candidate but unrealized compositions. The rule system deciding which candidate transformations are admissible remains deferred to Section 6.

5.1 Compositional Roles of Higher Cells

A higher cell has a topological role before it receives a social interpretation. For an n -cell $e^n \in \mathcal{E}_n(X_{Q,\sigma})$ with $n \geq 3$, its attaching map specifies how its boundary sphere enters the preceding skeleton. This attachment can support several substantive interpretations depending on the model: integration of primitive configurations, closure of a normative arrangement, coordination among already constituted structures, identification of a higher interface, or another higher-order relation. Cell dimension alone therefore does not determine whether a cell represents composition.

Let $\mathcal{T}_n$ be the type set introduced in Section 4.5. For each dimension $n \geq 3$, let $\mathcal{C}_n \subseteq \mathcal{T}_n$ denote the subset of types that a particular application designates as compositional. The classification in Equation 23 defines when a higher cell is treated as composition-bearing.

$$e^n \text{ is composition-bearing} \iff \tau_n(e^n) \in \mathcal{C}_n. \quad (23)$$

Equation 23 separates cellular dimension from social semantics. A 3-cell with a noncompositional type can record a higher relation without asserting that its boundary structures form a composite. Conversely, a composition-bearing 3-cell records an additional unity attributed to the attached configuration by the model. The same discipline applies at higher dimensions.

The sets $\mathcal{C}_n$ are model-specific. Their elements require operational or theoretical interpretation in an application and should not be populated by analogy alone. An institutional study might use compositional types for legally constituted organizational units; a study of scientific practice might use them for stabilized knowledge formations; a political model might use them for integrated governance arrangements. These examples illustrate possible roles and do not establish a universal correspondence between a social category and a cell dimension.

This typed treatment also preserves the possibility that a higher cell represents a relation whose importance is structural without being compositional. A higher-dimensional attachment can encode identification, obstruction resolution, compatibility, equivalence, constraint, or another organized relation

among lower structures. The CW architecture supplies a common carrier for these possibilities, while typing determines which interpretation is licensed in a particular model.

5.2 Attaching Support and Primitive Footprints

A compositional interpretation requires a precise account of the lower-dimensional structure involved in the attachment. The attaching trace in Equation 18 identifies the image of the boundary sphere, but the image can intersect lower cells in ways that are difficult to compare directly across models. It is therefore useful to associate each higher cell with a carrier support in the preceding skeleton.

Definition 5.1 (Attaching support). For a higher cell e^n with attaching trace $\text{Tr}(e^n) \subseteq X_{Q,\sigma}^{(n-1)}$, the *attaching support* $\text{Supp}(e^n)$ is the smallest subcomplex of $X_{Q,\sigma}^{(n-1)}$ containing $\text{Tr}(e^n)$, when this support is taken relative to the chosen CW decomposition.

Definition 5.1 records the cellular carrier required to contain the attachment. It can include cells that are lower-dimensional than $n-1$, especially when the attaching map traverses a nonregular configuration or when the smallest containing subcomplex must include cell closures required by the CW structure. The definition therefore avoids identifying a higher cell solely with a list of immediately adjacent $(n-1)$ -cells.

Because the event-coupling-primitive layer is fixed at the 2-skeleton, every higher attachment can also be projected to the primitive resolution through its intersection with that skeleton. The primitive footprint is defined in Equation 24.

$$\text{Foot}_2(e^n) = \text{Supp}(e^n) \cap X_{Q,\sigma}^{(2)}. \quad (24)$$

Equation 24 identifies the part of the base social-spacetime carrier involved in the higher attachment. It can contain primitive faces, their coupling edges, and event vertices. The footprint should not be interpreted as a complete description of the higher cell: two higher cells can share a primitive footprint while differing in their higher-dimensional support, attaching maps, types, or associated data.

The distinction between support and footprint becomes important above dimension three. A 3-cell attaches directly to the 2-skeleton, so its support is already contained in the primitive layer. A 4-cell can attach to a configuration containing 3-cells as well as lower cells; its primitive footprint records the base primitives implicated in the attachment, while its full support retains the intermediate compositional organization. The same pattern continues recursively.

This construction also accommodates changes in analytical resolution. If a primitive face in $\text{Foot}_2(e^n)$ is refined at a finer resolution, the corresponding attaching support may require reconstruction because the original map into a coarse face need not determine a unique map into the refined subcomplex. Inter-resolution comparison therefore depends on explicit refinement maps, as anticipated in Section 4.4. Higher composition and primitive refinement are related operations but should remain formally distinct.

5.3 Aggregation, Composition, and Constituted Unity

The principal social use of a composition-bearing higher cell is to distinguish a lower-dimensional assembly from a configuration to which the model assigns an additional constituted unity. Consider a CW complex Y containing the relevant lower-dimensional structure and an attaching map $\alpha : S^{n-1} \rightarrow Y$. Two models can share the same lower carrier while differing by the presence of an attached n -cell. The paired construction is stated in Equation 25.

$$X^- = Y, \quad X^+ = Y \cup_\alpha D^n. \quad (25)$$

If the added cell in X^+ has a type in $\mathcal{C}_n$, Equation 25 expresses a controlled representational contrast. The lower configuration is present in both models. The second model additionally records a compositional relation through the higher attachment. The distinction does not assert that the first configuration is socially insignificant or that the second is ontologically more real. It states that the second representation contains a specified higher-order unity as carrier-native structure.

Table 1: Carrier-level conditions for interpreting higher cellular structure. A higher cell receives a compositional interpretation only when its type and the application-specific semantics support that role.

Carrier configuration	Higher-cell condition	Permitted interpretation
Lower-dimensional assembly	No relevant higher cell specified	Coupled or organized lower structure; no additional compositional unity is represented by the carrier
Higher attachment	Cell type outside $\mathcal{C}_n$	Higher-order relation is represented, while composition remains unspecified
Composition-bearing attachment	Cell type in $\mathcal{C}_n$	The model represents an additional constituted unity associated with the attaching support
Candidate boundary without realized attachment	Candidate status supplied independently	Unrealized or unresolved composition can be studied, subject to a later admissibility criterion

This contrast is especially useful when pairwise or local coupling does not settle collective constitution. Several primitive faces can meet through typed edges, and those couplings can be dense or mutually compatible, without the model thereby asserting that the primitives form one higher composite. The presence of a designated compositional cell supplies an additional structural fact. The difference can be important for questions concerning collective agency, institutional formation, legal constitution, integrated normative orders, scientific formations, or coordinated infrastructures, provided the corresponding type receives an independently justified interpretation.

A stylized example makes the distinction explicit. Let f_A , f_B , and f_C be primitive faces whose interfaces are represented in a 2-skeleton Y . The incidence and edge typing in Y can represent all pairwise couplings relevant to the inquiry. A model $X^- = Y$ therefore contains the coupled primitive configuration. A second model can attach a composition-bearing 3-cell c^3 whose attaching map has support containing the three primitive faces and their relevant interfaces. The additional cell records a collective constitution that is not inferred merely from the existence of the pairwise couplings.

The example should be read formally rather than sociologically. The three faces need not represent three persons, and the 3-cell need not represent an organization. Faces are primitives selected relative to $\mathcal{Q}$ and σ , while the type of c^3 supplies the substantive interpretation. The same construction can therefore be reused across domains without fixing a universal vocabulary of social objects.

Table 1 summarizes the interpretive statuses produced by this distinction.

The table also clarifies why the CW extension provides more than a relabeling of social objects. A composition-bearing cell records a relation between a higher unit and the lower structure through which it is constituted. The model can therefore retain both the composite and its constituent support. This makes it possible to study changes in the composite together with changes in the structure of constitution, rather than assigning a single label to a preexisting collection of primitives.

5.4 Topological Filling and Substantive Closure

The language of filling provides a useful topological description of some compositional attachments, particularly in regular CW models, but its social interpretation requires a separate criterion. Suppose an oriented cellular model admits a cellular chain representation and let $c \in C_n(X)$ be an n -cell whose cellular boundary is the chain $z \in C_{n-1}(X)$. The relation is stated in Equation 26.

$$\partial_n c = z, \quad \partial_{n-1} z = 0. \quad (26)$$

The second relation follows from the cellular chain condition. Equation 26 therefore identifies z as a closed lower-dimensional chain that is realized as the cellular boundary of c in the model. When c is

composition-bearing, the attachment can be interpreted as a represented closure of the corresponding lower structure, provided the application justifies that semantics.

Two notions of closure should remain distinct. *Carrier closure* refers to the formal fact that a lower-dimensional configuration is filled or bounded by an explicit higher cell in the chosen CW representation. *Substantive closure* refers to an application-specific claim that a social process has acquired a relevant form of integration, constitution, settlement, or completion. Carrier closure is determined by the model's cellular structure. Substantive closure depends on typing, operationalization, evidence, and the research question.

This distinction prevents a homological metaphor from becoming an automatic sociological diagnosis. An unfilled cycle is not intrinsically an institutional failure, normative contradiction, social deficit, or governance problem. It is an unfilled structure relative to a specified cellular representation. A substantive interpretation becomes warranted only when the modeled domain supplies a reason to associate the relevant filling relation with the social phenomenon under study.

The converse caution also matters. A substantively constituted social arrangement need not be represented by filling one cycle with one cell. Its adequate representation can require several higher cells, nonregular attachments, a distributed subcomplex, or additional dynamical data. The formalism therefore treats filling as one compositional mechanism rather than a universal definition of social unity.

Within these limits, explicit filling has a useful analytical consequence. Attaching an n -cell can alter homological or homotopical structure because the new cell changes which cycles or maps remain nontrivial in the completed complex. The exact effect depends on the attaching map and the surrounding complex. This makes higher attachment suitable for inquiries in which the constitution of a composite changes the topology available to subsequent analysis. A label on a 2-complex can encode the same event, while an explicit cell makes the resulting boundary relation part of the native cellular algebra.

5.5 Shared Supports and Overlapping Composites

Social composites frequently share constituents, interfaces, histories, or lower-order structures. The CW model can represent such overlap without requiring the composites themselves to be collapsed into one cell. Let c and d be higher cells with attaching supports $\text{Supp}(c)$ and $\text{Supp}(d)$. Their overlap at skeletal level k is defined in Equation 27.

$$\mathcal{O}_k(c, d) = \text{Supp}(c) \cap \text{Supp}(d) \cap X_{Q, \sigma}^{(k)}, \quad 0 \leq k \leq \min\{\dim c - 1, \dim d - 1\}. \quad (27)$$

The family $\{\mathcal{O}_k(c, d)\}$ is an overlap profile across cellular dimensions. At $k = 2$, it records shared primitive-level support. At higher k , where defined, it can record shared intermediate composites or higher interfaces. The profile therefore distinguishes two higher cells that involve the same primitive faces but differ in which intermediate structures they share.

Overlap need not imply identity or compatibility. Two composition-bearing cells can share substantial primitive support while carrying different types, attaching maps, or associated data. They may represent nested, intersecting, competing, partially coordinated, or otherwise related composites depending on the application. The topology records the shared carrier; the substantive relation between the composites requires additional typing or grammatical conditions.

The overlap profile also permits a more precise treatment of membership. A primitive face can belong to the footprint of several higher composites without being duplicated in the base model. Likewise, an intermediate composition-bearing cell can participate in the support of a still higher attachment. This feature is useful for social structures in which one constituent participates simultaneously in several institutions, normative orders, markets, knowledge systems, or governance arrangements. The formalism records multiple participation as shared cellular support rather than forcing exclusive hierarchical nesting.

Nested organization can nevertheless be represented when it is appropriate. If the support of a higher cell d contains a composition-bearing cell c , then d can be interpreted as being constituted partly through the composite recorded by c , provided the cell types and attaching structure justify that reading. The nesting is then explicit in incidence rather than inferred from a naming convention. Partial overlap and nesting can coexist within the same complex.

5.6 Recursive Composition across Cellular Dimensions

The first higher-dimensional extension above the primitive layer occurs at dimension three because a 3-cell attaches to the 2-skeleton. A composition-bearing 3-cell can therefore record a constituted relation whose support is expressed entirely through events, couplings, and primitive faces. This gives dimension three a specific architectural role relative to the fixed base model, while leaving its social type open.

Dimensions above three permit explicit composition in which previously generated higher structures can themselves enter the attaching support. A 4-cell may have a support containing composition-bearing 3-cells; a 5-cell may involve 4-cells and lower structures; and the construction can continue when the inquiry requires further levels. This recursive organization is the feature that a fixed 2-skeleton can encode informationally but does not display through its own incidence relations.

Recursive composition should not be inferred from dimension alone. A 4-cell whose attaching map lands entirely inside the 2-skeleton does not thereby encode composition of 3-dimensional composites. Likewise, a high-dimensional cell can have a noncompositional type. The appropriate interpretation depends jointly on the attaching support and the typing introduced in Equation 23.

This observation supplies a disciplined reading of organizational depth. The model can distinguish the cellular dimension at which a relation is attached from the number or kind of compositional stages represented in its support. A high-dimensional cell can be structurally shallow with respect to a particular composition scheme, while a lower-dimensional family of many cells can encode a rich distributed organization. Cellular dimension is therefore one coordinate of structural organization rather than a scalar measure of social complexity.

The recursive architecture becomes especially relevant when the generation of one composite changes the structural environment in which later composites are formed. Once a composition-bearing cell is present, it can appear in the support of later attachments, participate in overlap profiles, alter topological invariants, or carry associated state data. The completed CW structure therefore retains a genealogy of constitutive dependence at the level of incidence, even though chronological generative history still requires the additional history structure developed in Section 7.

5.7 Candidate Attachments and Incomplete Constitution

A model of social formation often needs to distinguish structures that are realized from structures that are available, attempted, expected, proposed, or otherwise treated as candidates. The CW complex by itself records realized attachments. The absence of an n -cell therefore has no determinate meaning unless the model independently specifies that a corresponding attachment was a relevant candidate.

For the present section, let a *candidate attachment* be provisionally represented by attaching data (α, κ) , where $\alpha : S^{n-1} \rightarrow X^{(n-1)}$ is a proposed attaching map and $\kappa \in \mathcal{T}_n$ is a proposed type. Candidate status is supplied externally at this stage. Section 6 will replace this provisional designation with explicit grammatical conditions for admissibility, while Section 7 will distinguish admissibility from historical realization.

Given a specified candidate attachment, the model can distinguish a realized case, in which an n -cell of the relevant type is attached through the corresponding boundary data, from an unrealized case, in which the candidate remains without such a cell. The latter can represent incomplete constitution, failed integration, pending formation, prohibited attachment, or mere nonrealization only when the application supplies the corresponding interpretation. The formal distinction itself is neutral among these possibilities.

This neutrality is important because absence can arise for different reasons. A configuration may be structurally impossible under a grammar, historically possible but never realized, actively blocked, empirically unobserved, outside the chosen resolution, or simply irrelevant to the inquiry. Treating every unfilled configuration as a failed composite would conflate these cases. The grammar developed in Section 6 and the generative histories developed in Section 7 are required to separate them.

The candidate concept also shows why generated structure and generative law must remain distinct coordinates. The CW complex provides evidence about what the representation contains. It does not

enumerate every alternative structure that could have been generated. A theory of social possibility therefore requires a rule system over transformations, not only a richer terminal topology.

5.8 Interpretive Discipline for Social Composition

The preceding constructions support a conservative interpretation of higher cells. A higher cell contributes explicit attachment, incidence, and topological consequences to the carrier. A compositional type can additionally designate that attachment as a represented constituted unity. The substantive meaning of that unity remains dependent on the research question, primitive selection, operational definitions, and evidence.

Several restrictions follow. First, a named social object should not be assigned a fixed cell dimension across models. The same institution, person, legal order, or scientific field can be primitive at one resolution and distributed across many cells at another. Second, topological closure should not be identified automatically with normative, legal, institutional, or causal closure. Third, shared support records structural overlap without establishing cooperation, conflict, hierarchy, or equivalence. Fourth, absence of a higher cell becomes analytically meaningful only relative to a specified candidate or admissibility structure.

These restrictions preserve the central advantage of the CW representation while limiting overinterpretation. The model can make higher-order constitution native to the carrier without claiming that the carrier exhausts the represented social phenomenon. It can retain recursive composition without turning cellular dimension into an ontology of social ranks. It can also compare alternative compositional structures while keeping primitive choice and substantive typing explicit.

Section 5 therefore supplies the semantic bridge between the generated CW architecture and the generative theory developed next. The CW complex can record assemblies, composition-bearing attachments, shared supports, recursive organization, and realized or unrealized candidate structures. What it does not yet determine is which transformations are permissible, under what contextual conditions a candidate attachment is admissible, how history changes those conditions, or whether generated structures can modify the rule system itself. Those questions define the task of the generative grammar in Section 6.

6 Generative Grammars of Cellular Structure

Section 5 specified what higher cellular attachments can represent once they are present in a social CW complex. The present section introduces the separate rule system governing which cellular transformations are available and admissible. Its objective is to define a grammar that operates natively on typed CW structure while preserving the distinction among structural matching, topological constructibility, social admissibility, and historical realization. The construction adapts the interface-preserving logic familiar from algebraic graph transformation to the present cellular setting [2], while treating the CW-specific rewrite semantics as a proposal of this paper rather than assuming that general graph-transformation theorems transfer automatically to CW complexes. The section then defines the structural language generated by such a grammar, compares its expressive power with string-language classifications, and identifies local, contextual, historical, and reflexive forms of grammatical dependence.

6.1 Grammar as a Rule System over Typed CW States

Let $X_{Q,\sigma}$ be a social CW extension equipped with the cell typing introduced in Sections 2 and 4. For grammatical purposes, collect the type sets and typing maps used by the model into a type system $T_{Q,\sigma}$. This collection includes the edge types required by the 2-complex background, any event or primitive types used by the application, and the higher-cell maps τ_n of Equation 20. The type system records which distinctions can affect matching or admissibility; it does not require a type for every descriptive attribute carried by the model.

The grammar is defined independently from the CW state. Definition 6.1 gives the core object used throughout the remainder of the paper.

Definition 6.1 (Cellular generative grammar). A *cellular generative grammar* at question Q and resolution σ is a triple

$$\Gamma_{Q,\sigma} = (\mathbf{T}_{Q,\sigma}, \mathcal{R}, \mathbf{A}), \quad (28)$$

where $\mathbf{T}_{Q,\sigma}$ is the model's type system, $\mathcal{R}$ is a family of typed cellular rewrite rules, and $\mathbf{A} = \{A_r : r \in \mathcal{R}\}$ is a family of admissibility predicates associated with those rules.

Equation 28 separates three roles. The type system determines which cellular distinctions are available to the grammar. The rule family determines which structural changes have been specified. The admissibility predicates determine when a structurally specified rule can be used in a particular modeled situation. The CW complex itself remains the generated carrier. Consequently, the same $X_{Q,\sigma}$ can be paired with different grammars, and one grammar can act on several states sharing the type system required by its rules.

The grammar is indexed by Q and σ because primitive selection and the relevance of types depend on the inquiry. This indexing does not require every rule to change when the research question changes. It records that a rule whose left-hand pattern treats a face as primitive may require reformulation after that face is refined into a subcomplex. Section 9 will examine the compatibility of grammars across resolutions.

The term *grammar* is used here in a formal generative sense. It denotes a rule system that generates admissible structural transformations. Its native objects are typed cellular configurations rather than symbol strings. Classical string grammars remain relevant as a comparison for encoded versions of the generated structures, but they do not supply the primary syntax of Definition 6.1.

6.2 Interface-Preserving Cellular Rewrite Rules

A structural rule must specify what is recognized, what is preserved, and what is produced. For each $r \in \mathcal{R}$, let L_r , B_r , and R_r be finite typed CW patterns. The rule is represented by the span in Equation 29.

$$r : L_r \xleftarrow{\ell_r} B_r \xrightarrow{\rho_r} R_r. \quad (29)$$

The maps ℓ_r and ρ_r are type-preserving cellular embeddings in the core rule class developed here. The left pattern L_r is the configuration to be matched in the current state; B_r is the preserved interface; and R_r is the configuration supplied after the rewrite. The interface-span form makes explicit which cells survive the transformation and which cellular structure is removed or introduced. Algebraic graph transformation uses closely related left–interface–right rule architectures to control replacement while preserving an interface [2]. Equation 29 adopts that organizational principle without asserting that the category of CW complexes satisfies every categorical condition used in a particular graph-transformation framework.

A *match* of r in a current state X_t is a type-preserving cellular embedding $m : L_r \hookrightarrow X_t$. A match identifies a candidate occurrence of the left pattern. It does not yet establish that rewriting is topologically well defined or socially admissible.

The core replacement semantics is defined when the matched copy of L_r can be separated from the retained carrier through the preserved interface. Let D_m denote a residual CW complex satisfying the first relation in Equation 30. The transformed state is then constructed by the second relation.

$$X_t \cong D_m \cup_{B_r} L_r, \quad X_{t+1} \cong D_m \cup_{B_r} R_r. \quad (30)$$

The gluing in Equation 30 is understood through the interface maps transported by the match. A rule application belongs to the core semantics only when the required residual object and gluings exist in the chosen model class, produce a CW complex, preserve the fixed structures designated by the application,

and respect the required typing. These requirements make topological constructibility a condition of rule application rather than an automatic consequence of the symbolic span.

The construction covers several important transformations without assigning each one a separate primitive operator. A pure higher-cell attachment can take $B_r = L_r$ and let R_r contain the same interface together with the new attached cell. A controlled deletion can take $R_r = B_r$. A replacement can remove cells in $L_r \setminus B_r$ and introduce a different pattern in $R_r \setminus B_r$. Refinement can be represented when the coarse and refined patterns expose a common interface through B_r . More general identifications, quotient operations, noninjective boundary maps, or transformations outside this interface-preserving class require additional rewrite semantics and are not assumed by Definition 6.1.

The interface also prevents a common ambiguity in social rewriting models. A transformation need not rebuild the entire social spacetime. The rule declares which local structure is changed and which boundary with the surrounding carrier is preserved. The adequacy of that locality remains an application-specific assumption: a rule that omits socially relevant long-range dependence can be formally well defined while still being a poor model of the phenomenon.

6.3 Structural Matching and Grammatical Admissibility

The existence of a match in Equation 29 provides only a structural opportunity for transformation. Social generation can depend on conditions that are not recoverable from the matched subcomplex alone. The admissibility component **A** in Definition 6.1 supplies this additional layer.

For each rule r , let c_t denote the modeled contextual information relevant at stage t , and let H_t denote a history record available to the rule. Section 7 will specify the structure and equivalence of such histories; the present section uses H_t only as an argument on which a rule is permitted to depend. The admissibility predicate is defined in Equation 31.

$$A_r(m; X_t, c_t, H_t) \in \{0, 1\}. \quad (31)$$

A value of one means that the matched and topologically constructible rule is admissible under the modeled conditions. A value of zero means that the grammar does not authorize that application in the specified state, context, and history. Equation 31 is deliberately abstract: a concrete study must define the information entering c_t and H_t , the observable conditions used by A_r , and the empirical or theoretical basis for the predicate.

The admissibility predicate can incorporate several distinct requirements. A *cellular condition* can require that the replacement in Equation 30 is well formed in the chosen CW model class. A *typing condition* can require compatibility among the types of the matched cells, preserved interface, and generated cells. A *state-context condition* can depend on parts of X_t outside the matched copy of L_r or on externally represented circumstances in c_t . A *historical condition* can depend on the path by which the current state was reached. These conditions should be stated separately in applications when their empirical meanings differ, even if they are combined into the single Boolean output of Equation 31 for formal economy.

This distinction refines the candidate-attachment concept of Section 5.7. A pair (r, m) can be structurally matched while failing the CW construction. It can be topologically constructible while failing a typing, contextual, or historical condition. It can also be grammatically admissible while remaining historically unrealized. Table 2 summarizes these levels.

The four levels prevent possibility from being inferred directly from topology. A hole, boundary, or candidate higher attachment in X_t does not by itself imply that a corresponding cell can or should be generated. Topological structure supplies part of the domain on which the grammar acts; admissibility supplies a rule-governed restriction on the transformations supported by that domain.

6.4 Admissible Successors and Structural Generation

The grammar determines a set of possible successor structures rather than a unique future unless additional selection dynamics are specified. For a current state X_t , context c_t , and history record H_t , define

Table 2: Statuses of a proposed cellular transformation. The levels distinguish structural availability from topological constructibility, grammatical admissibility, and historical realization.

Status	Formal requirement	Interpretation within the model
Matched	A type-preserving match $m : L_r \hookrightarrow X_t$ exists	The left pattern occurs in the current carrier
Constructible	A residual carrier and valid CW gluing satisfy Equation 30	The specified structural replacement is topologically available in the core rule semantics
Admissible	$A_r(m; X_t, c_t, H_t) = 1$	The grammar authorizes the constructible transformation under the modeled conditions
Realized	The admissible application occurs in the represented generative history	The successor state is part of the historical trajectory rather than merely an available alternative

the admissible successor set by Equation 32.

$$\text{Succ}_\Gamma(X_t; c_t, H_t) = \left\{ X' \mid \begin{array}{l} \text{there exist } r \in \mathcal{R} \text{ and a match } m : L_r \hookrightarrow X_t, \\ \text{Equation 30 yields } X', \\ A_r(m; X_t, c_t, H_t) = 1 \end{array} \right\}. \quad (32)$$

Equation 32 is the paper’s basic formalization of a grammar of structural possibility. The set contains structures that can be generated in one admissible grammatical step from the represented state under the supplied conditions. It need not contain structures that are physically, legally, politically, or historically likely unless the corresponding conditions are encoded in the model. Probability, amplitude, utility, power, or selection mechanisms can be defined over the successor set when an application requires them, but they remain distinct from grammatical admissibility.

Repeated application generates a derivation relation. Write $X \Rightarrow_\Gamma X'$ when X' is an admissible one-step successor under the contextual and historical arguments supplied for that step. Its finite transitive-reflexive closure is written $\Rightarrow_\Gamma^*$. The notation records generative reachability; it does not identify different derivations that happen to terminate at isomorphic carriers.

The distinction is important because the grammar supplies a space of alternatives while history selects a trajectory within that space. A current state can have several admissible successors, one admissible successor, or none under a particular grammar and context. Conversely, the same terminal CW complex can be reached through distinct sequences of rules. Section 7 will retain those sequences explicitly rather than collapsing them into the terminal topology.

The successor construction also clarifies the relation between grammar and topology. The topology of X_t constrains which matches and gluings are available. Rule application changes the topology or cellular decomposition when R_r differs structurally from L_r . The resulting X_{t+1} then becomes the carrier on which subsequent rules are matched. Structural generation is therefore recursive: generated topology changes the domain of later grammatical application even when the grammar itself remains fixed.

6.5 Structural Languages of Reachable CW Complexes

A generative grammar can be associated with a language once the generated objects and the relevant equivalence are specified. Classical formal-language theory commonly treats a grammar as a device generating strings; Chomsky’s hierarchy classifies important families of such grammars by restrictions on their productions and the resulting string languages [3]. The cellular grammar has a different native output. Its generated objects are typed CW states and their derivations.

