

A Preliminary Field-Theoretic Formalism of Power in Generative Relational Theory

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Abstract

This paper develops a preliminary field-theoretic formalism of power within Generative Relational (GR) theory. The central problem is dynamical: once power is treated as a historically constituted relational structure, a further account is required of how its effects compose across local relations, propagate through history-bearing configurations, generate system-level characteristics, and modify the structure of admissible continuations. The paper adopts an algebra-first strategy. Rather than presupposing a single global state space for a potentially open, heterogeneous, and indefinitely extensible relational system, it associates local relational regions with algebras of admissible observables and transformations, and treats state-like evaluations and Hilbert-space representations as secondary constructions introduced where analytically useful. A field-theoretic representation is then developed over a discrete GR spacetime represented provisionally by a labelled 2-complex. Local power-conditioned transformations compose along paths and cycles, permitting the study of sequence dependence, holonomy-like residual effects, curvature-like structures, invariant and approximately invariant quantities, and the emergence of system-level properties that are not recoverable from selected isolated local or pairwise diagnostics. The formalism permits applications to distinguish changes in declared continuation evaluations, changes in relational connectivity or admissibility, and transformations that modify the topology of the represented spacetime. It also separates a fixed-past regime, retained as a practical modeling restriction, from stronger history-reconfiguring regimes in which the relational topology assigned to already-instantiated history can change. Finite constructions provide a route to logical and computational checking, while quasi-local algebraic formulations retain compatibility with open and potentially unbounded systems. The paper treats field, algebraic, topological, and spacetime language as formal representations under ontological suspension. Later applications examine multi-field interaction, systemic emergence, balance dynamics, and near-critical transitions without identifying these mathematical structures with a final ontology of political or social power.

Keywords: Generative Relational theory; power; field theory; operator algebra; relational spacetime

Discussion Paper Note

This manuscript develops a formal research programme within Generative Relational (GR) theory. Its immediate object is the dynamics of power once relational entities are represented as history-bearing configurations whose local relations, admissible transformations, and continuation structures can change through interaction.

The paper uses mathematical resources from operator algebras, algebraic field theory, discrete complexes, topology, and spin-foam-inspired history representations. These resources are introduced as modeling languages. Their use does not establish that social, political, institutional, or economic relations possess the ultimate ontology of a quantum field, a C^* -algebra, a spin foam, or a topological complex. GR retains ontological suspension while asking whether these structures provide precise and revisable representations of relational dynamics.

The formal programme follows an algebra-first strategy. A single global state space is not assumed as a primitive object. Local relational regions are instead associated with algebras of admissible quantities and transformations, with state-like evaluations and Hilbert-space representations introduced where required for a particular construction. This choice is motivated by the need to represent open, historically evolving, and potentially unbounded relational systems without requiring one globally tractable phase or Hilbert space.

A second modeling distinction concerns history. Earlier GR constructions often preserved already-instantiated history while allowing operators to deform future continuations. The present paper treats that restriction as a useful regime rather than as an ontological commitment. It also develops stronger transformations in which the represented relational structure of past history can be reconfigured, including cases in which topological invariants of the historical complex change. Epistemic reconstruction, relational-historical reconstruction, and stronger ontological readings are kept analytically distinct.

The manuscript aims to make the formalism sufficiently explicit that finite instances can be checked and calculated. Formal consistency, computational evaluation, and empirical validation remain separate standards. A finite toy construction can demonstrate that a definition composes correctly or that an invariant changes under a specified transformation without thereby establishing that the same construction is an empirically adequate model of a concrete political or social system.

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Status. This working draft records an evolving stage of the author’s position and is circulated for discussion. The present release contains a revised Section 1 and complete first-draft Sections 2 through 15. The manuscript is undergoing a section-by-section revision pass; definitions, notation, figures, section structure, formal statements, and applications remain subject to revision.

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Statement on the use of language models. The exploratory discussions and preparation of this paper involved OpenAI’s ChatGPT. ChatGPT supported exploratory dialogue, formal reconstruction, source discovery followed by website verification, argumentative criticism, drafting in $\text{\LaTeX}$, and compilation checks. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, and bears sole responsibility for the manuscript, including its definitions, formal constructions, calculations, arguments, conclusions, and errors. Authorship credit remains with the human author.

1 Introduction

This section establishes the research object, formal boundary, and methodological architecture of the paper. The paper develops a field-theoretic formalism of power within Generative Relational (GR) theory at a level where local effects can be composed, finite instances can be calculated, and system-level consequences can be distinguished from the properties of isolated relations. The construction begins from history-bearing relational configurations and proceeds through region-indexed deformation records, field realizations, local dynamical maps, algebraic evolution, cyclic composition, and structural diagnostics. The argument is formal and reconstructive. Mathematical structures are treated as candidate representations whose analytical standing depends on explicit typing, compatibility conditions, calculable consequences, and a separately specified empirical interpretation.

1.1 Dynamical Object of the Power Formalism

Power has been analyzed through relational, dispositional, structural, and productive vocabularies. Dahl's influential formulation made relational comparison explicit by defining power through a relation between persons [4]. Foucault emphasized mobile power relations embedded in practices and fields of possible action [5]. Barnett and Duvall distinguished several mechanisms through which power operates in international politics, including direct interaction, institutional relations, structural positions, and socially productive processes [2]. These traditions differ in ontology, method, and explanatory purpose, yet each makes it difficult to reduce power to one scalar attribute attached to an isolated actor.

GR represents the relevant entities as history-bearing relational configurations at a declared analytical scale. An actor may therefore be modeled through a structured historical object whose present capacities depend on accumulated relations, prior transformations, inherited constraints, and available continuations. A dyadic comparison between actors is consequently embedded in a wider coupled configuration. The present paper begins after a power-relevant relational structure has been substantively identified and asks how its dynamical consequences can be represented with sufficient precision for composition, comparison, and calculation.

That task requires several distinctions. A power-conditioned transformation may alter the relative support assigned to already-admissible continuations, modify which continuations remain reachable, change the family of admissible transformations, or reconfigure the relational complex on which future operations are defined. These effects have different domains and need not share one scalar measure. The formal object of interest is therefore the structured deformation of relational evolution. Scalar actor-level magnitudes, where useful, enter later through explicitly declared comparison rules.

The central advance beyond an abstract operator-level treatment is to resolve this deformation across scales. Local consequences must be assigned to declared regions; neighboring assignments must satisfy compatibility conditions; local transformations must possess well-typed compositions; cycles must support residual diagnostics; and global properties must be tied to explicit algebraic, spectral, or topological constructions. This layered structure supplies the basis for the field-theoretic formalism developed below.

1.2 Algebra-First Relational Architecture

A finite model can begin from a global state space and define operators acting on that space. The present paper retains that possibility as a calculational specialization. Its general architecture begins earlier, because GR systems can be open, heterogeneous, historically extensible, and only partially specified. Their relational components, boundaries, and admissible operations can change as the represented history changes. A fixed global phase or Hilbert space would therefore impose a substantial structural assumption before the local relational organization has been established.

Operator-algebraic physics provides a mature example of a different order of construction. Haag and Kastler formulated local observable algebras without identifying the theory with one privileged Hilbert-space representation [6]. In quantum statistical mechanics, C^* - and W^* -algebraic methods likewise organize observables, states, symmetries, and infinite-system structures [3]. The present paper imports neither

the physical ontology nor the empirical interpretation of those theories. It adopts the narrower methodological lesson that local algebraic structure can precede a global state-space representation when openness, composition, and representation dependence are analytically central.

Let K denote a provisional discrete representation of a GR spacetime and let R denote an admissible region of K . The formalism associates retained regions with local algebras $\mathfrak{A}_K(R)$ whose substantive interpretation is supplied by the model. Their elements may encode locally admissible quantities, transformations, constraints, or other algebraic data required by the analysis. Compatible embeddings among nested regions permit larger algebraic structures and, under the assumptions stated later, a quasi-local completion $\mathfrak{A}_K$.

States and Hilbert-space operators enter at a later representation layer. In an appropriate C^* -algebraic specialization, a state can be introduced as a positive normalized functional, and the GNS construction can then produce a representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$. Finite matrix models remain valuable for explicit computation, while their convenience does not determine the general architecture. This ordering also accommodates topology-changing or carrier-changing dynamics: a transformation may act within one algebra, or it may be represented by a typed map between algebras associated with distinct complexes.

1.3 GR Power-Field Realizability

The term *field* acquires a specific meaning in this paper only after a relational domain and local realization rules have been defined. The substantive GR power structure is first represented through region-indexed deformation data. A field-level representation is reached only when those data admit compatible local carriers whose restriction, overlap, comparison, and transport rules reproduce the selected deformation record. Field structure is therefore an additional formal achievement with its own realizability conditions.

A labelled 2-complex supplies one candidate discrete representation of GR spacetime. Spin-foam models provide a developed example in which histories are represented by labelled two-dimensional complexes and local data are associated with faces, edges, and vertices [1]. GR uses this architecture solely as a structural resource; its quantum-gravitational ontology remains outside the claims of the present formalism. The relevant feature is that local relational data can be attached to a complex whose gluing and composition encode how local structures participate in a larger history.

The paper consequently distinguishes the substantive power structure $\mathcal{P}^R$, its region-indexed deformation record $\mathbf{D}_\mathcal{P}$, and a GR power-field realization $\mathbf{P}_\mathcal{P} = \{p_R\}$. The local carriers containing the p_R may vary across regions, provided the model supplies the restriction and comparison structure needed to test coherence. This permits finite-dimensional, operator-algebraic, or other typed realizations without forcing all regions into one common vector space.

Once local field realizations exist, dynamical laws can associate them with transformations $\Phi_R[p_R]$ of local generative-possibility structures. Their algebraic components can then be represented by maps $\alpha_R[p_R]$ acting within or between local algebras. Ordered composition along relational paths produces history-dependent transformations; closed compositions produce cycle residuals; and compatible regional algebra maps can, under additional boundedness and coherence assumptions, induce quasi-local evolution. The field-theoretic content of the paper lies in these local-to-global compatibility and composition rules.

1.4 Historical Reconfiguration Regimes

The treatment of realized history defines an important regime distinction. A history-preserving model can retain an embedded historical subcomplex K^- while allowing power-conditioned transformations to alter its continuation family. This fixed-past regime supports ordinary prospective and counterfactual analysis while leaving the ultimate ontology of the past suspended.

A stronger regime permits the represented historical carrier itself to change. Such change can describe epistemic reconstruction, reorganization of the relationally operative past, or, under a stronger formal reading, a topology-changing map between historical complexes. These interpretations carry different commitments and remain separate throughout the paper. The formalism therefore distinguishes changes in

evaluation, incidence structure, algebraic organization, homological diagnostics, and stronger topological equivalence classes.

Topology is useful because it distinguishes deformation within a retained structure from transformation of the structure. A map $K \rightarrow K'$ may preserve selected topological invariants while altering labels, weights, or local incidence data. Another map may change homology, Betti numbers, Euler characteristic, or other declared diagnostics. Such changes provide sufficient evidence for a detected topological difference relative to the chosen diagnostic, while equality of selected invariants supplies no general proof of topological equivalence. This one-sided use of invariants keeps the topological claims commensurate with what is actually calculated.

The same regime distinction sharpens the generative interpretation of power. Reweighting an existing continuation family, changing reachability, changing admissible operations, and altering the relational carrier correspond to increasingly structural forms of deformation. The paper keeps these mechanisms formally distinct so that finite examples can identify which kind of change a given transformation produces.

1.5 Compositional System Emergence

System-level structure becomes accessible when local power-conditioned transformations are composed. Ordered products can encode sequence sensitivity, while closed cycles can reveal residual transformations that are absent from the inspection of any single local map. The paper develops cycle return maps, base-point and representation covariance conditions, curvature-like residuals, fixed structures, and mixed-field cycles as calculable objects derived from the local dynamics.

The corresponding notion of emergence is diagnostic-relative. A system characteristic is treated as emergent relative to a declared local diagnostic family when systems that agree on those local diagnostics differ in a specified cycle-level, algebraic, spectral, or topological diagnostic. This criterion avoids a stronger metaphysical claim of absolute irreducibility while still allowing local data and compositional structure to carry demonstrably different information.

Multiple GR power fields add a second compositional layer. The paper distinguishes ordered coexistence of separately defined field-induced maps, joint dynamical laws depending on several fields simultaneously, and non-factorizable interactions that cannot be reproduced by any admissible ordering of the single-field maps. These distinctions make reinforcement, cancellation, frustration, mixed-cycle residuals, and higher-order overlap structure formally testable.

The same machinery supports the later analysis of criticality. Critical-transition research motivates diagnostics involving susceptibility, recovery, amplification, and structural transition under declared perturbations [8]. The Criticality Hypothesis of International Politics proposes that some international relational systems may, under specified conditions, occupy near-critical regimes combining persistence, heightened susceptibility, and continuing reconfigurability [7]. The present paper treats that hypothesis as a conditional application: near-criticality is defined only relative to a stated configuration family, perturbation family, diagnostic set, scale, and threshold rule.

1.6 Formal Architecture of the Research Problem

The paper addresses a connected sequence of formal problems. It identifies the history-bearing object on which power-conditioned deformation is represented; specifies the conditions under which a region-indexed deformation record admits a field realization; derives local dynamical action and ordered composition; develops algebraic and represented operator realizations; separates history-preserving and history-reconfiguring regimes; constructs cycle-level and multi-field diagnostics; and examines balance and near-criticality through quantities generated by the formalism itself.

The central representation ladder is summarized in Equation 1. Each transition in the ladder is conditional on independently stated assumptions developed in later sections.

$$\begin{aligned} K &\longmapsto \{\mathfrak{A}_K(R)\}_{R \in \mathcal{R}(K)} \longmapsto \mathfrak{A}_K, \\ \mathcal{P}^{\mathcal{R}} &\longmapsto \mathbf{D}_{\mathcal{P}} \longmapsto \mathbf{P}_{\mathcal{P}} = \{p_R\} \longmapsto \{\Phi_R[p_R]\} \longmapsto \{\alpha_R[p_R]\}. \end{aligned} \tag{1}$$

Equation 1 separates the history-bearing carrier, the local and quasi-local algebraic architecture, the substantive GR power structure, its deformation record, a compatible field realization, and the dynamical maps induced by that realization. When the regional algebra maps satisfy the compatibility and boundedness conditions developed later, they can induce a quasi-local map $\alpha_K : \mathfrak{A}_K \rightarrow \mathfrak{A}_{K'}$. The source and target may coincide in a fixed-carrier regime or differ when the represented spacetime changes.

A further representation step becomes available only under additional algebraic and state assumptions. In an appropriate GNS-admitting specialization, the sequence in Equation 2 produces a representation-dependent operator implementation.

$$(\mathfrak{A}_K, \omega) \longmapsto (\pi_\omega, \mathcal{H}_\omega, \Omega_\omega), \quad \alpha_K \longmapsto V_\alpha \text{ when the required compatibility conditions hold.} \quad (2)$$

Equation 2 keeps relational power, field realization, algebraic dynamics, and Hilbert-space operator implementation at distinct levels. Different evaluations can induce different representations, and topology-changing dynamics can remain well typed at the algebraic level even when no common Hilbert-space implementation is selected.

The resulting contribution is methodological as well as formal. A proposed field structure must satisfy declared local realization and coherence conditions. A proposed noncommutative effect must arise from defined composable transformations whose ordering changes a retained result. A cycle residual must be tied to a typed closed composition and a declared equivalence. A topological transition must identify the represented complex and the invariant or structural property that changes. A criticality claim must specify the perturbations and diagnostics used to establish susceptibility or structural proximity. These requirements make the formalism open to finite verification and revision.

