

Completion Probabilities in Finite Public-Service Referral Models with Uncertain Routing

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Abstract

A public-service referral route can be available while its probability of completion remains unspecified. This paper develops a finite, fixed-policy probability enrichment that preserves that distinction. Normalized transition rows define a path law over a time-bounded referral graph. A route-refinement example shows the sensitivity of uniform route counts to descriptive choices. Standard hitting-probability recursion and a finite derivation of the Doob h-transform explain how conditioning on completion changes observed route shares. For uncertain routing, independently selectable compact row sets support backward lower and upper recursions; a shared-parameter example shows how rectangular relaxation can widen the bounds through incompatible local choices. A fictional referral dossier connects these results to missing terminal records, admission denominators and participant-defined outcomes. Verified probability and robust dynamic-programming sources supply mathematical antecedents. The proposed contribution concerns the assumptions needed for interpretable completion estimates. Empirical calibration, causal policy effects, interacting service capacity and the normative value of completion remain open.

Keywords: public-service referrals; completion probability; uncertain routing; conditioning; robust dynamic programming; route representation

Research Scope and Formal Status

This paper develops a restricted conceptual application and conditional mathematical results from established project state. Its fictional cases are analytical constructions. Source-dependent claims were checked online before bibliography and manuscript writing, with abstract-only access retained as an explicit limitation in the accompanying literature records. The mathematical arguments specify the assumptions under which their conclusions follow. Empirical adequacy, normative authority and publication-level originality remain open to investigation. The project files preserve the argument, source verification, objections, formal records and validation procedure.

Statement on AI Assistance

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1 Introduction

A public-service referral model can describe which routes are available before it supplies a probability that a participant completes a route. This paper studies the assumptions needed to add that probability layer in a finite, fixed-policy setting. It derives path and completion calculations, examines route refinement and completed-case selection, develops coherent uncertainty bounds, and discusses their interpretation through a fictional referral dossier.

Consider a local information service that directs an admitted participant to one of two appointment channels. Both channels can lead to a completed appointment. A diagram of available routes establishes a form of attainability. Estimating how often completion occurs additionally requires a routing procedure, conditional completion behavior, a defined cohort and an observation scheme. Those ingredients can change while the route diagram remains the same.

The distinction becomes consequential in three ordinary modeling choices. First, splitting one route into several descriptive labels can change a probability obtained by counting labels uniformly. Second, a dataset containing only completed appointments can overrepresent routes with higher completion probabilities. Third, uncertainty about two stages may arise from one shared parameter, so selecting the worst value independently at each stage can describe an impossible joint situation.

The paper develops each issue within a small mathematical scope. A finite directed acyclic graph represents a bounded referral episode. Nonterminal states have normalized transition rows under a fixed policy. Terminal states distinguish the declared completion target from other termination. The graph may be obtained by unrolling a process to a fixed horizon; the horizon and terminal interpretation are part of the model.

The basic equations are established probability constructions. Hitting probabilities satisfy a first-step recursion. Conditioning on eventual completion gives a finite form of the Doob h -transform. Independently selectable row uncertainty supports backward lower and upper recursions. The paper supplies self-contained derivations and credits verified antecedents. Its proposed contribution concerns the interpretation and evidence requirements of these constructions in a referral model.

The normative target also requires care. Completing an appointment can be an administrative outcome while a participant reasonably declines, finds another service or judges the appointment unhelpful. Completion probability therefore has a limited referent. Claims about welfare, access, fairness or the causal value of a policy require additional definitions and evidence. The model keeps those questions visible without attempting to settle them through one number.

The sustained dossier separates admitted participants from the broader population, records missing terminal outcomes and compares independent with shared-parameter uncertainty. Its numerical values are constructed. They demonstrate logical effects and supply finite verification fixtures; they are not estimates of any actual service.

The resulting research proposal is a disciplined probability enrichment of an existing feasible-route model. It asks which assumptions make a completion estimate interpretable, which observations could support those assumptions, and how to report the remaining uncertainty. Empirical calibration and causal policy evaluation remain open obligations.

2 Probability Antecedents and the Application Boundary

This section identifies the verified mathematical antecedents and separates the proposed application from earlier feasible-route work. It also states the limits of source access. The ensuing proofs are complete for the paper's finite scope and do not rely on uninspected general theorem conditions.

Pitman's lecture on hitting probabilities derives the first-step harmonic equation with prescribed target and other boundary values [3]. The source also discusses escape and the conditions relevant to uniqueness in broader countable-state settings. The present paper avoids those general complications by using a finite acyclic episode graph and proving uniqueness by backward induction.