Fix an initial state X_0 and, for the moment, a specified family of contextual and historical inputs. Let $\cong_T$ denote typed cellular isomorphism. The structural language generated from X_0 is defined in Equation 33.

$$\mathcal{L}_{\text{str}}(\Gamma; X_0) = \{[X]_{\cong_T} : X_0 \Rightarrow_{\Gamma}^* X\}. \quad (33)$$

Equation 33 treats isomorphic typed carriers as the same generated structure. Other equivalence relations can be substituted when an application regards homotopy-equivalent, observationally equivalent, or historically distinguished states differently. Section 11 will examine such alternatives. The present definition uses typed cellular isomorphism because it preserves the native incidence structure whose transparency motivates the CW representation.

The structural language does not require a terminal/nonterminal distinction. Social structures can remain available for further transformation after they are generated. If an application requires accepted or terminal configurations, it can specify a distinguished subset of states and restrict Equation 33 accordingly. The absence of that distinction in the core formalism reflects the paper's interest in continuing generativity rather than a claim that terminal structures never occur.

Higher-dimensional approaches to rewriting and word problems already show that generative formalisms need not be confined to one-dimensional strings. Burroni, for example, introduced n -dimensional word problems with an explicitly geometrical motivation [4]. The present proposal is narrower in one respect and broader in another: it does not identify the social CW grammar with a specific established higher-dimensional rewriting calculus, while it permits the generated carrier itself, its question-relative primitives, contextual conditions, and historical dependence to enter the modeling object. A future categorical reconstruction may relate the grammar more tightly to polygraphic or other higher-dimensional rewriting frameworks.

6.6 String Encodings and Chomsky-Style Expressiveness

The question of where a cellular grammar lies in the Chomsky hierarchy is meaningful only after a string encoding and an effective semantics have been chosen. Let $\mathfrak{X}_{\text{fin}}$ be a class of finite typed CW states used by the application, let Σ be a finite alphabet, and let $E : \mathfrak{X}_{\text{fin}} \rightarrow \Sigma^*$ be an injective effective encoding. The induced string language is defined in Equation 34.

$$\mathcal{L}_E(\Gamma; X_0) = \{E(X) : [X]_{\cong_T} \in \mathcal{L}_{\text{str}}(\Gamma; X_0)\}. \quad (34)$$

A Chomsky-style classification applies to the language in Equation 34, not directly to the unencoded CW structures in Equation 33. Different encodings can make different structural features syntactically explicit, and restrictions on A_r can materially change the computational properties of the generated language. A statement that the native cellular grammar “belongs to Type 1” or “exceeds Type 0” would therefore be underspecified without these choices.

The distinction also prevents structural richness from being confused with computational power. Higher-dimensional attachment can make incidence, filling, and recursive composition native to the generated objects even when an effective string encoding of the same derivations falls within an ordinary class of computable languages. The structural representation can therefore be more transparent for the inquiry without being computationally stronger than a Turing-complete string formalism.

Proposition 6.2 (Effective encodings and recursive enumerability). *Assume that the modeled states are finite and effectively encodable, the rule family $\mathcal{R}$ is finite or recursively enumerable, matches and the admissibility predicates can be effectively enumerated, and every permitted rewrite construction is effective. Then the encoded reachable language $\mathcal{L}_E(\Gamma; X_0)$ is recursively enumerable.*

Proof. Under the stated assumptions, finite derivations can be enumerated by dovetailing over applicable rules, matches, and derivation lengths from X_0 . Whenever a derivation produces a state, its effective encoding can be emitted. Every reachable finite state appears after a finite derivation and is therefore eventually enumerated. The procedure need not decide nonreachability, so the conclusion is recursive enumerability rather than decidability. $\square$

Table 3: Dependency classes for cellular generative grammars. The classes organize what rule admissibility may depend on and should not be read as a direct analogue of the Chomsky hierarchy.

Grammar form	Additional dependency	Analytical role
Local	Matched pattern, interface, and rule-bounded neighborhood	Generation governed by locally represented cellular conditions
State-contextual	Ambient CW state and specified context c_t	Admissibility depends on structural or contextual conditions outside the matched pattern
History-sensitive	Generative history H_t	Current structural similarity need not imply identical generative possibilities
Reflexive	Current grammar Γ_t and a grammar-update operation	Generated events or structures can alter the rules governing later generation

Proposition 6.2 gives a conservative answer to the strongest version of the expressiveness question. Under ordinary effectivity assumptions, the cellular grammar does not acquire a string-language class beyond recursively enumerable languages merely because its native objects are higher-dimensional. Its distinctive expressive contribution lies in the structure retained by the native objects and in the dependencies admitted by the rule system. Noncomputable predicates could invalidate the proposition, but introducing them would amount to adding a noncomputable oracle or assumption to the model rather than deriving extra computational power from CW topology itself.

6.7 Dependencies of Cellular Grammars

The dependence structure of the admissibility predicates provides a more informative native classification for the present framework than a direct import of the Chomsky hierarchy. Four forms are especially relevant. They describe which information a rule is allowed to consult; they are not claimed to form a strict hierarchy of computational complexity.

A *local grammar* evaluates admissibility from the matched cellular pattern, its preserved interface, and a bounded neighborhood specified by the rule. Such a grammar is appropriate when the relevant generative conditions are local to the represented carrier. A *state-contextual grammar* can additionally inspect the ambient state X_t or specified contextual data c_t . It can therefore distinguish two occurrences of the same local pattern when they are embedded in different larger structures. A *history-sensitive grammar* can additionally depend on H_t , allowing structurally similar current states to support different transformations because they were generated through different histories. A *reflexive grammar* permits a realized transformation to modify the grammar that governs later transformations.

Table 3 summarizes these dependency classes.

The classes can overlap. A reflexive grammar can also be local in the conditions triggering an update, and a history-sensitive grammar can use only a limited summary of the past. Their purpose is to expose modeling commitments that would be hidden if all dependencies were absorbed into one unrestricted predicate. A concrete application should use the weakest dependency structure that supports the target explanation.

The distinction also sharpens the comparison with an enriched 2-complex. A 2-complex can encode the information consulted by any of these rule classes if sufficient auxiliary data are supplied. The CW grammar makes a different contribution: it defines transformations over a carrier whose higher incidence is itself explicit, allowing rule matching and admissibility to refer directly to higher structural organization rather than first decoding that organization from face or edge metadata.

6.8 Reflexive Change of the Grammar

Some social processes concern changes in the rules by which later structures can be constituted. The formalism can represent such cases by allowing the generative state to include the current grammar. Let the full generative state at stage t be the pair (X_t, Γ_t) . A reflexive rule can carry, in addition to its cellular rewrite, a grammar-update map U_r defined on the information required by the application. The coupled transition is stated in Equation 35.

$$(X_t, \Gamma_t) \xrightarrow{r, m} (X_{t+1}, \Gamma_{t+1}), \quad \Gamma_{t+1} = U_r(\Gamma_t, X_t, m, c_t, H_t). \quad (35)$$

For an ordinary nonreflexive rule, U_r is the identity on the grammar. For a reflexive rule, the update may add, remove, restrict, enable, disable, or modify rules or their admissibility conditions, provided the new grammar remains well defined under the application's formal requirements.

Equation 35 supplies a precise distinction between structural change and rule change. A new higher cell can alter the domain on which an unchanged grammar subsequently acts. A reflexive transition additionally changes the grammar itself. The distinction is useful for representing processes in which a generated legal, institutional, scientific, technical, or normative structure changes the conditions for later valid generation. The formalism does not assume that every such process is reflexive; the interpretation requires evidence that the relevant rule system has itself changed.

Reflexivity also introduces a potential source of open-endedness. A grammar that can alter its own rule family can generate new spaces of admissible successors over time. This feature should not be described as computational power beyond the Chomsky hierarchy without an explicit encoding and effectivity analysis. Under effective update rules, Proposition 6.2 continues to provide the appropriate computational caution for finite encoded states, provided the evolving grammars themselves are effectively representable and enumerable.

6.9 Scope of the Grammar Formalism

Definition 6.1 is intended as a minimal architecture for the paper rather than a claim of categorical completeness. It specifies typed CW states, interface-preserving rewrite patterns, admissibility predicates, reachable structural languages, and an optional reflexive update mechanism. This architecture is sufficient to distinguish the generated carrier from the law of generation and to make higher cellular structure available directly to the matching process.

Several extensions remain possible. Stochastic or amplitude-weighted grammars can assign weights to admissible rule applications. Concurrent semantics can formalize partially independent transformations. A categorical formulation could seek a class of CW-like objects and morphisms with pushout and complement properties adequate for a full double-pushout or related rewriting theory. Higher-dimensional rewriting could represent rules between rules or coherence among transformations, building on traditions in geometrical rewriting such as higher-dimensional word problems [4]. These extensions require additional assumptions and are therefore left outside the core grammar.

The present formalism also preserves the methodological status of primitives established earlier. A grammar acts on the primitives and higher structures supplied by the chosen representation. It does not determine which faces are ontologically primitive. Changing $\mathcal{Q}$ or σ can alter the primitive decomposition, the available type system, and the cellular patterns appearing in the rule family. Grammar comparison across such changes requires explicit translation or refinement maps rather than an assumption that the same production rule survives unchanged.

Section 6 therefore supplies the generative counterpart to the CW architecture. A generated CW state records higher structural organization natively; the grammar specifies typed interface-preserving transformations and the conditions under which they are admissible. The resulting structural language is a language of reachable CW configurations rather than primarily a language of strings. Chomsky-style classification becomes meaningful after effective encoding, while local, contextual, historical, and reflexive dependence provide a more direct description of the grammar's native structural expressiveness. Section

7 next makes generative history explicit, allowing two structurally similar states to remain distinguishable when their paths of formation differ.

7 Generative Histories and Cellular Filtrations

Section 6 defined a cellular grammar whose admissibility predicates may depend on a history record H_t . The present section specifies that record and separates three structures that are easily conflated: the cellular dimension filtration intrinsic to a CW complex, the chronological sequence through which a modeled structure is generated, and the additional memory retained by a history-sensitive or reflexive grammar. The objective is to make path dependence formally inspectable without assuming that every sequence of rewrites forms a filtration. The discussion first defines realized generative histories and endpoint equivalence, then identifies the special class of monotone histories that induce cellular filtrations, develops a two-parameter organization by generative stage and cellular dimension, and finally introduces history summaries sufficient for future admissibility.

7.1 Realized Derivations as Generative Histories

A grammar describes admissible structural transformations, whereas a history records transformations that are represented as having occurred. This distinction is needed because a state can admit several successor structures while only one, several, or none of those successors are realized in a particular modeled trajectory. It is also needed for reflexive grammars, where a realized transformation may modify the rule system governing later transformations.

Let the stage- i model state be $\mathfrak{M}_i = (X_i, \Gamma_i)$, where X_i is a typed social CW complex and Γ_i is the grammar effective at that stage. For a fixed grammar, $\Gamma_i = \Gamma$ for all i . A *realized rule event* is a tuple $\varepsilon_i = (r_i, m_i, c_{i-1})$ recording the applied rule, its match, and the modeled context used in evaluating admissibility. The rewrite semantics of Section 6 determine the structural transition from X_{i-1} to X_i , while a reflexive rule may additionally determine Γ_i from Γ_{i-1} .

Definition 7.1 (Generative history). A finite *generative history* of length t is a realized derivation

$$H_t = (\mathfrak{M}_0, \varepsilon_1, \mathfrak{M}_1, \dots, \varepsilon_t, \mathfrak{M}_t), \quad (36)$$

in which every ε_i is matched, topologically constructible, grammatically admissible under the relevant prefix history, and represented as historically realized.

The prefix ending at stage i is denoted $H_i \preceq H_t$. Equation 36 records more than the terminal complex. It retains the ordered rule events, the intermediate CW states, and, when relevant, changes in the grammar itself. The full record is intentionally generous. Later subsections identify conditions under which a smaller history summary is sufficient for the analytical purpose.

The context c_i is not included as a permanent coordinate of $\mathfrak{M}_i$ because contextual variables can have a different empirical status from the generated cellular carrier and grammar. A particular application may nevertheless internalize some contextual variables as cell data, external state variables, or additional components of $\mathfrak{M}_i$. The formal distinction is therefore organizational rather than ontological.

Definition 7.1 also avoids treating admissibility as realization. If $Y \in \text{Succ}_{\Gamma_i}(X_i; c_i, H_i)$ according to Equation 32, then Y is available under the model. It enters the realized history only when the modeled trajectory includes the corresponding rule event. Counterfactual or unrealized admissible successors remain outside H_t unless the application separately records them.

7.2 Endpoint Equivalence and Generative Path Dependence

Alternative histories can terminate in cellular structures that are indistinguishable at the resolution of the model while retaining different implications for subsequent generation. The relevant comparison therefore requires separate notions of endpoint equivalence and historical equivalence.

Let $\cong_{\mathbf{T}}$ denote typed cellular isomorphism for the type system used by the inquiry. Two histories H_t and H'_s are *structurally endpoint-equivalent* when their terminal carriers satisfy the relation in Equation 37.

$$H_t \equiv_X H'_s \iff X_t \cong_{\mathbf{T}} X'_s. \quad (37)$$

Equation 37 deliberately ignores the sequence of rules that produced the endpoints and, in reflexive systems, can also ignore differences between Γ_t and Γ'_s . It is therefore a weak equivalence notion suited to the question of whether two realized histories presently exhibit the same typed cellular structure.

A stronger comparison can be defined stagewise. Two histories are *derivationally equivalent* when there is an order-preserving correspondence between their realized rule events together with typed cellular isomorphisms between corresponding intermediate states that preserve the matched patterns, preserved interfaces, generated patterns, and rule types. Derivational equivalence permits renaming of cells and rule instances while preserving the organization of the derivation. It is consequently stronger than endpoint equivalence and weaker than literal identity of the history records.

For future behavior, let $\text{Fut}_1(H_t; c_t)$ denote the one-step admissible successor family generated from the terminal model state using the history record carried by H_t . The family is defined in Equation 38.

$$\text{Fut}_1(H_t; c_t) := \text{Succ}_{\Gamma_t}(X_t; c_t, H_t). \quad (38)$$

Two endpoint-equivalent histories are *one-step future-equivalent* when an endpoint isomorphism transports the admissible successor family of one history to that of the other, up to typed cellular isomorphism of the successor states. This definition treats differences caused either by explicit history dependence in the predicates or by reflexive changes in the terminal grammar as differences in modeled future possibility.

Definition 7.2 (Generative path dependence). A modeled process exhibits *generative path dependence* at the chosen question, resolution, and context when there exist structurally endpoint-equivalent histories $H_t \equiv_X H'_s$ that are not one-step future-equivalent.

Definition 7.2 isolates a specific form of path dependence: the path matters because it changes what the model permits next, even when the present typed cellular carrier is the same. This condition can arise through a history-sensitive admissibility predicate, through a grammar modified by earlier reflexive rules, or through both mechanisms. It does not assert that all social path dependence should be represented in this way.

The complementary case is also useful. A fixed grammar is *state-sufficient* for one-step generation when every admissibility predicate depends on the current typed state and current context but is invariant to the remaining contents of H_t . Under this condition, typed endpoint isomorphism and corresponding contexts determine the same one-step successor structure, provided the rule semantics themselves are invariant under typed cellular isomorphism. Historical records may still matter for interpretation, evidence, attribution, or longer-run empirical analysis, but they do not add further information to one-step grammatical admissibility.

7.3 Cellular Filtrations and General Rewrite Histories

CW complexes already possess an intrinsic filtration by cellular dimension. For a complex X , its skeletal form the nested sequence $X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \dots \subseteq X$ used throughout standard treatments of CW complexes [1]. This *skeletal filtration* is indexed by cell dimension. It should be distinguished from a chronological filtration indexed by generative stage.

A general history generated by the grammar of Section 6 need not be monotone. Replacement can remove cells in $L_r \setminus B_r$, controlled deletion can reduce the carrier, and refinement can replace one cellular decomposition with another without preserving the earlier complex as a literal subcomplex. The state sequence in Equation 39 therefore records a derivation without assuming inclusions.

$$X_0 \xRightarrow{\varepsilon_1} X_1 \xRightarrow{\varepsilon_2} \dots \xRightarrow{\varepsilon_t} X_t. \quad (39)$$

Calling Equation 39 a filtration would impose a property that many rewrite histories do not possess. The distinction becomes important when later sections use topological invariants across generated states: constructions requiring nested inclusions should be applied only when the modeled history supplies them or when an additional comparison map has been justified.

A useful special case is an attachment-monotone history.

Definition 7.3 (Attachment-monotone history). A generative history is *attachment-monotone* when each realized step admits a type-preserving cellular embedding $\iota_i : X_{i-1} \hookrightarrow X_i$ whose image is a subcomplex identified with the retained earlier state, and the embeddings compose consistently across stages.

After choosing the consistent identifications supplied by Definition 7.3, the history induces the chronological filtration in Equation 40.

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t. \quad (40)$$

Pure cell attachment is the clearest example. Some extensions that add labels or auxiliary structures while retaining all cells can also be represented monotonically, depending on what the carrier regards as part of the CW state. By contrast, deletion and replacement typically require the general history representation in Equation 39.

The distinction between Equations 39 and 40 is methodological. It preserves the ability to use filtration-based mathematics where the social model genuinely has cumulative attachment while avoiding the assumption that social generation is intrinsically cumulative. Institutional dissolution, legal repeal, reclassification, merger, fragmentation, and replacement can require nonmonotone rewrite histories even when some information about their predecessors is retained in a separate provenance record.

7.4 Cellular Dimension and Generative Stage as Independent Indices

An attachment-monotone history carries two independent orderings. The first is the cellular dimension ordering inside each X_i ; the second is the generative-stage ordering among the X_i . Keeping these indices separate avoids interpreting high cellular dimension as late historical formation or low cellular dimension as historical priority.

For stage i and cellular dimension n , define the two-parameter family in Equation 41.

$$F_{i,n} := X_i^{(n)}. \quad (41)$$

For an attachment-monotone history whose embeddings respect cell dimension, the family satisfies the inclusion relations in Equation 42.

$$F_{i,n} \subseteq F_{i,n+1}, \quad F_{i,n} \subseteq F_{i+1,n}. \quad (42)$$

Equations 41 and 42 provide a two-parameter cellular-history filtration. The horizontal direction can be read as increasing explicit cellular dimension at one generative stage; the vertical direction can be read as accumulation of generated structure at a fixed dimension. This organization makes it possible to ask whether a topological feature first appears because the model resolves a higher-dimensional attachment or because a later social-generative event adds structure at a dimension already represented.

The two indices should also remain distinct from analytical resolution σ . Refining the model can alter primitive selection and replace a face by a lower-level complex, as discussed in Section 4. Section 9 will treat scale and refinement explicitly. The present two-parameter construction therefore concerns cellular dimension and generative stage within a fixed question and resolution.

For nonmonotone histories, the family $X_i^{(n)}$ remains well defined at each stage, but the second inclusion in Equation 42 can fail. The result is a sequence of skeletal families rather than a two-parameter filtration. This failure is itself informative: it identifies where a later model state deletes, replaces, or reorganizes earlier cellular structure rather than merely extending it.

7.5 Cellular Lineage and Rule-Level Provenance

A terminal CW complex can retain higher-order incidence without retaining the chronological order in which its cells were introduced. Generative history therefore requires a separate account of lineage when the origin of current structure matters. The interface-preserving rewrite rule of Equation 29 already supplies a minimal basis for such a record because it distinguishes preserved cells from removed and newly generated cells.

At a realized rule event ε_i , cells represented through the matched image of B_{r_i} have counterparts before and after the rewrite and can be treated as preserved occurrences under the selected identification. Cells in the matched image of $L_{r_i} \setminus B_{r_i}$ terminate at that event, while cells introduced from $R_{r_i} \setminus B_{r_i}$ begin at that event. This rule-level distinction defines a conservative lineage relation without asserting that every newly generated cell is causally produced by every cell in the matched left pattern.

For an attachment-monotone history, the birth stage of a cell $a \in X_t$ is defined in Equation 43.

$$b_H(a) := \min\{i \in \{0, \dots, t\} : a \in X_i\}. \quad (43)$$

Equation 43 is well defined because cells persist once introduced. In a general rewrite history, a single permanent cell identity cannot be assumed across replacement. The appropriate object is then a lineage of cell occurrences connected through preserved interfaces. An occurrence can have a birth event, a termination event, and one or more represented successors only when the rule semantics explicitly provide the relevant correspondence.

Rule-level provenance can be recorded more coarsely by associating each generated cell occurrence with the realized rule event that introduced it and with the matched support of that event. This record answers a genealogical question that terminal incidence alone may not answer: which represented transformation introduced the structure now under examination? More detailed causal attribution requires additional modeling commitments and should not be inferred solely from rewrite membership.

The distinction between incidence and provenance is especially important for the paper's central argument. A CW representation preserves higher-order structure as native topology, but native topology does not by itself reconstruct the full history that generated it. Structural preservation and historical preservation are related yet distinct objectives. A model can therefore require both an explicit higher-dimensional carrier and a provenance or history record.

7.6 History Summaries and Admissibility-Sufficient Memory

The complete record in Equation 36 can be much larger than the information actually needed by a grammar. A history-sensitive model should therefore state which aspects of the past affect admissibility rather than appealing to unrestricted historical dependence whenever a smaller sufficient representation is available.

Let $\mathfrak{H}$ be the class of history records admitted by the model and let $\mathcal{M}_H$ be a chosen memory space. A *history summary* is a map $\eta : \mathfrak{H} \rightarrow \mathcal{M}_H$. The summary can record, for example, selected rule occurrences, cell-lineage information, counts, ordered milestones, provenance classes, or other theoretically justified historical variables.

Definition 7.4 (Admissibility-sufficient history summary). A history summary η is *admissibility-sufficient* for a grammar Γ when, for histories ending in the same typed state and evaluated in the same modeled context, equality of their summaries implies equality of every rule-admissibility judgment for corresponding matches. Formally, whenever $X_t \cong_{\mathbf{T}} X'_s$, the contexts correspond under the comparison, and $\eta(H_t) = \eta(H'_s)$, the relation in Equation 44 holds for every corresponding rule and match.

$$A_r(m; X_t, c_t, H_t) = A_r(m'; X'_s, c'_s, H'_s). \quad (44)$$

Definition 7.4 permits a history-sensitive grammar to be represented through a finite or otherwise compressed memory state when the theory supports such compression. A state-sufficient grammar is the limiting case in which the summary carries no information beyond the current state and context. A

grammar can also be sensitive to a bounded recent trace, to selected historical milestones, or to an accumulated memory variable. None of these forms should be presumed adequate universally; the appropriate summary is part of the substantive model.

History compression also clarifies the relation between generated topology and grammatical memory. Some information initially stored in H_t can become recoverable from X_t because earlier transformations leave persistent structural traces. If a function $\eta(H_t)$ can be reconstructed from the current typed CW state, then that part of historical dependence can be internalized as structural state for the purpose of admissibility. Other information, including deleted structures, unrealized alternatives, temporal ordering among commuting attachments, or external contextual events, may remain unavailable from X_t and require an explicit history coordinate.

7.7 Historical Equivalence and Structural Memory

The preceding definitions yield several distinct levels at which two modeled trajectories can be considered equivalent. Structural endpoint equivalence in Equation 37 compares only the terminal typed CW carriers. Derivational equivalence compares the organization of the realized rewrite sequence. Equality of an admissibility-sufficient history summary compares only the historical information relevant to the chosen grammar. One-step future equivalence compares the successor possibilities generated from the terminal states. These relations answer different questions and need not coincide.

A particularly important case occurs when $H_t \equiv_X H'_s$ while the histories have different sufficient summaries or different terminal grammars. The present carrier then exhibits the same typed topology at the chosen resolution, yet the modeled structures have different generative futures. Conversely, histories can differ derivationally while converging to the same terminal carrier and the same future-admissibility class. The model can therefore distinguish historical difference from historically consequential difference.

The CW carrier itself can preserve a form of *structural memory*. Higher cells can retain the incidence of previously constituted structures, shared supports can preserve traces of composite participation, and nested attachments can make some constitutive dependencies visible in the terminal topology. This structural memory is one reason to resist unnecessary flattening into a carrier that externalizes higher incidence. It remains a partial record: the terminal topology generally does not determine the chronological order of independent attachments, the rules that failed to realize, the contexts under which decisions were made, or deleted structures whose traces were not retained.

The resulting formal separation can be summarized by three coordinates: X_t records the generated cellular structure presently represented; Γ_t records the rule system governing admissible transformations; and H_t , or an admissibility-sufficient summary $\eta(H_t)$, records historical information required beyond the present structure and grammar. These coordinates can coincide partially when history is internalized into structure or grammar, but the framework does not assume such reducibility.

This separation prepares the analysis of topological invariants in Section 8. Invariants of a terminal CW complex answer questions about generated structure. Invariants evaluated across an attachment-monotone filtration answer questions about cumulative structural development. General rewrite histories require comparison methods that respect deletion and replacement rather than silently imposing monotonicity. The historical architecture developed here therefore determines which topological comparisons are well posed before any social interpretation is assigned to them.

8 Topological Invariants of Social Generative Structure

Sections 4–7 established three distinct formal objects: a generated CW carrier, a grammar of admissible transformations, and a realized generative history. The present section identifies topological quantities that can be extracted from the generated carrier and, under suitable historical conditions, from sequences of carriers. Its objective is to distinguish decomposition-sensitive descriptions of higher incidence from topological invariants that survive specified equivalences. The discussion proceeds through cellular

chain data, homology, topology relative to the retained 2-skeleton, homotopy classes of higher attachments, rule-induced topological change, and persistent structure along attachment-monotone histories. Throughout the section, mathematical invariance is kept separate from substantive social interpretation.

8.1 Invariance Relative to a Modeling Equivalence

An invariant is meaningful only relative to a stated class of objects and a stated equivalence relation. The present framework contains several plausible equivalences, including typed cellular isomorphism, homeomorphism, homotopy equivalence, equivalence of pairs that preserves the distinguished 2-skeleton, and stronger notions that also preserve grammar or history. A quantity can be invariant under one of these relations while changing under another.

Definition 8.1 (Model-relative topological invariant). Let $\mathcal{C}$ be a class of social CW carriers and let $\sim_{\mathcal{E}}$ be an equivalence relation on $\mathcal{C}$ chosen for an inquiry. A map $I : \mathcal{C} \rightarrow \mathcal{S}$ is an *invariant relative to $\sim_{\mathcal{E}}$* when Equation 45 holds.

$$X \sim_{\mathcal{E}} Y \implies I(X) = I(Y). \quad (45)$$

Definition 8.1 prevents the word “invariant” from carrying more force than the mathematics supports. Ordinary homology is invariant under homotopy equivalence, whereas an incidence matrix written in a chosen cellular basis depends on the cellular decomposition. Typed social models can require still stronger preservation conditions: a homeomorphism that exchanges cells carrying substantively different types may be topologically valid while failing to preserve the typed structure relevant to the inquiry. When type information is part of the target object, the chosen equivalence must preserve it explicitly.

The same caution applies to primitives. Because $X_{\mathcal{Q},\sigma}$ depends on the research question and resolution, an invariant computed on one carrier is initially an invariant of that modeled carrier. Comparison across σ requires the refinement or coarse-graining maps anticipated in Section 4.4. A stable topological quantity across several resolutions can become analytically important, but its cross-resolution status must be demonstrated rather than inferred from the word “topological.”