1.7 Verification Architecture

The paper separates three evidential layers. Formal verification concerns typing, composition, compatibility, invariance, and propositions derived from explicit assumptions. Computational checking concerns finite realizations in which local maps, ordered products, cycle residuals, spectra, homology, balance defects, or susceptibility diagnostics can be calculated. Empirical validation adds the problem of mapping observations to the formal objects, estimating or constraining their parameters, and comparing the resulting model with plausible alternatives.

Calculability therefore establishes consequences of a chosen model. Empirical adequacy requires an additional evidential layer. A finite complex can be assigned matrices, a cycle product can be evaluated, and homological quantities can be compared before and after a transformation. Empirical use additionally requires a justified relational boundary, coding of local algebraic elements, evidence for power-conditioned transformations, temporal and historical scales, uncertainty treatment, and criteria capable of rejecting the proposed mapping.

The same discipline governs physical analogy. Algebraic quantum field theory, quantum statistical mechanics, spin foams, gauge-theoretic holonomy, and algebraic topology supply mathematical structures whose physical interpretations remain domain-specific. Their role here is selective and structural. A GR application retains a borrowed construction only when the GR problem independently requires the corresponding mathematical property and when the resulting assumptions and failure conditions can be stated explicitly.

The present paper therefore establishes a formally inspectable research architecture and a collection of calculable finite specializations. Empirical identification remains a separate stage. In particular, mathematical coherence alone does not establish the empirical existence of a GR power field, a curvature-like residual, a topology-changing historical process, or a near-critical international configuration.

1.8 Manuscript Architecture

Section 2 supplies the mathematical background needed for the later construction, including 2-complexes, spin-foam-inspired histories, local and quasi-local algebras, C^* - and W^* -algebraic specializations, dynamical

ical maps, ordered composition, and topological diagnostics. Section 3 defines GR spacetime and history-bearing configurations. Sections 4 and 5 construct the deformation record and derive the additional realizability conditions required for a GR power-field representation.

Sections 6 and 7 develop local dynamical action, ordered composition, algebraic evolution, quasi-local lifts, state/evaluation transport, and representation-dependent operator implementations. Sections 8 and 9 separate history-preserving dynamics from historical reconfiguration and topology-changing regimes. Sections 10 and 11 derive cyclic system structure, diagnostic-relative emergence, and compositional dynamics for multiple GR power fields.

Sections 12 and 13 use those constructions to analyze balance and near-criticality in international relations. The balance analysis distinguishes corridor proximity, order sensitivity, cycle residuals, and higher-order overlap structure. The criticality analysis introduces finite-scale susceptibility, spectral recovery and amplification, compositional sensitivity, and structural margins under a declared criticality specification. Section 14 states the epistemic status, robustness requirements, empirical calibration interface, and formal limits of the programme. Section 15 concludes.

The paper is preliminary in the precise sense that it develops a formal research programme whose mathematical objects can be defined, composed, and checked while their substantive calibration remains open to empirical and historical investigation. Its intended contribution is a more fine-grained GR account of power dynamics in which local deformation, field realizability, algebraic evolution, cyclic composition, historical reconfiguration, and critical susceptibility occupy distinct formal levels within one compatible architecture.

2 Formal Background for GR Spacetime Dynamics

This section supplies the mathematical background required for the field-theoretic construction developed later in the paper. Its role is pedagogical and definitional: readers should be able to follow the subsequent GR-specific formalism without prior training in operator algebras, spin foams, or algebraic topology. The section proceeds from local algebraic operations to discrete complexes, field assignments, operator-algebraic representations, open-system maps, ordered composition, and topological diagnostics. Standard mathematical objects are distinguished throughout from their later GR interpretation. The discussion therefore introduces only the structures that will be used in the paper and states explicitly where a physical construction serves as mathematical precedent rather than as an ontological model of social or political reality.

2.1 Relational Domains and Algebraic Operations

An algebra provides a way to collect quantities or transformations that can be added, scaled, and multiplied according to declared rules. Let $\mathfrak{A}$ be an algebra over $\mathbb{C}$. For $a, b \in \mathfrak{A}$, the product ab represents a composition internal to the algebra. The interpretation of the product depends on the model: it may encode sequential transformation, combination of observables, or another formally specified operation. Multiplication need not commute, so ab and ba may represent different compositions. This elementary possibility becomes important later because history-dependent power transformations may be order sensitive.

A $*$ -algebra additionally has an involution $a \mapsto a^*$ satisfying the usual conjugate-linearity and reversal of multiplication. A C^* -algebra is a complete normed $*$ -algebra whose norm and involution satisfy the C^* identity stated in Equation 3 [3].

$$\|a^*a\| = \|a\|^2, \quad a \in \mathfrak{A}. \quad (3)$$

Equation 3 is a mathematical condition on the algebra. Its use later in the paper will provide analytic control and access to established representation theory; it does not imply that GR relations are quantum observables.

The term *operator* will likewise be used in a formally limited sense. An operator is a map acting on a specified mathematical domain. Depending on the construction, that domain may be a vector space, a Hilbert space, an algebra, or another structured object. An operator representation of a GR transformation

therefore requires an independently defined domain and a rule for composition. The word itself carries no substantive claim about the ontology of power.

For a local relational region R , the notation $\mathfrak{A}(R)$ will later denote an algebra assigned to that region. Section 2 does not yet determine what substantive objects belong to $\mathfrak{A}(R)$. The later GR construction must specify whether its elements represent observables, admissible transformations, constraints, generators, or another class of locally supported formal objects. This separation prevents the algebraic vocabulary from settling substantive questions by notation alone.

2.2 Discrete Complexes and Two-Dimensional Cell Structure

A graph records vertices and edges. A cell complex can additionally encode higher-dimensional incidence structure. For the purposes of this paper, a finite 2-complex K may be written schematically as $K = (V, E, F, \partial)$, where V is a set of vertices, E a set of edges, F a set of faces, and ∂ records how the boundary of each higher-dimensional cell is attached to lower-dimensional cells. This notation is intentionally more general than a simple graph because a face can record a relation among several transitions or a closed compositional structure that is not reducible to one edge.

The underlying topological idea can be stated through cell attachment. A CW complex is constructed from cells of increasing dimension by attaching each n -cell along a map from its boundary sphere into the existing $(n - 1)$ -skeleton [10]. A 2-dimensional complex stops after vertices, edges, and faces. The paper will work mainly with finite or locally finite instances for which incidence and boundary data can be represented explicitly.

Figure 1 provides a schematic example. The drawing is not intended to encode a physical spacetime. It illustrates only the distinction among vertices, edges, faces, and a selected local region that can carry its own algebraic data.

A labelled complex carries additional data. We write such a complex as $K = (V, E, F, \partial, \lambda)$. Here λ denotes a labelling rule for types, parameters, representations, empirical attributes, or other model-dependent information assigned to vertices, edges, or faces. The later GR construction will use labels to distinguish relational roles and admissible transformations. The topological incidence structure and the labels should remain conceptually separate: changing a label can preserve topology, whereas attaching, deleting, identifying, or splitting cells can change the complex itself.

A region $R \subseteq K$ will mean a declared subcomplex or another explicitly admissible local portion of the complex. This modest notion of locality is sufficient for the algebra-first construction. It does not yet impose relativistic causal locality, metric distance, or a preferred temporal foliation. Those additional structures may be introduced only when a particular model requires them.

2.3 Spin-Foam-Inspired Histories

Spin foams provide one physical precedent for using labelled 2-complexes to represent histories. In the standard spin-foam setting, a spin network is a graph whose edges carry representation labels and whose vertices carry intertwining data; a spin foam is a two-dimensional complex whose faces carry representation labels and whose edges carry intertwining data. Spin-foam models then associate amplitudes with local parts of the complex and use the labelled 2-complex to represent a transition between boundary spin networks [1]. The construction is part of research on quantum gravity and related gauge-theoretic models.

The present paper uses this precedent at the level of formal architecture. A 2-complex can record a history with several levels of incidence, and local data can be assigned to its cells. GR does not inherit the physical meanings of spin, intertwiner, quantum geometry, or transition amplitude. A GR label may instead represent a relational type, constraint, generative capacity, institutional condition, or other quantity defined by the substantive model. Likewise, a GR history weight need not be a quantum amplitude.

The distinction is important because two separate ideas are often combined in informal discussion. The first is combinatorial: a history may be represented by a complex assembled from local events and relations. The second is physical: a specific quantum-gravity model assigns quantum labels and amplitudes to that

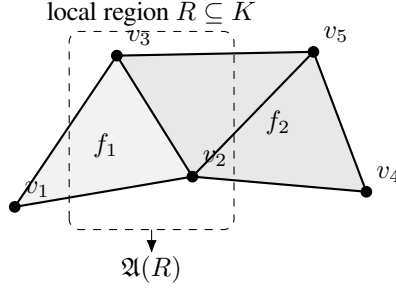


Figure 1: Schematic 2-complex and a selected local region. Vertices, edges, and faces provide distinct incidence levels; a local algebra can be associated with a chosen region without assigning physical meaning to the cells at this background stage.

complex. GR adopts the first possibility as a candidate representation and leaves the second behind unless a later formal argument independently justifies an analogous mathematical structure.

The 2-complex representation also permits refinement. One model may treat an institution, state, or organization as a coarse vertex, while another may replace that vertex with a subcomplex containing internal histories. Such refinement changes representational resolution without requiring a claim that one scale is ontologically fundamental. This flexibility will later support history-bearing entities whose internal structure can be opened when a power relation depends on details hidden by a coarser representation.

2.4 Field Assignments on Discrete Structures

A field is used here in a broad mathematical sense: it is an assignment of local data across a domain together with whatever comparison, transport, or dynamical rules the model requires. In continuum physics a field is commonly defined over spacetime points or regions. On a discrete complex, the analogous assignment can be made to vertices, edges, faces, or composite regions. The choice of support determines what kind of locality the field can express.

Let $\mathcal{C}_p(K)$ denote the chosen set of p -cells of K and let X_c be an admissible value space associated with a cell c . A general discrete field assignment is stated in Equation 4.

$$\varphi : c \in \mathcal{C}_p(K) \mapsto \varphi(c) \in X_c. \quad (4)$$

Equation 4 leaves the codomain open deliberately. A scalar-valued field, a vector-valued field, an algebra-valued field, and a field of linear maps are different mathematical constructions. The later power formalism must select a codomain because its intended transformation law requires it; the term *field* alone does not determine that choice.

A second ingredient becomes necessary when information is transported or transformations are composed across neighboring cells or regions. If an oriented edge e connects local domains associated with its source and target, a transport-like map may be written as $U_e : X_{s(e)} \rightarrow X_{t(e)}$. For a path $\gamma = (e_1, \dots, e_n)$ whose edge maps are composable, the induced transport is the ordered composition stated in Equation 5.

$$U_\gamma = U_{e_n} \circ \dots \circ U_{e_2} \circ U_{e_1}. \quad (5)$$

Equation 5 already shows why a field theory can carry more information than a list of local magnitudes. The effect associated with a path depends on the local maps and their composition. When the maps fail to commute, changing their order can change the resulting transformation.

Later sections will derive a power-conditioned field from GR premises. Section 2 therefore does not define power as a field. It only establishes the mathematical vocabulary needed to ask whether distributed power effects can be represented through local assignments and compositional transport.

2.5 Operator Algebras and Derived Representations

The algebra-first architecture separates an abstract algebra from any particular realization of that algebra as operators on a Hilbert space. This distinction is central in algebraic approaches to quantum field theory and quantum statistical mechanics, where one can formulate observable structure abstractly and then study different representations [6, 3]. For the present paper, the mathematical lesson is that a Hilbert space need not be fixed before the relevant algebraic relations are known.

A state on a unital C^* -algebra $\mathfrak{A}$ is a positive normalized linear functional $\omega : \mathfrak{A} \rightarrow \mathbb{C}$. Positivity means $\omega(a^*a) \geq 0$ for $a \in \mathfrak{A}$, and normalization means $\omega(1) = 1$. The Gelfand–Naimark–Segal construction associates such a state with a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ satisfying Equation 6 [3].

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a) \Omega_\omega \rangle_{\mathcal{H}_\omega}, \quad a \in \mathfrak{A}. \quad (6)$$

Equation 6 explains the order of construction adopted later in the paper. One may first specify an algebra and, when an evaluation context is needed, choose a suitable state and obtain a Hilbert-space representation. The representation is therefore available for calculations without being elevated to the unique primitive state space of the whole GR system.

A W^* -algebra can be characterized abstractly as a C^* -algebra that is the dual of a Banach space; in a concrete Hilbert-space representation, the corresponding objects are von Neumann algebras. For a represented algebra $\pi_\omega(\mathfrak{A}) \subseteq \mathcal{B}(\mathcal{H}_\omega)$, the double commutant $\pi_\omega(\mathfrak{A})''$ provides a standard weakly closed operator algebra associated with that representation [3]. The paper will use W^* - or von Neumann-algebra language only where weak closure, normal states, spectral structure, or representation-dependent limiting behavior is analytically required.

This layered architecture also accommodates inequivalent representations. Haag and Kastler emphasized that an abstract algebraic formulation can separate common local algebraic structure from a commitment to one unitarily equivalent Hilbert-space realization [6]. GR will use this possibility cautiously. Distinct representations may become useful when different evaluation contexts, asymptotic regimes, or emergent sectors cannot be captured economically within one representation. The existence of this mathematical option does not itself establish that a substantive GR system exhibits physically analogous superselection or quantum phases.

2.6 Locality and Quasi-Local Construction

Local algebraic structure is organized by assigning an algebra to each admissible region. If R_1 is contained in R_2 , a basic compatibility condition is that the smaller region's algebra can be embedded into the larger one. Equation 7 states this inclusion structure in the form used later.

$$R_1 \subseteq R_2 \implies \iota_{R_1 R_2} : \mathfrak{A}(R_1) \hookrightarrow \mathfrak{A}(R_2). \quad (7)$$

Equation 7 is analogous to the isotony principle used in algebraic quantum field theory, although the admissible regions and their substantive interpretation in GR will be defined differently [6]. The paper does not automatically impose relativistic commutativity between spacelike separated regions because a GR model may lack a relativistic causal metric or may represent relations whose dependence structure is different.

For an increasing directed family of finite regions, the compatible local algebras can be assembled into an inductive union and, when a normed C^* setting is chosen, completed to a quasi-local algebra. The construction is stated schematically in Equation 8.

$$\mathfrak{A}_K = \overline{\bigcup_{R \in K} \mathfrak{A}(R)}^{\|\cdot\|}, \quad (8)$$

where $R \in K$ denotes the declared family of finite or relatively compact regions. Equation 8 makes the global algebra a completion of locally generated structure. This is one reason the formalism can avoid beginning with a global vector of infinitely many components.

The term *quasi-local* should therefore be read structurally. It expresses construction from compatible local pieces and a suitable completion. Whether a particular GR application admits a directed family of regions, injective embeddings, or a useful norm completion is a model question. Later sections will state these as assumptions where they are required rather than treating quasi-locality as automatic.

2.7 Dynamical Maps for Open Subsystems

A transformation of an algebra need not be an invertible automorphism. This flexibility is important for open subsystems, coarse-grained descriptions, and irreversible effective dynamics. If $\mathfrak{A}$ and $\mathfrak{B}$ are algebras associated with two regions or two stages of a model, a map $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ can preserve different amounts of structure. A $*$ -homomorphism preserves multiplication and involution; an automorphism is an invertible $*$ -homomorphism from an algebra to itself. Positive and completely positive maps preserve positivity under progressively stronger matrix-level conditions and play a central role in the mathematical description of open quantum dynamics.

For Markovian quantum systems with finite-level state spaces, Gorini, Kossakowski, and Sudarshan, and independently Lindblad, characterized generators of completely positive dynamical semigroups [9, 11]. The physical assumptions of those theorems are not imported into GR. Their relevance here is narrower: they demonstrate that effective subsystem dynamics can be represented by structure-preserving maps that are more general than reversible unitary evolution.