Bovier’s treatment of the Doob h -transform describes a change of Markov law through a ratio of harmonic-function values [1]. The paper applies the established construction to conditioning a finite referral episode on completion. A direct Bayes calculation and a telescoping path product make the meaning of the transformed law explicit. This is a conditional description of completed episodes, not an automatically implementable service policy.

Iyengar’s work on robust dynamic programming identifies rectangularity of uncertain conditional measures as a condition supporting robust recursion [2]. The verified publisher abstract supports that antecedent claim. The present result concerns compact row sets that can be chosen independently at the states of a finite acyclic graph under a fixed policy. Its proof is given directly, with shared-parameter dependence treated as a countercondition.

The repository’s existing route model already separates available mechanisms, prerequisites, resource constraints and information available to a selecting agent. It also treats probability as an optional enrichment requiring additional structure. The current paper retains those distinctions. Its focus is the inferential effect of route descriptions, outcome conditioning and coherent parameter constraints after a probability law has been proposed.

Several boundaries follow. The model concerns one declared episode and cohort. If people compete for capacity, an individual transition law may depend on the joint queue state; that dependence must be represented or justified as negligible for the chosen question. If policy adapts to earlier observations, the state must retain the relevant history or include time and information explicitly. The phrase “fixed policy” refers to a specified decision rule, which can include such declared state dependence.

The finite horizon also has substantive meaning. Ending an episode after three referrals differs from ending it after thirty days. The first is a step limit; the second requires time information or duration assumptions. The acyclic representation can incorporate either when properly constructed, but a step count alone supplies no calendar-time guarantee.

The proposed application is therefore modest. It brings standard probability tools to a concrete interpretive problem and makes their assumptions reviewable. It leaves broader novelty, empirical routing behavior and institutional effectiveness open. A useful contribution would show that these distinctions prevent a consequential misinterpretation of service records or support a better justified estimate.

The remaining sections define every symbol locally, derive the finite results and preserve their counterconditions. This arrangement allows the mathematical argument to be assessed independently from the eventual choice of a service, target or estimation method.

3 A Probability Layer over Feasible Referral Episodes

This section defines the episode graph and its probability law. It separates feasibility from positive probability and establishes normalization before introducing completion inference. These steps specify what the later numerical results refer to.

Let $G = (V, E)$ be a finite directed acyclic graph with a declared initial state s_0 . Let T and F be disjoint terminal sets whose union contains every sink. States in T satisfy the selected completion definition; states in F terminate the episode otherwise. Terminal states have no outgoing episode transitions. Every nonterminal state has a nonempty successor set $N(s)$.

The graph records feasible transitions under the declared scope. A state can include referral stage, relevant eligibility information, remaining resources and time when those variables affect the intended law. Two situations may share a state only if the conditional transition description is adequate for the chosen episode question. This adequacy is an assumption to be assessed, not a consequence of drawing the graph.

Assumption 3.1 (Normalized fixed-policy transition law). For each nonterminal s , a row $p(s, \cdot)$ assigns nonnegative values to $N(s)$ with $\sum_{v \in N(s)} p(s, v) = 1$. Conditional on the retained state, these values describe the next transition under the specified policy and cohort. Infeasible edges receive no probability.

An available edge may receive probability zero under a policy. Thus feasibility and positive support remain distinct. In a finite model, an entire feasible path has positive probability precisely when every edge on it has positive assigned probability. A claim that all feasible routes have positive probability requires an additional strict-positivity assumption.

Let Ω_s contain directed paths from s to the first terminal state. Acyclicity and finiteness make this set finite and ensure termination. For $\omega = (s_0, s_1, \dots, s_m)$ beginning at a declared state, define

$$P(\omega) = \prod_{j=0}^{m-1} p(s_j, s_{j+1}).$$

The empty path at a terminal state has probability one. The product follows from the chain rule together with the retained-state conditional-law assumption.

Proposition 3.2 (Finite path normalization). *For every state s , $\sum_{\omega \in \Omega_s} P(\omega) = 1$.*

Proof. At a terminal state, the empty path has mass one. At a nonterminal state, group paths by their first successor. If each successor's continuation paths have total mass one, the total from s is $\sum_v p(s, v) \cdot 1 = 1$. Reverse topological induction covers the finite graph. $\square$

For any declared outcome map $Y : \Omega_{s_0} \rightarrow \mathcal{Y}$, the induced outcome law is

$$P_Y(y) = \sum_{\omega: Y(\omega)=y} P(\omega).$$

This finite pushforward is the probability enrichment of the route representation. The map Y can report completion, route sequence or a participant-defined outcome, provided its meaning is specified. Changing Y changes the question even when the path law remains fixed.