This distinction also separates three forms of description used below. A *cellular descriptor* can depend on the chosen decomposition while directly exposing higher incidence. A *topological invariant* is constant under the selected topological equivalence. A *historical invariant* or persistence object additionally refers to maps across a realized sequence of states. Each form answers a different question.

8.2 Cellular Boundary Operators and Homology

CW structure provides a direct algebraic representation of higher incidence through the cellular chain complex. Fix a coefficient field $\mathbb{F}$ for the present subsection. For a CW complex X , the cellular chain group in dimension n is given by Equation 46 and is generated by the n -cells after orientations are chosen [1].

$$C_n^{\text{cell}}(X; \mathbb{F}) = H_n(X^{(n)}, X^{(n-1)}; \mathbb{F}). \quad (46)$$

The attaching maps determine cellular boundary homomorphisms $\partial_n : C_n^{\text{cell}}(X; \mathbb{F}) \rightarrow C_{n-1}^{\text{cell}}(X; \mathbb{F})$ satisfying the chain condition in Equation 47.

$$\partial_{n-1} \circ \partial_n = 0. \quad (47)$$

For a finite complex and chosen cellular bases, ∂_n can be represented by an incidence matrix D_n . Its entries summarize how oriented n -cells attach to $(n-1)$ -cells at the chain level. The matrices D_n are therefore useful native descriptors of the higher structure introduced after the 2-skeleton. They are not, by themselves, invariants of homotopy type: subdivisions, basis changes, or alternative CW decompositions can change the matrices while leaving the underlying space homotopy equivalent.

Homology removes part of this decomposition dependence. The n th cellular homology group is defined in Equation 48.

$$H_n(X; \mathbb{F}) = \ker(\partial_n) / \text{im}(\partial_{n+1}). \quad (48)$$

Over the field $\mathbb{F}$, the corresponding Betti number is $\beta_n(X; \mathbb{F}) = \dim_{\mathbb{F}} H_n(X; \mathbb{F})$. In particular, H_0 records connected-component structure, while higher groups distinguish cycles that are not boundaries at the corresponding dimension. Homology is invariant under homotopy equivalence [1].

The relation between cycles and fillers is especially relevant to the present paper. If $z \in C_n^{\text{cell}}(X; \mathbb{F})$ satisfies $\partial_n z = 0$, then z is an n -cycle. The class $[z]$ vanishes in $H_n(X; \mathbb{F})$ precisely when an $(n+1)$ -chain c exists with $\partial_{n+1} c = z$. A nonzero class therefore identifies a cycle that is not filled by the currently represented higher cellular chains. This is a precise chain-level statement. Its social interpretation remains question-dependent: a nonzero homology class does not by itself establish an institutional gap, normative conflict, coordination failure, or other substantive diagnosis.

Higher cells can both create and eliminate homological features. An attachment can kill a previously nontrivial class by providing a higher-dimensional boundary, and it can also generate new classes in other dimensions depending on the attaching map. The invariant therefore summarizes consequences of the full attachment structure rather than counting cells. Two models with the same number of n -cells can have different homology, while two models with different cellular decompositions can have the same homology.

8.3 Higher-Extension Topology Relative to the 2-Skeleton

The central comparison of this paper keeps the established social-spacetime 2-complex as a distinguished base. Let $K = X^{(2)}$ denote the event-coupling-primitive 2-skeleton of a social CW extension. The topology of X can then be studied together with the topology of the pair (X, K) . Relative homology is useful because it distinguishes information carried by the full extension from information already present in the retained base [1].

The pair (X, K) has a long exact homology sequence containing the segment in Equation 49.

$$\cdots \longrightarrow H_n(K; \mathbb{F}) \longrightarrow H_n(X; \mathbb{F}) \longrightarrow H_n(X, K; \mathbb{F}) \xrightarrow{\delta_n} H_{n-1}(K; \mathbb{F}) \longrightarrow \cdots \quad (49)$$

The relative groups alone should not be read as a complete description of how higher cells attach to the base. The connecting maps δ_n are also informative. For example, when $K = X^{(2)}$, the map $\delta_3 : H_3(X, K; \mathbb{F}) \rightarrow H_2(K; \mathbb{F})$ can record how relative three-dimensional classes meet homology already present in the 2-skeleton. This relation is directly relevant to the paper's distinction between data stored above the primitive layer and the structure through which those data attach to it.

For bookkeeping, define the *base-relative homological profile* of the extension by Equation 50.

$$H_{\mathbb{F}}(X \mid K) := (\{H_n(X; \mathbb{F})\}_{n \geq 0}, \{H_n(X, K; \mathbb{F})\}_{n \geq 0}, \{\delta_n\}_{n \geq 1}). \quad (50)$$

The profile in Equation 50 is an invariant of the pair up to the corresponding homotopy equivalence of pairs. It is deliberately richer than a vector of Betti numbers because the connecting homomorphisms carry relational information between the base and extension. It remains coarser than the full CW attachment data. Distinct attaching maps can induce identical homology and relative homology, so the profile should be treated as a topological summary rather than a complete reconstruction of generative structure.

This limitation is analytically useful. Section 3 argued that dimensional reduction can hide native structure even when enough information remains encoded for reconstruction. Equation 50 supplies one family of quantities that can be compared before and after a proposed reduction. If a reduction preserves an encoded reconstruction of X but computes topology only on the two-dimensional carrier, then the homology of the carrier can fail to expose the base-relative profile of the reconstructed extension without an explicit decoding step.

8.4 Homotopy Classes of Higher Attachments

Homology is intentionally coarse. Higher attachments can differ in ways that homology does not distinguish, while the attaching maps themselves carry homotopically relevant information. For a path-connected component with chosen basepoint, an attaching map $\alpha_{n,\lambda} : S^{n-1} \rightarrow X^{(n-1)}$ determines a

homotopy class when the ordinary based hypotheses are satisfied. The class is represented schematically in Equation 51.

$$[\alpha_{n,\lambda}] \in \pi_{n-1}(X^{(n-1)}, x_0). \quad (51)$$

Equation 51 should be adapted to the relevant component or unbased formulation when the carrier is not connected. Its analytical role is to retain how a boundary sphere maps into the lower skeleton up to homotopy, rather than retaining only the image of that boundary or its incidence counts.

The example used in Section 3 already illustrates why this matters: a common 2-skeleton can support higher-cell attachments with different homotopy classes and therefore different resulting topological types. More generally, homotopy groups $\pi_n(X, x_0)$ can detect higher-dimensional structure that ordinary homology may miss [1]. They are consequently candidates for analyzing higher generative organization when the research question turns on attachment classes rather than only on cycle ranks.

Homotopy information also has practical limits. Higher homotopy groups can be difficult to compute, and the full homotopy type of a complex is usually much richer than a small collection of groups. A social interpretation must additionally explain why the corresponding homotopy distinction matters for the represented phenomenon. The framework therefore treats homotopy as an available structural resource rather than as a mandatory descriptor for every application.

A useful hierarchy follows from these observations. Cell counts and incidence matrices expose the chosen decomposition directly. Homology compresses that structure into cycle-and-boundary invariants. Homotopy can retain distinctions invisible to homology. The grammar and realized history can still distinguish models whose terminal carriers are homotopy equivalent. No one level therefore subsumes the others for the purposes of the present research programme.

8.5 Topological Effects of Generative Rules

The grammar of Section 6 acts on CW carriers and can therefore preserve or change their topological invariants. This fact supplies a direct interface between generative law and generated topology. Let I be an invariant defined on the relevant carrier class and let a realized rule application transform X_t into X_{t+1} . The application is *I-preserving* when $I(X_t) = I(X_{t+1})$ and *I-changing* otherwise.

For a rule schema r , a stronger property can be defined over all of its admissible matches.

Definition 8.2 (Invariant-conservative rule). A rule $r \in \mathcal{R}$ is *I-conservative* on a model class $\mathcal{C}$ when every admissible application of r between states in $\mathcal{C}$ preserves I .

Definition 8.2 permits grammars to be classified by structural effects rather than only by syntactic form. A rule can be conservative for connectedness while changing higher homology, conservative for homology while changing cell incidence, or conservative for the homotopy type while changing the grammar itself through a reflexive update. Conversely, a rule that attaches or removes cells can alter several topological quantities at once.

For finite complexes over a field, Betti-number change provides a simple transition summary. Define the finite-dimensional Betti-change vector through Equation 52 for a chosen maximum dimension N .

$$\Delta_{\beta}^{(N)}(X_t, X_{t+1}) := (\beta_0(X_{t+1}) - \beta_0(X_t), \dots, \beta_N(X_{t+1}) - \beta_N(X_t)). \quad (52)$$

Equation 52 is a transition descriptor, not a complete invariant of the rewrite. A zero vector does not imply that the two states are homeomorphic or homotopy equivalent, and a nonzero entry does not by itself identify the substantive meaning of the change. Its utility is comparative: it provides a compact record of homological change that can be related to rule type, context, or history in an empirical or computational application.

The rule perspective also clarifies the difference between topology and grammar. Two grammars can generate the same family of topological types through different rule systems, and one grammar can admit several transformations with distinct topological effects. Topological invariants characterize generated carriers; they do not reconstruct the generative law that produced them. Section 7 likewise showed that endpoint-equivalent structures can arise from different histories. Topological equivalence is therefore one axis of comparison within a broader generative model.

8.6 Persistent Structure along Attachment-Monotone Histories

Attachment-monotone histories provide an additional source of topological information because the stage sequence forms a filtration. Let a realized history satisfy $X_0 \subseteq X_1 \subseteq \dots \subseteq X_t$ as in Definition 7.3. For $i \leq j$, the inclusion $X_i \hookrightarrow X_j$ induces a linear map on homology with coefficients in $\mathbb{F}$. The persistent homology group between stages i and j is defined by Equation 53 [5].

$$P_n^{i,j} := \text{im} (H_n(X_i; \mathbb{F}) \longrightarrow H_n(X_j; \mathbb{F})) . \quad (53)$$

The dimension of $P_n^{i,j}$ records how many n -dimensional homology classes represented at stage i remain represented at stage j . Under the usual finite-type hypotheses, the associated persistence module can be summarized through interval or barcode representations. In the present framework, such persistence is a property of a specified filtration of modeled CW states; it should not be identified automatically with social durability, institutional stability, or historical importance.

Ordinary persistent homology requires compatible maps across the indexed sequence. A general rewrite history of Section 7.3 can contain deletion or replacement and therefore need not supply inclusions $X_i \hookrightarrow X_{i+1}$. Equation 53 should consequently be used only when a monotone filtration, or another explicitly justified persistence construction, is available. Applying ordinary persistence to an arbitrary rewrite sequence would erase precisely the distinction established in Section 7.

Persistence also differs from the multiscale analysis developed in Section 9. A historical filtration indexes realized generative stages, whereas a resolution family indexes analytical scale or refinement. The two indices can coincide in a particular model, but they need not. Keeping them separate allows the later framework to ask whether a topological feature persists through historical generation, through changes of resolution, or through both.

8.7 Interpretive Scope of Topological Signatures

The preceding constructions provide several levels of topological description, summarized in Table 4. The table distinguishes quantities that depend directly on a chosen cellular decomposition from invariants of a carrier or pair and from persistence objects defined across a filtration. The distinctions are important because the central contribution of the paper concerns structural transparency, not the replacement of social interpretation by topology.

Three interpretive constraints follow. First, topological invariants are coarse. Equal homology groups, equal Betti numbers, or even homotopy equivalence can coexist with different cell types, attaching data, grammars, and generative histories. Second, primitive selection remains methodological. A feature that appears topologically nontrivial at one resolution can be refined, decomposed, or represented differently at another resolution. Third, mathematical persistence and substantive persistence are different claims. A class persisting through a filtration is a formal fact about the selected sequence of carriers and maps; interpreting it as a durable norm, institution, identity, conflict, or other social phenomenon requires an independently stated correspondence and evidence.

These constraints also indicate why topology remains useful. The CW extension makes selected higher attachment structure native to the carrier, so chain complexes, homology, relative topology, homotopy classes, and persistence can be applied directly to that generated organization. The resulting invariants provide comparison tools that are difficult to see when the same information is stored only as auxiliary metadata on a 2-complex. Their value lies in exposing which structural properties survive specified transformations and which change when the generative architecture changes.

Section 9 develops the complementary problem of scale. A single CW carrier supplies topology at one chosen representation, while a family of refinements can reveal whether structural patterns recur, persist, or transform across resolutions. Fractal or multifractal interpretations will therefore require additional metric, measure, or scaling structure rather than following from cellular dimension alone.

Table 4: Topological and cellular quantities available in the CW framework. The final column states the principal interpretive restriction rather than a substantive social meaning.

Object	Mathematical quantity	Stability class	Interpretive restriction
Chosen cellular decomposition	Boundary maps ∂_n and incidence matrices D_n	Basis- and decomposition-sensitive	Exposes native incidence but is not a homotopy invariant
Generated carrier X	$H_n(X; \mathbb{F})$, β_n , and when justified $\pi_n(X)$	Topological or homotopy invariant at the stated level	Does not identify the social meaning of cycles or higher structure
Extension relative to 2-skeleton	$H_n(X, K)$ and connecting maps δ_n	Invariant of the pair under the corresponding pair equivalence	Summarizes attachment effects without reconstructing all attaching maps
Realized monotone history	Persistence module and $P_n^{i,j}$	Stable under isomorphism of the indexed persistence structure	Requires justified maps across stages and does not equal social persistence by definition
Grammar-induced transition	$\Delta_\beta^{(N)}$ or preservation of a chosen I	Transition-dependent descriptor	Does not reconstruct the rule, context, or history that produced the change

9 Multiscale Topology of Generated Structure

Sections 4 and 7 kept analytical resolution separate from cellular dimension and chronological generation. The present section makes that third coordinate explicit. Its objective is to specify when a family of CW representations can support claims about scale dependence, recursive organization, or fractal and multifractal signatures. The argument proceeds through scale-indexed cellular families, inter-resolution maps, cell-growth exponents, metric fractal dimensions, measure-dependent multifractal quantities, and scale-dependent topology. A central restriction is maintained throughout: the integer dimension of a CW cell is a property of a cellular decomposition, whereas fractal dimension is a scaling quantity that requires additional metric, measure, or covering structure. A single CW complex therefore does not establish fractality merely by containing cells of several dimensions.

9.1 Scale-Indexed Cellular Families

Fix a research question $\mathcal{Q}$. Let $\sigma_0, \sigma_1, \dots, \sigma_m$ denote analytical resolutions ordered from coarser to finer, and let $X_k := X_{\mathcal{Q}, \sigma_k}$ be the CW representation selected at resolution σ_k . To discuss scaling quantitatively, associate each resolution with a positive scale parameter ε_k satisfying $\varepsilon_{k+1} < \varepsilon_k$. The value ε_k can represent spatial diameter, temporal window, organizational granularity, interaction radius, or another application-specific notion of resolved scale. Its empirical meaning must be supplied by the inquiry.

A multiscale model requires more than the list of carriers X_k . It also requires maps that state how information at one resolution relates to information at another. The present paper uses coarse-graining maps $\rho_{k+1,k} : X_{k+1} \rightarrow X_k$ from a finer representation to the preceding coarser representation. When several scales are compared, coherence of coarse-graining is expressed by Equation 54.

$$\rho_{k+2,k} = \rho_{k+1,k} \circ \rho_{k+2,k+1}. \quad (54)$$

Equation 54 is an additional modeling requirement rather than a theorem of CW theory. Different empirical constructions can supply different maps, and some pairs of resolutions may admit no defensible map at all.

Definition 9.1 (Scale-indexed cellular family). A *scale-indexed cellular family* for an inquiry $\mathcal{Q}$ is a collection

$$\mathfrak{X}_{\mathcal{Q}} = (\{(X_k, \varepsilon_k)\}_{k=0}^m, \{\rho_{\ell,k}\}_{\ell > k}) \quad (55)$$

for which the X_k are CW representations at ordered resolutions, the ε_k encode the associated scales, and the available inter-resolution maps satisfy the stated composition relations wherever those maps are defined.

Equation 55 records the data of Definition 9.1. The definition separates three indices that recur in the paper. Cellular dimension n indexes the skeleton $X_k^{(n)}$. Generative stage t indexes historical transformation. Resolution k indexes what the model chooses to resolve. These coordinates can interact, but none can be inferred from another. Finer analytical resolution can replace one coarse face by many cells without increasing the largest CW dimension, while a higher-dimensional attachment can be introduced at a fixed resolution and historical stage.

A refinement can preserve a coarse external interface while resolving a primitive internally. If a coarse primitive face $f \subset X_k$ is represented at the next scale by a subcomplex $R_f \subset X_{k+1}$, the map $\rho_{k+1,k}$ can collapse R_f to the coarse representation of f while retaining the selected interface with the remainder of X_k . This construction makes the methodological character of primitiveness operational: a face is primitive only relative to the resolution at which its internal organization is intentionally suppressed.

9.2 Recursive Generation across Refinement

Multiscale organization becomes especially informative when refinement is generated by recurring rule schemas. Let Γ^{ref} denote the subset of the cellular grammar whose rules alter analytical resolution by replacing a coarse cell or subcomplex with a finer configuration while respecting a specified interface. A refinement step can be written as Equation 56.

$$(X_k, \Gamma_k^{\text{ref}}) \implies (X_{k+1}, \Gamma_{k+1}^{\text{ref}}). \quad (56)$$

The use of $\implies$ in Equation 56 records a generative transformation; it does not assert that X_k is a subcomplex of X_{k+1} . A refinement may replace cells, change attaching maps, or alter types while still admitting a coarse-graining map back to the earlier representation.

A recurring refinement schema provides a structural notion of self-similarity that precedes any fractal-dimension claim. Suppose a typed subcomplex pattern $P_k \subseteq X_k$ is replaced at the next resolution by a family of subcomplexes $P_{k+1}^{(1)}, \dots, P_{k+1}^{(b_k)}$ whose coarse images reproduce the role of P_k . Recurrence of the same typed rewrite schema across several k values constitutes *grammar-level scale recurrence*. It identifies repeated generative organization even when the geometric realizations are only approximately similar or when the relevant scale is organizational rather than spatial.

This distinction is useful because exact geometric self-similarity is stronger than recurring cellular production. A legal procedure, institutional replication process, organizational template, or scientific training structure can recur across analytical scales without admitting a literal similarity transformation in Euclidean space. The cellular grammar can represent that recurrence structurally, while fractal geometry becomes appropriate only after a defensible metric or scale relation has been supplied.

When a refinement does have approximately uniform branching b and a common linear scale ratio $r \in (0, 1)$, one can define the candidate similarity exponent in Equation 57.

$$d_{\text{sim}}^{\text{cand}} = \frac{\log b}{\log(1/r)}. \quad (57)$$

Equation 57 is a diagnostic quantity in the present framework. Under the familiar hypotheses of a genuinely self-similar geometric construction, related expressions can coincide with standard fractal dimensions; those conclusions depend on the metric realization and separation conditions, not on the CW structure alone [6].

9.3 Structural Cell-Growth Exponents

A scale-indexed family supports combinatorial scaling measurements even when a canonical metric on the represented social structure is unavailable. Let $\mathcal{C}_k$ be a specified family of cells or typed structural units in X_k , and let $N_{\mathcal{C}}(\varepsilon_k) = |\mathcal{C}_k|$. The upper and lower cell-growth exponents are defined in Equation 58 when the indicated limits are meaningful.

$$\bar{d}_{\mathcal{C}} = \limsup_{k \rightarrow \infty} \frac{\log N_{\mathcal{C}}(\varepsilon_k)}{-\log \varepsilon_k}, \quad \underline{d}_{\mathcal{C}} = \liminf_{k \rightarrow \infty} \frac{\log N_{\mathcal{C}}(\varepsilon_k)}{-\log \varepsilon_k}. \quad (58)$$

When the two values agree, their common value is denoted $d_{\mathcal{C}}$.

The exponent $d_{\mathcal{C}}$ should initially be called a *cell-growth* or *structural scaling exponent*. It is sensitive to primitive selection, cell type, refinement policy, and the scale variable. It becomes a standard box-counting dimension only when the counted cells implement an appropriate controlled covering of a metric set and satisfy the conditions required by that dimension theory. This naming discipline prevents a combinatorial count from acquiring an unjustified geometric interpretation.

The same framework permits typed scaling. If $\mathcal{E}_n^{\tau}(X_k)$ denotes the n -cells of type τ at scale k , define $N_{n,\tau}(\varepsilon_k) = |\mathcal{E}_n^{\tau}(X_k)|$. The resulting family of exponents can reveal whether some forms of higher-order organization proliferate more rapidly under refinement than others. Such a profile can be analytically useful even when no single scalar “dimension of society” is meaningful.

A multiscale CW representation therefore retains information that a single flattened carrier can easily obscure: the branching history of refinement, the level at which new incidence appears, the recurrence of attachment patterns, and the rates at which selected structural families proliferate. An enriched 2-complex could encode all of these quantities, but the multiscale family makes their generative organization explicit through the resolution maps and refinement grammar.

9.4 Metric Conditions for Fractal Dimensions

Fractal dimension becomes a mathematically standard object only after the model supplies a metric or another admissible covering structure. Let (Y, d) be a metric realization associated with the modeled system, and let $N_Y(\varepsilon)$ denote the minimum number of sets of diameter at most ε required to cover Y . The upper and lower box-counting dimensions are defined through the corresponding limsup and liminf of the logarithmic covering rate; the upper form is shown in Equation 59 [6].

$$\overline{\dim}_B Y = \limsup_{\varepsilon \downarrow 0} \frac{\log N_Y(\varepsilon)}{-\log \varepsilon}. \quad (59)$$

The lower box-counting dimension replaces the limsup in Equation 59 with a liminf. Hausdorff dimension provides another metric notion with different covering and limiting requirements [6]. None of these dimensions is determined by the abstract CW topology alone.

This point matters for the motivating claim concerning higher-dimensional and fractal information. A finite CW complex has an integer-valued cellular dimension, and an abstract CW decomposition does not canonically determine the metric data required by Equation 59. A noninteger scaling dimension can therefore enter the framework through a metric realization, a controlled sequence of refinements, or another scale-dependent construction. The resulting fractal quantity characterizes that enriched multiscale object, not the bare integer index of its cells.

If the cells of X_k furnish a controlled cover of Y at scale ε_k , with cell diameters comparable to ε_k and with suitable bounds preventing arbitrary over-refinement, then the cell-growth exponent of Section 9.3 can approximate or, under stronger assumptions, recover a metric covering dimension. Those

assumptions have to be stated in each application. Without them, d_C remains a representation-dependent structural signature. Remark 9.2 records the corresponding dimensional distinction explicitly.

Remark 9.2 (Dimensional quantities in the multiscale model). For the present framework, the statement $\dim_{CW} X = n$ and a statement such as $\dim_B Y = d$ answer different questions. The first concerns the largest dimension of cells required by a chosen CW decomposition. The second concerns asymptotic covering rates in a metric space. Either value can vary while the other remains fixed.

9.5 Multifractal Signatures of Measure Scaling

Some social structures can carry highly uneven distributions of activity, attention, authority, value, probability, or other modeled weights. A single scaling exponent can then conceal substantial heterogeneity. To study such cases, augment the scale-indexed family with a normalized measure μ_k on a selected partition or cover $\mathcal{C}_k$. For $q \in \mathbb{R}$, define the partition sum in Equation 60.

$$Z_q(\varepsilon_k) = \sum_{c \in \mathcal{C}_k} \mu_k(c)^q. \quad (60)$$

If the partition sums exhibit a stable power-law regime, a mass exponent $\tau(q)$ can be estimated from the scaling relation in Equation 61.

$$Z_q(\varepsilon) \asymp \varepsilon^{\tau(q)}. \quad (61)$$

For $q \neq 1$, the associated generalized-dimension candidate is given by Equation 62.

$$D_q = \frac{\tau(q)}{q - 1}. \quad (62)$$

Multifractal analysis studies heterogeneous scaling of measures through families of exponents and, in a standard formulation, through singularity spectra such as $f(\alpha)$ [7]. The present framework can use these tools only after the measure μ_k , the scale variable, and the partition scheme have substantive interpretations.

The measure is therefore not an innocuous add-on. Choosing μ_k to represent frequency of events, transaction volume, institutional authority, probability amplitude, scholarly attention, or another quantity produces different multifractal objects. A claim of multifractality must identify which measure scales, over which range, and under which refinement procedure. A heterogeneous distribution on one finite cellular decomposition is insufficient by itself.

A useful connection with the generative grammar arises when the grammar distributes weights during refinement. A rule can specify both a structural replacement and a weight-transfer law. Repeated application then generates a sequence of measures together with the sequence of CW structures. In such a model, multifractal behavior is potentially a property of the coupled structural-generative process rather than a statistic appended after the topology has been constructed.

9.6 Scale-Dependent Topology

Topological organization can also vary with analytical resolution. For each k , Section 8 provides invariants such as $H_n(X_k; \mathbb{F})$. The coarse-graining maps of Definition 9.1, when continuous and compatible with the chosen coefficients, induce homomorphisms on homology. The resulting scale-indexed algebraic object is displayed in Equation 63.

$$\cdots \longrightarrow H_n(X_{k+1}; \mathbb{F}) \xrightarrow{H_n(\rho_{k+1,k})} H_n(X_k; \mathbb{F}) \longrightarrow \cdots. \quad (63)$$

Equation 63 allows the inquiry to distinguish features that persist under coarse-graining from features visible only after finer resolution. It also makes explicit when an apparent higher-order hole or closure is created by the modeling scale rather than stable across scales.

Table 5: Levels of multiscale evidence and the claims they can support.

Observed structure	Additional requirement	Supported interpretation
Higher CW cells	Explicit attaching data	Higher incidence or composition at the chosen resolution; no fractal claim follows
Repeated refinement motif	Inter-resolution maps and recurring rule schema	Grammar-level scale recurrence or structural self-similarity
Power-law cell growth	Defensible numerical scale ε and stable scaling range	Structural scaling exponent; geometric dimension requires further assumptions
Metric covering law	Metric realization and controlled covers	Box-counting or related fractal-dimension analysis
Heterogeneous measure scaling	Defined measure, partitions, and stable q -dependent scaling	Candidate multifractal spectrum or generalized dimensions
Stable topology across scales	Continuous comparison maps or nested filtration	Scale-persistent topological structure

When the scale family is genuinely nested by inclusions, persistent homology can summarize the birth and persistence of homological classes across the scale filtration, using the same mathematical logic introduced for attachment-monotone histories in Section 8 [5]. General coarse-graining maps need not form such an inclusion filtration. The appropriate object is then the mapped system itself, or another comparison construction justified for the application. Ordinary filtration persistence should not be imposed on a nonnested resolution family merely for convenience.

Scale recurrence can also be examined at the level of attaching maps and typed incidence, not only homology. Two scales can have identical Betti numbers while differing substantially in how higher cells attach or in how types are distributed across the incidence structure. Multiscale analysis therefore inherits the hierarchy established in Section 8: native cellular data are finer than homological summaries, and both can be studied across resolution.

9.7 Criteria for Fractal Interpretation

The term “fractal” is analytically useful in this framework only when a scale-dependent claim is supported by specified structure. Table 5 summarizes the levels of evidence introduced in this section.

Three cautions follow from Table 5. First, approximate power laws over a short finite range can arise without genuine asymptotic fractality. Second, changing the primitive selection can alter cell counts and apparent scaling. Third, social data usually have finite resolution and finite observation windows, so empirical fractal claims should report the scale range, estimator, uncertainty, and sensitivity to alternative cellularizations.