A one-parameter semigroup of maps $\{\Phi_t\}_{t \geq 0}$ satisfies the composition law stated in Equation 9.

$$\Phi_{t+s} = \Phi_t \circ \Phi_s, \quad \Phi_0 = \text{id}. \quad (9)$$

Equation 9 supplies one possible dynamical architecture when an ordered parameter t and a Markov-type composition rule are justified. GR does not assume that every relational history satisfies this property. Historical memory, topology change, delayed effects, or changing domains may require non-Markovian maps, cocycles, partial morphisms, or maps between different algebras.

The last possibility is particularly important for the present paper. If a transformation changes the represented complex from K to K' , the natural algebraic object may be a map $\Phi^{K \rightarrow K'} : \mathfrak{A}_K \rightarrow \mathfrak{A}_{K'}$. Such a map is not an endomorphism of one fixed algebra unless an additional identification between $\mathfrak{A}_K$ and $\mathfrak{A}_{K'}$ is supplied. This distinction will later separate topology-preserving dynamics from historical and topological reconfiguration.

2.8 Ordered Composition and Noncommutative Structure

Noncommutativity expresses order sensitivity. For two elements a, b of an algebra, their commutator is defined by Equation 10.

$$[a, b] = ab - ba. \quad (10)$$

Equation 10 vanishes when the two products agree. A nonzero commutator means that the ordering of the two algebraic operations matters. In the present paper, noncommutativity will be introduced only when distinct orders of defined GR transformations produce distinct formal consequences. It will not be used as a stylistic synonym for complexity.

The same idea applies to maps. If Φ_1 and Φ_2 have compatible domains and codomains, the two compositions $\Phi_2 \circ \Phi_1$ and $\Phi_1 \circ \Phi_2$ can be compared when both are defined on the same relevant object. History dependence can therefore enter through composition itself: the first transformation changes the object on which the second transformation acts.

Closed compositional paths provide a mesoscopic object between local transformations and global structure. Consider a composable cycle of maps that returns to its starting algebra. The cycle composite is defined in Equation 11.

$$\mathcal{H}_c = \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1 : \mathfrak{A}_0 \longrightarrow \mathfrak{A}_0. \quad (11)$$

Equation 11 will later motivate a holonomy-like diagnostic. If $\mathcal{H}_c$ equals the identity, the selected cycle produces no residual algebraic transformation in the chosen representation. If it differs from the identity,

a residual effect remains after composition around the cycle. Calling this quantity *holonomy-like* records a structural analogy with transport around closed paths; a genuine gauge-theoretic holonomy requires additional geometric assumptions that will be stated separately if used.

Cycles are also relevant to emergence. A system-level quantity may depend on $\mathcal{H}_c$ even when no individual local transformation has the same property. The later formalism will use this observation to distinguish local power effects from compositional characteristics of a relational system.

2.9 Homological Structure and Topological Transformation

Topology concerns structural properties preserved under continuous deformation, while algebraic topology assigns algebraic objects to spaces so that structural differences can be studied calculably. Homology is particularly useful for finite complexes because it can be computed from boundary matrices. Hatcher provides a standard development of cell complexes and homology that underlies the definitions used here [10].

For a cell complex K , let $C_q(K)$ be the q -th chain group generated by the oriented q -cells. Boundary maps $\partial_q : C_q(K) \rightarrow C_{q-1}(K)$ satisfy the chain-complex condition stated in Equation 12.

$$\partial_q \circ \partial_{q+1} = 0. \quad (12)$$

Equation 12 implies that every boundary is a cycle. The q -th homology group is then defined by Equation 13.

$$H_q(K) = \ker \partial_q / \text{im } \partial_{q+1}. \quad (13)$$

Equation 13 identifies cycles that differ by boundaries. Over a field of coefficients, the Betti number $\beta_q(K) = \dim H_q(K)$ counts the number of independent homology classes in degree q . For a finite complex, the Euler characteristic can be written as the alternating sum of cell counts and, under the usual finite-dimensional conditions, also as the alternating sum of Betti numbers.

These quantities provide useful diagnostics of structural change, with an important limitation. If a transformation $T : K \rightarrow K'$ changes a Betti number or another genuine topological invariant, the represented topology has changed. Equality of a selected set of invariants does not establish that two complexes have the same topology because homology is not a complete topological invariant. Later claims of topology preservation will therefore state the level at which preservation has actually been established.

Elementary modifications of a complex include attaching or deleting cells, identifying previously distinct cells, splitting a cell, refining a cellulation, and gluing complexes along a common subcomplex. Some such operations preserve homotopy type or homology; others change them. A power-conditioned historical transformation can therefore be classified more finely than by asking whether individual edge weights changed. The formalism can distinguish a change of labels on a fixed complex, a change of incidence structure that preserves selected invariants, and a transformation detected by changed homological or other topological data.

This distinction will also govern the treatment of the past. A fixed-past regime can keep a realized subcomplex K^- unchanged while modifying admissible continuations. A stronger history-reconfiguring regime can permit transformations of the structure assigned to K^- itself. Homology supplies one calculable diagnostic for the strongest cases, while the ontological interpretation of such a transformation remains separate from the mathematics.

2.10 Modeling Status of the Mathematical Background

The mathematical resources introduced in this section enter the paper at different epistemic levels. Table 1 summarizes their standard role and the narrower role for which they are imported into the GR formalism. The table is intended to prevent a recurring category error: the availability of a mathematical representation does not establish that its source domain and its GR application share an ontology.

Table 1 also clarifies the paper's use of the phrase *field-theoretic*. The paper does not begin by declaring power to be a physical field. It begins with a history-bearing relational complex and local algebraic structure. A field representation is introduced only if distributed local data and their composition provide an

Table 1: Mathematical background and its restricted role in the GR power formalism.

Structure	Standard mathematical or physical role	Role adopted in this paper
2-complex	Incidence structure with vertices, edges, and faces; used in several topological and physical constructions	Candidate discrete representation of history-bearing relational structure
Spin foam	Labelled 2-complex used to represent quantum-gravitational histories in spin-foam models	Architectural precedent for labelled histories; quantum interpretation is not transferred
C^* -algebra	Norm-complete $*$ -algebra supporting states and representation theory	Algebra-first container for locally generated observables or transformations when its assumptions are justified
W^* / von Neumann algebra	Weakly closed operator-algebraic structure with strong analytic representation theory	Representation-dependent completion used only where weak limits, normality, or spectral analysis require it
GNS construction	Representation of a C^* -algebra and state on a Hilbert space	Derived route to Hilbert-space calculations after an algebra and evaluation state have been chosen
Positive or completely positive map	Structure-preserving map central to operator-algebraic and open-system dynamics	Candidate effective map for GR subsystems when positivity assumptions have a declared interpretation
Homology	Algebraic invariant of topological spaces and complexes	Calculable diagnostic for selected forms of structural and topology change

economical formal description of power-conditioned dynamics. Operator representations are then derived where useful, and topological diagnostics are applied only to transformations that act on the represented complex.

The resulting hierarchy is methodological. Discrete complexes supply a domain, local algebras supply compositional structure, field assignments distribute formal data, algebraic maps encode dynamics, states supply evaluations when needed, Hilbert-space representations support particular calculations, and homology supplies selected global diagnostics. Later GR sections will specify the substantive meaning of these objects and test whether the formal relations among them are coherent.

The background also leaves several choices open. A GR application may use real rather than complex algebras, may require partial rather than total maps, may use categories of local algebras rather than one quasi-local completion, or may employ invariants stronger than homology. The preliminary formalism developed in this paper will select a tractable core architecture while marking these alternatives as extensions. This scope discipline is important because mathematical generality and empirical usefulness need not increase together.

3 GR Spacetime and History-Bearing Configurations

This section specifies the GR-specific kinematic objects on which the later power formalism will act. Its role is to translate the mathematical background of Section 2 into a representation of history-bearing relational configurations while preserving ontological suspension. The construction proceeds from a labelled relational complex to history-bearing subconfigurations, local regions and interfaces, realized-history exten-

sions, local algebraic assignments, and changes of representational resolution. Power is deliberately absent from the definitions in this section. The resulting kinematic layer provides a domain on which power-conditioned fields and algebraic transformations can subsequently be defined without building power into the representation of spacetime itself.

3.1 GR Spacetime as a Labelled Relational Complex

Within the present paper, a *GR spacetime* is a formal representation of historically extended relational structure. It is represented provisionally by a labelled 2-complex. The basic object is stated in Definition 3.1.

Definition 3.1 (GR spacetime representation). A finite or locally finite GR spacetime representation is a tuple $K = (V, E, F, \partial, \lambda)$ in which V , E , and F are respectively vertices, edges, and faces, ∂ records incidence and attachment data, and λ assigns model-dependent labels to selected cells or regions. The tuple is a representational object used to encode historical and relational structure; its cells do not receive fixed substantive meanings independently of the application.

Definition 3.1 leaves the semantic role of each cell type partially open. A vertex may represent an event, a localized relational update, or another elementary unit selected by the model. An edge may represent a transition, dependence, interaction, or continuation relation. A face may encode a compositional relation among several transitions, a compatibility condition, or another higher-order relational structure. The paper therefore uses the same combinatorial architecture across applications while permitting the labels and interpretation maps to vary.

The label map λ may itself have several components. For a given application one can write the labelling structure as in Equation 14.

$$\lambda = (\lambda_V, \lambda_E, \lambda_F, \lambda_R), \quad (14)$$

where the first three components assign admissible labels to cells and λ_R denotes optional labels on composite regions. Equation 14 does not prescribe a universal list of GR variables. It separates incidence structure from model-dependent attributes so that a change in relational meaning, empirical parameter, or local constraint need not automatically be treated as a change in topology.

The word *spacetime* is used because the represented object carries both relational extension and historical structure. The formalism does not assume a differentiable manifold, Lorentzian metric, or physically fundamental temporal coordinate. Metric, causal, probabilistic, or geometric data can be added as further structure when an application requires them. This restraint permits the same basic representation to describe heterogeneous systems whose relevant relations are historical but lack a natural embedding in ordinary physical space.

3.2 History-Bearing Entities and Coupled Configurations

GR comparisons concern entities whose present relational capacities depend on accumulated structure. Representing such an entity by one vertex would generally erase the history through which its current relations became possible. The present formalism therefore represents an entity A by a history-bearing subconfiguration $K_A \subseteq K$ at the chosen resolution. Another entity B is represented by $K_B \subseteq K$. These subconfigurations may contain internal vertices, edges, faces, labels, and local algebras.

The coupled configuration relevant to a relation between A and B is written K_{AB} . It contains the portions of K_A and K_B required for the comparison together with the relational structure through which they interact. The construction is summarized in Equation 15.

$$K_A, K_B \subseteq K_{AB} \subseteq K. \quad (15)$$

Equation 15 is an inclusion statement at a selected representational scale. It does not require K_A and K_B to be disjoint, exhaustive, or uniquely decomposable. Shared institutions, common histories, jointly constituted interfaces, and other overlapping structures can be represented by common subcomplexes or by additional cells whose meaning is defined only at the coupled level.

This representation changes the formal meaning of an expression such as “ A affects B .” The relevant transformation may act on K_B , on an interface between K_A and K_B , or on the coupled configuration K_{AB} itself. A later power relation will therefore be allowed to modify the historical and relational conditions under which both entities are represented. Power need not be attached to an isolated point or stored as an intrinsic scalar property of one actor.

Figure 2 illustrates the distinction between point-like actors and coupled history-bearing configurations. The figure is schematic; its purpose is to show that the formal unit of analysis can include internal histories and a relational interface before any power field is introduced.

A history-bearing entity can also be represented at different resolutions. A coarse model may collapse a stable internal subcomplex into one effective cell, while a finer model exposes the internal relations relevant to a particular comparison. The object called A is therefore associated with an equivalence of useful representations rather than with one immutable cell decomposition. Section 3.7 formalizes this representational flexibility.

3.3 Local Regions, Boundaries, and Relational Interfaces

The field-theoretic construction requires a notion of locality that is meaningful for a relational complex. Let $\mathcal{R}(K)$ denote a declared family of admissible regions of K . A region $R \in \mathcal{R}(K)$ may be a subcomplex or another finite local structure for which incidence and labels are well defined. The family should be rich enough to cover the portions of K used by the model and structured enough that compatible inclusions can be tracked.

For two regions $R_1, R_2 \in \mathcal{R}(K)$, their overlap $R_1 \cap R_2$ represents shared relational structure when the intersection is defined in the chosen category of regions. A boundary ∂R records the interface through which the region connects to its complement in the modeled domain. The boundary is not assumed to be a physical spatial boundary. It may encode an institutional interface, a communication channel, a legal jurisdictional boundary, a market interface, or another declared relation between local structures.

This notion of interface is important because many effective dynamics are neither wholly internal to an entity nor reducible to an external force acting on it. A transformation supported near ∂R can change which local operations remain compatible with the surrounding system. Later power fields can therefore be supported on cells, interfaces, or composite regions depending on the substantive mechanism.

The minimal compatibility requirement for local structure is inclusion consistency. If $R_1 \subseteq R_2$, data defined on R_1 should admit a declared relation to data on R_2 . At the algebraic level this relation will be represented by an embedding or another structure-preserving map. At the level of labels, restriction maps can recover the local part of a broader assignment. The paper does not assume that every local quantity extends uniquely to every larger region; nonunique extension is one source of genuine open-system behavior.

3.4 Realized Histories and Continuation Structures

A history-bearing spacetime representation requires a distinction between structure treated as already instantiated for a particular model and structure treated as a possible continuation. The paper introduces this distinction through extensions rather than through a universal decomposition into past, present, and future slices.

Let K^- denote the complex that a given modeling stage treats as realized history. A continuation of K^- is represented by an admissible extension pair (K', ι) in which K' is a candidate enlarged or modified complex and $\iota : K^- \hookrightarrow K'$ is an incidence- and label-compatible embedding under the assumptions of the selected regime. The continuation family is defined in Equation 16.

$$\text{Ext}(K^-) = \{(K', \iota) \mid \iota : K^- \hookrightarrow K' \text{ is an admissible extension embedding}\}. \quad (16)$$

Equation 16 treats the fixed-past construction as a family of embeddings. It is particularly convenient for finite modeling and for interventions whose intended effects concern possible continuations while preserving the realized subcomplex.

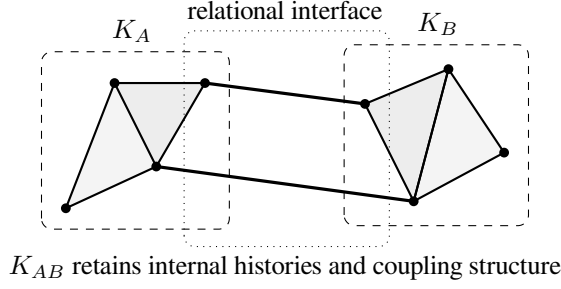


Figure 2: History-bearing subconfigurations within a coupled GR spacetime representation. K_A and K_B retain internal relational histories, while the coupled configuration K_{AB} includes the interface through which transformations can propagate.

The extension family is not an ontological definition of the past. It is a modeling device. A later section will permit transformations that change the representation assigned to realized history itself, including transformations for which no embedding $K^- \hookrightarrow K'$ preserves the relevant historical structure. The distinction between these regimes allows practical models to use a stable realized history while the broader GR framework remains open to stronger forms of historical reconfiguration.

Continuations may differ at several levels. Two extensions can preserve the same incidence structure while changing labels or weights. They can add cells while preserving the homology of the realized part. They can also introduce new relations that alter global topology. The extension formalism therefore separates the existence of a continuation from the later question of how a power field weights, enables, suppresses, or transforms that continuation.