The construction leaves estimation open. Observed frequencies may inform the rows under appropriate sampling and stability conditions. Expert ranges or mechanistic assumptions may support a weaker set of laws. An arbitrary uniform assignment supplies a law mathematically, while its interpretation as actual routing requires an independent justification.

4 Route Refinement and Probability Conservation

This section examines how descriptive refinement affects a route-based probability. It gives a small counterexample to uniform counting and states a sufficient conservation condition. The analysis distinguishes relabeling a route from actually changing the routing procedure.

Consider a one-step model with two route labels: one leads to completion and one to other termination. Uniform probability over the two labels gives completion probability $1/2$. Now describe the completing route through k labels that refer to equivalent implementations of that same outcome path, while leaving one noncompleting label. Uniform probability over the new labels gives

$$c_k = \frac{k}{k+1}.$$

The underlying distinction between completion and other termination has stayed fixed, yet the number changes with the descriptive multiplicity.

The example applies directly to uniform counting of complete paths in this one-step graph. In larger graphs, uniform choice among outgoing edges and uniform choice among complete paths define different laws, adding another modeling choice. Neither convention follows merely from the existence of the routes.

Suppose an original route group has total mass a and conditional outcome distribution K on a finite outcome set. Replace it with labels $1, \dots, k$ having masses a_j and the same conditional distribution K . Its refined contribution to each outcome is

$$\sum_{j=1}^k a_j K(y) = \left(\sum_{j=1}^k a_j \right) K(y).$$

Proposition 4.1 (Sufficient refinement invariance). *If all unchanged route contributions remain fixed, each refined label preserves the original conditional outcome law, and $\sum_j a_j = a$, then the refinement preserves the overall outcome distribution.*

Proof. The displayed equality makes the refined route group's contribution equal to $aK(y)$ for every outcome. Adding the unchanged contributions establishes equality of the complete outcome laws. $\square$

The result is local and conditional. Labels with different downstream effects require their actual conditional laws, and a refinement that changes information or behavior can change the process itself. More general state aggregation requires an additional compatibility analysis. The proposition supplies a simple check for a declared purely descriptive refinement.

In the two-route example, preserve the completing group's mass $1/2$ by assigning its k labels masses summing to $1/2$. The completion probability then remains $1/2$. If an actual dispatch procedure selects uniformly among displayed labels, adding labels really changes that procedure. The resulting change is behavioral under the specified rule, rather than evidence that the original probability was representation-invariant.

Feasibility expansion presents a related distinction. Suppose an initial policy assigns completion probability 0.8 through its available routes. A new route becomes available, and a revised policy sends half the participants through the original route mixture and half through a route with completion probability 0.1 . The resulting probability is $0.5(0.8) + 0.5(0.1) = 0.45$. Attainability has expanded while the particular completion probability has fallen. Other allocations could preserve or raise it.

This example supplies no general evaluation of route expansion. A new route can improve autonomy, reach people excluded from the original cohort or serve another valuable outcome. The calculation establishes that feasible-route inclusion alone cannot determine a completion comparison. A defensible report should state which route set, routing law, cohort and outcome map changed.

5 Completion Recursion and Boundary Interpretation

This section derives the probability of reaching the completion set and proves uniqueness within the finite model. It then examines the role of terminal definitions, state adequacy and time horizons. The derivation is the standard first-step calculation in the stated scope.

Let $h(s)$ be the probability that an episode beginning at s terminates in T . The boundary values are $h(t) = 1$ for $t \in T$ and $h(f) = 0$ for $f \in F$. For a nonterminal state, partition the completion event by the first successor:

$$\begin{aligned} h(s) &= \sum_{v \in N(s)} P(X_1 = v, \text{ completion} \mid X_0 = s) \\ &= \sum_{v \in N(s)} p(s, v) h(v). \end{aligned}$$

The second equality uses the retained-state conditional-law assumption. It is the hitting-probability equation discussed by Pitman, specialized here to a terminating finite graph.

Proposition 5.1 (Unique finite completion solution). *The boundary values and first-step recursion determine exactly one function $h : V \rightarrow [0, 1]$. That function equals the total probability of completion paths from each state.*

Proof. Set the terminal values as specified. Reverse topological order determines each nonterminal value from already determined successors, giving existence and uniqueness. A convex combination of values in $[0, 1]$ stays in that interval. Grouping completion paths by first successor gives the same recurrence and terminal values, so the path sum equals the constructed function. $\square$

The acyclic restriction does substantive work in this proof. A finite graph with cycles can represent indefinite referral or waiting. A bounded-horizon unrolling can still be acyclic, but its terminal interpretation includes the horizon. Claims about eventual completion without a horizon need a separate analysis of nontermination and the appropriate solution of the boundary equations.