The strongest object developed in the present section is therefore not a single “fractal CW complex.” It is a coupled multiscale system consisting of scale-indexed CW carriers, inter-resolution maps, a refinement grammar, and optional metric or measure data. A compact notation for this object is given in Equation 64.

$$\mathfrak{M}_Q^{\text{scale}} = (\mathfrak{X}_Q, \Gamma^{\text{ref}}, \mathcal{D}, \mu), \quad (64)$$

where $\mathcal{D}$ denotes the optional metric or scale geometry and μ denotes optional measures across resolutions. Equation 64 makes clear which information belongs to the cellular topology, which belongs to the law of refinement, and which is introduced to support geometric or multifractal analysis.

The multiscale extension sharpens the central argument of the paper. Reduction to an enriched 2-complex can preserve the data needed to reconstruct a hierarchy, yet a one-carrier encoding can relocate the refinement lineage, repeated attachment patterns, scale maps, and scaling laws into auxiliary metadata.

Explicit scale-indexed CW structure keeps those relations inspectable as part of the modeled organization. The gain concerns structural transparency rather than absolute representational capacity. Section 10 uses this distinction to examine reduction to enriched 2-complexes directly and to state conditions under which such reduction is analytically safe.

10 Reduction to Enriched 2-Complexes

This section evaluates dimensional reduction as a representation choice rather than as a deficiency test for the 2-complex formalism. Its objective is to determine when a higher-dimensional CW model can be reduced to an enriched 2-complex without compromising the analyses required by a specified research question. The discussion first classifies common enrichment channels, then distinguishes exact reconstruction from analytical sufficiency, and finally states criteria for reduction safety across observables, transformations, history, and scale. The method is comparative and formal: the section assumes that a reduction can carry arbitrarily rich auxiliary data and asks which structural operations remain native, which remain recoverable only through decoding, and when the auxiliary encoding effectively reconstructs the higher-dimensional structure that the reduction nominally suppresses.

10.1 Reduction Packages and Enrichment Channels

Section 3 introduced a reduction map $\mathfrak{R}_2$ whose carrier is the 2-skeleton and whose auxiliary datum D_X may contain information about higher cells. The present section makes the enrichment channels more explicit. For a CW model X , write the reduced representation as in Equation 65.

$$\mathfrak{R}_2(X) = \left(X^{(2)}, \tau_2, D_X^{\text{lab}}, D_X^{\text{int}}, D_X^{\text{path}}, D_X^{\text{aux}} \right). \quad (65)$$

The four auxiliary components in Equation 65 are schematic classes rather than mandatory fields. Label enrichment D_X^{lab} includes scalar, vector, representation-theoretic, operator-valued, or categorical labels attached to vertices, edges, or faces. Internal enrichment D_X^{int} includes manifolds, complexes, state spaces, or other structured objects associated with a cell. Path enrichment D_X^{path} includes strings, words, trajectories, provenance traces, or other ordered data associated with a carrier element. Auxiliary-structural enrichment D_X^{aux} includes explicit records of higher incidence, higher attaching maps, nested subcomplexes, or independent complexes linked back to the 2-skeleton.

These channels can carry substantial information. A face treated as primitive at resolution σ may, for example, be associated with an internal manifold that resolves its internal organization without changing its status as a face of the exterior 2-complex. An edge can carry a string recording a sequence of couplings or transformations. A face or edge can carry an operator whose action summarizes a family of internal transitions. None of these constructions is excluded by the 2-complex architecture developed in Section 2.

The representational issue therefore concerns the organization of auxiliary information. Internal manifolds preserve structure inside individual cells but do not automatically specify how internal structures belonging to several different cells participate in one higher incidence relation. Strings preserve order or provenance when a suitable alphabet and composition rule are supplied, yet relations among several strings may themselves require an additional structure. Auxiliary higher-incidence tables can record those relations directly, but once they acquire their own incidence maps, composition laws, and topology they function as an additional structural layer rather than as ordinary labels. The distinction is one of model architecture, not raw storage capacity.

10.2 Exact Reconstruction through Complete Auxiliary Encoding

Exact reconstructibility is a weak requirement when the auxiliary data are unrestricted. Let $\text{Enc}(X)$ be any lossless serialization of a finite or effectively representable CW model X . A reduction can then retain the 2-skeleton while storing the serialization as auxiliary data, as stated in Equation 66.

$$\mathfrak{R}_2^{\text{full}}(X) := \left(X^{(2)}, \tau_2, \text{Enc}(X) \right). \quad (66)$$

Whenever Enc admits a decoding map Dec satisfying $\text{Dec}(\text{Enc}(X)) \cong X$, Equation 66 is exactly reconstructive in the sense of Equation 7. This observation establishes that representational sufficiency alone cannot justify the CW extension. A sufficiently permissive enriched 2-complex can carry a complete description of the higher-dimensional object.

The analytical question begins after this information-theoretic observation. If every calculation involving higher incidence first executes Dec and reconstructs X , then the reduced carrier has not eliminated the need for the higher-dimensional structure in that analysis. It has changed where the structure is stored. The workflow is represented in Equation 67.

$$\mathfrak{R}_2^{\text{full}}(X) \xrightarrow{\text{Dec}} X \xrightarrow{\mathcal{A}} \mathcal{A}(X), \quad (67)$$

where $\mathcal{A}$ denotes a structural analysis such as computing a relative homology group, inspecting higher attaching maps, comparing higher incidence, or tracing a refinement hierarchy. Equation 67 can be entirely adequate computationally. It nevertheless differs architecturally from applying $\mathcal{A}$ directly to a model in which the relevant structure is native.

This distinction also applies when the auxiliary datum is itself a higher-dimensional complex. Suppose D_X^{aux} contains an explicit cellular object Y_X together with maps identifying its lower-dimensional interfaces with $X^{(2)}$. The resulting model may still be described as an enriched 2-complex because its primary carrier is two-dimensional, but the analytical architecture already contains a second structured object carrying the higher relations. In that case, the substantive comparison is between a single CW carrier and a coupled pair consisting of a 2-complex plus an auxiliary higher structure. Terminology alone does not remove the higher-dimensional organization.

10.3 Inquiry-Relative Observational Sufficiency

A reduction can be evaluated more usefully relative to the quantities required by a research question. Let $\mathcal{I}_Q$ denote the family of observables, invariants, structural statistics, or decision quantities used to answer question Q . Definition 10.1 strengthens the recoverability idea of Section 3.2 by treating the entire analytical family at once.

Definition 10.1 (Observationally sufficient reduction). A reduction $\mathfrak{R}_2$ is *observationally sufficient* for Q on a model class $\mathcal{C}$ when, for every $I \in \mathcal{I}_Q$, there exists a reduced-representation procedure $\hat{I}$ such that Equation 68 holds for all $X \in \mathcal{C}$.

$$I(X) = \hat{I}(\mathfrak{R}_2(X)). \quad (68)$$

Definition 10.1 permits $\hat{I}$ to inspect auxiliary data. Observational sufficiency therefore concerns correctness of the requested outputs, not structural transparency. If $\mathcal{I}_Q$ contains only vertex incidence, primitive couplings, face amplitudes, or selected operator expectations, an enriched 2-complex can remain sufficient even when X contains higher cells. If $\mathcal{I}_Q$ contains higher attaching-map classes, higher incidence intersections, or base-relative topology, sufficiency requires the enrichment to retain enough corresponding structure.

A stronger property combines sufficiency with native access. Let $\mathcal{I}_Q^{\text{nat}} \subseteq \mathcal{I}_Q$ contain the quantities that the inquiry intends to treat as direct properties of the modeled carrier. A reduction is *native-sufficient* for this family when those quantities can be evaluated from the declared native structures of the reduced model without reconstructing suppressed higher organization. Native sufficiency is therefore representation-sensitive by design. It formalizes the difference between storing a higher incidence relation and making that incidence relation part of the topology on which the analysis directly operates.

10.4 Transformation Compatibility of Reduced Models

Static sufficiency does not guarantee that a reduction preserves generative analysis. Section 6 models admissible change through rules acting on CW structures. A reduced model is dynamically useful when transformations of the full carrier have well-defined counterparts on the reduced representation.

Let r be an admissible realized rule application taking X to X' . A reduced transition $\hat{r}$ is compatible with r when Equation 69 commutes up to the equivalence chosen for the reduced representation.

$$\hat{r}(\mathfrak{R}_2(X)) \simeq \mathfrak{R}_2(r(X)) = \mathfrak{R}_2(X'). \quad (69)$$

Equation 69 requires more than endpoint reconstructibility. The reduced representation must carry enough information to determine the transformed auxiliary data, including changes in higher attaching structure when the original rule modifies that structure.

Definition 10.2 (Transformation-compatible reduction). Let $\mathcal{R}_Q$ be the family of grammar rules whose effects are relevant to question Q . A reduction $\mathfrak{R}_2$ is *transformation-compatible for Q* when each admissible realized application of every $r \in \mathcal{R}_Q$ admits a reduced counterpart $\hat{r}$ satisfying Equation 69 on the model class under study.

Transformation compatibility exposes another form of hidden reconstruction. If $\hat{r}$ operates by decoding the entire higher CW structure, applying r , and re-encoding the result, the reduction remains correct but the generative computation is still performed in the higher-dimensional representation. By contrast, a genuinely closed reduced dynamics would update the enriched 2-complex directly. Whether such closure exists depends on the chosen grammar and enrichment scheme; it should be demonstrated rather than assumed.

This distinction matters especially for reflexive grammars. When a generated higher structure modifies the grammar governing subsequent transformations, the reduced representation must preserve both the structure that triggered the update and the rule-level consequence. Otherwise two reduced states that appear identical at the carrier level can incorrectly be assigned the same future admissibility set even when their decoded higher structures or histories differ.

10.5 Historical and Multiscale Sufficiency

Sections 7 and 9 introduced two forms of organization that are not determined by one terminal carrier: realized generative history and analytical scale structure. Reduction safety therefore requires separate tests for both.

Let η_Q be a history summary sufficient for every history-sensitive admissibility test or observable used by question Q , in the sense developed in Section 7. A reduction is *history-sufficient* when the reduced representation carries a summary $\hat{\eta}_Q$ satisfying Equation 70.

$$\hat{\eta}_Q(\mathfrak{R}_2(X_t), \hat{H}_t) \equiv \eta_Q(H_t) \quad (70)$$

for the history-dependent decisions relevant to the inquiry. Equation 70 does not require preservation of the complete derivation history when a smaller sufficient summary exists. It requires preservation of exactly the historical distinctions that can change the answer to Q .

Multiscale sufficiency is analogous. Let $\mathfrak{X}_Q$ be a scale-indexed CW family with coarse-graining maps, refinement grammar, and optional metric or measure data as in Equation 64. A one-carrier reduction is multiscale-sufficient only when it preserves the scale-indexed quantities used by the inquiry and the maps required to compare them. Storing each scale independently without the inter-resolution maps can preserve snapshots while losing the refinement genealogy. Storing the maps as metadata can restore recoverability, but scale-dependent topology becomes native only if the reduced architecture treats those maps and indexed structures as part of the analytical object.

The same principle applies to fractal or multifractal signatures. A scalar estimate of a dimension can be stored as a face label, but such a label does not preserve the refinement process, covering law, measure distribution, or scale range from which the estimate was derived. When those generative and multiscale structures are themselves part of the research question, preserving only the final number is observationally insufficient for richer inquiries even though it may be sufficient for a narrow downstream calculation.

Table 6: Reduction criteria for choosing between an enriched 2-complex and an explicit CW representation.

Analytical requirement	Enriched 2-complex can remain adequate when	Explicit CW structure becomes useful when
Primitive-level observables	Required quantities depend on events, couplings, faces, and attached labels	Higher incidence or attachment itself enters the observable
Higher structural reconstruction	Auxiliary data preserve attaching maps and decoding is acceptable	Repeated decoding obscures or complicates the main analysis
Generative dynamics	Reduced update rules are closed and satisfy Equation 69	Rule applicability or effects depend natively on higher-cell structure
Historical analysis	A sufficient historical summary is retained	Generative lineage or alternative derivations are themselves analytical objects
Multiscale analysis	Scale summaries and required comparison maps are preserved	Refinement genealogy, inter-scale topology, or recurrent attachment patterns are central
Computational economy	Two-dimensional carrier plus compact enrichment lowers modeling or computational cost	Auxiliary metadata approaches a serialized reconstruction of the full CW model
Interpretive transparency	Users can inspect the structures relevant to $\mathcal{Q}$ directly in the reduced architecture	The principal objects of interpretation are hidden inside labels, code, or auxiliary records

10.6 Inquiry-Safe Reduction

The preceding criteria can be combined into a question-relative notion of reduction safety. Let the analytical specification for $\mathcal{Q}$ be the tuple stated in Equation 71.

$$\mathfrak{A}_{\mathcal{Q}} := (\mathcal{I}_{\mathcal{Q}}, \mathcal{R}_{\mathcal{Q}}, \mathcal{H}_{\mathcal{Q}}, \mathcal{S}_{\mathcal{Q}}), \quad (71)$$

where $\mathcal{I}_{\mathcal{Q}}$ is the required observable family, $\mathcal{R}_{\mathcal{Q}}$ the relevant transformation family, $\mathcal{H}_{\mathcal{Q}}$ the required historical distinctions, and $\mathcal{S}_{\mathcal{Q}}$ the required scale-dependent structure.

Definition 10.3 (Inquiry-safe reduction). A reduction $\mathfrak{R}_2$ is *inquiry-safe for $\mathcal{Q}$* on a model class $\mathcal{C}$ when it is observationally sufficient for $\mathcal{I}_{\mathcal{Q}}$, transformation-compatible for $\mathcal{R}_{\mathcal{Q}}$, history-sufficient for $\mathcal{H}_{\mathcal{Q}}$, and multiscale-sufficient for $\mathcal{S}_{\mathcal{Q}}$ to the level of equivalence specified by the inquiry.

Definition 10.3 is deliberately relative. A reduction can be safe for one question and unsafe for another. A model studying local event incidence and primitive-level amplitudes may have little use for explicit cells above dimension two. A model studying whether several primitive complexes participate in one higher compositional closure can require higher attaching data. A model studying the genealogy of such closures across scale can require still more structure. The appropriate representation is therefore chosen by the analytical burden rather than by a general preference for either higher or lower dimension.

Table 6 summarizes the resulting decision logic.

10.7 Representation Economy and the Role of the CW Extension

Inquiry safety still leaves an engineering choice. Two representations can be equally sufficient while differing in complexity, computational cost, interpretability, or compatibility with available methods. A compact enriched 2-complex can be preferable when higher structure is peripheral, sparse, or efficiently summarized. Explicit CW structure can be preferable when higher incidence, composition, topology, or refinement is repeatedly queried and transformed. The present framework therefore treats representation choice as a problem of analytical economy rather than as a hierarchy in which the higher-dimensional model is automatically superior.

A useful limiting case clarifies the argument. If the enrichment required for safe reduction contains a lossless encoding of all higher cells, all attaching maps, all generative histories relevant to future rules, and all inter-resolution relations, then the reduced representation can be fully sufficient. Its auxiliary layer, however, carries much of the structure that the explicit CW model places directly in the carrier or in its associated grammar and history. At that point the difference between the two models concerns organization, interfaces, and available operations more than information content.

This limiting case returns to the core contribution stated in Section 3. Reduction to a 2-complex can preserve information while relocating the topology of structural generation into auxiliary encodings. The CW extension is useful when the research question benefits from keeping higher incidence, attachment, compositional closure, or multiscale organization as native structure. The enriched 2-complex remains useful when those features are peripheral or can be summarized without impairing the required analyses.

Section 11 develops the corresponding equivalence problem. Once several representations can be inquiry-safe, the next question is which sense of “the same model” is intended: informational equivalence, topological equivalence, generative equivalence, historical equivalence, or a more limited equivalence relative to selected observables.

11 Structural Equivalence across Representations

Section 10 established that a higher-dimensional CW model and an enriched 2-complex can be adequate for the same inquiry even when they organize their information differently. The present section develops the corresponding equivalence theory. Its objective is to replace an undifferentiated claim that two representations are “the same” with a family of explicit comparison relations. The discussion first defines a common representation package, then distinguishes informational, observable, carrier-topological, generative, historical, and multiscale equivalence, and finally combines these axes into an inquiry-relative equivalence profile. The method is formal and comparative. Each equivalence notion is defined by the structure it is required to preserve, and no implication between different axes is assumed without an explicit argument.

11.1 Representation Packages and Equivalence Axes

A comparison across representations requires an object broad enough to include both an explicit CW carrier and a reduced 2-complex with auxiliary data. For the present purpose, a *representation package* is written as in Equation 72.

$$\mathfrak{P} = (C, D, \Gamma, H, \mathfrak{X}), \quad (72)$$

where C is the primary cellular carrier, D is any auxiliary datum attached to that carrier, Γ is the generative grammar when generative analysis is included, H is the realized history or an admissibility-sufficient history summary when historical analysis is included, and $\mathfrak{X}$ is the scale-indexed structure when multi-scale analysis is included. Components irrelevant to a particular inquiry may be omitted. An explicit CW representation can therefore use $C = X$ with modest auxiliary data, whereas an enriched 2-complex can use $C = X^{(2)}$ with a more substantial D .

Equation 72 is an organizational device rather than an ontological decomposition. The same substantive information can be located in different coordinates under different architectures. A higher attaching

map can be part of the carrier in one representation and part of D in another. A history summary can be stored externally, attached to cells, or encoded into grammar state. The equivalence question therefore concerns which distinctions survive a translation between packages, not which coordinate happens to store them.

Let $\mathfrak{P}$ and $\mathfrak{P}'$ be two packages intended to represent the same domain instance at the resolution relevant to a research question Q . The paper distinguishes six principal comparison axes:

1. informational equivalence, concerning mutual recoverability of the declared model information;
2. observable equivalence, concerning agreement on the quantities required by Q ;
3. carrier-topological equivalence, concerning the topology of the primary carriers themselves;
4. generative equivalence, concerning admissible structural futures under the associated grammars;
5. historical equivalence, concerning realized derivations or history summaries relevant to the inquiry;
6. multiscale equivalence, concerning refinement structure and inter-resolution relations.

These axes answer different questions. A pair of representations can agree on one axis and disagree on another without inconsistency.

11.2 Informational Equivalence

Informational equivalence is the strongest notion used here for representational payload, but it remains neutral about where that payload is structurally located. Let $\cong_D$ denote identity of the declared model data up to admissible renaming, reindexing, and lossless serialization conventions. Two representation packages are *informationally equivalent* when there exist lossless translations T and U satisfying Equation 73 on the model class being compared.

$$U(T(\mathfrak{P})) \cong_D \mathfrak{P}, \quad T(U(\mathfrak{P}')) \cong_D \mathfrak{P}'. \quad (73)$$

Equation 73 expresses mutual recoverability. It does not require the carriers of $\mathfrak{P}$ and $\mathfrak{P}'$ to be homeomorphic, homotopy equivalent, or even of the same dimension. It also does not require the same operations to be native in both representations. A complete serialization of a CW complex stored as auxiliary data on a 2-complex can therefore satisfy informational equivalence while retaining a different carrier topology.

Informational equivalence is representation-relative because the declared data schema determines what counts as complete information. If one package records empirical provenance, uncertainty, or historical attribution and the other does not, a translation that reconstructs only the cellular carrier is not informationally complete under the richer schema. The equivalence claim should therefore identify the model class and the data specification to which Equation 73 applies.

11.3 Observable Equivalence

Many research questions require less than mutual reconstruction of the entire model. Let $\mathcal{I}_Q$ be the family of observables, invariants, statistics, classifications, or decision quantities required by question Q , as introduced in Section 10.3. Two representation packages are *observable-equivalent for Q* when the identified instances agree on every quantity in that family. The condition is stated in Equation 74.

$$\mathfrak{P} \equiv_{\mathcal{I}_Q}^{\text{obs}} \mathfrak{P}' \iff I(\mathfrak{P}) = I(\mathfrak{P}') \text{ for every } I \in \mathcal{I}_Q. \quad (74)$$

Equation 74 is deliberately inquiry-relative. Two models can be equivalent for a question about primitive-level amplitudes while differing on higher homology, generative history, or refinement genealogy. Expanding $\mathcal{I}_Q$ can therefore destroy an equivalence that held for a narrower inquiry.

Observable equivalence also differs from informational equivalence in analytical burden. If the quantities in $\mathcal{I}_Q$ are directly available in both packages, the two representations can be observationally equivalent even when neither can reconstruct the other. Conversely, informational equivalence can coexist with poor structural transparency when one representation requires substantial decoding before the relevant observable can be evaluated. Section 10 treats that difference as part of representation economy rather than as a failure of correctness.

11.4 Carrier-Topological Equivalence

Topological comparison concerns the primary carriers C and C' in Equation 72. The phrase *topological equivalence* is too coarse unless the retained structure is specified. For CW complexes, several standard relations form a useful hierarchy of strength. Homeomorphism implies homotopy equivalence, and homotopy equivalence implies isomorphic homology groups in every degree with fixed coefficients [1]. The implication chain is recorded in Equation 75.

$$C \cong_{\text{homeo}} C' \implies C \simeq C' \implies H_n(C; \mathbb{F}) \cong H_n(C'; \mathbb{F}) \text{ for every } n. \quad (75)$$

The converses in Equation 75 do not hold in general. An inquiry should therefore state whether it requires homeomorphic carriers, homotopy-equivalent carriers, agreement of selected homology groups, agreement of relative invariants such as $H_n(X, X^{(2)})$, or another explicitly defined relation.

Carrier-topological equivalence concerns the carrier rather than the total informational content of a package. This distinction is central to dimensional reduction. If a higher CW complex X is encoded losslessly into auxiliary data attached to $X^{(2)}$, the packages can be informationally equivalent while the carriers X and $X^{(2)}$ fail even to have the same homotopy type. Section 3 supplied an explicit example with a 3-cell attached to S^2 .

The topology of the carrier can also agree while other comparison axes differ. Two packages can use the same CW complex X but associate different grammars, histories, empirical labels, or scale relations with it. Carrier-topological equivalence consequently establishes a structural commonality without determining the generative or historical identity of the modeled system.

11.5 Generative Equivalence

Generative equivalence concerns what the representations permit to happen next and, recursively, what futures remain reachable. Let $\text{TS}(\mathfrak{P})$ denote the typed transition system induced by the grammar associated with $\mathfrak{P}$. Its states are the reachable representation states, and its labeled transitions are admissible rule applications of the form developed in Section 6. A relation $\mathcal{B}$ between the states of $\text{TS}(\mathfrak{P})$ and $\text{TS}(\mathfrak{P}')$ is *generatively matching* when paired states can match each other's admissible labeled transitions in both directions. The back-and-forth condition is stated in Equation 76.

$$\begin{aligned} (s, s') \in \mathcal{B}, s \xrightarrow{r} u &\implies \exists u' (s' \xrightarrow{r'} u' \text{ and } (u, u') \in \mathcal{B}), \\ (s, s') \in \mathcal{B}, s' \xrightarrow{r'} u' &\implies \exists u (s \xrightarrow{r} u \text{ and } (u, u') \in \mathcal{B}), \end{aligned} \quad (76)$$

where r and r' must correspond under the rule-type identification chosen for the comparison. Two packages are *generatively equivalent* when their designated initial or current states are related by a generatively matching relation satisfying Equation 76.

This definition strengthens the one-step future equivalence of Section 7.2. One-step equivalence compares immediate successor families. Generative equivalence requires the matching to continue recursively through reachable futures. The relation can be restricted when the inquiry concerns only a selected rule family $\mathcal{R}_Q$, producing a question-relative generative equivalence.

Generative equivalence does not require literal identity of grammars. Two different rule systems can generate matching transition structures after a translation of rule types and states. Conversely, identical current carriers do not guarantee generative equivalence. History-sensitive predicates, reflexive grammar updates, or different rule sets can assign different future possibilities to the same typed cellular structure.

11.6 Historical Equivalence

Historical equivalence compares realized generation rather than admissible possibility. Section 7 already distinguished structural endpoint equivalence, derivational equivalence, equality of admissibility-sufficient history summaries, and one-step future equivalence. The present section organizes those notions for cross-representation comparison.

Let $\mathcal{H}_Q$ be the family of historical functionals required by the inquiry. These can include rule provenance, order of realized transformations, cell birth stages, lineage relations, attribution data, or a compressed history summary η_Q . Two packages are *history-equivalent for Q* when every required historical functional agrees after translation, as stated in Equation 77.

$$\mathfrak{P} \equiv_{\mathcal{H}_Q}^{\text{hist}} \mathfrak{P}' \iff J(H) = J'(H') \text{ for every identified } J \in \mathcal{H}_Q. \quad (77)$$

Equation 77 permits the historical burden to match the question. A study of future admissibility may require only equality of an admissibility-sufficient summary. A study of institutional provenance may require the realized rule sequence and source attribution. A study of generative genealogy can require a substantially richer correspondence among intermediate states.

Derivational equivalence from Section 7.2 is therefore one strong form of historical equivalence, while equality of a compressed summary is a weaker form. Structural endpoint equivalence alone is generally insufficient for historical equivalence because alternative histories can terminate in the same typed carrier. This distinction preserves the path-dependent cases formalized in Definition 7.2.

11.7 Multiscale Equivalence

Multiscale equivalence concerns families of representations linked across analytical resolutions. Let $\mathfrak{X} = (\{X_k\}, \{\rho_{k+1,k}\})$ and $\mathfrak{X}' = (\{X'_k\}, \{\rho'_{k+1,k}\})$ be two scale-indexed systems over a common scale index, where the maps $\rho_{k+1,k}$ and $\rho'_{k+1,k}$ are the coarse-graining or comparison maps introduced in Section 9. A family of scale-wise correspondences $\{\phi_k\}$ is compatible with the scale structure when the comparison square commutes to the equivalence required by the inquiry. The condition is given in Equation 78.

$$\phi_k \circ \rho_{k+1,k} \sim_{S_Q} \rho'_{k+1,k} \circ \phi_{k+1} \quad \text{for each compared scale } k. \quad (78)$$

The relation $\sim_{S_Q}$ in Equation 78 can require literal equality, cellular isomorphism, homotopy equivalence, agreement of selected invariants, or another scale-specific relation. A monotone reparameterization of the scale index can also be admitted when two models use different but comparable resolution conventions.

Multiscale equivalence requires more than similarity of snapshots. Two families can contain equivalent carriers at each isolated scale while encoding different refinement genealogies because the inter-scale maps differ. The maps in Equation 78 therefore carry part of the structure. When fractal or multifractal signatures are used, the comparison should additionally specify whether it preserves the metric, measure, scaling range, partition functions, or dimension estimates required by the inquiry.

11.8 Inquiry-Relative Equivalence Profiles

The preceding relations can be assembled into a profile rather than forced into one universal equivalence. For a research question Q , let the equivalence profile be the tuple in Equation 79.

$$\mathbf{E}_Q(\mathfrak{P}, \mathfrak{P}') := (e_{\text{info}}, e_{\text{obs}}, e_{\text{top}}, e_{\text{gen}}, e_{\text{hist}}, e_{\text{scale}}), \quad (79)$$

where each entry records whether the pair satisfies the corresponding relation at the strength selected by Q . An entry can be binary for a sharply specified criterion or graded when the inquiry compares several nested strengths, such as homeomorphism, homotopy equivalence, and homology agreement.