3.5 Extension Order and Temporal Representation

Historical order can be represented without assuming that the full system is parameterized by one primitive scalar time. Define a relation $\preceq$ on admissible history representations by $K_i \preceq K_j$ when there exists an admissible embedding from K_i into K_j that preserves the structure declared relevant to the regime. The relation is stated in Equation 17.

$$K_i \preceq K_j \iff \exists \iota_{ij} : K_i \hookrightarrow K_j \text{ admissible under the selected history regime.} \quad (17)$$

Equation 17 defines an extension relation. Identity embeddings and composition make this relation reflexive and transitive when the admissibility rules are closed under composition. Antisymmetry requires an additional condition: if mutual admissible embeddings identify the two histories under the model's chosen equivalence, the resulting quotient relation is a partial order. In finite constructions such a partial order can be displayed as a directed acyclic diagram when the admissibility rules exclude cycles of strict extension. Branching records multiple possible continuations, while incomparable elements record histories for which the model supplies no extension relation.

A scalar parameter t may still be introduced when observations, institutional calendars, physical time, or another application provides a useful ordering variable. In that case a family K_t can be chosen so that $t_1 \leq t_2$ is compatible with $K_{t_1} \preceq K_{t_2}$ for the histories being modeled. The scalar parameter is then additional structure on the extension order. It is not required for the general kinematic construction.

This distinction will matter for operator composition. A one-parameter semigroup can be used in a regime with a justified scalar parameter and stable algebraic domain. A branching history or topology-changing process may instead require maps indexed by composable extensions. The later dynamics will therefore be formulated first in terms of local and composable transformations, with continuous-time generators introduced only for models that support them.

3.6 Local Algebraic Assignment

The algebra-first strategy associates each admissible region with a local algebra before a global state representation is selected. Let $R \in \mathcal{R}(K)$. The assignment $R \mapsto \mathfrak{A}_K(R)$ associates R with an algebra of locally supported formal objects. At the kinematic stage these objects can include declared observables, admissible transformations, constraints, generators, or other elements whose multiplication and involution have specified meanings. Power-conditioned elements will be introduced only in the subsequent sections.

For nested regions the basic compatibility condition is stated in Equation 18.

$$R_1 \subseteq R_2 \implies \iota_{R_1 R_2} : \mathfrak{A}_K(R_1) \longrightarrow \mathfrak{A}_K(R_2), \quad (18)$$

where $\iota_{R_1 R_2}$ is an injective $*$ -homomorphism when the C^* -algebraic specialization is used. Equation 18 is an isotony-type assumption rather than an automatic property of every GR application. If local descriptions overlap imperfectly or require quotienting, gluing, or contextual translation, a weaker categorical construction may be required.

When the family of regions is directed and the embeddings are compatible, the local algebras can generate a quasi-local algebra $\mathfrak{A}_K$ through the completion described in Section 2.6. The global algebra remains derived from local structure. A state ω on $\mathfrak{A}_K$ is then an evaluation functional chosen for a particular analytical purpose. Different states can support different GNS representations without changing the underlying decision to treat the algebraic net as the primary kinematic object.

This architecture is useful for history-bearing entities because K_A and K_B can possess their own local algebraic structure while remaining embedded in K_{AB} . A transformation relevant to A and B can therefore be represented on $\mathfrak{A}_{K_{AB}}(R)$ for a suitable coupled region R , with restrictions to the entity-level algebras when those restrictions are defined. The formalism does not require a universal tensor-product factorization $\mathfrak{A}_A \otimes \mathfrak{A}_B$; shared histories and common interfaces can prevent such a decomposition from being appropriate.

3.7 Refinement, Coarse-Graining, and Representation Dependence

A GR spacetime representation is chosen at a resolution adequate to the research question. Let $K^{(f)}$ be a finer complex and $K^{(c)}$ a coarser representation of overlapping relational content. A coarse-graining map $q : K^{(f)} \rightarrow K^{(c)}$ can identify or aggregate cells according to a declared rule. Conversely, a refinement replaces a coarse cell or region with a more detailed subcomplex while preserving the interface information required by the surrounding model.

A useful formal quantity should therefore state its behavior under resolution change. Some quantities may be invariant under a selected class of refinements. Others may transform covariantly, while some exist only at a particular resolution. The relation between fine and coarse descriptions is summarized in Equation 19.

$$K^{(f)} \xrightarrow{q} K^{(c)}, \quad \mathfrak{A}_{K^{(f)}} \xrightarrow{Q} \mathfrak{A}_{K^{(c)}}, \quad (19)$$

where the algebraic map Q must be specified by the model and need not be invertible. Equation 19 records the distinction between a representational reduction and a reversible symmetry transformation.

Resolution dependence is especially important for power. A relation that appears bilateral at a coarse scale may decompose into several institutionally mediated transformations at a finer scale. A cycle absent from the coarse representation may become visible after refinement, while another apparent cycle may disappear when redundant edges are identified. Later claims about curvature-like residuals, topology, or invariants will therefore specify the complex and resolution on which they are calculated.

Refinement also limits the interpretation of “entity.” A state, institution, organization, person, or technological system can be treated as one effective object for one question and as an internally structured history for another. GR therefore treats entity boundaries as modeling commitments that should remain explicit, especially when a proposed power effect depends on hidden internal structure.

3.8 Kinematic Interface for Power Dynamics

The constructions in this section provide the domain on which the paper's power formalism will operate. Table 2 summarizes the principal objects and the questions they leave to later sections.

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Table 2: Kinematic objects supplied before the introduction of power-conditioned dynamics.

Object	Kinematic role	Later dynamical question
$K = (V, E, F, \partial, \lambda)$	History-bearing relational spacetime representation	Which parts of incidence, labels, and admissibility are deformed by power?
K_A, K_B, K_{AB}	Entity-level and coupled history-bearing configurations	Where is a power relation supported, and which coupled structures does it modify?
$\mathcal{R}(K)$ and ∂R	Local regions and relational interfaces	How do local field values propagate across interfaces?
$\text{Ext}(K^-)$	Family of admissible continuations in the fixed-past regime	How are continuations weighted, enabled, suppressed, or structurally transformed?
$\preceq$	Extension order on history representations	How does ordered composition depend on historical path?
$\mathfrak{A}_K(R)$	Local algebraic structure	Which power-conditioned maps act on or between local algebras?
q and Q	Resolution-changing maps	Which power observables or invariants survive refinement and coarse-graining?

Table 2 makes the separation between kinematics and power explicit. The present section specifies the relational domain, history structure, locality, algebraic support, and representational resolution. It does not define power as a field, operator, scalar, or intrinsic attribute. The subsequent power-representation section will begin from the GR account of power developed conceptually in the preceding paper and ask which features of that account can be represented on the kinematic architecture established here.

Several formal constraints follow immediately. First, a power representation must respect the fact that the relevant entities are history-bearing configurations. Second, a local power representation should identify its support region and the algebra on which it acts. Third, transformations between different complexes should be represented as maps between their corresponding algebraic structures unless an explicit identification justifies an endomorphism on one fixed algebra. Fourth, quantities presented as system-level characteristics should state their dependence on resolution and on the selected extension regime.

These constraints narrow the admissible form of the next construction. A power field can be introduced only after its local domain, value structure, and transformation rule have been specified. An operator representation can be introduced only after the algebraic object on which it acts has been defined. Historical topology change can be claimed only when a transformation acts on the complex itself and a declared topological diagnostic changes. The resulting sequence keeps the field-theoretic formalism connected to calculable objects while preserving the distinction between a useful representation and an ontological assertion.

4 Power Representation in Generative Relational Theory

This section supplies the substantive and representational interface required before a field of power can be constructed on GR spacetime. Its role is to identify the object being represented, the historical and regional support on which that object is instantiated, the local consequences through which its dynamical action can be interrogated, and the conditions under which those consequences can be organized into a distributed mathematical representation. The construction begins from power as a history-bearing relational organization inside a coupled GR spacetime. It then introduces region-indexed deformation records extracted through admissible perturbative comparisons. The resulting object is deliberately weaker than a field: it

records where and how a power structure deforms generative possibilities without yet assuming that those deformations admit values in one fixed vector space, Lie algebra, operator algebra, or scalar codomain. The final subsection states the additional coherence requirements that a later field representation must satisfy.

4.1 Relational Power as the Substantive Starting Object

The field-theoretic construction requires a substantive object prior to its mathematical lift. Let K_{AB} be a coupled history-bearing GR spacetime containing admissible subconfigurations K_A and K_B in the sense of Section 3. Let $\alpha_\alpha(A, B \mid K_{AB})$ denote a typed relational asymmetry along a declared relational dimension α . The notation records a historically situated failure of interchangeability or equivalence inside the coupled configuration; it does not assign a scalar difference to the two entities.

The phrase *relationally real* is used here as a model-status predicate inherited from the GR framework rather than as an assertion about ultimate ontology. A retained structure is treated as relationally real when the adopted identification criteria regard it as sufficiently persistent across the relevant historical comparisons and causally efficacious within the represented relational domain. The criteria, comparison family, and evidential basis must be declared by the application. This restricted use is consistent with the ontological suspension maintained throughout the paper.

A typed asymmetry becomes relevant to power when its organization has directional consequences for the generative possibilities of another history-bearing entity or of the coupled spacetime itself. For the present paper, the required generative possibilities are represented locally. Let $R \in \mathcal{R}(K_{AB})$ be an admissible region. Reusing the extension construction of Equation 16 with the current coupled carrier treated as the anchor, let $\text{Ext}(K_{AB})$ denote its admissible extension pairs. For an extension pair $(K', \iota) \in \text{Ext}(K_{AB})$, let $\text{Supp}_{\text{cont}}(K', \iota)$ denote the continuation-specific support declared by the representation: the cells, relations, or incidence changes in K' that are additional to, or modify the attachment of, the embedded anchor $\iota(K_{AB})$. The regional extension family $\text{Ext}_R(K_{AB})$ is then the subset of extension pairs whose continuation-specific support is incident to $\iota(R)$ or to a model-declared interface neighborhood of $\iota(R)$. This support criterion derives the regional family from the primitive extension pair and the retained regional incidence structure rather than from an undefined notion of “change intersecting” a region. Let $\text{Adm}_{K_{AB}}(R)$ be the declared family of admissible local transformations whose domains contain the retained local data associated with R . The local generative-possibility structure is stated in Equation 20.

$$\text{Poss}_{K_{AB}}(R) := (\text{Ext}_R(K_{AB}), \text{Adm}_{K_{AB}}(R), \mathfrak{A}_{K_{AB}}(R)). \quad (20)$$

Equation 20 combines three structures that later power effects may modify: admissible continuation, admissible local transformation, and the local algebraic carrier. The third component does not imply that every power effect changes the algebra itself. Many applications retain $\mathfrak{A}_{K_{AB}}(R)$ while changing only a distinguished admissible subset or the maps acting on it. Other applications can require a different local algebra because the represented relations, labels, or topology have changed.

The present paper adopts the following local reconstruction of relational power. The definition is intentionally model-relative because persistence, causal efficacy, and generative efficacy require domain-specific identification.

Definition 4.1 (Relational power structure). Let K_{AB} be a coupled history-bearing GR spacetime and let $\alpha_\alpha(A, B \mid K_{AB})$ be a relationally real typed asymmetry. A directional relational power structure, denoted $\mathcal{P}_{A \curvearrowright B}^{\mathcal{R}}[K_{AB}]$, is a historically situated relational organization associated with that asymmetry when the adopted model establishes: (i) a directionally consequential deformation of at least one declared generative-possibility structure; (ii) expected persistence or recurrent retention across a declared family of admissible historical extensions; (iii) causal efficacy under at least one well-typed comparative intervention or dependence test; and (iv) generative efficacy through contribution to the production, modification, stabilization, or removal of conditions governing later relational continuation.

For a fixed coupled carrier K_{AB} , let $\mathfrak{P}^{\mathcal{R}}(K_{AB})$ denote the class of directional relational power structures satisfying Definition 4.1 under the declared entity, history, and comparison conventions. This class is

introduced only as the domain of later representation maps; it is not a vector space and carries no primitive addition or norm.

Definition 4.1 identifies the substantive object represented in the remainder of the paper. The superscript $\mathcal{R}$ marks relational-reality status inside the adopted GR model. It does not denote a mathematical field value. The direction $A \curvearrowright B$ identifies one directional component of a potentially reciprocal or distributed power configuration. The bracketed K_{AB} records that the component is constituted inside a coupled history-bearing spacetime and need not be an intrinsic possession stored in A .

The definition also separates power from several later mathematical devices. A scalar magnitude, local field value, generator, operator, spectrum, attractor, or topological invariant can represent some aspect of the power structure under additional assumptions. None of these objects is part of Definition 4.1. This separation is necessary because the present paper aims to derive a field-theoretic representation from relational structure rather than identify power with a field by stipulation.

4.2 Directional Deformation of Generative Possibilities

The first representational task is to specify what changes when a power structure is dynamically consequential. The paper uses admissible perturbation as an epistemic probe. Let $\Delta_{\mathcal{P}}(K_{AB})$ denote a declared family of comparison transformations that modify, attenuate, remove, redirect, or otherwise vary a represented component of $\mathcal{P}_{A \curvearrowright B}^{\mathcal{R}}[K_{AB}]$ while retaining enough structure for the before-and-after configurations to remain comparable. For $\delta \in \Delta_{\mathcal{P}}(K_{AB})$, write K_{AB}^{δ} for the comparison configuration when the transformation is defined.

The local response object associated with region R and perturbation δ is defined in Equation 21.

$$\mathcal{R}_{\mathcal{P}}(R; \delta) := \left(\text{Poss}_{K_{AB}}(R), \text{Poss}_{K_{AB}^{\delta}}(R^{\delta}) \right), \quad (21)$$

where R^{δ} is the region in the comparison configuration identified by the model's declared correspondence rule. Equation 21 is an ordered comparison, not a subtraction. The two possibility structures can differ in set membership, admissible transformations, algebraic carrier, incidence relations, or another declared feature for which a numerical difference has not been defined.

The perturbation family serves an epistemic role. A large response to one arbitrarily chosen perturbation is insufficient to establish a scalar amount of power, and a perturbation that changes the entity boundary beyond recognition is not automatically informative about the original relation. The model must state which structural features are held comparable and which are intentionally varied. This requirement is recorded in Assumption 4.2.

Assumption 4.2 (Power-perturbation comparability). For each retained $\delta \in \Delta_{\mathcal{P}}(K_{AB})$ and each region used to diagnose a power effect, the model supplies a correspondence between the relevant pre- and post-perturbation entities, regions, and relational scales sufficient to compare the declared components of $\text{Poss}_{K_{AB}}(R)$ and $\text{Poss}_{K_{AB}^{\delta}}(R^{\delta})$.

Assumption 4.2 allows a response to be structural rather than metric. One perturbation may alter the continuation family while preserving the local algebra. Another may preserve the continuation set while changing admissible local maps. A stronger transformation may change the local complex and therefore require a map between distinct algebras. These possibilities are kept separate because later field realizations, dynamical laws, and operator representations depend on which layer is actually deformed.

A local deformation can therefore be typed through the components on which the perturbative response is nontrivial. Table 3 records the principal targets used later in the paper.

Table 3 gives a hierarchy of possible effects without ranking them by strength. A change in admissibility can be consequential while topology remains fixed, and a topological transformation can be analytically important without implying a larger scalar quantity of power. The relevant ordering depends on the research question and on the comparison structure supplied by the application.

Table 3: Principal deformation targets available to a GR representation of power.