The completion set also requires an explicit meaning. A scheduled appointment, attendance, receipt of a service and a participant's assessment that a need was met are different events. A record system may observe only one of them. Selecting T should identify the operational target and the evidence through which it is recognized. Replacing attendance with scheduling changes h even if the referral transitions before scheduling remain identical.

State adequacy is another condition. If two participants at the same recorded stage have different next-step probabilities because of an omitted variable, the row can represent a cohort mixture under a declared distribution. That mixture may change across settings. A finer state or an explicit mixture model may be required for transfer. The equation remains a conditional identity for an adequate representation; it supplies no guarantee that a proposed state is sufficient.

Capacity dependence can also change the unit of analysis. When one person's appointment consumes a slot needed by another, separate independent episodes may fail to describe the joint system. A joint state containing the relevant queue and capacity can restore a Markov description in principle, while increasing modeling and evidential demands. The current paper assumes a fixed conditional law for its declared episode scope and flags such interactions as an extension.

Finally, a probability summarizes repeated or hypothetical episodes under a law. It does not determine any individual's obligation to complete. A participant's refusal, alternative choice or changed need can terminate an episode outside T while remaining reasonable. The boundary label F denotes other termination within the calculation and carries no automatic moral judgment.

6 Completion-Conditioned Route Distributions

This section derives the law seen when only completed episodes are retained. It uses Bayes' rule to obtain the finite success-conditioned transition and then checks the entire path law. The result explains a selection effect without treating the conditioned law as a causal intervention.

For a nonterminal state s with $h(s) > 0$, define

$$p^+(s, v) = \frac{p(s, v)h(v)}{h(s)}.$$

The numerator is the probability of moving to v and subsequently completing, conditional on starting at s . Division by $h(s)$ conditions on completion. States with $h(s) = 0$ have no completed episodes under the model, so this conditional row is undefined there. Terms with $h(v) = 0$ receive zero mass from a positive- h state.

Proposition 6.1 (Finite completion conditioning). *On states reachable under completion conditioning, the rows p^+ sum to one. For a completed path ω from s_0 with $h(s_0) > 0$, their path product equals $P(\omega)/h(s_0)$.*

Proof. The first-step equation gives $\sum_v p^+(s, v) = \sum_v p(s, v)h(v)/h(s) = 1$. Along a completed path $(s_0, \dots, s_m)$ with positive original path probability, all encountered h values are positive and $h(s_m) = 1$. Hence

$$\prod_{j=0}^{m-1} \frac{p(s_j, s_{j+1})h(s_{j+1})}{h(s_j)} = P(\omega) \frac{h(s_m)}{h(s_0)} = \frac{P(\omega)}{h(s_0)}.$$

Zero-probability completed paths retain zero mass. Thus the construction agrees with the original path law conditioned on completion. $\square$

This is a finite application of the established Doob h -transform described by Bovier. The proof makes its interpretive status clear: it reweights the law by a future event. Implementing these transition probabilities prospectively would be another policy, requiring access to the relevant probabilities and an account of how behavior and capacity would respond.

Consider two initial channels, A and B , chosen with probability $1/2$ each. Channel A completes with probability 1, while channel B completes with probability $1/4$. Then

$$h(s_0) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{5}{8}.$$

Among completed episodes, the initial shares become

$$P(A \mid \text{completion}) = \frac{4}{5}, \quad P(B \mid \text{completion}) = \frac{1}{5}.$$

A database of completed appointments therefore contains a different route distribution from the admitted cohort, despite the original equal routing shares.

Recovering the original routing law from the completed-case law requires additional information. If the completion probabilities by route are known and positive, an appropriate inverse weighting can be derived. If a route has zero completion probability, completed cases contain no observations from it. Its original share is then invisible to that dataset without external information. The paper leaves estimation under such missingness to a separately specified design.

The selection distinction also applies to explanations. A route frequently observed among successful episodes may be common initially, effective at completion, or both. The conditional frequency alone cannot separate those contributions. Comparing policies adds further causal assumptions because the policy can change who enters, which route is chosen and how downstream services operate.

The practical reporting implication is to name the conditioning event. An account of completed appointments should identify itself as such and retain a separate denominator for admitted episodes when available. This permits the conditional description to remain useful while keeping its inferential limits visible.

7 Coherent Bounds under Uncertain Routing

This section replaces each transition row by a set of possible rows. It derives exact completion bounds when rows can be selected independently, then gives a shared-parameter counterexample. The distinction determines whether local worst cases can coexist in one admissible model.