A pair of packages is *inquiry-equivalent for Q* when every axis designated as necessary by the analytical specification $\mathfrak{A}_Q$ in Equation 71 is satisfied at its required strength. This definition aligns Section 11

Table 7: Equivalence axes for comparing CW representations with enriched 2-complex representations.

Equivalence axis	Primary test	Distinctions that can remain outside the test
Informational	Mutual lossless translation of the declared model data	Carrier topology, native operations, computational cost, interpretive transparency
Observable	Agreement on $\mathcal{I}_Q$	Unqueried data, unqueried topology, unused history, unused future possibilities
Carrier-topological	Chosen relation on primary carriers, such as homeomorphism or homotopy equivalence	Grammar, realized history, auxiliary provenance, multiscale maps
Generative	Back-and-forth matching of admissible typed transitions	Realized path, unused metadata, carrier details ignored by the transition relation
Historical	Agreement on required history functionals or derivational organization	Counterfactual futures outside the historical test, unused carrier invariants
Multiscale	Compatible scale-wise correspondences and inter-resolution maps	Single-scale metadata outside the scale inquiry, unrelated historical variables
Inquiry-relative	Satisfaction of every axis selected by $\mathfrak{A}_Q$	Any distinctions explicitly declared irrelevant to Q

with the inquiry-safe reduction criterion of Definition 10.3. An inquiry-safe reduction supplies the equivalences demanded by the selected observables, transformations, historical distinctions, and multiscale structures; it does not thereby establish stronger equivalences that the inquiry did not require.

Table 7 summarizes the comparison axes and the distinctions each one preserves.

11.9 Non-Collapse of the Equivalence Axes

The equivalence notions above do not form one linear hierarchy. Their independence is part of the paper’s central argument because dimensional reduction can preserve one kind of sameness while changing another.

Proposition 11.1 (Non-collapse of representation equivalence). *Within the representation class developed in this paper, informational equivalence does not generally imply carrier-topological equivalence, and carrier-topological equivalence does not generally imply generative or historical equivalence.*

For the first statement, consider the complete auxiliary encoding of Equation 66. Let X be the CW complex obtained by attaching a 3-cell to S^2 through the identity map, as in Section 3. The resulting carrier is homeomorphic to a 3-ball, whereas the reduced carrier is $X^{(2)} = S^2$. Storing a lossless encoding of X on the reduced carrier yields mutual reconstructibility, hence informational equivalence under the declared schema, while the primary carriers have different homotopy types and homology.

For the second statement, fix one typed CW carrier X and form two packages with the same carrier but different grammars Γ and Γ' . If the grammars assign different admissible successor structures to X , the carriers are identical and therefore carrier-topologically equivalent, while the packages fail the generative matching condition in Equation 76. The same construction can hold the carrier and grammar fixed while assigning two generative histories that differ on a historical functional required by $\mathcal{H}_Q$, producing carrier-topological equivalence without historical equivalence. These constructions establish Proposition 11.1.

Observable equivalence is even more explicitly relative. If $\mathcal{I}_Q$ contains only a primitive-level quantity that both packages compute identically, Equation 74 can hold while the packages differ on higher topol-

ogy, grammar, history, and scale organization. Observable equivalence therefore reports sufficiency for a question, not identity of the representations.

The non-collapse result supplies a precise vocabulary for the paper’s motivating claim. An enriched 2-complex and an explicit CW model can contain the same recoverable information while presenting different carrier topology; they can share carrier topology while generating different futures; they can generate matching futures while recording different realized histories; and they can match at each isolated scale while differing in the maps that organize those scales. A claim that two representations are “equivalent” is therefore incomplete until the preserved axis and its required strength are stated.

This taxonomy also prepares the dynamical analysis of Section 12. Once equivalence is decomposed by axis, a transformation can be studied according to which equivalence classes it preserves, breaks, or transports. The resulting dynamics concern both changes of social CW structure and changes in the representational relations through which that structure is compared across models.

12 Dynamics on CW Social Spacetime

Sections 6–11 separated generated CW structure, admissible structural transformation, realized history, topological invariants, multiscale organization, and equivalence across representations. The present section introduces dynamics on that architecture. Its objective is to distinguish three forms of change that can coexist in one model: evolution of cell-associated dynamical data, transformation of the cellular carrier, and modification of the grammar that constrains later transformations. The discussion first defines a layered dynamical state, then specifies realization dynamics over grammatically admissible successors, develops data transport across cellular rewrites, and finally couples carrier, data, history, and grammar evolution. The construction is intentionally general. Probability, complex amplitude, utility, power, optimization, or other selection mechanisms can be supplied by an application, while the core formalism requires only that their roles be distinguished from grammatical admissibility.

12.1 Layered Dynamical State

The structural state X_t records the CW carrier represented after the t th realized update. The index t is an ordering parameter for realized model updates; it need not coincide with continuous physical time, calendar time, or analytical resolution. Section 7 already distinguished generative stage from cellular dimension, and Section 9 distinguished both from resolution. The dynamical formalism retains those distinctions.

Cellular topology alone does not specify all quantities that may evolve while the carrier remains unchanged. Let Ξ_t denote the collection of dynamical data assigned to the represented state at stage t . The data may include cell-associated amplitudes, representation labels, operator-valued variables, empirical attributes, local state variables, coupling strengths, probability parameters, or other quantities whose semantics are supplied by the application. A dynamical social CW state is defined in Equation 80.

$$\mathfrak{Z}_t = (X_t, \Xi_t, \Gamma_t, H_t; c_t). \quad (80)$$

Here Γ_t is the cellular generative grammar active at the represented stage, H_t is the realized generative history or an admissibility-sufficient summary, and c_t denotes contextual information represented outside the carrier and its dynamical data. The semicolon in Equation 80 separates the modeled endogenous package from the contextual coordinate without asserting that the division is ontologically fixed.

The decomposition in Equation 80 permits several kinds of change to be represented separately. A data update can change Ξ_t while preserving X_t and Γ_t . A cellular rewrite changes X_t and can require a corresponding update of Ξ_t . A reflexive grammatical event can modify Γ_t . Historical state changes whenever a realized event is appended to H_t . Table 8 summarizes these channels and their principal interpretive restrictions.

This separation is methodologically important because a model can exhibit substantial dynamics without changing its topology. Coupling strengths can vary, amplitudes can redistribute, institutional

Table 8: Distinct change channels in the CW dynamical framework. Analytical refinement is included to prevent a change of representation from being misidentified as a historical transformation of the represented social structure.

Change channel	Formal locus	Typical role	Principal restriction
Dynamical-data update	Ξ_t	Evolution of amplitudes, labels, local states, weights, or empirical attributes	Does not imply a change of carrier topology
Carrier rewrite	X_t	Attachment, deletion, replacement, or another specified cellular transformation	Must satisfy the rewrite and admissibility semantics of the grammar
Grammar update	Γ_t	Modification of later structural possibilities	Distinct from merely realizing a rule already contained in the grammar
History update	H_t	Addition of the realized event and its provenance	Records realization rather than the whole space of alternatives
Analytical refinement	$(\mathcal{Q}, \sigma)$ and associated representation	Change of primitive decomposition or resolution	Is not a historical social transformation unless the application separately models it as one

variables can change, and event-associated states can evolve on a fixed carrier. Conversely, a cellular attachment can change topology even when most cell-associated dynamical data are transported unchanged. Structural and dynamical change therefore remain distinct axes of the model.

12.2 Realization Dynamics over Admissible Successors

Section 6.4 defined $\text{Succ}_\Gamma(X_t; c_t, H_t)$ as the set of one-step structures admitted by the grammar. That set specifies structural possibility, while a realized history requires a mechanism that selects or weights particular admissible transformations. The present section calls such a mechanism *realization dynamics*.

Let $\mathcal{E}_t^\Gamma$ be the set of admissible rule events at stage t . An element $\varepsilon = (r, m)$ consists of a rule r , an admissible match m , and the corresponding well-defined successor construction. The set can depend on the full state in Equation 80 when dynamical data enter admissibility. A realization law assigns a status to the elements of $\mathcal{E}_t^\Gamma$ without enlarging that set beyond the grammar unless the model explicitly includes grammar modification.

A deterministic realization law can be represented by a selector S_t that chooses one admissible event when the model supplies a unique selection criterion. A stochastic realization law can instead assign a conditional probability distribution as in Equation 81.

$$P_t(\varepsilon \mid \mathfrak{Z}_t) \geq 0, \quad \sum_{\varepsilon \in \mathcal{E}_t^\Gamma} P_t(\varepsilon \mid \mathfrak{Z}_t) = 1. \quad (81)$$

Equation 81 applies only when the admissible event set is discrete and the model intends a probabilistic interpretation. Continuous or mixed event spaces require the corresponding measure-theoretic formulation.

An amplitude-weighted model can assign complex or otherwise nonprobabilistic weights $\mathcal{A}_t(\varepsilon \mid \mathfrak{Z}_t)$ to admissible events. Such amplitudes should remain formally distinct from probabilities. A map from

amplitudes to probabilities requires an additional rule specified by the application. The existence of a complex amplitude alone does not license a probabilistic interpretation, and the present framework does not impose a particular amplitude calculus.

More generally, a realization weight can be written as $W_t(\varepsilon \mid \mathfrak{Z}_t)$ with codomain chosen for the application. The grammar answers whether an event belongs to the admissible structural possibility set; W_t answers how the model compares, selects, or weights events inside that set. This distinction permits power, utility, institutional capacity, empirical hazard, optimization criteria, stochastic intensity, or amplitude to influence realization without being conflated with grammatical permission.

The distinction also clarifies historical contingency. Two states can have the same admissible event set while differing in realization weights, and therefore differ dynamically without differing grammatically. Conversely, two states can share the same realization mechanism while possessing different grammars and therefore different sets of events to which that mechanism can apply. Grammar and dynamics are coupled coordinates rather than interchangeable descriptions.

12.3 Structural Rewrite Dynamics

A realized admissible event $\varepsilon_t = (r_t, m_t)$ induces a carrier transformation through the rewrite construction of Section 6.2. Denote that partial transformation by Φ_{ε_t} . The carrier update is stated in Equation 82.

$$X_{t+1} = \Phi_{\varepsilon_t}(X_t). \quad (82)$$

The map in Equation 82 is partial because a rule event can be applied only when its match is topologically constructible and grammatically admissible. The equation therefore records realized structural dynamics rather than an unconstrained endomorphism of a fixed state space.

The interface span $L_r \leftarrow B_r \rightarrow R_r$ supports several important special cases. A pure attachment preserves the matched structure and introduces new cells. A deletion removes the designated cells while retaining the interface. A replacement combines removal and introduction. A refinement substitutes a more resolved pattern while preserving a declared external interface. These cases are structurally defined through the rule rather than by assigning a universal social meaning to any one operation.

The term *collapse* requires additional care. In algebraic topology, several technically specific notions of collapse and contraction exist, while social-scientific coarse-graining can identify many lower-level distinctions through a model-dependent map. The present framework therefore uses a reduction or collapse operation only when its semantics are explicitly supplied. If a model introduces a map $q : X_t \rightarrow \overline{X}_t$ that identifies or removes structure, it must state whether q is a cellular quotient, a coarse-graining map between resolutions, an elementary collapse under stronger topological assumptions, or another transformation. A generic many-to-one simplification is not treated as a canonical CW operation.

Structural rewrites can also preserve the homeomorphism or homotopy type of the carrier while changing its cellular decomposition. Conversely, an attachment or deletion can change selected topological invariants. The invariant-change profile developed in Section 8 can therefore be evaluated along a realized rewrite sequence. Carrier dynamics and topological dynamics coincide only for the properties actually changed by the realized rewrite.

A useful special case is a topology-preserving dynamical step in which the carrier remains fixed. It is represented by Equation 83.

$$X_{t+1} = X_t, \quad \Xi_{t+1} \neq \Xi_t. \quad (83)$$

Equation 83 permits the framework to represent substantial social or physics-inspired dynamics without manufacturing a cellular rewrite at every update. The explicit CW architecture is therefore compatible with both fixed-topology dynamics and topology-changing structural generation.

12.4 Dynamical Data on the Social-Spacetime 2-Skeleton

The earlier 2-complex remains the base social-spacetime carrier within the CW architecture. At stage t , write $K_t = X_t^{(2)}$. The dynamical data restricted to this layer are denoted $\Xi_t^{(2)}$. A model can decompose

these data by cell dimension as in Equation 84.

$$\Xi_t^{(2)} = \left(\{\xi_v(t)\}_{v \in K_t^{(0)}}, \{\xi_e(t)\}_{e \in K_t^{(1)}}, \{\xi_f(t)\}_{f \in K_t^{(2)}} \right). \quad (84)$$

The symbols in Equation 84 are placeholders for model-specific state spaces. In a spin-foam-inspired specialization, face data can include representation labels such as spins, edge data can include coupling or intertwining information, and vertex data can contribute event-associated amplitudes. In another application, the same slots can contain empirical states, legal statuses, economic quantities, learned representations, or operator-valued variables. The cellular role and the semantics of the attached data should remain separately specified.

The base 2-skeleton can therefore support dynamics that are largely independent of the higher CW extension. Let $\mathcal{A}_t$ be an application-specific amplitude or weight functional evaluated on $(K_t, \Xi_t^{(2)})$, as stated in Equation 85.

$$\mathcal{A}_t = \mathcal{A}\left(K_t, \Xi_t^{(2)}; c_t, H_t\right). \quad (85)$$

Equation 85 deliberately leaves the functional form open. The paper's contribution does not depend on a universal amplitude law. The purpose of the notation is to preserve compatibility with earlier physics-inspired 2-complex constructions while allowing the CW extension to carry a different kind of information: explicit higher incidence and compositional topology.

When a rewrite changes K_t , the dynamical data on the 2-skeleton must be transported, removed, or initialized consistently with the carrier transformation. Let B_{r_t} be the preserved interface of the realized rule. A rule-specific data-update map $\mathcal{U}_{\varepsilon_t}$ acts on the dynamical data and returns data defined on the new carrier. On preserved cells, the update should satisfy the interface-compatibility condition in Equation 86 whenever the application declares those cell identities to persist.

$$\Xi_{t+1}|_{B_{r_t}} = T_{\varepsilon_t}\left(\Xi_t|_{B_{r_t}}\right), \quad (86)$$

where T_{ε_t} is the transport rule specified for preserved data. New cells require initialization rules, and deleted cells cease to carry active state in the successor carrier unless their values are retained separately in history. Equation 86 avoids assuming that cellular identity alone determines how every dynamical quantity should be transported.

12.5 Dynamical Data on Higher Cells

Higher cells can carry dynamical data without changing their topological role. For each $n \geq 3$ and each n -cell $e^n \in X_t$, let $\mathcal{S}(e^n)$ be an application-specific state space and let the associated variable satisfy Equation 87.

$$\xi_{e^n}(t) \in \mathcal{S}(e^n). \quad (87)$$

The state in Equation 87 can represent a weight, activation level, empirical confidence, stability parameter, institutional strength, legal status, effective coupling, local operator, or another defined quantity. None of these interpretations follows from cellular dimension itself. The higher cell supplies the structural locus on which the variable is defined; the substantive theory supplies the meaning of the variable.

This distinction allows a constituted higher-order structure to persist while its dynamical condition changes. A composition-bearing cell can remain present as part of the modeled topology while a strength or validity parameter evolves. If the application treats loss of that parameter as sufficient for dissolution of the represented composition, the transition from dynamical weakening to cell deletion must be specified by the grammar or another explicit threshold rule. The topology does not delete itself when an attached scalar crosses an unspecified value.

Higher-cell data can also influence lower-dimensional dynamics. Let e^n have primitive footprint $\text{Foot}_2(e^n)$ as defined in Section 5.2. An application can introduce an influence operator $\mathcal{J}_{e^n}$ acting on dynamical data associated with that footprint. The existence and form of $\mathcal{J}_{e^n}$ are model commitments.

The formal possibility is important because higher-order organization can thereby have dynamical effects on the primitive social-spacetime layer without requiring that the higher cell be reinterpreted as an additional physical spacetime coordinate.

The converse dependence is equally possible. Dynamical conditions on the 2-skeleton can affect the admissibility, weight, or persistence of higher attachments. Such dependence should enter through an explicit rule or admissibility predicate. The architecture therefore supports bidirectional coupling between higher structural organization and lower-layer dynamics while keeping the carrier hierarchy, dynamical variables, and rule semantics formally distinct.

12.6 Coupled Evolution of the Full Generative State

The preceding constructions can be combined into one staged update without collapsing their roles. Let ε_t be the realized rule event selected or weighted by the realization dynamics. A full update of Equation 80 is represented by Equation 88.

$$\mathfrak{Z}_t \xrightarrow{\varepsilon_t} \mathfrak{Z}_{t+1} = (X_{t+1}, \Xi_{t+1}, \Gamma_{t+1}, H_{t+1}; c_{t+1}). \quad (88)$$

Equation 88 is an organizational schema. Its coordinates can be updated by different mechanisms. The carrier follows Equation 82; the dynamical data follow a rule-specific update map; the history appends the realized event; the grammar can remain fixed or change under a reflexive rule; and the context can be externally supplied or endogenously updated when the application models that feedback.

The data update is written in Equation 89.

$$\Xi_{t+1} = \mathcal{U}_{\varepsilon_t}(\Xi_t; X_t, X_{t+1}, c_t, H_t). \quad (89)$$

Equation 89 permits transport on preserved cells, initialization on generated cells, deletion of active data associated with removed cells, and state changes on cells whose carrier identity persists. It also permits a pure data update by taking $X_{t+1} = X_t$.

For a reflexive grammar, the grammar update can depend on realized structure and dynamical state as in Equation 90.

$$\Gamma_{t+1} = \mathcal{G}(\Gamma_t, X_t, \Xi_t, H_t, c_t, \varepsilon_t). \quad (90)$$

A nonreflexive model takes $\Gamma_{t+1} = \Gamma_t$. Equation 90 is therefore optional rather than a universal law. It becomes relevant when realized structures modify the conditions under which later structures can be generated, for example when a constitutional, institutional, technical, or epistemic transformation changes subsequent admissibility rules.

Dynamical variables can also enter admissibility itself. Section 6.3 used $A_r(m; X_t, c_t, H_t)$ as the core predicate. In a dynamical application, this predicate can be refined to $A_r^{\text{dyn}}(m; X_t, \Xi_t, c_t, H_t)$ when rule admissibility depends on current amplitudes, resources, strengths, states, or other dynamical data. The earlier predicate is recovered when Ξ_t is irrelevant to admissibility. This extension preserves the conceptual order: topology defines the carrier, the grammar defines admissibility, dynamical data can parameterize that admissibility, and realization dynamics selects or weights events among the admissible possibilities.

The coupled architecture yields three analytically distinct sources of future difference. Two states can have different carriers and the same grammar; the same carrier and different dynamical data; or the same carrier and data but different grammars or history summaries. A model that identifies these cases loses information about whether divergence arises from present structure, present state, historical memory, or the rules of future generation. Equation 88 preserves those distinctions while allowing them to interact.

The resulting dynamical formalism extends the CW framework without making higher-dimensional cells responsible for all social dynamics. The 2-skeleton can retain the amplitude-bearing or operator-bearing social-spacetime dynamics already available in the earlier model. Higher cells make selected organization explicit, dynamical data can evolve on every represented layer, and the grammar constrains changes of the carrier itself. Section 13 uses this layered architecture to examine social structure formation, including the distinction between coupled aggregates, constituted structures, and historically stabilized formations.

13 Social Structure Formation

Sections 4–12 developed a formal architecture for generated CW structure, cellular composition, generative grammar, realized history, topological invariants, multiscale organization, representation reduction, equivalence, and dynamics. The present section examines how that architecture can be used to represent social structure formation without assigning fixed sociological meanings to cellular dimensions. Its objective is application-oriented but still formal: to identify a common modeling pattern through which institutions, normative orders, social facts, and knowledge structures can be represented as historically generated configurations whose social status depends on domain-specific criteria. The discussion first defines formation as an inquiry-relative status change, then distinguishes aggregation from constituted and stabilized structure, and finally develops four application families. The examples are schematic. They illustrate how the formalism can organize an inquiry; they do not establish that the named social phenomena possess a unique CW representation.

13.1 Formation as an Inquiry-Relative Status Change

A social structure is rarely identified by topology alone. A connected subcomplex can fail to constitute an institution, a filled cellular boundary can lack legal force, and a recurrent event pattern can remain socially unrecognized. The CW carrier records incidence and attachment; the type system records modeled distinctions; the grammar records admissible structural transformations; the history records realized derivations; and the dynamical data can record weights, recognition levels, capacities, amplitudes, or other application-specific quantities. A formation claim should therefore state which combination of these coordinates is being used to identify the relevant social structure.

Let D denote a domain-specific structure type, such as an institution, normative order, recognized social fact, or knowledge structure. A *formation criterion* for D is a scoped predicate on the dynamical social CW state introduced in Equation 80. The criterion is stated in Equation 91.

$$F_D(\mathfrak{Z}_t; \theta_D) \in \{0, 1\}. \quad (91)$$

The parameter package θ_D contains the definitions, thresholds, observables, evidential rules, and resolution choices required by the particular inquiry. Equation 91 is consequently a modeling interface rather than a universal sociological law. Its role is to make explicit the additional conditions under which a generated cellular configuration is interpreted as an instance of D .

A formation transition occurs when a realized update changes the predicate from zero to one. For a realized event ε_t carrying $\mathfrak{Z}_t$ to $\mathfrak{Z}_{t+1}$, the transition condition is given in Equation 92.

$$F_D(\mathfrak{Z}_t; \theta_D) = 0, \quad F_D(\mathfrak{Z}_{t+1}; \theta_D) = 1. \quad (92)$$

Equation 92 separates formation from the mere occurrence of a cellular rewrite. A rewrite can alter the carrier without satisfying the formation criterion, while a change in dynamical or recognition data on a fixed carrier can, in some applications, satisfy the criterion without changing carrier topology. The distinction is important for social phenomena whose status depends partly on recognition, authority, convention, or persistence rather than solely on incidence structure.

The criterion also permits one named structure to be represented differently across inquiries. At coarse resolution an organization can be treated as one primitive face, while at finer resolution its internal roles, procedures, documents, decisions, and relations can be represented by a distributed subcomplex. Section 4.4 formalized such changes of primitive resolution. The formation criterion must therefore be indexed implicitly or explicitly by the research question and resolution. A claim that a structure has formed is meaningful only relative to those declared choices.

13.2 Constitutive Thresholds in Aggregated Structure

Section 5.3 distinguished lower-dimensional aggregation from composition-bearing higher attachment. The distinction becomes operational in formation analysis when the inquiry asks whether a collection

of primitives merely co-occurs or has acquired a modeled form of constituted unity. Let $Y_t \subseteq X_t$ be the subcomplex carrying the candidate structure. A formation criterion can inspect at least four different aspects of Y_t : its primitive footprint, its higher attaching structure, its persistence through realized history, and its relation to the active grammar.

A useful schematic profile is stated in Equation 93.

$$\mathcal{C}_D(Y_t) = (\mathcal{P}_D(Y_t), \mathcal{A}_D(Y_t), \mathcal{H}_D(Y_t), \mathcal{G}_D(Y_t)). \quad (93)$$

Here $\mathcal{P}_D$ records the domain-relevant primitive participation, $\mathcal{A}_D$ records composition-bearing attachments and higher incidence, $\mathcal{H}_D$ records the required historical persistence or provenance, and $\mathcal{G}_D$ records the relation between the structure and the grammar, including rules that reproduce, constrain, or modify it. The four coordinates are analytical placeholders whose concrete definitions belong to the application. Equation 93 does not require every domain to use every coordinate.

This profile permits several distinct cases. A large primitive footprint with weak or absent composition-bearing attachment can represent extensive aggregation without modeled constitution. A compact structure with strong higher incidence can satisfy a compositional criterion even when few primitives are involved. A structure can also possess the required cellular organization while failing a historical criterion because the relevant persistence interval, provenance condition, or sequence of authorizing events has not been realized. Finally, a structure can be historically realized yet remain unstable because the current grammar no longer reproduces the relations on which its formation criterion depends.

Formation and stabilization should therefore remain distinct. Let $I = [t, t+m]$ be a declared interval of realized stages. A simple persistence condition for D is given in Equation 94.

$$F_D(\mathfrak{Z}_s; \theta_D) = 1 \quad \text{for each } s \in I. \quad (94)$$

Equation 94 defines persistence only with respect to the discrete stage index of the model. A substantive study must justify why the selected interval and sampling of stages constitute meaningful evidence of stabilization. More demanding criteria can require robustness under a family of admissible perturbations, recurrent regeneration after local disruption, or persistence across analytical scales. The core framework leaves those choices open.

The constitutive profile also makes it possible to distinguish a social structure's existence from its maintenance mechanism. A structure can satisfy $F_D = 1$ at a given stage while the rules that originally generated it are no longer active. Conversely, a grammar can continue to admit formation even when no currently realized instance satisfies the criterion. The model can therefore represent constitution, persistence, reproducibility, and generative availability as different properties.

13.3 Institutional Formation in Distributed Subcomplexes

Institutional formation provides a useful application because an institution is difficult to represent faithfully as a single cell across resolutions. Within the present framework, an institution can instead be modeled as a typed distributed subcomplex $I_t \subseteq X_t$ together with a domain-specific formation criterion. The subcomplex may contain primitive faces representing roles, procedures, resources, documents, facilities, or other analytically selected units; edges represent relevant couplings among those primitives; vertices remain events; and higher cells can represent composition-bearing structures when the inquiry requires explicit higher incidence.

A minimally specified institutional criterion can depend on three classes of evidence: organized composition, rule-governed reproduction, and historical continuity. This dependence is expressed schematically in Equation 95.

$$F_{\text{inst}}(\mathfrak{Z}_t; \theta_{\text{inst}}) = \Phi_{\text{inst}}(I_t, \Gamma_t|_{I_t}, H_t|_{I_t}, \Xi_t|_{I_t}). \quad (95)$$

The restrictions in Equation 95 mean “the information relevant to the modeled institutional substructure” and need not be literal mathematical restrictions when the corresponding coordinate is represented exter-

nally. The function Φ_{inst} must be defined by the study. It can require, for example, specified composition-bearing attachments, repeated authorization events, continuity of membership or resources, or the existence of rules under which institution-specific acts remain admissible.

The grammar is especially important here because institutional structure can influence later admissibility. Once a role, office, procedure, or authorization structure has formed, later events may be admitted only when they occur through the newly constituted interfaces. A nonreflexive model can represent this by rules whose applicability changes with the carrier state. A reflexive model from Section 6.8 can additionally represent cases in which a realized institutional transformation modifies the rule system itself. The distinction depends on whether the change is encoded as state-dependent application of an unchanged grammar or as an actual update of Γ_t .

Institutional dissolution can be represented without treating it as the exact inverse of formation. A later state may fail the institutional criterion because constituent cells are removed, higher attachments are replaced, authorization rules change, relevant dynamical capacities decay, or a historical condition ceases to be satisfied. These paths need not be topologically equivalent. Formation and dissolution can therefore display hysteresis: the route by which a structure loses institutional status can differ from the route by which it acquired that status. The history formalism of Section 7 retains that difference even when two terminal carriers are isomorphic.