Target	Represented change	Later formal consequence
Continuation family	Addition, removal, redirection, or reclassification of admissible historical extensions	Reachability and history-structure analysis
Admissible transformations	Change in which local maps can participate in later relational generation	Algebraic and operator-dynamical analysis
Local algebraic carrier	Change in relations, constraints, or supported observables requiring $\mathfrak{A}_K(R)$ and $\mathfrak{A}_{K'}(R')$ to be distinguished	Morphisms between local algebras rather than one fixed endomorphism
Incidence and boundary structure	Change in vertices, edges, faces, attachments, or interfaces of the represented complex	Geometric and topological deformation
History evaluation or weight	Change in a declared weighting, propensity, cost, or evaluation on retained continuations	Weighted-history and criticality calculations when semantics are supplied

4.3 Constitutive Support and Domains of Effect

A field representation becomes informative only if the model can distinguish where the power structure is constituted from where its consequences are registered. The two domains can overlap, but their equality is not assumed. This distinction is especially important for institutions, infrastructures, symbolic systems, and other distributed structures whose effects can propagate far beyond the regions containing the relations that sustain them.

Let $\mathcal{S}_{\mathcal{P}} \subseteq \mathcal{R}(K_{AB})$ be a model-specified *constitutive support family*. A region $R \in \mathcal{S}_{\mathcal{P}}$ contains relational structure whose admissible modification can alter the identity, persistence, causal efficacy, or generative efficacy of $\mathcal{P}_{A \cap B}^{\mathcal{R}}[K_{AB}]$ under the declared identification criteria. The family need not have a unique minimal member. Distributed or nonlocal organization can require several overlapping regions, and different resolutions can yield different support descriptions.

The corresponding *effect family* is defined relative to the retained perturbations. Let $\equiv_R^{\text{cmp}}$ denote the model's declared equivalence relation for comparing the retained components of two local possibility structures at region R . The construction is stated in Equation 22.

$$\mathcal{E}_{\mathcal{P}}^{\Delta} := \left\{ R \in \mathcal{R}(K_{AB}) \mid \exists \delta \in \Delta_{\mathcal{P}}(K_{AB}), \text{Poss}_{K_{AB}}(R) \not\equiv_R^{\text{cmp}} \text{Poss}_{K_{AB}^{\delta}}(R^{\delta}) \right\}. \quad (22)$$

Equation 22 identifies regions in which a retained change to the represented power structure alters at least one declared component of local generative possibility. The comparison equivalence $\equiv_R^{\text{cmp}}$ remains application-specific: it may compare continuation sets, admissible-map families, algebra morphisms, incidence data, or another formally specified feature.

The distinction between $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{E}_{\mathcal{P}}^{\Delta}$ blocks a common simplification. A power structure can be sustained in one set of relations while its effects appear across a larger or differently organized part of the spacetime. Conversely, a region can participate in sustaining power without exhibiting a large locally visible response under the perturbations selected for one analysis. The later field construction will therefore treat source-like support and distributed field effect as separately represented structures.

Remark 4.3 (Support is representation-relative). The support family $\mathcal{S}_{\mathcal{P}}$ depends on entity boundaries, historical scope, resolution, and the identity criteria used for the power structure. It should therefore be interpreted as support within a declared GR representation rather than as a unique ontological location of power.

Remark 4.3 is particularly important under refinement. A region represented as one institutional relation at coarse resolution can decompose into several internal subregions at finer resolution, while the relational power structure remains recognizable under a declared coarse-graining map.

4.4 Region-Indexed Deformation Records

The substantive power structure and the perturbative responses can now be connected through a representation map that remains weaker than a field. Let $\mathfrak{D}(K_{AB})$ denote the class of region-indexed deformation records that satisfy the typing and comparability rules of the adopted model. A deformation record assigns to each retained region a family of well-typed response objects generated by the admissible perturbations. For later local realizations, define the regional record class by $\mathfrak{D}_R(K_{AB}) := \{\mathbf{D}(R) \mid \mathbf{D} \in \mathfrak{D}(K_{AB}), R \text{ lies in the domain of } \mathbf{D}\}$. The deformation lift is then a partial representation map on the class of substantive power structures, as stated in Equation 23.

$$\Lambda_{\text{def}}^{K_{AB}} : \mathfrak{P}^{\mathcal{R}}(K_{AB}) \rightarrow \mathfrak{D}(K_{AB}), \quad \Lambda_{\text{def}}^{K_{AB}}(\mathcal{P}_{A \cap B}^{\mathcal{R}}[K_{AB}]) = \mathbf{D}_{\mathcal{P}}. \quad (23)$$

where the local component is defined by Equation 24.

$$\mathbf{D}_{\mathcal{P}}(R) := \{\mathcal{R}_{\mathcal{P}}(R; \delta) \mid \delta \in \Delta_{\mathcal{P}}(K_{AB}), \mathcal{R}_{\mathcal{P}}(R; \delta) \text{ is defined}\}. \quad (24)$$

Equations 23 and 24 define the first mathematical lift used specifically for this paper's power dynamics. The record $\mathbf{D}_{\mathcal{P}}$ is an epistemically constructed representation of how a relationally real power structure responds to declared variations across regions. It is not yet a field because no common codomain, local generator, restriction law, connection, or action principle has been assumed.

The deformation lift need not be injective. Distinct relational power structures can produce the same retained deformation record at a coarse resolution or under an impoverished perturbation family. It need not be surjective because many mathematically possible deformation records may fail to correspond to a relationally real power structure. The relation between substantive power and formal representation is therefore many-to-one or partially defined in some applications.

This non-uniqueness has an epistemic consequence. Calculating $\mathbf{D}_{\mathcal{P}}$ from a model does not by itself identify the complete relational organization that produced the record. Additional historical, causal, institutional, symbolic, or other evidence can be required to distinguish observationally similar power structures. The field formalism developed later should therefore be read as a representation of dynamical consequences under declared assumptions, not as a lossless encoding of relational reality.

4.5 Locality, Restriction, and Representation Coherence

A region-indexed record becomes suitable for field-theoretic development only when its local components can be compared coherently across overlapping or nested regions. Let $R_1 \subseteq R_2$ be retained regions of K_{AB} . The algebraic net in Section 3.6 supplies an embedding $\iota_{R_1 R_2} : \mathfrak{A}_{K_{AB}}(R_1) \rightarrow \mathfrak{A}_{K_{AB}}(R_2)$ under the isotony-type assumptions of the model. A compatible deformation description should also specify how a response represented on R_2 restricts, projects, or induces a response on R_1 .

Let $\rho_{R_2 R_1}^{\mathcal{P}}$ denote such a model-defined restriction or projection when it exists. The minimum coherence condition is stated in Equation 25.

$$\rho_{R_2 R_1}^{\mathcal{P}} : \mathbf{D}_{\mathcal{P}}(R_2) \rightarrow \mathbf{D}_{\mathcal{P}}(R_1), \quad R_1 \subseteq R_2. \quad (25)$$

Equation 25 is deliberately partial. A deformation visible only at a coarser relational organization may fail to possess a meaningful restriction to an arbitrarily chosen subregion, while some local responses can be defined only after information from neighboring regions is retained. The later field formalism will therefore state locality assumptions explicitly rather than assume that every systemic power effect decomposes into independent pointwise values.

When three nested regions $R_1 \subseteq R_2 \subseteq R_3$ admit the required restriction maps, a coherent representation should satisfy the composition condition in Equation 26 on the common domain of definition.

$$\rho_{R_3 R_1}^{\mathcal{P}} = \rho_{R_2 R_1}^{\mathcal{P}} \circ \rho_{R_3 R_2}^{\mathcal{P}}. \quad (26)$$

Equation 26 supplies a consistency test that can be checked in a finite implementation. Failure of this condition can indicate that the selected local description loses context required by the larger region, that the proposed region correspondence is inadequate, or that a different mathematical architecture is needed for the application.

The same requirement applies under coarse-graining. If $q : K^{(f)} \rightarrow K^{(c)}$ and $Q : \mathfrak{A}_{K^{(f)}} \rightarrow \mathfrak{A}_{K^{(c)}}$ represent an admissible reduction, a power representation should state whether the fine deformation record induces a coarse record and which features are preserved. Exact equality need not hold. A coarse-grained model can intentionally discard cycle structure, internal asymmetry, or high-frequency relational variation. The retained invariants and approximation error therefore become part of the later calculational problem.

4.6 Power Comparison and the Absence of a Primitive Magnitude

The distributed deformation record should not be confused with a scalar power ranking. A judgment such as $A \succ_P B$ requires a comparison rule whose domain, criteria, and historical context are independently specified. The comparison may use selected features of $\mathbf{D}_{\mathcal{P}}$, such as breadth of affected regions, persistence across extensions, topological reach, or a spectral response in a finite model. Different comparison rules can produce different orderings from the same underlying deformation record.

The paper therefore does not introduce a universal norm $\|\mathcal{P}\|$ or a primitive local scalar $P(x)$. Such quantities can be defined in applications where a justified codomain and comparison rule exist. Their meaning remains derivative. In particular, a later field value should be interpreted first as a local formal carrier of transformation structure, source data, connection data, or another explicitly defined dynamical role. Its norm, when one exists, does not automatically equal an amount of social or political power.

This restraint is also important for reciprocal configurations. Two directional components $\mathcal{P}_{A \curvearrowright B}^{\mathcal{R}}$ and $\mathcal{P}_{B \curvearrowright A}^{\mathcal{R}}$ can coexist without admitting a total order. Their local deformation records can affect different regions, act on different transformation families, or generate distinct cycle structures. The later analysis of multiple fields will therefore examine composition and systemic effects before asking whether any scalar comparison is warranted.

Remark 4.4 (Field values and power rankings). A mathematical value assigned to a region in the subsequent field construction is a representation of locally specified dynamical structure. Any interpretation of its norm, spectrum, trace, or other numerical characteristic as a ranking of substantive power requires a separate comparison model and empirical justification.

Remark 4.4 preserves the distinction between calculability and substantive measurement. A finite model can yield exact numerical outputs while the empirical meaning of those outputs remains conditional on the mapping from observed relations to the formal objects.

4.7 Representation Ladder for the Field-Theoretic Construction

The constructions above establish a representation ladder without yet assuming a power field. Figure 3 summarizes the levels. Relational power occupies the substantive GR level. Admissible perturbations and local response objects provide an epistemic interface. The region-indexed deformation record organizes those responses into a distributed model. A field representation will be introduced only when the record admits sufficiently coherent local values and transformation laws. Algebraic dynamical maps and operator representations then provide later realizations of field action.

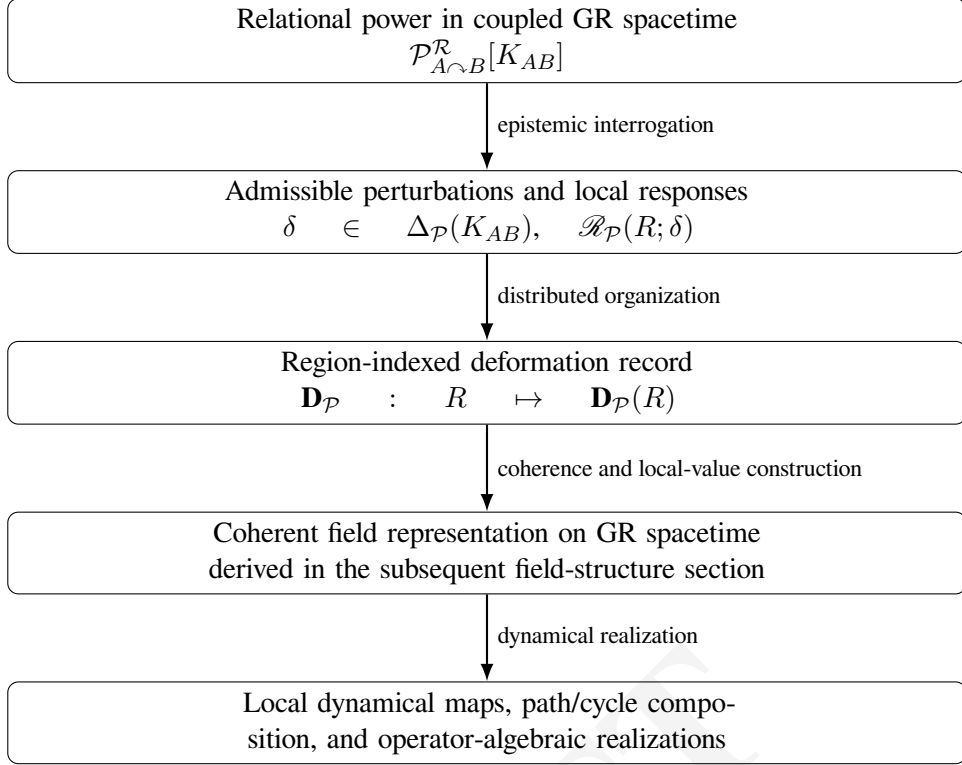


Figure 3: Representation ladder from relational power to later field and operator formalisms. Each downward step is a modeling or analytical construction rather than an ontological identity. The field level is introduced only after the region-indexed deformation record satisfies the additional local-coherence conditions developed in the subsequent field-structure section.

Figure 3 also clarifies why an operator is not the immediate representation of power in the present paper. An operator representation presupposes a specified algebraic or Hilbert-space realization and a domain on which the operation is defined. The field level adds spatially and historically distributed structure before such an operator lift is selected. This order is especially important when topology changes, because the relevant dynamical map can connect $\mathfrak{A}_K$ and $\mathfrak{A}_{K'}$ rather than act as an endomorphism on one fixed algebra.

The next construction can now be stated precisely. A GR power field must provide local mathematical values or local generating data for the distributed record $\mathbf{D}_{\mathcal{P}}$, specify their behavior under restriction and overlap, distinguish constitutive support from domains of effect, and define enough compositional structure for local transformations to be propagated along paths and around cycles. The existence of $\mathbf{D}_{\mathcal{P}}$ alone does not guarantee that such a field representation exists or is unique. The subsequent field-structure section therefore treats field structure as an additional formal achievement rather than a terminological relabeling of relational power.

5 Field Structure on GR Spacetime

This section constructs the field level of the power formalism. Its role is to determine when the region-indexed deformation record of Section 4 can be represented by compatible local mathematical values distributed over a history-bearing GR spacetime. The objective is narrower than a complete dynamical theory: the section defines local field carriers, realization maps, restriction and overlap laws, interface transport, representation equivalence, and quasi-local assembly. The construction remains algebra-first and allows the local carriers to vary from region to region. A field is therefore introduced as a coherent family of local realizations whose values reproduce the selected deformation record and admit controlled comparison across relational interfaces. Dynamical action on observables, histories, and complexes is reserved for the

subsequent sections on local dynamical composition and operator-algebraic spacetime evolution.

5.1 Field-Realization Requirement

The deformation record $\mathbf{D}_{\mathcal{P}}$ records local responses to declared perturbations, while a field representation must supply local mathematical data from which those responses can be generated or reconstructed. Let $R \in \mathcal{R}(K_{AB})$ be a retained region. The class of well-typed local deformation records admissible at R is denoted $\mathfrak{D}_R(K_{AB})$. Its typing follows the comparison rules of Section 4.2. A field construction assigns to R a local carrier $\mathcal{V}_{\mathcal{P}}(R)$ together with a realization map into the local record class. The local realization requirement is stated in Equation 27.

$$\Xi_R : \mathcal{V}_{\mathcal{P}}(R) \longrightarrow \mathfrak{D}_R(K_{AB}), \quad \Xi_R(p_R) = \mathbf{D}_{\mathcal{P}}(R). \quad (27)$$

Equation 27 defines $p_R \in \mathcal{V}_{\mathcal{P}}(R)$ as a local field datum only relative to a declared realization map. The carrier may be a vector space, an algebra, a module, a space of generators, a family of admissible maps, or another structure suited to the intended dynamics. The present formalism does not require one universal carrier shared by every region. This flexibility is important when different regions contain different relational interfaces, local algebras, or levels of refinement.