For every nonterminal state s , let K_s be a nonempty compact subset of the probability simplex on $N(s)$. A complete model selects one row $p_s \in K_s$ for each state. First assume that every such combination is admitted:

$$\mathcal{K} = \prod_{s \in T \cup F} K_s.$$

This product condition is the present finite form of rectangularity. It states a substantive independence of allowed parameter choices, not statistical independence of observed transitions.

Let h_p be the completion function of model $p \in \mathcal{K}$. Define lower and upper functions with the same terminal values as h , and recursively set

$$\begin{aligned} \underline{h}(s) &= \min_{q \in K_s} \sum_{v \in N(s)} q(v) \underline{h}(v), \\ \bar{h}(s) &= \max_{q \in K_s} \sum_{v \in N(s)} q(v) \bar{h}(v). \end{aligned}$$

Compactness and continuity of the finite linear expressions ensure attainment of the displayed extrema.

Proposition 7.1 (Rectangular finite-graph bounds). *Under the product-row assumption, every state satisfies*

$$\underline{h}(s) = \min_{p \in \mathcal{K}} h_p(s), \quad \bar{h}(s) = \max_{p \in \mathcal{K}} h_p(s).$$

One complete row selection attains all lower values simultaneously, and another attains all upper values simultaneously.

Proof. Process states in reverse topological order. At terminals every admissible model has the fixed boundary value. Suppose the successor lower bounds hold. For any row selection, nonnegative weights give $h_p(s) = \sum_v p_s(v) h_p(v) \geq \sum_v p_s(v) \underline{h}(v) \geq \underline{h}(s)$. At each state choose a minimizing row for the displayed lower recurrence. The product assumption admits the combination of all these choices. Reverse induction then makes every successor and current value equal to its lower bound. The upper result follows through the corresponding inequalities and maximizing choices. $\square$

The proof remains valid when different branches merge at a later state, because the same globally selected minimizing row at that state attains its scalar completion value for every predecessor. Its validity depends on a common completion target and independently selectable rows. Conflicting objectives or shared parameter constraints require renewed analysis.

Now consider two successive completion stages. Stage one succeeds with probability θ ; conditional on reaching stage two, its success probability is $1 - \theta$. Let $\theta \in [1/4, 3/4]$ be one shared parameter. Completion probability is

$$c(\theta) = \theta(1 - \theta) = \frac{1}{4} - (\theta - \frac{1}{2})^2.$$

The exact range is $[3/16, 1/4]$: the maximum occurs at $1/2$, and the minimum at either endpoint. If each stage is separately allowed any success probability in $[1/4, 3/4]$, the rectangular relaxation yields $[1/16, 9/16]$. Its extreme pairs $(1/4, 1/4)$ and $(3/4, 3/4)$ violate the shared relation that the two success probabilities sum to one.

The relaxed interval is a valid outer bound when the joint family is contained in the product family. It can be useful as a conservative calculation, provided its status is explicit. Calling its endpoints attainable under the shared-parameter model would be incorrect. A narrower interval requires retaining the joint constraint and optimizing over that family.

Uncertainty about missing terminal outcomes creates another bounded calculation. Suppose already accounted terminal mass contributes completion mass A , while a disjoint unclassified terminal mass u remains. If the unknown mass can have any completion fraction in $[0, 1]$, then

$$c \in [A, A + u].$$

The endpoints assign all unknown mass to other termination or completion. This result assumes the accounted and missing masses form disjoint parts of the same normalized episode distribution. Adding missing fractions at several stages can count the same episode repeatedly; a path-consistent model is then required.

These calculations separate epistemic uncertainty from the stochastic law within a model. A set of possible rows expresses uncertainty about which law is appropriate. It supplies no distribution over those laws. An average over parameter choices requires an additional weighting assumption, just as uniform route counting required a routing assumption.

8 A Finite Referral Dossier

This fictional dossier joins the probability layer, completed-case conditioning and coherent uncertainty in one service setting. It also separates admitted participants from the broader population. Every numerical value is constructed to illustrate an inferential distinction.