The distributed representation also permits institutional overlap. A primitive face or intermediate composite can participate in several institutional subcomplexes, as formalized by the overlap profiles of Section 5.5. The model can therefore represent shared personnel, shared infrastructure, joint authority, or overlapping jurisdiction without assigning each constituent to one exclusive hierarchy. The social meaning of the overlap remains a domain interpretation supplied by the study.

13.4 Normative Order Formation

Normative and legal structures require an additional distinction because some of their most important components function simultaneously as generated social structures and as conditions on later admissibility. A promulgated rule, recognized convention, judicial interpretation, or constitutional provision can be represented in the carrier as a historically generated object while also contributing to the grammar or context through which later events are evaluated. The formal architecture therefore allows a normative item to have several modeled roles without identifying those roles.

Let $N_t \subseteq X_t$ denote the carrier portion used to represent a candidate normative order. Its generated structure can include norm-creating events, authoritative texts treated as primitives at the selected resolution, relations among sources, and higher cells representing compositional or hierarchical organization when those relations are part of the inquiry. Separately, the active grammar Γ_t can include admissibility predicates whose evaluation depends on N_t or on dynamical data associated with it.

A normatively relevant generated structure can therefore alter later possibility while remaining distinct from the grammar that formalizes that effect. In a state-dependent but nonreflexive representation, the grammar contains a fixed rule whose admissibility predicate reads the current normative carrier. In a reflexive representation, the generated event updates the grammar itself. These two encodings can be observationally similar for some questions while differing historically and generatively. Section 11 provides the comparison vocabulary needed to state which equivalence matters.

Legal formation can be treated as a more specifically typed case when the inquiry supplies legal validity conditions. A legal status should not be inferred from cellular filling or topological closure alone. The domain criterion can require an authorized chain of events, competent institutional participation, a specified source relation, temporal validity, jurisdictional context, or other legally relevant data. The topology can make the organization of those dependencies explicit, while the legal interpretation comes from the criterion F_D and the corresponding type and admissibility system.

This division of labor permits the model to distinguish at least three questions: whether a normatively relevant structure has been generated, whether it satisfies the study's criterion for recognition or validity, and whether it currently constrains later admissible generation. These statuses can coincide in a simple model, but the formalism does not require them to do so. A historically generated norm can

lose operative force while remaining part of the historical carrier; a recognized rule can admit exceptions through contextual predicates; and a normative order can contain several partially overlapping suborders with different scopes.

13.5 Social Fact Stabilization

The distinction between historical occurrence and socially stabilized fact is particularly well suited to the separation among events, generated structure, history, and formation criteria. An event can occur in the represented history without the corresponding proposition, classification, status, or attribution becoming socially stabilized. Conversely, a socially effective classification can persist through repeated recognition and reproduction even when participants disagree about its interpretation.

Let h denote a historically represented event or event configuration contained in H_t and supported by the carrier. Let $S_h(t) \subseteq X_t$ denote the distributed structure through which the event is classified, recorded, cited, recognized, institutionalized, disputed, or otherwise socially mediated at stage t . A social-fact criterion can be written as in Equation 96.

$$F_{sf}(h, \mathfrak{Z}_t; \theta_{sf}) = \Phi_{sf}(S_h(t), H_t, \Xi_t, \Gamma_t; \theta_{sf}). \quad (96)$$

Equation 96 makes the historical occurrence and its social stabilization separate arguments. The construction does not imply that every social fact must be grounded in a single prior event; h can itself be an extended event configuration or other historical referent appropriate to the application.

The separation permits several analytically distinct cases. A historical event can be present while the relevant social-fact criterion remains unsatisfied. A socially stabilized attribution can later weaken or fragment if recognition structures, records, institutional endorsements, or reproduction rules change. Several competing social-fact structures can also coexist around the same historical referent. Their difference can be represented through distinct typed subcomplexes, different higher attachments, different dynamical recognition data, or different grammatical consequences.

The CW extension is useful when the inquiry concerns the organization of recognition itself. A reduced 2-complex can encode who recognized what, but explicit higher structure can preserve the topology of how recognition is composed across organizations, records, communities, or levels of authority. The question is therefore the same one developed in Sections 10 and 11: whether the higher organization is merely data needed for reconstruction or a recurrent object of the analysis.

The formation criterion also prevents a topological feature from being mistaken for collective acceptance. A highly connected recognition structure need not satisfy the substantive threshold chosen by a study, and a sparse but authoritative structure can satisfy it under a different institutional definition. Empirical operationalization must determine which observations support Φ_{sf} . The topology organizes those observations once the relevant social meaning has been specified.

13.6 Knowledge-Structure Formation

Knowledge production supplies a further application in which primitives, couplings, higher composition, grammar, and history can all change across resolution. At one resolution a published paper can be treated as a primitive face; at another its claims, methods, datasets, derivations, or experimental events can be resolved internally. Couplings can represent citation, evidential dependence, methodological reuse, conceptual translation, replication relation, or another typed dependence selected by the inquiry. Higher cells can represent explicitly modeled composites such as coherent argument structures, integrated methods, research programmes, or other higher-order formations when their incidence is analytically relevant.

A knowledge-structure criterion can distinguish the presence of information from the formation of an organized epistemic structure. Let $K_t \subseteq X_t$ be the relevant subcomplex. A study can define $F_{know}(\mathfrak{Z}_t; \theta_{know})$ using requirements concerning evidential support, inferential organization, uptake, reproducibility, disciplinary recognition, or other declared criteria. The present framework supplies the structural and generative coordinates but does not determine an epistemology of knowledge by itself.

The grammar can represent methodological and disciplinary constraints on what counts as an admissible continuation. A new method can create new possible couplings among previously separate primitives; a new evidential standard can remove or weaken admissible transformations; a new representational technique can refine the primitive decomposition; and a reflexive epistemic event can modify the grammar under which later work is evaluated. These possibilities make knowledge formation a useful example of the distinction between changing the generated corpus and changing the rules by which later contributions can be constituted.

Multiscale analysis can also be relevant. A concept can appear stable at the level of papers while fragmenting into heterogeneous lineages at the level of claims, or several fine-scale structures can aggregate into a robust coarse-scale research programme. The scale-indexed architecture of Section 9 permits these patterns to be compared without declaring one resolution ontologically privileged. Fractal or multifractal language would require the additional metric, measure, and scaling conditions developed there; repeated scholarly subdivision alone is insufficient.

This application also illustrates why history should remain explicit. Two fields can display similar present citation or conceptual topology while having different derivational histories, institutional memories, methodological grammars, and future successor sets. Structural similarity can therefore coexist with generative nonequivalence. The comparison requires the multi-axis equivalence framework of Section 11 rather than a single structural similarity score.

13.7 Comparative Formation Profiles

The preceding applications share a common formal pattern while preserving different substantive criteria. Table 9 summarizes the principal coordinates. The table is intended as a design aid for future empirical or computational models; it does not assign universal primitives or universal thresholds to any domain.

Table 9: Illustrative formation profiles across social domains. Each row identifies a possible use of the common CW/grammar architecture while leaving primitive selection, evidential thresholds, and substantive interpretation to the application.

Domain	Structural focus	Generative focus	Principal interpretive caution
Institution	Distributed subcomplex, higher composition, overlapping support	Reproduction of roles, procedures, authorization, and possible grammar update	Institution should not be identified with one cell or one dimension
Normative order	Sources, acts, typed dependencies, composition among rules	Conditions under which later acts become admissible, valid, or constrained	Topological closure does not establish normative or legal validity
Social fact	Historical referent plus distributed recognition and stabilization structure	Reproduction, contestation, recognition, recording, and possible decay	Historical occurrence and socially stabilized status remain distinct
Knowledge structure	Claims, works, methods, evidence, conceptual dependencies, higher synthesis	Methodological admissibility, evidential continuation, refinement, disciplinary reproduction	Structural coherence does not by itself establish epistemic truth

The common architecture can be summarized without imposing a common social ontology. For each

domain D , the analyst selects primitives relative to $(\mathcal{Q}, \sigma)$, constructs or infers the relevant CW carrier, specifies the type system and generative grammar, identifies the historical and dynamical information needed by the inquiry, and defines a domain formation criterion F_D . The resulting model can then distinguish candidate structure, grammatical admissibility, realized generation, satisfaction of the formation criterion, and subsequent stabilization.

This sequence also clarifies the role of higher cells. Their value is greatest when the inquiry repeatedly analyzes higher incidence, constituted composition, overlap among composites, recursive organization, or scale-dependent structure. When those properties are peripheral, Section 10 provides conditions under which an enriched 2-complex may remain more economical. Social structure formation therefore supplies an application domain for the representation choice rather than an argument that every social structure requires explicit high-dimensional cells.

The application families also prepare the next problem. Social structures rarely develop under one homogeneous grammar. Legal, institutional, cultural, technical, scientific, and interpersonal rule systems can overlap while authorizing different attachments, preserving different interfaces, or assigning different meanings to the same event configuration. A theory of generated social structure therefore requires a treatment of partially compatible and conflicting grammars. The next section develops that problem through structural mediation and grammar extension.

14 Structural Mediation of Generative Grammar Conflict

Section 13 applied the CW–grammar architecture to several domains in which generated structures can acquire institutional, normative, social, or epistemic status. Such domains can coexist within one social spacetime while relying on different type systems, admissibility predicates, historical memories, and rules of structural continuation. The present section develops a formal treatment of heterogeneity at that interface. Its objective is to distinguish mere grammatical difference from conflict, to define compatibility relative to a shared structural problem, and to model mediation as the construction of an interface through which previously incompatible generative structures can acquire jointly admissible continuations. The treatment remains structural: it does not assume that mediation produces agreement, that heterogeneous grammars should converge, or that a mediator is neutral by definition.

14.1 Conflict Conditions for Heterogeneous Grammars

Let two modeled domains, communities, institutions, or other relational formations be represented at stage t by generative packages

$$\mathfrak{P}_t^A = (X_t^A, \Xi_t^A, \Gamma_t^A, H_t^A; c_t^A), \quad \mathfrak{P}_t^B = (X_t^B, \Xi_t^B, \Gamma_t^B, H_t^B; c_t^B). \quad (97)$$

Equation 97 permits the two packages to use different primitive selections, cell types, admissibility predicates, histories, and dynamical data. Their difference therefore need not be reducible to disagreement over one proposition or one rule.

Grammar heterogeneity alone is analytically insufficient for conflict. Two grammars can authorize different transformations while operating on disjoint structures or while permitting mutually compatible outcomes. Conflict requires a specified domain of interaction. Let I_t be a typed interface complex together with cellular embeddings

$$\iota_A : I_t \hookrightarrow X_t^A, \quad \iota_B : I_t \hookrightarrow X_t^B. \quad (98)$$

The interface in Equation 98 identifies the structural material, boundary condition, resource, event configuration, jurisdictional locus, shared task, or other modeled object with respect to which compatibility is being assessed. The interface need not contain everything that the two formations share. It records the part relevant to the particular compatibility question.

For candidate successors $X'_A \in \text{Succ}_{\Gamma^A}(X_t^A; c_t^A, H_t^A)$ and $X'_B \in \text{Succ}_{\Gamma^B}(X_t^B; c_t^B, H_t^B)$, define an interface-relative compatibility predicate as stated in Equation 99.

$$C_{I_t}(X'_A, X'_B; \Theta_t) \in \{0, 1\}. \quad (99)$$

Here Θ_t collects the compatibility conditions declared by the inquiry, including the required preservation of I_t , type correspondences, admissibility conditions, and any contextual, historical, or dynamical restrictions that must hold jointly. A value of one means that the two candidate continuations admit a joint realization satisfying those declared conditions. A value of zero means that the pair fails at least one of them. A topological amalgam may therefore exist while the pair remains grammatically, legally, normatively, institutionally, or otherwise inadmissible under the modeled conditions.

Definition 14.1 (Interface-relative grammar conflict). The packages in Equation 97 exhibit *interface-relative grammar conflict* at I_t when each has at least one admissible successor relevant to the interface, while no relevant pair of admissible successors satisfies Equation 99.

Definition 14.1 makes conflict relative to a shared generative problem. Different grammars can coexist without conflict when their admissible successors can be jointly realized or when no shared interface requires coordination. Conversely, grammars with similar rule syntax can conflict because their admissibility predicates, historical conditions, or required interfaces differ. The definition therefore locates conflict in the relation among generative possibilities rather than in formal dissimilarity alone.

The definition also permits partial conflict. Let the task-relevant successor families be defined in Equation 100.

$$\mathcal{U}_A \subseteq \text{Succ}_{\Gamma^A}(X_t^A; c_t^A, H_t^A), \quad \mathcal{U}_B \subseteq \text{Succ}_{\Gamma^B}(X_t^B; c_t^B, H_t^B). \quad (100)$$

Some pairs in Equation 100 can satisfy $C_{I_t} = 1$ while others do not. A model can therefore represent localized incompatibility without declaring the complete grammars mutually inconsistent.

14.2 Partial Structural Alignment at Shared Boundaries

Mediation requires some representable relation between the structures under comparison. The interface I_t supplies one possibility, but direct identification can be too strong when the grammars use different primitive decompositions or cell types. A mediator can instead work through a third interface complex M_t equipped with partial cellular maps from selected subcomplexes $J_A \subseteq X_t^A$ and $J_B \subseteq X_t^B$. The alignment data are stated in Equation 101.

$$J_A \xrightarrow{\mu_A} M_t \xleftarrow{\mu_B} J_B. \quad (101)$$

The maps in Equation 101 need not be isomorphisms. They can forget distinctions, refine a common category, identify compatible boundary roles, or expose a correspondence that is only partial. Their analytical meaning must be specified by the application. A legal status, technical state, cultural category, or epistemic classification can therefore be aligned through a mediating interface without being asserted to be identical across the participating grammars.

A useful constraint is boundary compatibility. Suppose a cell or cellular pattern in J_A and one in J_B are proposed as mediated counterparts. Their images in M_t should preserve the incidence relations that the mediation declares relevant. A translation that maps a composite boundary in one grammar to an unrelated interior cell in the other may be formally writable while destroying the structural content that motivated the mediation. The required degree of preservation can range from typed cellular embedding to a weaker application-specific relation, but it should be stated explicitly.

The use of a third interface also permits mediation without full common ontology. The mediator can construct M_t from the smallest set of distinctions required for coordination while leaving substantial structures in X_t^A and X_t^B untranslated. This permits a form of structural partiality analogous to seeking sufficient common ground for a particular generative task rather than demanding global equivalence of the two formations.

14.3 Partial Translation between Cellular Grammars

Structural alignment of carriers does not by itself align the rules acting on them. Let

$$\Gamma^A = (\mathbf{T}^A, \mathcal{R}^A, \mathbf{A}^A), \quad \Gamma^B = (\mathbf{T}^B, \mathcal{R}^B, \mathbf{A}^B) \quad (102)$$

be the cellular generative grammars associated with the two packages. A mediation architecture can introduce an interface grammar

$$\Gamma^M = (\mathbf{T}^M, \mathcal{R}^M, \mathbf{A}^M), \quad (103)$$

whose rules operate on the mediating structure M_t or on a larger carrier containing it. Equations 102 and 103 separate the original rule systems from the rules used to coordinate them.

A *partial grammar translation* from A into the mediating grammar consists of mappings defined only on selected types, patterns, rules, and admissibility conditions of Γ^A . Denote such a translation by $\mathcal{T}_A : \Gamma^A \rightharpoonup \Gamma^M$, with $\mathcal{T}_B$ defined analogously. The partial arrow records that some rules or distinctions can remain outside the mediation. The translation is required to preserve only those structural and admissibility relations declared by the mediation problem.

The untranslated remainder is analytically important. For the rule component, define the remainder of A under $\mathcal{T}_A$ by Equation 104.

$$\text{Rem}_{\mathcal{R}}(\mathcal{T}_A) = \mathcal{R}^A \setminus \text{dom}_{\mathcal{R}}(\mathcal{T}_A), \quad (104)$$

with an analogous definition for B . Equation 104 prevents successful mediation of a selected interface from being interpreted as translation of the complete grammar. Large or substantively important remainders can remain even when a shared continuation has been constructed.

Translation should also preserve the difference between authorization and realization. If a rule r_A is translated into Γ^M , its image need not become admissible in every state in which r_A was admissible, because the mediating grammar can include additional joint conditions. Conversely, Γ^M can contain genuinely new rules that have no preimage in either original grammar. Mediation can therefore be generative rather than purely interpretive.

14.4 Mediator Structure for Interface Construction

The formal mediator is best treated as a structured role rather than a predetermined social actor. A human mediator, institution, treaty mechanism, technical protocol, translation practice, adjudicative procedure, or other relational formation can instantiate that role when it constructs and maintains the required interface. Definition 14.2 states the minimal formal object.

Definition 14.2 (Structural mediator). A *structural mediator* for two grammar-bearing packages is a tuple

$$\mathfrak{M}_t = (M_t, \Gamma_t^M, \mathcal{T}_A, \mathcal{T}_B, \mathcal{C}_t^M), \quad (105)$$

where M_t is a mediating cellular interface, Γ_t^M is an interface grammar, $\mathcal{T}_A$ and $\mathcal{T}_B$ are partial translations from the participating grammars, and $\mathcal{C}_t^M$ specifies the compatibility conditions for mediated continuations.

Equation 105 makes mediation structurally richer than transmission of messages. The mediator can modify which distinctions are exposed, how boundaries correspond, which rules can be expressed at the interface, and which joint continuations count as admissible. This capacity can open a successor that neither original grammar could realize directly while preserving substantial untranslated structure on both sides.

A mediator can operate at several strengths. *Coordinative mediation* leaves Γ^A and Γ^B unchanged and searches for compatible successors already available within them. *Translational mediation* constructs mappings that make latent compatibility expressible. *Extensional mediation* introduces new interface

rules or higher cells that enlarge the space of jointly admissible continuations. *Reflexive mediation* additionally permits updates of one or both participating grammars. These forms are analytically distinct because the last two change the generative conditions rather than merely selecting among existing possibilities.

The mediator need not occupy a position external to the social spacetime. If mediation is itself socially generated, the construction of M_t , the adoption of Γ_t^M , and the acceptance or rejection of its translations are events and structures within the larger CW model. The mediator can therefore be represented as part of the same generative history whose conflict it helps transform.

14.5 Mediated Extension of Generative Possibility

A successful mediation should be stated in terms of changed generative possibility rather than presumed consensus. Let $\widehat{X}_t$ denote a carrier that contains representations of the two participating structures and the mediating interface. Let Γ_t^* be the grammar obtained by adjoining the mediating rules and any explicitly accepted translations to the relevant rule environment. The mediated successor family is defined in Equation 106.

$$\text{MSucc}_{\mathfrak{M}}(\widehat{X}_t) = \left\{ Y \mid Y \in \text{Succ}_{\Gamma_t^*}(\widehat{X}_t; \widehat{c}_t, \widehat{H}_t), C_t^M(Y) = 1 \right\}. \quad (106)$$

Equation 106 contains jointly admissible continuations enabled by the mediation architecture. The set can remain empty, contain one continuation, or contain several alternatives. Mediation is therefore compatible with failure, underdetermination, and persistent plurality.

Definition 14.3 (Generatively effective mediation). A structural mediator is *generatively effective* for the specified conflict when the original interface-relative compatibility problem has no admissible joint continuation, while Equation 106 is nonempty under the mediated extension.

Definition 14.3 does not require the parties' complete grammars to merge. It requires the mediation to create at least one jointly admissible continuation for the specified problem. The distinction is particularly important for heterogeneous social orders. A treaty mechanism can create a shared procedure without making domestic legal systems identical; a technical protocol can enable interoperability without unifying internal architectures; a scholarly translation can permit joint reasoning without erasing disciplinary differences.

Mediated extension can also be represented by explicit higher-cell generation. Suppose the participating 2-skeletons expose compatible translated boundary structures after mediation. A new higher cell can then record a constituted joint organization whose boundary spans material from both sides. In such a case the mediator affects both grammar and generated topology: the grammar renders the attachment admissible, and the realized attachment becomes part of the CW carrier. This construction makes the relation between mediation and the paper's broader thesis precise. A mediating rule can alter the topology of possible composition, while the resulting higher structure records one realized outcome of that altered possibility space.

14.6 Grammar Revision through Reflexive Mediation

Some conflicts cannot be resolved within fixed participating grammars because the relevant admissibility conditions themselves block every joint continuation. A reflexive mediation can then propose updates of the grammars rather than only an interface extension. Let the post-mediation grammars be Γ_{t+1}^A and Γ_{t+1}^B . Their update is represented in Equation 107.

$$(\Gamma_{t+1}^A, \Gamma_{t+1}^B, \Gamma_{t+1}^M) = \mathcal{K}(\Gamma_t^A, \Gamma_t^B, \Gamma_t^M, \widehat{X}_t, \widehat{H}_t, \widehat{c}_t). \quad (107)$$

The operator $\mathcal{K}$ in Equation 107 is not assumed to be universal. It represents whatever constitutionally, institutionally, negotiated, computationally, or otherwise specified process is permitted to revise

the grammars. A model must therefore distinguish who or what can authorize $\mathcal{K}$, which rules it may modify, and which histories or constraints limit the update.

Reflexive mediation can generate a shared rule while retaining heterogeneous local rules. It can also modify only one side, modify the mediating grammar alone, or fail because no authorized revision is available. The formalism therefore permits asymmetric and partial transformation. This is more appropriate for many social settings than an assumption that mediation culminates in a common grammar.

The historical record should retain such revisions. Once a mediated rule has been accepted and used repeatedly, later admissibility can depend on the fact of its prior establishment. The mediator can thereby become unnecessary as an active actor while the generated interface grammar remains historically effective. Conversely, withdrawal, invalidation, or institutional collapse can remove or weaken those mediated rules in later stages. Mediation is consequently representable as a temporally extended transformation of generative conditions rather than a single static correspondence.

14.7 Structural Asymmetry in Mediation

A structural mediation formalism should not infer neutrality from the existence of a shared interface. The construction of M_t selects which distinctions become representable, and the translations $\mathcal{T}_A$ and $\mathcal{T}_B$ determine which rules can enter the mediated domain. These choices can preserve one grammar more faithfully than another, exclude important remainders, or impose a type system that is easier for one side to satisfy. The formal architecture can expose such asymmetry even when it cannot determine its normative acceptability by topology alone.

Let $\mathcal{D}_A \subseteq \mathcal{R}^A$ and $\mathcal{D}_B \subseteq \mathcal{R}^B$ denote the rule domains considered materially relevant to the mediation. A finite-rule application can record translation coverage by the ratios in Equation 108.

$$\text{Cov}_A = \frac{|\mathcal{D}_A \cap \text{dom}_{\mathcal{R}}(\mathcal{T}_A)|}{|\mathcal{D}_A|}, \quad \text{Cov}_B = \frac{|\mathcal{D}_B \cap \text{dom}_{\mathcal{R}}(\mathcal{T}_B)|}{|\mathcal{D}_B|}. \quad (108)$$

Equation 108 is only a coverage diagnostic. Equal values do not establish fairness, and unequal values do not establish injustice, because rules can differ greatly in importance and the relevant domains themselves are modeling choices. The diagnostic is useful mainly for making representational asymmetry visible and for preventing a partially translated grammar from being described as fully represented.

Mediation can fail in several structurally distinct ways. *Alignment failure* occurs when no adequate mediating interface can be constructed. *Translation failure* occurs when relevant types or rules cannot be mapped while preserving the required structure. *Admissibility failure* occurs when a shared representation exists but no mediated successor satisfies the joint conditions. *Authorization failure* occurs when a possible grammar revision cannot be legitimately enacted within the modeled system. *Realization failure* occurs when a mediated successor is admissible but is not historically realized under the dynamics of Section 12. These failure modes should remain distinct because they call for different explanatory accounts.

The same distinction also prevents mediation from being evaluated only by terminal structure. Two mediations can generate isomorphic joint carriers while differing substantially in translation remainders, rule revision, historical authorization, and future successor sets. Their comparison therefore requires the multi-axis equivalence framework of Section 11 rather than carrier topology alone.

14.8 Scope of Structural Mediation

The present construction gives grammar conflict a relational and generative interpretation. Heterogeneous grammars conflict relative to a specified interface when their admissible continuations cannot be jointly realized under the declared compatibility conditions. A mediator can construct a partial interface, translate selected structures and rules, introduce an interface grammar, and under suitable conditions enlarge the family of jointly admissible continuations. Successful mediation can therefore create new generative possibility without requiring global equivalence or homogenization of the participating grammars.

The formalism also clarifies the relation between mediation and higher CW structure. The mediator does not supply higher cells by definition. It changes the conditions under which particular joint attachments or rewrites can become admissible. When such a transformation is realized, the resulting higher cell or composite structure belongs to the generated CW carrier, while the mediating grammar remains part of the rule environment that made the structure possible. This preserves the paper's central separation between generated topology and generative law.

The analysis remains limited in several respects. Compatibility predicates, translation maps, rule domains, and authorization mechanisms require substantive interpretation and empirical or doctrinal support in concrete applications. The existence of a mathematically coherent mediating structure does not establish political legitimacy, legal validity, ethical acceptability, or epistemic adequacy. Mediation can also reproduce power asymmetries through selective representation or control of grammar revision. These questions motivate the ontological and methodological restraint developed in the next section, which locates the CW and grammar constructions within the three-level account of reality used by the broader Generative Relational framework.

15 Three Levels of Reality in the CW Social-Spacetime Formalism

The CW and grammar constructions developed in the preceding sections are intended as formal resources for relational analysis. Their usefulness depends on a clear account of what kind of reality they are meant to represent and what kinds of ontological claims they do not establish. This section locates the event-coupling-primitive architecture, higher CW cells, generative grammars, historical records, and manifestation-dependent observations within the three-level distinction used by the broader Generative Relational framework. The objective is methodological as well as philosophical: the distinction prevents formal structure from being reified as ultimate ontology, while also preventing context-dependent descriptions from being conflated with the relational structures that make those descriptions possible. The discussion proceeds through ontological suspension, relational reality, manifestation reality, formal observation, primitive selection, and a set of safeguards for interpreting the framework.

15.1 Ontological Suspension and Formal Scope

The first level concerns ontological reality in the strongest sense: whatever ultimately exists independently of the present formal description. The framework suspends a positive determination of that level. It therefore does not assert that ultimate reality is cellular, topological, grammatical, event-based, spin-foam-like, manifold-like, or exhausted by relational description. Ontological suspension is a restriction on claim strength rather than a denial of reality. The formalism can represent persistent relational structures and their consequences without converting the representational vocabulary into a metaphysical proof.

This restriction is particularly important for the CW extension. A higher cell can make a higher-order incidence or composition explicit in the model, but its existence in the carrier does not establish that social reality contains a literally n -dimensional object in an ultimate ontological sense. Likewise, an attaching map can encode how a modeled composite is constituted from lower-dimensional boundary structure without establishing that social constitution is fundamentally a topological attachment process. The analytical claim is weaker and more precise: under the specified research question, resolution, typing, and interpretation, the CW construction can preserve structural relations that would otherwise be relocated into auxiliary data.