A collection of local solutions to Equation 27 is still weaker than a field. Independent local fits can reproduce each regional record while contradicting one another on overlaps. The additional requirement is coherence across the regional structure of K_{AB} . Definition 5.1 records the preliminary field object used in the remainder of the paper.

Definition 5.1 (GR power-field representation). Let $\mathcal{P}_{A \curvearrowright B}^{\mathcal{R}}[K_{AB}]$ admit a deformation record $\mathbf{D}_{\mathcal{P}}$. A GR power-field representation on a retained region family $\mathcal{R}_{\mathcal{P}} \subseteq \mathcal{R}(K_{AB})$ is a family $\mathbf{P}_{\mathcal{P}} = \{p_R\}_{R \in \mathcal{R}_{\mathcal{P}}}$ with $p_R \in \mathcal{V}_{\mathcal{P}}(R)$ such that: (i) each p_R realizes the selected local deformation record through Ξ_R ; (ii) nested-region restrictions are compatible wherever the adopted model defines them; (iii) overlapping local values satisfy the declared consistency or gluing rule; and (iv) neighboring values admit the comparison or transport structure required by the later dynamical model.

Definition 5.1 makes the word *field* a formal achievement. The definition does not identify $\mathcal{P}_{A \curvearrowright B}^{\mathcal{R}}$ with $\mathbf{P}_{\mathcal{P}}$. The former is the substantive GR power structure and the latter is one mathematical realization of selected dynamical consequences. Distinct carriers or realization maps can represent the same deformation record, while a coarse deformation record can remain compatible with more than one field representation.

5.2 Restriction Compatibility and Nested Regions

Nested regions require compatibility between field values and the restriction structure already defined for deformation records. Let $R_1 \subseteq R_2$ and suppose the carrier family provides a restriction map $\sigma_{R_2 R_1} : \mathcal{V}_{\mathcal{P}}(R_2) \rightarrow \mathcal{V}_{\mathcal{P}}(R_1)$. The local field values satisfy the nested-region condition in Equation 28 whenever the relevant maps are defined.

$$\sigma_{R_2 R_1}(p_{R_2}) = p_{R_1}. \quad (28)$$

The realization maps must also agree with the deformation-record restriction $\rho_{R_2 R_1}^{\mathcal{P}}$ introduced in Equation 25. Their compatibility is stated in Equation 29.

$$\Xi_{R_1} \circ \sigma_{R_2 R_1} = \rho_{R_2 R_1}^{\mathcal{P}} \circ \Xi_{R_2} \quad (29)$$

on the common domain on which both compositions are defined. Equation 29 gives a calculable consistency test. A local field value on a larger region can be restricted first and then interpreted, or interpreted first and then restricted at the level of deformation records. A coherent representation requires the two procedures to agree within the declared exact or approximate tolerance.

For three nested regions $R_1 \subseteq R_2 \subseteq R_3$, the field restrictions satisfy the composition rule in Equation 30 whenever all maps exist.

$$\sigma_{R_3 R_1} = \sigma_{R_2 R_1} \circ \sigma_{R_3 R_2}. \quad (30)$$

Equation 30 is stronger than merely fitting local data independently. Its failure can indicate that the selected carrier omits context retained at the larger scale, that the region correspondence is inadequate, or that the power effect has an irreducibly contextual component under the chosen decomposition. Such failure is therefore diagnostically useful and should not automatically be repaired by adding free parameters.

5.3 Overlap Coherence and Local Gluing

Overlapping regions provide a second consistency requirement. Let R_i and R_j have an admissible intersection $R_{ij} = R_i \cap R_j$. When both local values restrict to the overlap, exact overlap coherence is stated in Equation 31.

$$\sigma_{R_i R_{ij}}(p_{R_i}) = \sigma_{R_j R_{ij}}(p_{R_j}). \quad (31)$$

Equation 31 means that two regional descriptions assign the same retained field information to their common relational content. The equality concerns the chosen representation. It does not require that the two larger regions have identical internal structure.

A stronger local-to-regional construction is available when a compatible cover can be glued. Let $\{R_i\}_{i \in I}$ cover a retained region R and suppose the local values satisfy Equation 31 on every defined pairwise overlap. An admissible gluing operation Glue_R produces a regional value as stated in Equation 32.

$$p_R = \text{Glue}_R(\{p_{R_i}\}_{i \in I}), \quad \sigma_{R R_i}(p_R) = p_{R_i}. \quad (32)$$

Equation 32 is imposed only when the selected carrier admits such a reconstruction. Some GR applications can preserve overlap consistency while lacking unique gluing because contextual information is lost when regions are separated. The formalism therefore distinguishes overlap coherence from unique local reconstruction.

Finite implementations also permit an approximate version. Suppose the overlap carriers are normed and the two restricted values lie in the same normed space. The overlap discrepancy for a pair of regions is defined in Equation 33.

$$\varepsilon_{ij} := \|\sigma_{R_i R_{ij}}(p_{R_i}) - \sigma_{R_j R_{ij}}(p_{R_j})\|. \quad (33)$$

Equation 33 creates a practical verification statistic when subtraction and a norm are meaningful. A model may declare a tolerance ε_{tol} and retain a field approximation when the relevant overlap discrepancies remain below that threshold. The tolerance is part of the model specification rather than a universal constant of GR.

5.4 Constitutive Support and Distributed Field Effect

The field representation preserves the distinction between the constitutive support family $\mathcal{S}_{\mathcal{P}}$ and the perturbatively identified effect family $\mathcal{E}_{\mathcal{P}}^{\Delta}$ introduced in Section 4.3. A local field value can therefore be nontrivial in a region that does not belong to the constitutive support of the power relation. Distributed consequences can propagate through relational interfaces, institutional chains, informational channels, or other admissible paths encoded by the model.

When each carrier $\mathcal{V}_{\mathcal{P}}(R)$ contains a distinguished neutral datum 0_R , one may define a representation-level support. Equation 34 states the corresponding family.

$$\text{Supp}_{\text{rep}}(\mathbf{P}_{\mathcal{P}}) := \{R \in \mathcal{R}_{\mathcal{P}} \mid p_R \not\sim_R 0_R\}, \quad (34)$$

where $\sim_R$ is the representation-equivalence relation declared for the carrier. Equation 34 is optional because some carriers do not possess a canonical neutral element. Even when it is defined, the representation-level support remains distinct from the constitutive support $\mathcal{S}_{\mathcal{P}}$ and the effect family $\mathcal{E}_{\mathcal{P}}^{\Delta}$. Equality among these three objects requires an additional model argument.

This three-way distinction is important for later source and propagation equations. A constitutive support region identifies relational structure needed to sustain the substantive power configuration. A representation-level support identifies where the chosen field description carries non-neutral data. An effect region identifies where the retained perturbation family detects a change. The three families can overlap substantially while remaining conceptually distinct.

5.5 Cell-Supported Specializations

The region-based field is general enough to accommodate heterogeneous relational domains. Finite calculation often benefits from a cell-supported specialization. Let $\mathcal{C}_p(K_{AB})$ be a chosen family of p -cells, with the dimension p selected according to the modeled relational role. A local field specialization assigns a carrier $\mathcal{V}_c$ and a datum $p_c \in \mathcal{V}_c$ to each retained cell c . The cell assignment is stated in Equation 35.

$$\mathbf{P}_{\mathcal{P}}^{(p)} : c \in \mathcal{C}_p(K_{AB}) \mapsto p_c \in \mathcal{V}_c. \quad (35)$$

Equation 35 permits several specializations. Vertex-supported data can represent local attributes of event regions; edge-supported data can represent transformation or comparison structure across relational interfaces; face-supported data can represent higher-order compatibility or cycle-related information. The assignment of social or political meaning to these cell dimensions is application-specific. The spin-foam precedent motivates the use of labeled two-dimensional complexes as calculable combinatorial carriers, while the present interpretation remains GR-specific [1].

Regional values can then be assembled from cell-supported data through a model-defined map $\mathcal{A}_R$. Equation 36 states the finite assembly rule.

$$p_R = \mathcal{A}_R(\{p_c\}_{c \subseteq R}). \quad (36)$$

Equation 36 is intentionally open with respect to addition. A nonlinear, ordered, constrained, or algebra-valued assembly may be required. A sum of cell values becomes justified only when the carriers and the intended interpretation support linear superposition.

The cell specialization provides a direct route to computation. A finite complex supplies a finite list of local carriers, local values, overlaps, and incidence relations. Restriction and overlap tests can therefore be evaluated explicitly before any continuum or infinite-system limit is considered.

5.6 Comparison Transport Across Relational Interfaces

Local field values can live in different carriers, so comparison across neighboring regions requires an explicit transport or translation rule. Let e represent an oriented relational interface from a source region R_s to a target region R_t . A comparison transport is a partially defined map stated in Equation 37.

$$\tau_e : \mathcal{V}_{\mathcal{P}}(R_s) \rightarrow \mathcal{V}_{\mathcal{P}}(R_t). \quad (37)$$

Equation 37 compares field representations across an interface. It is not yet the substantive power-induced evolution of the local algebra. The subsequent section on local dynamical composition will introduce dynamical maps whose action changes relational possibilities. The present τ_e belongs to the representational infrastructure required to compare local data and to compose compatible local descriptions.

For a composable path $\gamma = (e_1, \dots, e_n)$, the comparison transport is the ordered composition in Equation 38.

$$\tau_\gamma = \tau_{e_n} \circ \dots \circ \tau_{e_1}. \quad (38)$$

Equation 38 permits sequence dependence even before a Hilbert-space operator representation is chosen. If the local transports fail to commute or connect different carriers, two paths with the same endpoints can induce different comparisons. Closed-path residuals and their relation to emergent system characteristics will be developed in the later section on cycles, invariants, and emergent system characteristics; the present section supplies only the transport structure required for that later construction.

A field representation should also distinguish transport failure from substantive transformation. If τ_e is undefined, the immediate conclusion is that the chosen local carriers lack a declared translation across that interface. A later empirical or substantive claim about inaccessible influence requires additional evidence. This separation prevents representational incompleteness from being interpreted automatically as a property of the modeled system.

5.7 Local Representation Freedom

The local carrier coordinates need not be unique. Let $g_R : \mathcal{V}_P(R) \rightarrow \mathcal{V}'_P(R)$ be an invertible reparameterization on a retained region. Two local descriptions are representation-equivalent when their realization maps assign the same deformation record. Equation 39 states this condition.

$$\Xi'_R \circ g_R = \Xi_R. \quad (39)$$

Equation 39 treats g_R as a change of formal description rather than a substantive change in power. When neighboring regions also possess comparison transports, consistency requires the transformed transport law in Equation 40.

$$\tau'_e = g_{R_t} \circ \tau_e \circ g_{R_s}^{-1}. \quad (40)$$

Equation 40 is a gauge-like covariance condition in a limited formal sense. The paper uses this analogy to identify representational redundancy. A genuine gauge theory would require additional geometric structure, a specified transformation group, and a connection with corresponding curvature. Those stronger ingredients are not assumed by Definition 5.1.

The distinction matters for calculation. A numerical characteristic intended to represent a substantive system property should either remain invariant under the admitted family $\{g_R\}$ or state explicitly how it transforms. Otherwise the quantity may measure the chosen coordinates of the field representation rather than a stable feature of the represented power dynamics.

5.8 Quasi-Local Assembly and Resolution Change

The local algebraic net of Section 3.6 motivates a parallel compatibility structure for the field. Let $\mathcal{R}_P$ be a directed family of retained regions. Define $\mathcal{I}_{\text{res}}$ to be the set of ordered pairs (R_1, R_2) in $\mathcal{R}_P^2$ for which $R_1 \subseteq R_2$ and the retained model declares a restriction map $\sigma_{R_2 R_1}$. The compatible field-family space is defined in Equation 41.

$$\begin{aligned} \text{Fld}_P(K_{AB}) := \left\{ (p_R)_{R \in \mathcal{R}_P} \in \prod_{R \in \mathcal{R}_P} \mathcal{V}_P(R) \mid \right. \\ p_{R_2} \in \text{Dom}(\sigma_{R_2 R_1}), \\ \left. \sigma_{R_2 R_1}(p_{R_2}) = p_{R_1} \quad \forall (R_1, R_2) \in \mathcal{I}_{\text{res}} \right\}. \end{aligned} \quad (41)$$

Equation 41 is a projective-limit-like construction of field data on the declared restriction diagram, complementary to the inductive construction of the quasi-local algebra in Equation 8. Because the restriction maps may be partial, membership includes the domain condition explicitly; the displayed family therefore does not silently extend an undefined restriction. The index set $\mathcal{I}_{\text{res}}$ is derived from the retained region family and its declared restriction arrows, rather than introduced as an independent structure. The paper uses the explicit compatible-family form because it remains meaningful even when the application does not satisfy the topological assumptions required for a completed inverse limit.

The architecture therefore separates two kinds of global object. Local algebras can generate a quasi-local algebra through compatible embeddings, following the algebra-first strategy motivated by algebraic field theory [6, 3]. Local field data form a compatible family through restriction maps. A state or Hilbert-space representation may subsequently evaluate or realize selected parts of this structure without becoming the primitive container for the entire system.

Resolution change introduces a further covariance requirement. Let $q : K^{(f)} \rightarrow K^{(c)}$ be a declared coarse-graining, and let $R^{(f)}$ map to $R^{(c)} = q(R^{(f)})$. Suppose $\mathcal{Q}_R^\mathcal{V}$ maps fine field data to coarse field data and $\mathcal{Q}_R^\mathcal{D}$ maps fine deformation records to the coarse record retained by the model. The shared symbol $\mathcal{Q}_R$ records that both maps are induced by the same declared coarse-graining q , while the superscripts keep their source structures distinct. Field coarse-graining is compatible with realization when Equation 42 holds.

$$\Xi_{R^{(c)}}^{(c)} \circ \mathcal{Q}_R^\mathcal{V} = \mathcal{Q}_R^\mathcal{D} \circ \Xi_{R^{(f)}}^{(f)}. \quad (42)$$

Equation 42 allows exact or approximate versions depending on the application. It supplies a formal test for whether a claimed field characteristic survives a change of resolution. Later invariants and systemic quantities will state whether they are invariant, covariant, approximately preserved, or resolution-specific.

5.9 Field Realizability and Formal Obstructions

The preceding structures permit a concise existence statement. Proposition 5.2 is constructive and applies only to the selected region family and carrier architecture.

Proposition 5.2 (Field realizability). *Let $\mathbf{D}_{\mathcal{P}}$ be a region-indexed deformation record on $\mathcal{R}_{\mathcal{P}}$. Suppose each retained region has a carrier $\mathcal{V}_{\mathcal{P}}(R)$ and a realization map Ξ_R with at least one preimage of $\mathbf{D}_{\mathcal{P}}(R)$. Suppose further that one can choose local preimages p_R satisfying the nested-region restriction condition, the declared overlap-coherence condition, and the required comparison-transport compatibility. Then the chosen family $\{p_R\}$ defines a GR power-field representation in the sense of Definition 5.1. If the local preimages and admissible gluing are unique up to the declared local representation equivalence, the resulting field representation is unique up to that equivalence.*

Proposition 5.2 follows directly from the construction. Its value lies in identifying where a field representation can fail. A local realization can fail because the selected carrier cannot reproduce an observed deformation record. Nested restrictions can fail because a coarse regional value contains irreducible contextual information. Overlap coherence can fail because neighboring local fits assign incompatible values to shared relational content. Transport can fail because no declared comparison exists across an interface. Gluing can fail or remain nonunique even when pairwise overlaps agree.

These failures are formal obstructions under the selected architecture. They can motivate a richer carrier, a different region family, contextual transition data, higher-order overlap information, or a categorical construction that does not require one globally compatible field. The present paper therefore treats non-realizability as informative. The existence of a substantive power structure does not guarantee that one field representation will capture all of its relevant relational consequences.