A neighborhood information desk offers admitted participants two appointment channels for an administrative support session. The target is attendance at one session within the declared episode. Channel

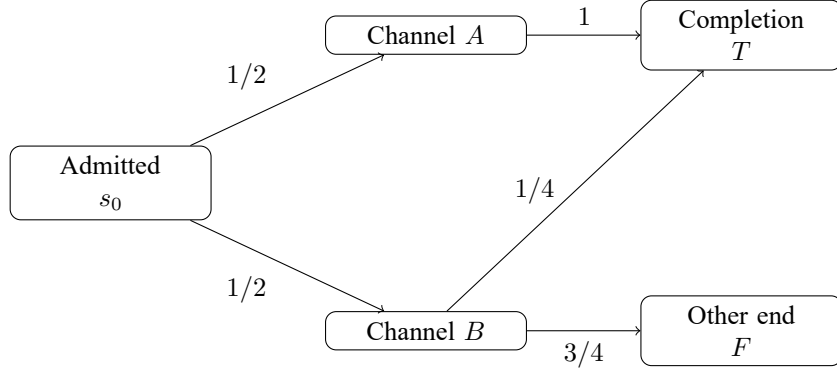


Figure 1: Constructed referral channels with different completion probabilities

Table 1: Distinct quantities in the constructed referral account

Quantity	Constructed value	Interpretation
Initial channel A share	$1/2$	Among admitted episodes
Completion probability	$5/8$	Under the specified routing law
Channel A share after completion conditioning	$4/5$	Among completed episodes only
Shared-parameter completion range	$[3/16, 1/4]$	Separate two-stage uncertainty case
Rectangular outer range	$[1/16, 9/16]$	Relaxed stage choices in that case

A is an immediate booking channel; channel B involves a further contact step. The model assigns initial shares $1/2$ each and conditional completion probabilities 1 and $1/4$. These values define an analytic case, with actual service behavior left open.

Figure 1 represents the law. Among an illustrative mass of 80 admitted episodes, the expected masses are 40 completed through A , 10 completed through B and 30 other terminations through B . These are model expectations, not a claimed sample. The completed mass of 50 contains route shares $40/50$ and $10/50$, reproducing the conditional $4/5$ and $1/5$ result.

Table 1 keeps the denominators and model variants separate. The two-stage parameter example is a sensitivity case, not an alternative estimate secretly substituted into the two-channel model. A report combining them should explain which structural assumptions changed.

Suppose the service database retains attendance records but omits initial referrals that ended before attendance. The conditional route shares remain valid descriptions of those recorded completions under the model. Estimating initial shares requires additional denominator records. If the initial log itself contains duplicate referrals, record co-reference becomes a further evidence-unit question before frequency estimation. The companion event-record work supplies a conceptual dependency for that situation; it leaves probability calibration to this project's own evidence design.

Admission introduces another denominator. Let a be the probability that a person in a declared target population enters the modeled referral episode, and let c be completion conditional on admission. If completion through this program requires admission, then population-level program completion is ac by the multiplication rule. With constructed $a = 1/2$ and $c = 5/8$, it is $5/16$. This says nothing about completion through other programs unless those outcomes are included explicitly.

The dossier also permits an outcome disagreement. A participant may receive useful information at the desk and decline an appointment. The attendance target classifies the episode outside T , while the participant may judge the encounter successful. A plural report can retain both outcomes with their definitions. Changing the target should be recorded as a change in the outcome map, not treated as a numerical correction to the same question.

The case's practical value lies in making these choices inspectable. Routing, completion, admission, recording and value each contribute a different condition. A single percentage becomes interpretable when its law, denominator, target and evidence are stated together.

9 Possibility, Propensity and the Value of Completion

The probability layer raises philosophical questions about possibility, agency and evaluation. This section clarifies the meaning of the model's modal and probabilistic statements, then examines the relation between completion and practical value. Its purpose is to prevent a well-defined number from carrying unsupported conclusions.

A feasible path states that the declared constraints admit a sequence. A positive path probability states that a specified law assigns positive mass to it. An estimated frequency describes a dataset under a sampling and recording procedure. A range of probabilities describes uncertainty across admitted laws. These statements can support one another through explicit assumptions, but their meanings remain distinct.

The route-refinement example shows why that distinction matters. Counting descriptions treats a choice of vocabulary as a basis for mass. A probability law can be invariant under a declared refinement when mass is conserved, while uniform recounting can change it. The issue is epistemic: an analyst should explain why a descriptive multiplicity has, or lacks, behavioral significance in the application.

Probabilistic representation also leaves room for agency. A transition probability can summarize behavior under specified conditions without representing a participant as compelled to follow a route. An individual can refuse, deliberate, seek clarification or choose an unmodeled alternative. If those possibilities matter to the target, the state and outcome domain should acknowledge them. A model that assigns all unexplained endings to one residual category may be useful for accounting while remaining thin as an explanation of action.

The distinction between law uncertainty and randomness is equally important. The interval $[3/16, 1/4]$ in the shared-parameter case represents a set of possible completion probabilities. It does not say that every value is equally likely, nor that a person's completion chance changes randomly between those values. A probability distribution over parameter uncertainty would be an additional epistemic commitment requiring its own rationale.