The same restraint applies to the grammar. Section 6 defined a cellular generative grammar as a formal object governing admissible structural transformations. Its rules can model historically effective conditions of generation, authorization, compatibility, or constitution. They are not thereby asserted to be metaphysical laws governing the ultimate production of reality. A grammar can be empirically, doctrinally, historically, or theoretically well supported at the relational level while the ontological basis of that regularity remains open.

For clarity, let $\mathcal{O}$ denote the ontological level, $\mathcal{R}$ the relational level considered below, and $\mathcal{M}_c$ a manifestation level indexed by context c . The present framework adopts the placement convention stated in Definition 15.1.

Definition 15.1 (Three-level placement convention). The social CW carrier, its typed cells, the generative grammar, realized history, and associated dynamical data are formal representations directed primarily at selected structures of relational reality. Context-dependent observations, descriptions, interfaces, classifications, and presentations belong to manifestation reality or to formal representations of manifestation reality. No formal identification between the model and ontological reality is assumed.

Definition 15.1 prevents two opposite errors. Ontological reification would treat the model's cells or grammar as the ultimate furniture or laws of the social world. Manifestational reduction would treat whatever appears in a particular description, database, institutional record, legal classification, or observer perspective as exhausting the relational structure being modeled. The three-level account permits a middle position in which relational structures can be sufficiently stable and causally or generatively consequential to warrant formal representation while remaining distinct from both ultimate ontology and any single manifestation.

15.2 Relational Reality of Generated Social Structure

Relational reality is the principal target level of the present formalism. It concerns historically constituted structures whose persistence, couplings, constraints, and generative effects are sufficient for the research problem even when their ultimate ontology is left open. Persons, institutions, legal statuses, currencies, scientific fields, norms, social facts, and other social beings can therefore be modeled as relationally real when their historically organized relations have consequences that cannot be reduced, for the inquiry at hand, to a momentary description by an observer.

The CW carrier X_t developed in Section 4 is best understood as a model of a selected relational structure at historical stage t . Its 2-skeleton retains the event–coupling–primitive architecture of Section 2, while higher cells make selected higher incidence and composition explicit. The grammar Γ_t models conditions under which structural transformations are admissible, and the history H_t records the realized derivational path required by the analysis. The dynamical package of Section 12 therefore has a relational interpretation rather than an ontological identity claim.

To make the distinction explicit, let $\mathcal{R}_Q$ denote the portion of relational reality relevant to research question Q . A model construction at resolution σ can be represented abstractly by the modeling relation in Equation 109.

$$m_{Q,\sigma} : \mathcal{R}_Q \rightsquigarrow \mathfrak{P}_{Q,\sigma}, \quad (109)$$

where $\mathfrak{P}_{Q,\sigma}$ is a representation package of the kind introduced in Section 11. The relation symbol in Equation 109 is intentionally weaker than a function or isomorphism. Model construction can be selective, many-to-one, uncertain, revisable, and dependent on evidence and interpretation. The equation therefore records directed representation without presupposing complete recoverability of relational reality from the formal package.

Relational reality also clarifies the status of generated structure. When a grammar-authorized transformation is historically realized, a new cell, rewritten subcomplex, altered coupling, or changed rule environment can represent a change in relational organization. The represented change can be socially consequential even when it is invisible from some manifestation contexts. An institutional rule can remain effective despite being unknown to many participants; a scientific priority structure can persist through citation and recognition practices even when an individual observer misidentifies it; a legal relation can exist despite incomplete public awareness. The framework therefore permits relational reality to exceed any single epistemic presentation.

At the same time, relational reality is not treated as automatically transparent to the modeler. A higher cell is justified only when the research design and evidence support treating the corresponding

composition or incidence as structurally relevant. The formation criteria of Section 13 remain domain-specific, and the mediation constructions of Section 14 require substantive interpretation of compatibility, translation, and authorization. Relational reality supplies the target level; it does not remove the need for evidential and conceptual justification.

15.3 Manifestation Reality and Context-Dependent Presentation

Manifestation reality concerns the context-dependent forms through which relational structures become presented, accessible, classified, expressed, measured, narrated, or otherwise available to an observer or participating system. A manifestation can be linguistic, legal, institutional, quantitative, visual, computational, archival, phenomenological, or practical. Different manifestations can expose different portions of the same relational structure and can impose distinct categories on what they present.

Let c index a manifestation context, such as a legal forum, administrative database, ethnographic encounter, scholarly literature, participant perspective, computational sensorium, or institutional reporting system. A manifestation relation can be represented abstractly by Equation 110.

$$\pi_c : \mathcal{R}_Q \rightsquigarrow \mathcal{M}_{c,Q}. \quad (110)$$

Equation 110 does not require π_c to be injective, surjective, deterministic, or lossless. A context can omit relationally important distinctions, combine several structures under one category, reveal only a selected interface, or generate a presentation partly through the observer's own practices. The relation can therefore be partial and many-to-many.

The context dependence in Equation 110 matters directly for the paper's reduction argument. A manifestation can make a higher-dimensional relational organization appear effectively two-dimensional because the observing system records only events, pairwise couplings, and selected primitive labels. Another context can expose the higher incidence among those primitives. The difference in presentation does not by itself establish that the relational structure changed. Conversely, a change in manifestation can sometimes be generated by an actual relational transformation, such as legal recognition, institutional reclassification, or revision of a scientific taxonomy. The formal analysis must therefore distinguish changes in $\mathcal{M}_{c,Q}$ from changes in $\mathcal{R}_Q$.

The distinction is also relevant to social facts. Section 13 separated historical occurrence from the stabilization of a social fact. A historical event can have occurred while its manifested attribution remains contested, suppressed, forgotten, or institutionally unrecognized. A later change in manifestation can alter recognition without changing the historical occurrence itself, while sufficiently extensive and persistent changes in recognition practices can also modify relational social-fact structure. The three-level account permits these possibilities to be represented separately rather than being collapsed into a single binary notion of reality.

15.4 Formal Observation and Manifestation-Sensitive Adequacy

A formal model is evaluated through manifestations, evidence, and observations even when its target is relational reality. The framework therefore requires an explicit distinction between the model package and the observation procedure used to compare the model with accessible material. Let $\hat{\pi}_c$ be a formal observation operator that maps a representation package to model-predicted or model-extracted manifestations. Equation 111 states the construction.

$$\hat{\pi}_c : \mathfrak{P}_{Q,\sigma} \longrightarrow \widehat{\mathcal{M}}_{c,Q}. \quad (111)$$

The output $\widehat{\mathcal{M}}_{c,Q}$ in Equation 111 is a modeled manifestation, not the manifestation itself. Empirical or interpretive adequacy concerns the relation between $\widehat{\mathcal{M}}_{c,Q}$ and accessible manifestations in $\mathcal{M}_{c,Q}$ under a declared comparison criterion.

This architecture yields the comparison pattern stated in Equation 112.

$$\mathcal{R}_Q \overset{\mathfrak{m}_{Q,\sigma}}{\rightsquigarrow} \mathfrak{P}_{Q,\sigma} \xrightarrow{\hat{\pi}_c} \widehat{\mathcal{M}}_{c,Q}, \quad \mathcal{R}_Q \overset{\pi_c}{\rightsquigarrow} \mathcal{M}_{c,Q}. \quad (112)$$

Equation 112 is not assumed to commute exactly. Differences between the modeled and accessible manifestations can arise from model error, measurement limitations, interpretive disagreement, contextual transformation, incomplete evidence, or instability in the relational structure itself. The pattern nevertheless clarifies where empirical and interpretive criticism enters: evidence bears on the adequacy of the representation and observation model without converting successful prediction into proof of ultimate ontology.

Definition 15.2 (Manifestation-sensitive adequacy). A representation package is *manifestation-sensitive adequate* for research question Q over a declared context family $\mathcal{C}$ when its formal observation operators preserve the distinctions and relations required by Q across the contexts in $\mathcal{C}$ to the degree specified by the inquiry's evidence and comparison criteria.

Definition 15.2 is deliberately weaker than global truth or complete structural recovery. A model can be adequate for one family of manifestations and inadequate for another. This context dependence is compatible with the inquiry-relative reduction criteria of Section 10: an enriched 2-complex can be observationally sufficient for the manifestations relevant to one question even when an explicit CW representation is preferable for another question concerned with higher incidence or generative genealogy.

15.5 Primitive Selection across Relational and Manifestation Levels

The status of faces as primitives requires special care within the three-level account. Section 2.2 treated primitiveness as relative to a research question and analytical resolution. The present section adds a level distinction: a primitive is selected within the formal representation of relational reality, while the evidence used to justify that selection is accessed through manifestation contexts. Neither step establishes ontological indivisibility.

Let $P_{Q,\sigma}$ denote the primitive family selected for question Q at resolution σ . A refinement can replace a primitive $p \in P_{Q,\sigma}$ by an internal complex when a finer question or resolution requires its internal organization. The availability of such refinement is one reason that primitive status should be interpreted methodologically. A university can be treated as a primitive in a model of international accreditation exchange and as a large structured subcomplex in a model of departmental governance. A scholarly paper can be primitive in one citation-level analysis and decomposed into claims, methods, evidence, and rhetorical acts in another. The relational target can remain continuous across these changes while the formal decomposition changes with the inquiry.

Manifestation contexts can also bias primitive selection. Administrative data may present legally registered organizations as natural units, bibliometric systems may present publications as basic units, and legal doctrine may foreground persons, acts, or instruments. Such units can be useful primitives, yet their convenience or institutional visibility does not establish their analytical adequacy for every question. The modeling relation in Equation 109 therefore should be informed by multiple manifestations when the question depends on structures that one context systematically obscures.

The same reasoning applies to higher cells. If a context explicitly records an institution, treaty regime, doctrinal category, or scientific field, the corresponding label is evidence for a relational structure but does not by itself determine its cellular dimension. Conversely, a higher-order organization can be relationally consequential even when no manifestation context supplies a ready-made name for it. The modeler may infer and test such organization through incidence, history, generative effects, or multiscale persistence, subject to the evidential restraint stated in the writing principles of the project.

15.6 Grammar, Dynamics, and Reality-Level Placement

The separation between generated CW structure and generative grammar becomes clearer when placed within the three-level account. A grammar is a formal model of relationally effective conditions of admissible generation. Its rules can be manifested through constitutions, statutes, technical standards, organizational procedures, linguistic conventions, scientific norms, informal practices, or recurring behavioral expectations. The manifested rule text or observed regularity is not identical to the relational grammar

itself, because interpretation, enforcement, historical precedent, institutional capacity, and interacting rules can alter which transformations are actually admissible.

The same distinction applies to dynamics. The realization laws of Section 12.2 can assign weights, probabilities, amplitudes, hazards, or other selection structures to admissible successors. Such quantities belong to the model of relational dynamics. Their empirical estimation uses manifestation data, while their successful fit does not establish that social reality literally computes the fitted quantity. A probabilistic law can be an adequate relational representation without being an ultimate stochastic ontology.

Reflexive grammar change also spans relational and manifestation levels. A constitutional amendment, protocol revision, institutional decision, or negotiated agreement can appear as a manifested event and simultaneously alter the relationally effective grammar when the relevant authorization and uptake conditions are satisfied. The manifested text alone may be insufficient if the change is invalid, unenforced, ignored, or overridden by other structures. Conversely, relational grammar can change through evolving practice before a formal text is updated. The distinction therefore helps explain why grammar evolution cannot be inferred mechanically from documents or observable statements alone.

For mediation, the same discipline is essential. The mediating interface of Section 14 can create a manifestation in which heterogeneous formations appear mutually translatable. Relational equivalence requires stronger conditions and may remain partial. A shared terminology, treaty clause, software schema, or negotiated category can facilitate coordination while leaving untranslated structures and divergent grammars intact. The manifestation of agreement should therefore be analyzed together with the relational successor sets and historical uptake that determine whether generative possibility has actually changed.

15.7 Interpretive Safeguards for the CW Framework

The three-level account can be operationalized as a set of safeguards against recurrent category errors. Table 10 summarizes the principal risks and the corresponding interpretive discipline.

Table 10 does not weaken the formalism. It identifies the level at which each formal claim earns its force. A proof about cellular homology can be mathematically strong while its social interpretation remains conditional. A verified legal rule can strongly support a grammar component while the ontology of legal normativity remains suspended. A robust empirical scaling law can justify a multiscale model while leaving open whether the same scaling is fundamental or historically contingent. Precision about claim level permits stronger local arguments because the framework does not require every successful representation to carry an unnecessary metaphysical burden.

The safeguards also clarify the status of structural equivalence. The multi-axis equivalence taxonomy of Section 11 concerns representation packages and the relational distinctions they are designed to preserve. Informational, topological, generative, historical, or multiscale equivalence between two packages does not establish ontological identity. Conversely, nonequivalent manifestations can still be generated by relational structures that are equivalent at the axis relevant to the research question. The choice of equivalence relation should therefore state both the formal axis and the reality level to which the interpretation is directed.

15.8 Scope of the Three-Level Account

The three-level account gives the CW framework a disciplined philosophical placement. Ontological reality remains suspended rather than denied or formally captured. Relational reality supplies the principal target for representing historically constituted social structures, couplings, generative conditions, and dynamical consequences. Manifestation reality records the context-dependent forms through which those structures become accessible, classified, measured, or expressed. Formal observation operators connect the model to manifestations without erasing the distinction between representation, evidence, and target.

This placement has several consequences for the paper's central thesis. The value of an explicit CW representation is a claim about structural transparency at the relational-modeling level, not a claim that higher-dimensional cells constitute ultimate social ontology. The value of the generative grammar is

Table 10: Reality-level safeguards for interpreting the CW and grammar formalism.

Risk	Overstatement	Required discipline
Cellular reification	A social structure literally is an n -cell because the model uses one.	Treat cellular dimension as inquiry-relative structural representation unless an independent ontological argument is supplied.
Primitive reification	A face is fundamentally indivisible.	Treat primitive status as relative to $\mathcal{Q}$ and σ and permit refinement when internal structure becomes relevant.
Grammar reification	Formal rewrite rules are ultimate laws of social generation.	Interpret Γ as a model of relationally effective admissibility conditions supported by the relevant evidence and domain theory.
Manifestational conflation	A recorded, named, measured, or institutionally recognized object exhausts the underlying relational structure.	Treat manifestations as context-dependent access paths that can omit, combine, or transform relevant distinctions.
Topological overinterpretation	A homology class, hole, or invariant has a fixed social meaning.	Supply an application-specific interpretation and preserve the mathematical/social distinction developed in Section 8.
Fractal overinterpretation	Higher CW dimension or visual complexity establishes fractality.	Require explicit scale, metric, measure, or refinement evidence as specified in Section 9.
Mediation reification	A common interface proves shared ontology, fairness, or complete agreement.	Retain translation remainders, authorization conditions, asymmetries, and the partiality of generative compatibility.

a claim about representing admissible relational transformations, not a claim that society literally operates as a formal rewriting machine. The question-relative choice of primitives is methodological, while higher incidence, history, and multiscale organization can become objects of relational analysis when the evidence and inquiry justify making them explicit.

The three-level distinction also explains why the reduction question remains inquiry-relative. A 2-complex can be fully adequate for a manifestation-centered question even when it hides higher incidence that would matter for a relational-structural question. An explicit CW representation can in turn be unnecessarily elaborate when the relevant manifestations and transformations are already preserved by a simpler model. Model choice should therefore be governed by the structure of the research question, the available evidence, the required equivalence axes, and the costs of explicit representation rather than by a presumption that higher-dimensional formalism is intrinsically closer to reality.

These constraints prepare the analysis of the framework's limitations in the next section. The main limitations arise not only from computational complexity or nonuniqueness of cellular decomposition, but also from the dependence of primitives, cell types, manifestation maps, admissibility conditions, and social interpretations on substantive modeling choices that topology alone cannot determine.

16 Limits of CW Representation

The CW formalism developed in the preceding sections increases structural transparency only under a specified modeling discipline. Its cells, attaching maps, grammars, histories, and scale structures do not determine their own empirical interpretation, and several distinct representational choices can support the same observations. This section therefore identifies the main limits of the framework before a broader research programme is proposed. The objective is to separate limitations that arise from cellular representation itself from limitations caused by primitive selection, empirical access, semantic interpretation, grammar specification, or computational cost. The discussion proceeds through nonuniqueness of cellular decomposition, resolution dependence, identifiability, interpretive ambiguity, noncellular and continuum structure, grammar uncertainty, computational scalability, and the explanatory limits of topology.

16.1 Nonuniqueness of Cellular Decomposition

A topological space can admit more than one CW decomposition. In the present framework this mathematical flexibility becomes a substantive modeling issue because the cells themselves carry social types, dynamical data, historical provenance, and roles in generative rules. Two cellular models can therefore have carriers with the same underlying topology while presenting different primitive partitions, incidence patterns, and rule interfaces. A claim that depends on the particular cells or their dimensions is consequently stronger than a claim that depends only on the topology of the realized carrier.

Let $|X|$ denote the underlying topological space represented by a CW complex X , and let $\mathcal{C}(|X|)$ denote a family of admissible cellular decompositions used for an inquiry. A decomposition-sensitive observable is a map J whose value can vary across members of $\mathcal{C}(|X|)$ even when the underlying space is unchanged. Equation 113 states this possibility.

$$|X| \cong |Y| \quad \text{while} \quad J(X) \neq J(Y). \quad (113)$$

Equation 113 does not indicate an error. It indicates that J is a property of the chosen cellular representation rather than a topological invariant of the underlying space. Cell counts, typed incidence patterns, primitive footprints, and many grammar interfaces can have this status.

This distinction limits strong interpretations of higher-dimensional structure. The presence of a particular 3-cell can be analytically useful because it makes a selected composition explicit. Another decomposition may represent the same underlying relational region through several cells or through a different attachment pattern. The modeled composition is therefore supported by the combination of topology, typing, primitive selection, and interpretation. Higher cellular dimension alone does not establish a unique decomposition of social structure.

A practical response is to specify the equivalence class relative to which a result is intended to be stable. Topological claims can be tested under homeomorphism, homotopy equivalence, or selected homological equivalence. Cellular claims can instead be tested under a narrower family of admissible subdivisions, refinements, or typed cellular isomorphisms. The choice should follow the research question rather than an expectation that one decomposition is intrinsically canonical.

16.2 Primitive Selection and Resolution Dependence

The framework treats faces as primitives relative to a research question Q and analytical resolution σ . This methodological choice avoids ontological reification, yet it introduces a second source of nonuniqueness. Different primitive selections can generate different 2-skeleta and different higher attachments even when they are intended to represent the same relational domain.

Let $P_{Q,\sigma}$ be the primitive family used at resolution σ . A change of resolution can replace one primitive by an internal complex, merge several primitives into one coarse unit, or alter which couplings are represented explicitly. Such changes need not yield a simple nested inclusion of carriers. The relation

between two resolutions is therefore represented more cautiously by a comparison map of the form stated in Equation 114.

$$\rho_{\sigma',\sigma} : X_{Q,\sigma'} \rightsquigarrow X_{Q,\sigma}, \quad \sigma' \text{ finer than } \sigma. \quad (114)$$

The relation symbol in Equation 114 allows partial, many-to-one, or interpretation-dependent comparison. A strict cellular map is appropriate only when the selected constructions support one.

Primitive selection can therefore change the apparent location of structure. What appears as a higher-cell composition at one resolution may become internal organization of a face at another. Conversely, a face treated as atomic in one inquiry may be decomposed into a substantial event–coupling–primitive structure in another. This flexibility is useful, but it means that statements about “the dimension” or “the number of higher compositions” of a social structure are incomplete without the associated Q , σ , typing system, and refinement convention.

A robust result should therefore be tested for resolution stability at the level appropriate to the claim. Some observables may remain approximately or exactly preserved under a specified refinement family; others are intentionally resolution-dependent. The framework does not require invariance under every change of primitive selection. It requires the dependence to be visible rather than silently treated as a property of the modeled social world.

16.3 Empirical Identifiability from Manifestations

Section 15 distinguished relational structure from its context-dependent manifestations. That distinction produces an empirical limitation: different CW packages can generate the same available observations. A highly articulated cellular model can therefore be formally coherent while remaining underdetermined by the evidence used to support it.

Let $\mathfrak{P}$ be a representation package and let $\mathcal{O}_{\text{emp}}$ denote the family of empirical observation procedures available to an inquiry. Each procedure $O \in \mathcal{O}_{\text{emp}}$ acts through a manifestation or measurement interface and returns an observable value. Definition 16.1 states a question-relative identifiability criterion.

Definition 16.1 (Empirical identifiability up to equivalence). Let $\sim_Q$ be the equivalence relation required by research question Q . A representation package $\mathfrak{P}$ is *empirically identifiable up to $\sim_Q$* within a model class $\mathcal{C}$ when every $\mathfrak{P}' \in \mathcal{C}$ satisfying Equation 115 also satisfies $\mathfrak{P}' \sim_Q \mathfrak{P}$.

$$O(\mathfrak{P}') = O(\mathfrak{P}) \quad \text{for every } O \in \mathcal{O}_{\text{emp}}. \quad (115)$$

Definition 16.1 makes identifiability relative to both evidence and analytical purpose. A model can be identifiable with respect to selected aggregate observables while remaining nonidentifiable with respect to higher attaching maps, generative history, or grammar. Additional higher cells do not solve this problem by themselves; they can increase the number of latent structures that require empirical support.

Proposition 16.2 (Observational underdetermination of stronger equivalence). *Suppose $\mathfrak{P}_1$ and $\mathfrak{P}_2$ satisfy Equation 115 for all available empirical observations but fail to satisfy the equivalence relation required by Q . Then the available observations are insufficient, by themselves, to identify the required equivalence class.*

Proposition 16.2 follows directly from Definition 16.1. Its importance is methodological. A higher-dimensional representation can be analytically illuminating without being uniquely recoverable from observed manifestations. In such cases the manuscript should distinguish empirically identified structure, theoretically motivated completion, and exploratory formalization.

16.4 Semantic Ambiguity of Higher Cells

Topology specifies incidence and attachment, while social interpretation specifies what those structures mean in a given domain. Section 5 therefore treated composition-bearing higher cells as a typed subclass

rather than assigning a universal social meaning to dimension. The same restraint remains necessary throughout the framework.

Let $\tau(e^n)$ denote the formal type of an n -cell and let $\text{Sem}_{Q,c}$ be an interpretation relation indexed by research question and context. A semantic assignment can be represented abstractly by Equation 116.

$$\text{Sem}_{Q,c} : (e^n, \tau(e^n), \partial e^n, H_t) \rightsquigarrow \mathcal{S}_{Q,c}, \quad (116)$$

where $\mathcal{S}_{Q,c}$ is the relevant domain of social interpretations. Equation 116 is not generally one-to-one. Similar attachment patterns can support different interpretations, and one social structure can require several cells of different dimensions.

This semantic openness limits claims such as “a filled boundary is an institution” or “a nontrivial cycle is a social conflict.” A filled boundary is first a topological fact about the model. Whether it represents legal closure, organizational constitution, normative integration, semantic compatibility, or another relation depends on typing and substantive evidence. Likewise, an unfilled cycle can be analytically relevant without automatically representing absence, dysfunction, contradiction, or incompleteness in the social domain.

The same problem applies to mediation and grammar. A formal translation between two typed interfaces can represent part of a mediating process, but normative legitimacy, political acceptability, interpretive fidelity, or fairness require additional criteria. The cellular formalism can expose where such criteria act; it does not supply them automatically.

16.5 Continuum, Metric, Measure, and Noncellular Structure

CW complexes are flexible topological objects, but the present social formalism is still a choice of representation. Some research problems are organized primarily by quantities that topology alone does not retain: duration, distance, intensity, probability, flow, measure, smoothness, order, causal direction, semantic content, or continuous fields. These quantities can be added to a CW carrier, as Sections 12 and 9 already permit, yet the need for substantial auxiliary structure can signal that topology is serving only as one layer of the model.

A continuum process can sometimes be approximated by increasingly fine cellular decompositions. Such approximation is analytically useful only when the refinement scheme preserves the quantities required by the inquiry and when the approximation error is controlled at the relevant level. A sequence of finer cellulations does not by itself establish convergence of social meanings, dynamical laws, or empirical estimates.

The framework should therefore remain open to hybrid representations. A CW carrier can be coupled to metric, measure, probabilistic, differential, operator-algebraic, categorical, linguistic, or agent-based structures when those are independently justified. In other cases a noncellular representation may be more economical. The criterion developed in Section 10 applies symmetrically: explicit CW structure is useful when cellular incidence and generation are central analytical objects, not merely because the formalism can be enriched until it contains everything needed by another method.

16.6 Grammar Specification and Rule Uncertainty

The generative grammar of Section 6 is a second major source of modeling discretion. A grammar can be specified from legal rules, institutional procedures, historical regularities, computational design, qualitative reconstruction, or theoretical hypotheses. These sources differ in evidential status. A formally precise rule does not become empirically warranted merely because its left and right cellular patterns are well defined.

For a rule r , let $\kappa(r)$ denote its evidential status and let A_r denote its admissibility predicate. The grammar can then be treated as carrying both formal and epistemic information. A rule inferred from repeated observed transitions may have a different status from a rule derived from an explicit statute or from a speculative mechanism introduced for exploration. The framework should preserve those distinctions when grammars are compared or updated.

History-sensitive and reflexive grammars create additional difficulty. If admissibility depends on a latent historical summary, incomplete records can make rule applicability uncertain. If realized structures modify the grammar itself, the rule system becomes nonstationary. A grammar learned from one historical interval may therefore fail to describe later transformation without revision. Reflexivity is a substantive feature of some social domains, yet it also weakens the assumption that a compact fixed rule set can summarize long-run generation.

Rule uncertainty suggests a probabilistic or set-valued extension in some applications. Instead of a single asserted admissibility predicate, an inquiry may work with a family of candidate predicates or confidence-weighted rules. Such an extension would alter the epistemic layer of the grammar without changing the distinction between grammar and generated CW structure.

16.7 Computational Scalability and Structural Explosion

The explicit preservation of higher structure has computational costs. The number of cells can increase rapidly under refinement, and a grammar can generate many candidate matches and successor structures from one state. Historical branching, multiscale representations, partial translations, and reflexive rule updates can multiply the number of objects that must be stored or compared. These burdens arise even when each individual cellular operation is conceptually simple.

A useful bookkeeping description separates four sources of cost: carrier size, rule-matching cost, branching of admissible successors, and retained historical or multiscale memory. Let the complexity profile at stage t be the tuple stated in Equation 117.

$$\mathfrak{C}_t := (N_t, M_t, B_t, R_t), \quad (117)$$

where N_t summarizes explicit carrier size, M_t the work required to identify rule matches under the chosen implementation, B_t the number or effective breadth of admissible successors considered, and R_t the retained memory needed for history and scale. Equation 117 is a diagnostic profile rather than a universal complexity measure.

The profile clarifies why the most structurally transparent representation need not be the most computationally practical one. A research workflow can retain an explicit CW model for selected subsystems while reducing other regions to enriched 2-complexes or summary statistics. It can truncate the maximal explicit cell dimension, restrict the rule family, retain only history summaries sufficient for current admissibility tests, or use coarse-to-fine analysis. Such approximations should be declared because each changes which equivalence axes remain available.

Computational economy also reinforces the paper's central comparison. If higher incidence is rarely queried, storing it explicitly can impose cost without analytical benefit. If higher incidence is repeatedly reconstructed from metadata, an explicit CW representation can reduce conceptual and implementation complexity despite increasing the nominal carrier dimension. Efficiency is therefore architecture-dependent rather than monotone in cell dimension.