For finite models, the obstruction tests are executable. One can solve the local realization equations, evaluate the restriction diagrams, compute the overlap discrepancies of Equation 33, test transport domains, and compare fine and coarse realizations through Equation 42. These checks establish formal consistency of a proposed field realization. Empirical validation still requires justified mappings from observed historical and relational evidence into the complexes, algebras, perturbations, and local carriers used by the model.

5.10 Interface for Local Dynamics

The field construction supplies the distributed data required for a dynamical theory without yet specifying the law of evolution. Table 4 summarizes the objects passed to the subsequent sections.

Table 4: Field-level objects supplied to the local dynamical construction.

Object	Field-level role	Subsequent dynamical use
$\mathcal{V}_{\mathcal{P}}(R)$ and p_R	Local carrier and local field datum	Parameterize or generate local transformations on relational/algebraic data
Ξ_R	Realization of field data as deformation records	Connect dynamical calculations to perturbatively identified effects
$\sigma_{R_2 R_1}$	Nested-region restriction	Test locality and consistency of dynamical laws across scales
Glue_R	Optional local-to-regional assembly	Construct regional dynamics from compatible patches
τ_e	Comparison transport across a relational interface	Support ordered path and cycle composition
$\{g_R\}$	Admitted local representation changes	Identify representation-invariant dynamical quantities
$\mathcal{Q}_R^{\mathcal{V}}$	Resolution-changing field map	Compare dynamics under refinement and coarse-graining

Table 4 marks the boundary between field structure and field action. The subsequent local-dynamics section will specify how a local datum p_R induces or parameterizes a transformation of the local GR possibility structure and how such transformations compose along histories. The following operator-algebraic section will then examine realizations of those transformations. The distinction is deliberate: a coherent field can be defined before one chooses a particular dynamical law, while a dynamical law that fails the restriction, overlap, or representation-equivalence tests cannot automatically be promoted to a coherent field theory on the selected GR spacetime.

The resulting architecture now has four distinct levels. Relational power supplies the substantive GR object. The deformation record supplies perturbatively interrogated regional consequences. The field representation supplies compatible local generating or descriptive data. Dynamical maps will supply the action of those data on histories, algebras, and eventually the represented spacetime itself. This separation allows later calculations of composition, cycles, invariants, and topology change to be traced back to explicit modeling assumptions at each level.

6 Local Transformations and Dynamical Composition

This section develops the first dynamical layer of the field-theoretic construction. Its role is to specify how a local GR power-field datum can induce or parameterize a well-typed transformation of the local generative-possibility structure introduced in Section 4. The objective is to obtain a compositional dynamics that remains meaningful without assuming a primitive global state space, a universal Hilbert space, or a single scalar time parameter. The construction proceeds from local dynamical realizations to restriction and overlap compatibility, interface dynamics, ordered composition along histories, sequence-sensitive diagnostics, history-conditioned dynamics, and infinitesimal generators in regimes where a differentiable parameterization is justified. The section remains at the level of maps and generators. Their realization inside C^* - or W^* -algebraic representations is reserved for the subsequent operator-algebraic section.

6.1 Local Dynamical Realization

Section 5 established a field value $p_R \in \mathcal{V}_{\mathcal{P}}(R)$ by requiring it to realize a selected local deformation record through Ξ_R . A dynamical theory requires an additional map from field data to admissible transformations. For a source pair (K, R) , let $\text{Dyn}_K(R)$ denote the model-specified class of typed partial or total transfor-

mations of the form $\text{Poss}_K(R) \rightarrow \text{Poss}_{K'}(R')$, with the admissible target pairs (K', R') retained as part of the map type. A local dynamical realization is stated in Equation 43.

$$\mathcal{D}_R : \mathcal{V}_{\mathcal{P}}(R) \longrightarrow \text{Dyn}_K(R), \quad p_R \longmapsto \Phi_R[p_R]. \quad (43)$$

Equation 43 makes the dynamical law an additional modeling choice. The field realization $\Xi_R(p_R) = \mathbf{D}_{\mathcal{P}}(R)$ constrains what the datum represents, while $\mathcal{D}_R$ specifies how that datum acts. To prevent these two constructions from becoming independent fits, let $\mathfrak{D}_R(K)$ denote the local deformation-record class obtained by the construction of Section 4.4 for source carrier K , and let $\text{Rec}_R^K : \text{Dyn}_K(R) \rightarrow \mathfrak{D}_R(K)$ be the model-declared response-extraction map that applies the same perturbation and comparison protocol used to construct the deformation record. A field-generated dynamics is record-consistent when Equation 44 holds.

$$\text{Rec}_R^K(\mathcal{D}_R(p_R)) = \Xi_R(p_R). \quad (44)$$

Equation 44 does not make the dynamical law unique. Distinct maps can induce the same retained perturbative record when the extraction protocol is coarse or incomplete. It does require every accepted law to reproduce the field datum's declared empirical or comparative meaning under the same protocol. Empirical or substantive selection among record-consistent laws requires evidence beyond formal realizability.

The source and target of $\Phi_R[p_R]$ should remain explicit. For a transformation from a represented complex K to a represented complex K' , with a declared correspondence from $R \subseteq K$ to $R' \subseteq K'$, the local action has the general type shown in Equation 45.

$$\Phi_R[p_R] : \text{Poss}_K(R) \rightarrow \text{Poss}_{K'}(R'). \quad (45)$$

Equation 45 uses a partial arrow because a dynamical rule can be defined only on a subset of admissible local configurations. The case $K' = K$ and $R' = R$ is an endomorphic specialization. A change of complex, region, or local algebra is retained at this stage as a typed possibility; the history-preserving and topology-changing cases will be separated in later dedicated sections.

The local possibility structure of Equation 20 has three principal components. Whenever the adopted model resolves the dynamical action componentwise, the local map can be represented by the typed family in Equation 46.

$$\Phi_R[p_R] \equiv \left(\Phi_R^{\text{ext}}[p_R], \Phi_R^{\text{adm}}[p_R], \Phi_R^{\text{alg}}[p_R] \right). \quad (46)$$

Equation 46 is bookkeeping for a model in which the action on the three components has been separately defined; it is not a canonical product decomposition of every admissible map on $\text{Poss}_K(R)$. It distinguishes action on the continuation family, on the family of admissible transformations, and on the local algebraic carrier. A model can retain one component while transforming another. For example, a power-conditioned rule can preserve the local algebra while changing which continuations remain admissible, or preserve the continuation family while changing the algebraic maps that can be applied within it. A stronger transformation can require $\Phi_R^{\text{alg}}[p_R] : \mathfrak{A}_K(R) \rightarrow \mathfrak{A}_{K'}(R')$ between distinct local algebras.

Definition 6.1 (Local GR power-conditioned dynamics). Given a GR power-field representation $\mathbf{P}_{\mathcal{P}}$ and a retained region R , a local GR power-conditioned dynamical law is a map $\mathcal{D}_R$ of the form in Equation 43 such that each $\Phi_R[p_R]$ is well typed on the declared local possibility structure, satisfies the record-consistency condition in Equation 44 whenever the response-extraction map is defined, and satisfies the locality, overlap, and representation-covariance conditions adopted by the model.

Definition 6.1 does not require $\Phi_R[p_R]$ to be linear, invertible, norm preserving, completely positive, or generated by an element of one fixed algebra. Those stronger properties belong to particular dynamical specializations. This level of generality is necessary because GR power can alter continuation structure, admissibility, or even the algebraic carrier itself.

6.2 Restriction and Overlap Compatibility of Dynamics

A field-theoretic dynamical law should respect the local compatibility structure that made the field representation possible. Let $R_1 \subseteq R_2$ and let $\rho_{R_2 R_1}^{\text{dyn}}$ denote the declared restriction of a dynamical map on R_2 to the retained structure on R_1 . Exact restriction covariance is stated in Equation 47.

$$\rho_{R_2 R_1}^{\text{dyn}}(\Phi_{R_2}[p_{R_2}]) = \Phi_{R_1}[\sigma_{R_2 R_1}(p_{R_2})]. \quad (47)$$

Equation 47 links the field restriction map $\sigma_{R_2 R_1}$ to dynamical restriction. A regional law that gives a different local action after restriction from the law constructed directly on the smaller region requires an explicit approximation rule, contextual correction, or revision of the field architecture. Exact equality is therefore a substantive coherence condition rather than an automatic property.

For finite or empirically fitted models, approximate restriction covariance can be measured when a distance or discrepancy functional d_{dyn} has been defined on comparable dynamical maps. The corresponding residual is given in Equation 48.

$$\varepsilon_{R_2 R_1}^{\text{res}} := d_{\text{dyn}}\left(\rho_{R_2 R_1}^{\text{dyn}}(\Phi_{R_2}[p_{R_2}]), \Phi_{R_1}[p_{R_1}]\right), \quad p_{R_1} = \sigma_{R_2 R_1}(p_{R_2}). \quad (48)$$

Equation 48 is calculable only after the application supplies a meaningful d_{dyn} . A norm difference is appropriate for some common linear realizations, while a structural edit distance, divergence of induced continuation weights, or another comparison may be more appropriate elsewhere.

Overlapping regions impose a related condition. If $R_{ij} = R_i \cap R_j$ is an admissible overlap, let $\rho_{i,ij}^{\text{dyn}}$ and $\rho_{j,ij}^{\text{dyn}}$ restrict the corresponding local dynamics to the shared region. Exact overlap coherence is stated in Equation 49.

$$\rho_{i,ij}^{\text{dyn}}(\Phi_{R_i}[p_{R_i}]) = \rho_{j,ij}^{\text{dyn}}(\Phi_{R_j}[p_{R_j}]). \quad (49)$$

Equation 49 prevents two locally fitted dynamical laws from assigning incompatible actions to the same retained relational content without recording that incompatibility explicitly. Approximate versions can be defined using the same principle as Equation 48. Together, restriction and overlap tests provide a direct route from local calculations to a checkable regional field theory.

6.3 Interface Dynamics and Field Transport

The comparison transport τ_e introduced in Section 5.6 and Equation 37 compares field representations across a relational interface. Dynamical action requires a distinct object. Let e connect a source region R_s to a target region R_t . The interface-level dynamical law may depend on the source datum, the target datum, and the declared comparison transport. Equation 50 records the resulting map.

$$\Phi_e = \mathcal{D}_e(p_{R_s}, p_{R_t}, \tau_e) : \text{Poss}_{K_s}(R_s) \rightarrow \text{Poss}_{K_t}(R_t). \quad (50)$$

Equation 50 permits several specializations. A source-driven rule can omit p_{R_t} . A target-conditioned rule can use both local values. A transport-mediated rule can depend on $\tau_e(p_{R_s})$ after the source datum has been translated into the target carrier. The chosen dependence should follow the substantive mechanism being represented.

Figure 4 separates comparison transport from dynamical action. The upper arrow belongs to representational infrastructure, while the lower arrow changes the represented generative possibilities.

Figure 4 also clarifies a practical diagnostic. Failure of τ_e indicates that the chosen field representations lack a declared comparison across the interface. Failure of Φ_e indicates that the adopted dynamical law lacks a well-typed action on the relevant local possibility structure. The two failures require different remedies.

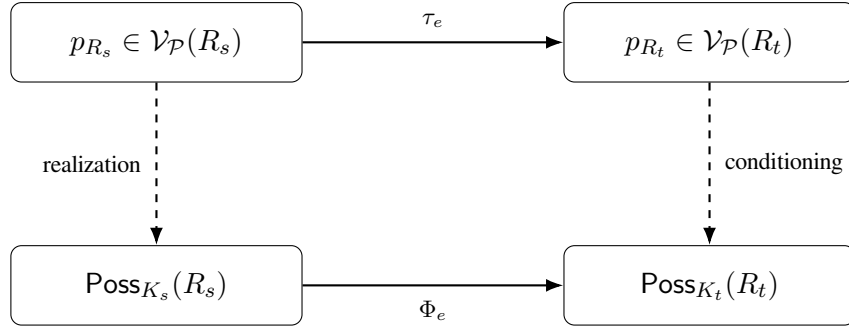


Figure 4: Comparison transport and substantive dynamical action across a relational interface. The field transport τ_e compares local representations, whereas Φ_e changes the modeled generative-possibility structure.

6.4 Ordered Composition Along Histories

A history-bearing GR spacetime supports composable sequences of local transformations. Let γ be the finite path with ordered interfaces $e_1, \dots, e_n$. Assume that each interface map is well typed so that the codomain of Φ_{e_i} lies in the domain of $\Phi_{e_{i+1}}$. The induced history transformation is defined in Equation 51.

$$\Phi_\gamma := \Phi_{e_n} \circ \Phi_{e_{n-1}} \circ \dots \circ \Phi_{e_1}. \quad (51)$$

Equation 51 is the basic compositional law of the present section. It requires neither a scalar time coordinate nor a global state vector. The ordering is inherited from the composable path or extension structure. When an external time parameter exists, the same construction can be indexed by time without changing its algebra-first basis.

Proposition 6.2 (Well-typed path composition). *Let $\gamma = (e_1, \dots, e_n)$ be a finite path of local dynamical maps. If every adjacent codomain-domain pair is compatible under the model's declared identifications, then Equation 51 defines a partial or total map from the source possibility structure of e_1 to the target possibility structure of e_n . Whenever all relevant compositions are defined, regrouping the maps leaves the composite unchanged by associativity of map composition.*

Proposition 6.2 is elementary, yet it isolates an important source of failure. A sequence can cease to be composable because an earlier transformation removes an option, changes an algebraic carrier, changes the entity correspondence, or alters another domain condition required by a later transformation. Such failure is itself a dynamical consequence of sequence order and should not be silently repaired by forcing all transformations into one pre-existing global domain.

Two paths with common endpoints need not induce the same transformation. Let γ_1 and γ_2 connect the same retained source and target structures. Path equivalence relative to a declared comparison $\equiv^{\text{dyn}}$ is stated in Equation 52.

$$\gamma_1 \sim_{\text{dyn}} \gamma_2 \iff \Phi_{\gamma_1} \equiv^{\text{dyn}} \Phi_{\gamma_2} \text{ on their common retained domain.} \quad (52)$$

Equation 52 allows path dependence to be formulated before a subtraction or norm has been defined. Later cycle and holonomy-like constructions will use this path-level distinction to identify mesoscopic and system-level residual effects.

6.5 Sequence Sensitivity and Noncommutative Diagnostics

Sequence dependence should be established through defined transformations. Let Φ and Ψ be two local or regional dynamical maps for which both ordered compositions are meaningful on a common retained set $\mathcal{X}$. The order-sensitive set is defined in Equation 53.

$$\Omega_{\Psi, \Phi} := \{x \in \mathcal{X} \mid (\Psi \circ \Phi)(x) \not\equiv^{\text{dyn}} (\Phi \circ \Psi)(x)\}. \quad (53)$$

Equation 53 supplies a representation-independent criterion relative to the declared equivalence. If $\Omega_{\Psi, \Phi}$ is nonempty, the two transformations are sequence-sensitive on the modeled domain. If one ordered composition is undefined while the other is defined, the sequence is also order-sensitive, although the situation is a domain asymmetry rather than an algebraic commutator.

When the two transformations are linear maps on a common vector or algebraic space, subtraction is defined and sequence sensitivity admits the commutator specialization in Equation 54.

$$[\Psi, \Phi] := \Psi \circ \Phi - \Phi \circ \Psi. \quad (54)$$

Equation 54 turns noncommutativity into a calculable property of a specified realization. A nonzero commutator on the retained domain records an order effect. The paper therefore uses noncommutativity only after the common linear structure required by Equation 54 has been established.