Completion has a limited normative role. An appointment may be administratively complete while failing to meet a need. A participant may reasonably decline a service, and a referral outside the modeled program may lead to a better outcome. The terminal definition can therefore support one component of evaluation while leaving other values visible. A decision to maximize attendance would require a separate argument about its consequences and priority relative to autonomy, burden and access.

Population boundaries can also carry normative significance. A program may report high completion among admitted participants while a difficult admission process excludes people with greater needs. The product ac exposes a denominator relation, but evaluating exclusion requires a richer account of who is affected and why. Conditional success alone gives a limited view of the institution's practical relation to its intended public.

Uncertainty reporting can improve accountability when it identifies its sources. A wide bound caused by missing terminal records calls for different work from a wide bound caused by shared behavioral parameters or unstable definitions. The report should connect each uncertainty to a possible investigation, a retained disagreement or an acknowledged limit. A range without that explanation can become another opaque institutional number.

The proposed probability layer thus supports a modest form of public reasoning. Its assumptions and derivations can be inspected, its conditioning can be stated, and its target can be challenged. The mathematics clarifies what follows within the model. Philosophical and institutional judgment remains necessary to determine whether the model answers a worthwhile question and how its results should guide action.

10 Objections and Model Counterconditions

This section tests the scope through objections concerning dependence, dynamics, data and decision use. Each objection identifies a point where the finite calculation can remain correct while its application becomes misleading. The responses specify revision obligations rather than extending the model by assertion.

The first objection is that fixed-policy Markov rows oversimplify referral behavior. Earlier experiences, staff judgment and information revealed during contact can affect later transitions. The response is to assess whether the retained state includes what the intended law requires. Time-unrolled history states can represent more dependence, at a cost in complexity and data demands. A convenient compact state needs empirical and conceptual justification.

A second objection concerns shared capacity. Two participants can compete for the same appointment, making their outcomes dependent. The single-episode model can be interpreted conditionally on an externally specified capacity situation, or expanded to a joint state. Treating separate episodes as independent requires an additional assumption and may fail under load. The present proofs concern the given row law, leaving system-level capacity effects open.

A third objection concerns stationarity. Routing probabilities may change with staffing, season, participant composition or policy. A row estimated from one period can remain a valid description of that period while transferring poorly. An application should identify its population and time scope, examine changes and report uncertainty accordingly. A formal state diagram alone cannot establish temporal stability.

A fourth objection concerns the outcome boundary. Attendance can be easier to record than whether a need was met. The proposed method can accommodate a different finite outcome map, but the evidence supporting it must be developed. Replacing the target silently would make comparisons ambiguous. Maintaining several named outcomes can be more informative than forcing them into one completion label.

A fifth objection concerns the robustness result. Independent row selection may be too permissive when all stages depend on the same staffing level or communication quality. The shared-parameter counterexample demonstrates this failure. The product model should be presented as an explicit assumption or outer relaxation. Joint constraints can narrow the family, but their validity also requires evidence.

A sixth objection concerns completed-case analysis. Researchers may have access only to attendance records. Such records can still support conditional descriptions, provided the target remains conditional. Recovering admitted-cohort behavior, population access or causal route effects requires additional information and design. The lack of that information should remain visible in the research question rather than be repaired through an arbitrary denominator.

A seventh objection concerns policy optimization. A route with higher observed completion may appear preferable. Selection into routes, different needs, capacity effects and changes caused by re-locating participants can all affect that inference. The present fixed-policy model calculates under an assigned law. Choosing a policy and predicting its consequences requires a causal or structural account that remains outside the established result.

A final objection concerns interpretive burden. Detailed uncertainty reports can overwhelm practitioners. A concise report can still identify the cohort, target, conditioning event, main uncertainty source and one consequential sensitivity case. The full derivation can remain available for review. Simplification should preserve those distinctions whose omission would materially change the intended inference.

The model should be revised when these counterconditions are central to the application. It should be abandoned as a basis for a particular decision when its target, law or evidence cannot be defended. Those limits preserve the value of the finite derivations while preventing their use beyond the argument they support.

11 Verification and Prospective Evidence Design

This section separates mathematical verification from probability estimation and policy evaluation. It specifies finite checks for the constructions, a prospective record design and criteria for assessing interpretation. The proposed work remains a design, with empirical results left open.

Finite verification should compare direct path enumeration with backward completion recursion on small acyclic graphs using exact rational probabilities. It should check total path mass, terminal boundary values and the success-conditioned path law. Graphs with zero-probability feasible edges and states with zero completion probability are important counterexamples because they expose the domain of the conditional transformation.