16.8 Limits of Topological Explanation

Topology can identify connectedness, cycles, fillings, higher attaching structure, and invariants under specified transformations. These properties can be important for social analysis, but they do not by themselves explain why a social structure formed, why a rule is legitimate, why one interpretation dominates another, why an actor exercises power, or why a particular outcome is normatively desirable. Such questions can require historical evidence, causal models, political theory, jurisprudence, economics, psychology, semantics, ethics, or other forms of inquiry.

The formalism is therefore strongest when topology identifies a structural problem that can be coupled to additional explanatory variables. A nontrivial homology class may locate a persistent structural feature; historical analysis is still needed to explain its formation. A higher-cell attachment may represent a constituted composite; institutional analysis is still needed to establish its authority and reproduction. A

Table 11: Principal limitations of the CW social-spacetime representation and corresponding methodological responses.

Limitation	Risk	Methodological response
Nonunique cellular decomposition	Decomposition-sensitive quantities are treated as intrinsic topology	State the admissible decomposition family and the intended equivalence relation
Primitive selection	Resolution-dependent structure is reified as an ontological primitive hierarchy	Index primitives by research question and resolution; test relevant refinement stability
Empirical nonidentifiability	Formal higher structure is presented as uniquely supported by observations	Distinguish identified structure, theoretically motivated completion, and exploratory alternatives
Semantic ambiguity	Cells, cycles, or fillings are assigned automatic social meanings	Require typing, interpretation relations, and domain-specific formation criteria
Non-topological quantities	Metric, measure, intensity, order, or meaning is lost or forced into topology	Use enriched or hybrid models when those quantities are primary
Grammar uncertainty	Formal precision obscures uncertain, contested, or changing rule evidence	Record rule provenance and epistemic status; allow revision or candidate grammars
Computational scalability	Explicit structure becomes too costly to construct, match, or evolve	Use inquiry-safe reduction, truncation, summaries, and multiresolution workflows
Topological overexplanation	Structural patterns are mistaken for causal, normative, or semantic explanations	Couple topology to the disciplinary methods required by the substantive question

grammar may specify admissible transformations; realization dynamics are still needed to account for which admissible path occurs. A mediation interface may open a shared continuation; normative and political analysis are still needed to evaluate the terms of that continuation.

This division of labor also limits causal interpretation. A change in topology between X_t and X_{t+1} records a structural transition in the model. The transition does not identify its cause unless the generative rule, dynamical mechanism, evidence, and relevant confounders or alternative explanations are separately specified. Likewise, the persistence of a topological invariant can coexist with major changes in social meaning, distribution, intensity, or power relations that the selected invariant does not detect.

16.9 A Discipline for Using the CW Extension

The limitations above support a disciplined rather than maximal use of the CW extension. Table 11 summarizes the principal limitation and the corresponding modeling response.

The resulting principle is one of representational proportionality. Explicit higher cells are warranted when higher incidence, attachment, composition, topology, or multiscale generation are themselves objects of analysis and when the available evidence and computational resources support the corresponding distinctions. Enriched 2-complexes remain preferable when those distinctions can be summarized safely. Other formalisms remain preferable when cellular organization is peripheral to the problem.

The limits therefore do not weaken the central contribution of the paper. They specify its domain. The CW extension is best understood as a formal resource for preserving and analyzing the topology of generated social structure when that topology would otherwise be relocated into auxiliary coding. Its validity in a particular application depends on explicit primitive selection, semantic interpretation, em-

pirical identifiability, grammar justification, and comparison with simpler or alternative representations. The next section develops a research programme organized around these open tasks rather than assuming that the present formalism closes them.

17 Research Programme for Social CW Generativity

The formal architecture developed in Sections 4–16 supports a research programme whose central task is to determine when explicit higher-dimensional cellular structure yields empirically and analytically useful information about social-spacetime generation. The objective of this section is to organize that programme into tractable lines of work while preserving the distinctions established throughout the paper among carrier structure, primitive selection, grammar, realization dynamics, history, manifestation, and scale. The programme is methodological and comparative. It emphasizes reconstruction from evidence, inference and testing of generative rules, multiscale analysis, representation comparison, computational implementation, domain applications, and formal extensions. Each direction is therefore framed as an open research task rather than as a completed consequence of the present formalism.

17.1 Empirical Reconstruction of Social CW Complexes

The first research direction concerns reconstruction. The present framework presupposes that a research question Q , an analytical resolution σ , a primitive family $P_{Q,\sigma}$, and a typed CW representation can be specified with sufficient empirical discipline. In applications, these objects must be recovered from heterogeneous manifestations such as archival records, legal documents, organizational traces, transaction logs, communication histories, interviews, databases, or experimentally generated observations. The reconstruction problem is therefore an inverse problem from manifestations to a family of admissible representation packages.

Let D denote an empirical record assembled under an observation design $\mathcal{O}$, and let $\mathcal{C}_Q$ denote a model class appropriate to the research question. The reconstruction correspondence is stated in Equation 118.

$$\text{Rec}_{Q,\sigma}(D, \mathcal{O}) \subseteq \mathcal{C}_Q. \quad (118)$$

Equation 118 deliberately returns a set rather than a unique model. Section 16 showed that empirical manifestations can underdetermine cellular decomposition, higher attaching maps, grammar, and history. A reconstruction method should therefore report ambiguity, confidence, and alternative admissible structures whenever the evidence does not identify one equivalence class.

A practical reconstruction pipeline should contain at least four separable decisions. First, the inquiry specifies which recorded occurrences count as events at the selected temporal and semantic resolution. Second, it selects primitive faces and typed couplings according to the operational meaning of $P_{Q,\sigma}$. Third, it tests whether higher incidence or compositional structure is supported strongly enough to justify explicit higher cells. Fourth, it records which assignments are directly observed, inferred from domain rules, reconstructed probabilistically, or introduced as theoretical completion. This provenance discipline is necessary for later comparisons between empirically identified CW structure and exploratory higher-dimensional extensions.

The programme should also develop inter-coder and inter-model stability tests. If two reconstruction procedures operating on the same record generate substantially different higher attachments while agreeing on the 2-skeleton, the disagreement identifies a region where the claimed higher structure requires additional evidence or a narrower interpretation. Stability across admissible primitive selections and moderate changes of resolution would provide a stronger basis for treating a recovered higher-order pattern as analytically robust.

17.2 Validation of Inferred Generative Grammars

The second research direction concerns the grammar Γ . Section 6 defined grammar as a typed, interface-aware rule system with separate admissibility conditions. Applications require methods for constructing such grammars from evidence while preserving rule provenance and epistemic status. The relevant task is richer than fitting a transition matrix because a rule can depend on local topology, type compatibility, ambient context, historical summaries, and reflexive modification of the rule system itself.

Let $\mathcal{H}^{\text{obs}}$ be a collection of observed derivation fragments and let $\mathfrak{G}$ be a candidate grammar class. A grammar-inference procedure can be represented by Equation 119.

$$\text{Infer}_{\mathfrak{G}}(\mathcal{H}^{\text{obs}}, \mathcal{Q}, \sigma) = \hat{\Gamma}, \quad (119)$$

where $\hat{\Gamma}$ may be a single fitted grammar, a posterior family, or a set of grammars that remain observationally indistinguishable under the available evidence. Equation 119 therefore leaves the statistical or logical inference method open.

The central evaluation object should be predictive and counterfactual admissibility rather than retrospective fit alone. A candidate grammar can be tested by whether it correctly distinguishes transformations that subsequently occur, transformations that remain available but unrealized, and transformations excluded by explicit institutional, legal, physical, or domain constraints. Such testing requires the realization dynamics of Section 12, because grammatical admissibility and historical realization are distinct events in the model.

A further research problem concerns rule minimality. Several large rule sets may reproduce the same observed histories. Future work can compare grammars by descriptive length, number of rule schemas, contextual dependence, historical memory, predictive performance, and robustness under changes of primitive resolution. The resulting comparison should avoid equating syntactic compactness with explanatory adequacy. A more complex grammar can be warranted when the additional rule structure corresponds to independently supported institutional or historical distinctions.

17.3 Comparative Benchmarks for Representation Reduction

The third research direction tests the paper's central representational claim directly. The relevant comparison is among explicit CW models, enriched 2-complexes, string- or path-enriched carriers, internal-manifold constructions, graph or hypergraph models, and other representations capable of storing comparable information. The objective is to identify which analytical tasks become simpler, more stable, or more transparent under each architecture.

A benchmark should begin from a common empirical or synthetic generative system and construct several representation packages $\mathfrak{P}_1, \dots, \mathfrak{P}_m$. Let $\mathcal{T}_{\mathcal{Q}}$ be the family of tasks required by the inquiry, including selected observables, transformations, history queries, and multiscale comparisons. The task-relative burden of representation i can be summarized abstractly by Equation 120.

$$\mathfrak{B}_i(\mathcal{Q}) := (E_i, C_i, R_i, V_i), \quad (120)$$

where E_i denotes empirical adequacy, C_i computational cost, R_i reconstruction or decoding burden, and V_i structural visibility for the task family. Equation 120 is a comparison profile rather than a scalar ranking.

Such benchmarks would make the distinction between informational reducibility and structural preservation empirically operational. A 2-complex encoding that reconstructs the full CW object exactly may still require repeated decoding before higher incidence, composition, or grammar-compatible transformations can be computed. An explicit CW model may carry a larger native state space while reducing that decoding burden. Other inquiries may show the reverse pattern. The research objective is therefore to identify the boundary conditions under which explicit higher-cell structure pays for its additional complexity.

The benchmark programme should also include deliberate negative cases. Some datasets or questions should be expected to show little or no advantage for an explicit CW extension. Demonstrating such cases

would strengthen the framework by locating the domain in which the enriched 2-complex remains the preferable representation.

17.4 Multiscale Diagnostics of Recursive Structure

Section 9 introduced scale-indexed CW families, refinement maps, measure data, and candidate fractal or multifractal diagnostics. A fourth research direction is to turn that architecture into a reproducible multiscale methodology. The primary empirical problem is to determine whether observed recurrence across resolutions reflects a stable generative mechanism, a measurement artifact, a chosen cellulation convention, or a genuine scale-dependent organization of the modeled social structure.

A multiscale study should therefore estimate several quantities together: cell-growth rates, persistence of topological features across resolution, stability of attaching-pattern distributions, recurrence of rule schemas, and measure-scaling functions where an interpretable measure is available. Evidence for self-similarity should require convergence or stable scaling over an explicitly reported range rather than visual resemblance across a small number of resolutions. Standard fractal dimensions remain metric or measure-dependent quantities and should be calculated only when their assumptions are satisfied [6, 7].

Persistent homology can provide one component of this programme when the refinement system forms a genuine filtration or can be represented through an appropriate filtered construction [5]. The resulting topological persistence should be compared with grammar persistence. A homology class can survive across scale while the production rules generating its local cellular realization change, and similar rule families can recur while the global topology changes. Joint analysis of these two persistence notions may help distinguish structural self-similarity from superficial geometric repetition.

A further open problem concerns scale-dependent primitive choice. The programme should investigate whether refinement maps can be learned or justified so that primitive replacement at one scale is compatible with higher attachments at adjacent scales. Successful solutions would connect the paper's methodological notion of primitiveness to an explicit multiresolution calculus rather than treating each $P_{Q,\sigma}$ as an isolated modeling decision.

17.5 Dynamical Coupling to the Amplitude-Bearing 2-Skeleton

The fifth direction develops the interface between the retained 2-complex dynamics and the higher CW architecture. The present paper preserves the event-coupling-primitive 2-skeleton as a valid base on which amplitudes, spins or representation labels, operators, and other physics-inspired data can be placed. Higher cells add explicit compositional and generative organization when the inquiry requires it. Future work should determine how these two layers interact dynamically without treating the higher cells as additional physical spacetime dimensions.

One family of models can retain a dynamical state $\Xi_t^{(2)}$ on $X_t^{(2)}$ while associating higher-cell data $\Xi_t^{(>2)}$ with composition, constraint, admissibility, or coarse structural organization. The coupled state is represented in Equation 121.

$$\Xi_t = \left(\Xi_t^{(2)}, \Xi_t^{(>2)} \right). \quad (121)$$

Equation 121 leaves open whether the two components obey deterministic, stochastic, amplitude-based, operator-algebraic, or hybrid dynamics. The research problem is to derive interaction laws that preserve the semantic distinction between base social-spacetime variables and higher organizational structure.

A particularly important question concerns coarse-graining. If a higher cell summarizes or constitutes a family of primitive configurations, an effective dynamical law on that cell should ideally be related to the dynamics of its primitive footprint. Future work can test whether such effective laws can be derived, estimated, or renormalized across cellular resolutions. Failure of such derivations would identify domains where higher cells are useful only as structural descriptors rather than as dynamical variables.

The spin-foam analogy also requires continued restraint. The earlier 2-complex architecture can remain physics-inspired while the higher CW structure serves a distinct social-compositional role. Research should therefore compare models in which higher cells merely constrain the amplitude-bearing

2-skeleton with models in which higher cells receive their own weights or transition amplitudes. Each extension should state which mathematical structure is inherited from physics, which structure is newly proposed for social analysis, and which interpretation remains heuristic.

17.6 Higher-Dimensional Rewrite Semantics

The sixth research direction concerns the mathematical foundations of the grammar. Section 6 used an interface-span form inspired by algebraic graph transformation while avoiding the assumption that established graph-rewriting theorems transfer automatically to CW complexes [2]. A fuller theory would identify a category or related higher-dimensional setting in which the intended cellular replacements, preserved interfaces, typing, and attachment constraints admit precise existence and uniqueness conditions.

Several questions arise. One concerns the conditions under which a cellular rewrite admits a well-defined residual carrier after removal of the replaceable region. Another concerns concurrency: transformations supported on sufficiently independent subcomplexes should admit a principled notion of commuting or partially ordered execution. A third concerns rules acting on rules, coherence among transformations, and higher-dimensional derivations. Higher-dimensional rewriting traditions provide relevant mathematical precedents without determining the social formalism uniquely [4].

The formal programme should therefore seek a hierarchy of rewrite semantics matched to modeling needs. A minimal semantics may require only finite typed CW complexes and explicitly checked replacement conditions. Stronger categorical semantics may provide pushout-based constructions, concurrency theorems, compositional verification, or higher coherences. The choice should remain proportional to the empirical and analytical tasks rather than treating maximal abstraction as a goal in itself.

17.7 Reproducible Computational Infrastructure

The seventh direction is computational. An empirically useful programme requires data structures capable of storing typed cells, attaching maps, primitive footprints, dynamic annotations, rule matches, histories, and scale relations while supporting comparison with reduced representations. A reference implementation would make the formal distinctions of the paper executable and expose computational bottlenecks that remain invisible in purely symbolic development.

A minimal software architecture should separate at least five layers: carrier storage, semantic typing, grammar and rule matching, history/provenance, and analysis. The analysis layer should support cellular boundary matrices, homology and relative homology, attaching-map metadata, reduction experiments, equivalence tests, and multiscale summaries. Where persistent homology is appropriate, existing computational principles can be used at the filtration layer rather than reimplemented as part of the grammar itself [5].

Reproducibility requires versioned primitive specifications and grammars. A change in $P_{Q,\sigma}$ can alter the carrier even when the source data remain unchanged, while a change in Γ can alter the reachable language without changing the current CW state. Computational workflows should therefore record data provenance, primitive-selection rules, grammar versions, interpretation mappings, and random seeds or probabilistic settings where applicable. Such provenance is part of the scientific object because it determines which structural comparisons can be repeated.

Synthetic benchmark generators would be valuable before large empirical deployment. They can produce systems with known higher-cell attachments, known rewrite grammars, controlled noise, and controlled reduction schemes. Recovery experiments can then test whether reconstruction procedures distinguish carrier topology, grammar, history, and multiscale structure under finite data and observation error.

17.8 Comparative Domain Programmes

The eighth direction evaluates the framework through domain-specific programmes. Section 13 identified institutions, normative and legal orders, social facts, and knowledge structures as candidate appli-

cations. These domains should be approached comparatively because they differ in the observability of events, explicitness of rules, persistence of records, and meaning of compositional closure.

Legal and administrative systems offer relatively explicit rule sources and durable event records, making them suitable for testing grammar reconstruction, rule conflict, mediation, and reflexive rule change. Knowledge production offers citation, authorship, publication, and conceptual-development traces that can support multiscale and social-fact studies, while also exposing ambiguity in primitive definition. Organizational systems can test institutional formation and dissolution, especially when formal procedures coexist with informal relational practices. Comparative cases involving heterogeneous normative orders can test the mediation formalism of Section 14 without presuming that successful coordination requires full grammatical integration.

Each domain programme should begin with a narrowly stated research question and an explicit reason for choosing the CW representation over an enriched 2-complex or another formalism. The empirical study should then report the selected primitives, evidence for higher attachments, grammar provenance, uncertainty, reduction comparison, and sensitivity to resolution. Cross-domain comparison becomes meaningful only after these modeling choices are visible.

A useful progression is to begin with small, well-documented cases whose relevant generative events can be reconstructed with high confidence. Larger observational corpora can follow once the formal pipeline has demonstrated stability. This sequence limits the risk that an elaborate higher-dimensional model is fitted to data whose semantic and historical quality is insufficient to identify the structures the model distinguishes.

17.9 Evaluation Criteria for Programme Progress

The research programme should be evaluated by more than the production of increasingly elaborate CW models. Progress occurs when the framework yields distinctions, predictions, reconstructions, comparisons, or explanations that survive explicit tests against simpler representations and alternative interpretations. Table 12 summarizes the principal research directions and the evidence that would constitute progress.

A particularly important criterion is comparative necessity. The programme advances when a higher-dimensional construction reveals a structural property that matters to the inquiry and that is materially harder to state, compute, test, or preserve in the relevant reduced representation. This criterion keeps the original 2-complex model as an active baseline throughout the research programme.

The programme also advances through principled failure. If an enriched 2-complex repeatedly performs as well as an explicit CW model on the required tasks, the result identifies a domain where higher cells provide little additional analytical value. If higher attachments cannot be empirically identified, the appropriate conclusion is underdetermination rather than forced completion. If a grammar fails predictive or counterfactual tests, the rule system should be revised. These outcomes refine the scope of the framework and help separate useful structural generalization from unnecessary formal elaboration.

The research programme therefore treats CW extension, grammar, dynamics, history, and multiscale structure as independently testable components of a broader representational architecture. The next section concludes the paper by consolidating the central contribution: a social-spacetime 2-complex can remain representationally sufficient while an explicit CW extension becomes analytically valuable when the topology of higher-order generation is itself part of the research object.

Table 12: Research directions and criteria for substantive progress in the CW social-spacetime programme.

Research direction	Primary task	Evidence of progress
Empirical reconstruction	Recover typed carriers and higher attachments from manifestations	Stable reconstruction across admissible coding choices, explicit uncertainty, and improved fit to structural observations
Grammar inference	Recover and test admissible cellular transformations	Out-of-sample discrimination of admissible, inadmissible, and unrealized transitions with interpretable rule provenance
Representation comparison	Compare CW and reduced encodings on common tasks	Identified domains in which explicit higher structure changes analytical burden, observability, or predictive performance
Multiscale analysis	Relate structures across resolution and scale	Reproducible scaling ranges, persistent structural features, and refinement relations supported by data
Dynamical integration	Couple higher organization to the amplitude-bearing 2-skeleton	Derived or empirically supported interaction laws with clear interpretation and controlled coarse-graining error
Rewrite foundations	Formalize cellular replacement, concurrency, and higher coherence	Well-defined semantics and theorems whose assumptions match the intended model class
Computational infrastructure	Implement reconstruction, rewriting, topology, and provenance	Reproducible open workflows, benchmark performance, and explicit failure diagnostics
Domain applications	Test the framework in substantive social-scientific settings	Case results that outperform or clarify simpler representations while preserving domain-specific interpretation

18 Conclusion

The present section consolidates the paper’s formal and methodological argument. Its role is to state the contribution at the level justified by the preceding construction and to identify the conditions under which the CW extension adds analytical value. The central conclusion is deliberately comparative. The event–coupling–primitive 2-complex remains a valid and often sufficient representation of social spacetime, including models enriched by amplitudes, spins, operators, internal manifolds, strings, and other auxiliary structures. The CW extension becomes useful when the research object includes the higher-order organization through which those primitives and their composites are constituted, related, transformed, and reproduced across history or scale.

The paper therefore locates the motivation for CW complexes in structural preservation rather than representational capacity. A sufficiently enriched 2-complex can encode information about higher cells, attaching maps, histories, grammars, and scale relations. Exact reconstruction from such an encoding is formally possible under suitable schemes. Section 3 showed, however, that information retained as auxiliary data need not remain native structure of the reduced carrier. Section 10 sharpened the distinction

by treating reduction as inquiry-relative: analytical adequacy depends on whether the reduced representation preserves the observables, transformations, historical distinctions, and multiscale relations that the research question actually uses. Definition 10.3 therefore provides the appropriate criterion for choosing between an explicit CW representation and an enriched 2-complex rather than a general preference for one carrier dimension.

The retained 2-skeleton gives the generalization a cumulative relation to the earlier social-spacetime model. Under the identification in Equation 15, vertices remain events, edges remain typed couplings or relational interfaces, and faces remain primitives selected relative to a research question and analytical resolution. Higher-dimensional cells introduce explicit attachment and incidence only where the inquiry requires higher-order structural organization. Their dimension carries no universal sociological meaning. An institution, legal order, scientific field, market, or other relational being can occupy a distributed substructure involving several cellular dimensions, while another inquiry may represent the same object at coarser resolution as one primitive face. The resulting formalism therefore treats primitiveness and cellular dimension as modeling commitments whose adequacy must be assessed relative to the problem being studied.

This resolution-relative treatment of primitives is also essential to the multiscale part of the framework. Refining a face into a more detailed complex changes what is resolved without implying that the earlier primitive was ontologically elementary or false. A family of such refinements can preserve information about recurrence, scale-dependent topology, and generative organization that a single fixed-resolution carrier does not display directly. The analysis in Section 9 accordingly separated integer-valued CW dimension from fractal or multifractal quantities. Fractal interpretation requires an explicit scale structure and, where relevant, metric or measure data together with evidence for a scaling regime. This distinction allows multiscale generativity to be studied without treating higher cellular dimension itself as evidence of fractality.

A second central contribution is the separation of generated structure from the law of generation. Equation 21 keeps the CW carrier and generative grammar as distinct coordinates. The carrier records realized cellular organization; the grammar specifies typed, interface-aware transformations and the conditions under which those transformations are admissible. Section 6 developed this grammar through interface-preserving rewrite rules supplemented by contextual, historical, and potentially reflexive admissibility conditions. The distinction between matched, constructible, admissible, and historically realized transformations prevents a formal production rule from being confused with an event that actually occurred. It also permits two isomorphic present structures to possess different spaces of possible continuation because their grammars, histories, or contextual states differ.

The expressive status of the grammar follows the same discipline. Its native language is a language of reachable structured CW states rather than a language whose basic objects are strings. A string encoding can nevertheless be used to compare computational expressiveness with classical formal-language theory. Under the effectivity assumptions stated in Proposition 6.2, the encoded reachable language is recursively enumerable. Higher-dimensional cellular structure therefore supplies no automatic computational class beyond Type-0 grammars. Its contribution lies in structural-generative expressiveness: attaching maps, shared interfaces, higher incidence, compositional closure, and rule dependence can remain explicit in the objects on which the grammar operates.

Generative history adds another axis that is not recoverable from terminal topology alone. Section 7 distinguished general rewrite histories from genuine filtrations, identified conditions under which attachment-monotone histories support nested CW states, and formalized path dependence through differences in admissible futures. Section 11 then showed why informational, observational, carrier-topological, generative, historical, and multiscale equivalence should remain separate notions. Proposition 11.1 captures a central consequence: lossless informational equivalence does not in general imply carrier-topological equivalence, while carrier-topological equivalence does not in general imply generative or historical equivalence. The formalism therefore provides a vocabulary for comparing models that contain comparable data while exposing different structures of generation.

The dynamical extension preserves these distinctions in time. The state package in Equation 80 sepa-

rates the CW carrier, cell-associated dynamical data, grammar, realized history, and context. This makes it possible to represent fixed-carrier dynamics, topology-changing rewrites, grammar revision, and historical accumulation without identifying them as one process. The distinction between grammatical admissibility and realization dynamics is particularly important for social applications. Probability, amplitude, utility, power, hazard, institutional capacity, or other dynamical quantities can influence which admissible transition is realized without thereby defining what the grammar permits. Reflexive cases can additionally modify the grammar itself, allowing generated structures to alter the conditions of later generation.

The application sections indicate how this architecture can support social analysis while preserving domain-specific interpretation. Institutional formation, normative and legal order, social-fact stabilization, and knowledge-structure formation can each be represented through domain-relative formation criteria acting on the CW, grammatical, historical, and dynamical package. Higher cells make some forms of compositional organization visible, yet a topological filling is not by itself legal validity, institutional existence, social recognition, or epistemic warrant. The mediation formalism extends the same restraint to heterogeneous grammars: partial translation and mediated extension can create jointly admissible continuations without requiring full equivalence, neutral translation, or merger of the participating orders.

The three-level treatment of reality in Section 15 places these constructions within the broader Generative Relational commitment to ontological suspension. CW complexes, grammars, amplitudes, and higher-dimensional structures are formal representations of relational organization under stated modeling assumptions. Their empirical adequacy is evaluated through manifestation and observation, and their success does not establish them as ultimate ontology. This restraint also clarifies the status of the paper's central claim. The claim concerns the analytical consequences of making generative topology explicit, rather than a metaphysical assertion that social reality is fundamentally cellular.

The limitations developed in Section 16 remain part of the result. Cellular decompositions can be nonunique; primitive choice can be resolution-sensitive; higher attachments and grammars can be empirically underidentified; computational cost can grow rapidly; and topological structure does not substitute for causal, semantic, normative, political, or historical explanation. These limitations support a comparative modeling discipline. Explicit higher cells should be retained when they make a task-relevant organization directly analyzable, and they should be omitted when an enriched 2-complex or another formalism answers the question with lower representational burden.

The paper's core contribution can therefore be stated in a narrow form. The generalization from a 2-complex to a CW complex is valuable when the topology of how social structures are generated is itself part of the research object. In that setting, flattening higher organization into auxiliary data may preserve recoverability while relocating higher incidence, attachment, compositional closure, generative genealogy, and multiscale structure away from the native topology of the carrier. An explicit CW representation keeps those relations available to topological and grammatical analysis, while a separate generative grammar specifies the admissible transformations that can produce, revise, or extend them.

This conclusion also preserves the success of the earlier model. For inquiries centered on events, primitive couplings, amplitudes, operators, or other dynamics that are naturally expressed on the 2-skeleton, the enriched 2-complex remains an appropriate endpoint rather than an incomplete approximation. For inquiries centered on higher-order constitution, structural generation, historical path dependence, or multiscale organization, a CW extension provides an additional level of structural transparency. The resulting research programme is therefore a theory of representational choice as much as a higher-dimensional formalism: it asks which structure should remain native to the model, which information can safely be encoded indirectly, and which generative relations must remain explicit for the question at hand.

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