The route-refinement fixture should compare uniform relabeling with conserved mass. The former changes $1/2$ to $k/(k + 1)$ in the one-step case; the latter preserves the outcome law. Tests should also distinguish an actual uniform dispatch procedure from a merely descriptive change, since they carry different substantive interpretations.

For uncertain rows, enumerate a finite collection of allowed row choices and compare the extreme completion values with the backward recursions. Include branching and merging paths so that shared successor states are tested. The compact-set theorem has a broader domain than these finite checks; its proof supplies the general argument. The shared-parameter example should be checked separately against its exact quadratic range and the rectangular relaxation.

A prospective service study should record the denominator before completion. Relevant records include admission into the episode, offered and chosen routes, the declared completion event, other termination and unresolved terminal status. Data minimization and participant burden should shape the record design. The purpose is to obtain the evidence required by a stated question, not to collect every possible detail.

Event identity deserves explicit attention. Several systems may report one referral or one attendance. Before using row frequencies as transition counts, the study should identify the unit of occurrence and handle duplicates, repeated events and uncertain matches. A record-identity audit can supply bounds or alternative count tables; propagating those alternatives into probability estimates requires a further statistical design.

Estimation should distinguish model fitting from assessment. A held-out period or appropriately separated set of episodes can test whether predicted completion aligns with observed outcomes in the declared scope. Evaluation should report calibration, subgroup and period variation when supported by adequate data, and the consequences of unresolved outcomes. Small or selected samples may support only limited conclusions.

Interpretive evaluation is also necessary. Practitioners and participants should be able to distinguish completion among admitted episodes from population access and completed-case route shares. A review can present the dossier and ask what each percentage warrants. Misunderstandings can reveal a reporting failure even when the calculations are correct.

Finally, policy-effect claims require an appropriate research design that addresses selection and changes in the system. A new route may change the admitted population or downstream capacity. The current paper's arithmetic can organize outcomes under specified laws, while the evidence for a law under intervention must be established separately. This division keeps probability estimation, model validation and causal evaluation from being silently combined.

12 Research Obligations and Further Development

The final section organizes the remaining work around the probability law, joint uncertainty and decision target. It then identifies extensions whose value depends on those foundations. The objective is a usable research program with explicit evidence gates.

Research Obligation 12.1 (Warrant for the probability enrichment). For each transition row or set of rows, specify the cohort, policy, retained state, observation procedure and evidence supporting the assigned values. Distinguish empirical estimates, expert constraints and constructed sensitivity values.

This obligation connects the formal probability layer to the earlier feasible-route model. Availability can be established through institutional or resource evidence while probabilities require another source of support. A project may validly stop at feasible routes when that support remains unavailable. It can also explore conditional probability scenarios while labeling them as constructed.

Research Obligation 12.2 (Coherent uncertainty family). Identify dependencies among uncertain transition rows and state whether the admitted model family is a product, a joint constrained family or an outer relaxation. Report which bound endpoints are attainable within the intended family.

The shared-parameter example makes this obligation concrete. A wide interval can reflect permissible uncertainty or incompatible local extremes. The distinction matters when a worst-case claim informs resource decisions. Joint parameter estimation, structural constraints and sensitivity analysis are possible avenues, each requiring its own evidence and validation.

Research Obligation 12.3 (Outcome and denominator interpretation). State the completion event, admission boundary, conditioning event and population scope alongside every reported probability. Establish separately the reasons for using that probability in a decision.

Formal extensions include bounded-duration models, explicit queues, multiple participant-defined outcomes and adaptive policies with retained information. Cyclic models can address repeated referral and indefinite waiting, provided termination and solution selection are analyzed. Partial observation can be represented through a richer state or a set of candidate laws. These developments should be driven by a consequential counterexample in an actual or carefully justified case.

The paper establishes several conditional consequences within its finite scope. Normalized rows induce a normalized path law. Conserved mass protects a declared descriptive refinement. Completion conditioning reweights routes through their downstream completion probabilities. Independent compact row sets support exact backward bounds, while a shared parameter can make independent extremes incompatible. Missing terminal mass and admission denominators require explicit accounting.

These results provide a structured way to question a completion percentage. The relevant questions concern what episode was modeled, how mass was assigned, what was observed and which target was selected. Their answers can make a limited estimate useful without turning it into a general assessment of a service or a person.

The continuing research should test that usefulness through prospective evidence and public interpretation. Mathematical analysis can clarify the consequences of assumptions. Empirical work can investigate whether those assumptions fit the service. Philosophical and institutional reasoning can assess the target and its decision role. The proposed probability enrichment remains open to revision through all three forms of inquiry.

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