A metric specialization can quantify order sensitivity without assuming linearity. If $d_{\mathcal{X}}$ is a declared distance on comparable outputs, the pointwise order discrepancy is defined in Equation 55.

$$\varepsilon_{\text{ord}}(x; \Psi, \Phi) := d_{\mathcal{X}}((\Psi \circ \Phi)(x), (\Phi \circ \Psi)(x)). \quad (55)$$

Equation 55 provides a finite-model diagnostic that can be estimated numerically when the output comparison has empirical or computational meaning. Its interpretation remains tied to the declared distance and retained representation.

6.6 History-Conditioned and Open-System Dynamics

The compositional rule does not require Markovianity. In a history-conditioned model, the map applied at interface e_i can depend on the previously realized path prefix $\gamma_{<i} = (e_1, \dots, e_{i-1})$. Equation 56 records this dependence.

$$\Phi_{e_i}^{\gamma} = \mathcal{D}_{e_i}(p_{R_{i-1}}, p_{R_i}, \tau_{e_i}; \gamma_{<i}). \quad (56)$$

Equation 56 permits institutional memory, accumulated commitments, historical path dependence, delayed constraints, and other mechanisms for which the present local data do not screen off the earlier history. A memoryless specialization removes the explicit dependence on $\gamma_{<i}$.

When a scalar parameter t is justified and the same algebraic domain is retained, a one-parameter or two-parameter evolution family becomes available. The semigroup law of Equation 9 is one special case. A nonautonomous evolution can instead use maps $\Phi(t_2, t_1)$ satisfying the composition relation in Equation 57.

$$\Phi(t_3, t_1) = \Phi(t_3, t_2) \circ \Phi(t_2, t_1), \quad t_1 \leq t_2 \leq t_3, \quad (57)$$

provided the intermediate domains are compatible. Equation 57 adds temporal parameterization to the more general compositional architecture. It does not replace the extension-based history order of Section 3.5.

Open-system dynamics also need not be invertible. A local algebraic component $\Phi_R^{\text{alg}}[p_R]$ can be an automorphism in a reversible specialization, an endomorphism when information is retained within one algebra without invertibility, a completely positive map in an appropriate C^* - or W^* -algebraic open-system specialization, or a morphism between distinct algebras when the represented local structure changes. The standard completely positive semigroup results reviewed in Section 2.7 provide a mathematical precedent for effective irreversible subsystem dynamics [9, 11]. Their quantum-physical assumptions are not transferred to GR by this use.

6.7 Infinitesimal Generators in Parameterized Regimes

A generator can be introduced when a local dynamical family possesses an appropriate differentiable parameterization and a common linear domain. Let s be a declared local evolution or intervention parameter

with $\Phi_{R,0}[p_R] = \text{id}$. The local generator is defined by Equation 58 whenever the limit exists in the selected topology.

$$\mathcal{L}_R[p_R] := \left. \frac{d}{ds} \Phi_{R,s}[p_R] \right|_{s=0}. \quad (58)$$

Equation 58 is derivative structure. It does not make $\mathcal{L}_R[p_R]$ the substantive power object. The generator summarizes the first-order action of a chosen parameterized realization. Its analytic properties depend on the adopted specialization; boundedness, domain density, derivation properties, dissipativity, or complete-positivity conditions require separate assumptions.

For a differentiable family acting linearly on a common local algebra, the first-order expansion is stated in Equation 59.

$$\Phi_{R,\epsilon}[p_R] = \text{id} + \epsilon \mathcal{L}_R[p_R] + o(\epsilon). \quad (59)$$

Equation 59 makes local calculations possible without assuming that the exact finite transformation has a closed form.

The generator formalism also yields a controlled connection between sequence dependence and non-commutativity. Let p_R and q_R parameterize two differentiable local transformation families on a common linear domain. Applying Equation 59 to both orderings gives the leading order difference shown in Equation 60.

$$(\Phi_{R,\epsilon}[q_R] \circ \Phi_{R,\epsilon}[p_R] - \Phi_{R,\epsilon}[p_R] \circ \Phi_{R,\epsilon}[q_R])(a) = \epsilon^2 [\mathcal{L}_R[q_R], \mathcal{L}_R[p_R]](a) + o(\epsilon^2) \quad (60)$$

for a in the common retained domain. Equation 60 shows that the commutator acquires a concrete dynamical meaning: it measures the leading-order discrepancy between two small transformations applied in opposite orders. The result is conditional on the differentiability, common-domain, and linearity assumptions used in the expansion.

6.8 Representation Covariance of Dynamical Laws

Section 5.7 allowed local field data to be reparameterized by invertible maps g_R . A substantive dynamical prediction should transform coherently under such changes of representation. Let C_R and $C_{R'}$ denote the induced comparison maps on the source and target possibility descriptions. Dynamical representation covariance is stated in Equation 61.

$$C_{R'} \circ \Phi_R[p_R] = \Phi_{R'}[g_R(p_R)] \circ C_R. \quad (61)$$

Equation 61 is a commutative-diagram requirement. It says that transforming the representation before applying the corresponding transformed law gives the same retained result as applying the original law and then translating its output. A quantity extracted from the dynamics can therefore be interpreted as representation-stable only when it is invariant or covariant under the admitted family of such transformations.

The same logic applies under coarse-graining. If Q maps fine algebraic data to coarse algebraic data and $Q_R^\vee$ maps fine field data to coarse field data, a compatible coarse dynamical law should make the fine and coarse routes agree exactly or within a declared approximation tolerance. This requirement will become particularly important for cycle residuals and topological diagnostics because coarse-graining can remove internal paths that contribute to the fine-scale composition.

6.9 Finite Dynamical Checks

The local dynamics defined above support a finite verification workflow. The workflow is formal rather than empirical: it checks whether a proposed model is internally executable under its own assumptions. Table 5 summarizes the principal checks available before an operator-algebraic representation is selected.

Table 5: Finite checks for a proposed local GR power dynamics.

Check	Calculation	Failure indicated
Typing	Verify every source, target, and intermediate domain in Φ_γ	Undefined composition or unrecorded structural change
Restriction covariance	Evaluate Equation 48 on nested regions	Inconsistent local and regional dynamical laws
Overlap coherence	Compare restricted maps on $R_i \cap R_j$	Incompatible local fits on shared relational content
Order sensitivity	Evaluate Equation 53, 54, or 55 as appropriate	Sequence-dependent transformation structure
History dependence	Compare maps with identical present local data and different path prefixes	Memory or omitted historical conditioning
Representation covariance	Test Equation 61	Coordinate- or representation-dependent prediction
Resolution behavior	Compare fine and coarse dynamical routes under the declared reduction	Scale-specific effect or inadequate coarse law

Table 5 turns several qualitative claims into executable obligations. A finite implementation can represent the local algebras and admissible maps explicitly, enumerate selected paths, test domain compatibility, evaluate compositions, and compute norm- or metric-based discrepancies where those structures have been justified. The same implementation can retain symbolic output when numerical comparison would impose an unsupported metric.

The checks also separate formal consistency from empirical validity. Passing the typing and covariance tests shows that a proposed dynamics is coherent under the chosen model. It does not establish that the selected complex, field carrier, dynamical law, or empirical parameterization correctly describes a political or social system. That stronger claim requires independently justified correspondence between the formal objects and historical or observational evidence.

6.10 Interface for Operator-Algebraic Spacetime Evolution

The constructions in this section supply a dynamical layer without making a Hilbert-space state vector primitive. Table 6 summarizes the objects passed to the next section.

Table 6: Dynamical objects supplied to the operator-algebraic evolution layer.

Object	Current role	Next formal use
$\Phi_R[p_R]$	Local action on generative possibilities	Algebraic realization on $\mathfrak{A}_K(R)$ or between local algebras
Φ_e	Interface-conditioned transformation	Ordered algebraic evolution across relational interfaces
Φ_γ	Composite along a history path	Path operators, cycle composites, and later holonomy-like diagnostics
$\Omega_{\Psi, \Phi}$ and $[\Psi, \Phi]$	General and linear diagnostics of sequence sensitivity	Noncommutative operator structure where a common representation exists
$\Phi_{e_i}^\gamma$	History-conditioned local law	Non-Markovian or cocycle-like operator families
$\mathcal{L}_R[p_R]$	Infinitesimal generator in a differentiable regime	Spectral, stability, and criticality analysis under additional assumptions

Table 6 fixes the division of labor between the present section and the subsequent operator-algebraic section. The present section defines how field data acquire dynamical action and how that action composes. The next section will examine how the algebraic component $\Phi_R^{\text{alg}}[p_R]$ can be represented on local or quasi-local C^* - and W^* -algebraic structures, how state-like evaluations transform when they are introduced, and when generators or represented operators support additional spectral calculations.

The resulting hierarchy preserves the paper’s algebra-first architecture. A substantive GR power structure gives rise to a deformation record under declared perturbations. A coherent local field realizes that record. A dynamical law maps field data to transformations. Ordered compositions of those transformations generate history-sensitive effects without a primitive global phase space. Operator-algebraic representations then become derived tools for calculation, comparison, and spectral analysis. This order keeps the calculational advantages of operator methods while retaining explicit control over the assumptions required to obtain them.

7 Operator-Algebraic Representation of Spacetime Evolution

This section develops the operator-algebraic realization of the local dynamics constructed in Section 6. Its role is to identify the additional assumptions under which a power-conditioned transformation of GR generative possibilities can be represented as a map between local C^* - or W^* -algebraic structures and, when an evaluation context is selected, as an operator between derived Hilbert-space representations. The objective is calculational rather than ontological. The construction proceeds from typed local algebra maps to quasi-local lifting, dual evaluation transport, GNS-derived implementations, W^* -level normal dynamics, generator and spectral structures, finite matrix specializations, and representation-covariant diagnostics. This order preserves the algebra-first architecture developed in Sections 2–6: operator representations enter after the relational spacetime, local algebras, field realization, and dynamical law have been specified.

7.1 Algebraic Realization of Local Dynamics

Section 6.1 allows $\Phi_R[p_R]$ to act on continuation structure, admissible transformations, and algebraic data. The present section isolates the algebraic component and records its source and target explicitly. Let ϱ denote the region correspondence induced by the selected spacetime transformation from K to K' . The power-conditioned algebraic realization is defined in Equation 62.

$$\alpha_R[p_R] := \Phi_R^{\text{alg}}[p_R] : \mathfrak{A}_K(R) \longrightarrow \mathfrak{A}_{K'}(\varrho(R)). \quad (62)$$

Equation 62 is typed independently of whether $K' = K$. The fixed-complex case permits an endomorphism or automorphism of one local algebra. A changing-complex case permits a morphism between distinct algebras. The stronger historical and topological interpretations of $K \rightarrow K'$ are deferred to the dedicated history-preserving and history-reconfiguring analyses; the present section establishes the algebraic machinery shared by those regimes.

The map class carried by $\alpha_R[p_R]$ is a modeling choice constrained by the substantive interpretation of the local algebra. A $*$ -automorphism is appropriate for a reversible specialization that preserves the full algebraic product and involution. A $*$ -endomorphism retains those operations while allowing a proper image. A positive or completely positive map becomes available when the local algebra carries an interpreted positive cone and an open-system description requires a weaker, potentially irreversible transformation. A general bounded linear map may be sufficient for some calculational approximations, while a change in the available operations can require a more general typed morphism whose preserved structure is stated separately. Table 7 summarizes these choices.

Table 7 keeps the mathematical classes separate because each supports different deductions. The completely positive semigroup results reviewed in Section 2.7 provide one precedent for irreversible operator-algebraic dynamics [9, 11]. Their physical assumptions remain specific to their source domain. Within GR, positivity, complete positivity, reversibility, and norm continuity acquire meaning only after the modeled algebra and its interpretation have been specified.

7.2 Regional Compatibility and Quasi-Local Lift

A family of local algebra maps represents a spacetime-level dynamics only when it respects the regional inclusion structure used to assemble the GR algebra. Let $R_1 \subseteq R_2 \subseteq K$, and let $\iota_{R_1 R_2}^K$ and $\iota_{\varrho(R_1)\varrho(R_2)}^{K'}$ denote the corresponding local embeddings. Regional compatibility is stated in Equation 63.

$$\alpha_{R_2}[p_{R_2}] \circ \iota_{R_1 R_2}^K = \iota_{\varrho(R_1)\varrho(R_2)}^{K'} \circ \alpha_{R_1}[p_{R_1}]. \quad (63)$$

Equation 63 says that an algebraic quantity supported in the smaller region receives the same transformed image whether it is evolved locally before inclusion or included first and evolved in the larger region. This is the operator-algebraic counterpart of the restriction and overlap requirements imposed on the field and dynamical layers in Sections 5 and 6.

Proposition 7.1 (Compatible quasi-local lift). *Let $\{\mathfrak{A}_K(R)\}_{R \in K}$ and $\{\mathfrak{A}_{K'}(\varrho(R))\}_{R \in K}$ be directed local systems whose inductive unions admit the norm completions used in Equation 8. Assume that the retained region correspondence ϱ is inclusion preserving on this directed family, so $R_1 \subseteq R_2$ implies $\varrho(R_1) \subseteq \varrho(R_2)$ and the target embeddings in Equation 63 are defined. Suppose the local maps α_R satisfy Equation 63 and possess a uniform bound $\|\alpha_R\| \leq C$ on the directed family. Then the local maps define a unique bounded map on the algebraic inductive union and extend continuously to a map $\alpha_K : \mathfrak{A}_K \rightarrow \mathfrak{A}_{K'}$.*

Proposition 7.1 follows by defining the image of a locally represented element through any region that contains its support. Equation 63 makes this definition independent of the selected larger region, and the uniform bound supplies continuity at completion. When every local map is a compatible $*$ -homomorphism, the induced map preserves the corresponding algebraic structure. For weaker local map classes, the lift preserves only the properties carried consistently by the family.

The quasi-local map supplied by Proposition 7.1 is recorded in Equation 64.

$$\alpha_K[\mathbf{P}] : \mathfrak{A}_K \longrightarrow \mathfrak{A}_{K'}. \quad (64)$$

Equation 64 is derived from compatible local dynamics rather than postulated as a transformation of a pre-existing global state vector. A finite model can stop at the local or finite-union level, while an open or indefinitely extensible model can use the quasi-local completion when its analytic assumptions are satisfied.

Table 7: Algebraic realizations of GR local dynamics and the assumptions associated with each class.

Map class	Structural requirement	GR use in the present formalism
*-automorphism	Same algebra; multiplicative, involution preserving, invertible	Reversible algebraic specialization on a retained local structure
*-endomorphism	Same algebra; multiplicative and involution preserving	Algebra-preserving evolution with a potentially proper image
Positive / completely positive map	Interpreted ordered *-algebraic setting; positivity conditions declared	Effective open-subsystem or coarse-grained dynamics where positivity has substantive meaning
Bounded linear map	Common linear and normed structure	Computational approximation with explicitly limited preserved structure
Typed inter-algebra morphism	Declared source, target, and preserved operations	Evolution between different regional or spacetime algebraic carriers

7.3 Dual Evaluation Transport

Algebraic evolution and evaluation of algebraic quantities are distinct operations. Once a state-like evaluation has been selected, an algebra map induces a dual pullback of that evaluation. Let $\alpha : \mathfrak{A} \rightarrow \mathfrak{B}$ be a unital positive map and let $\omega_{\mathfrak{B}}$ be a state on $\mathfrak{B}$. The pulled-back state is defined in Equation 65.

$$\alpha^*(\omega_{\mathfrak{B}}) := \omega_{\mathfrak{B}} \circ \alpha \in \mathcal{S}(\mathfrak{A}), \quad (65)$$

where $\mathcal{S}(\mathfrak{A})$ denotes the state space of the selected C^* -algebraic realization. Equation 65 is well defined because unitality preserves normalization and positivity preserves the positive cone. The construction is contravariant: an algebra map from source to target pulls a target evaluation back to the source.

The direction in Equation 65 should be distinguished from a substantive claim about forward evolution of a social or political state. A GR model that requires forward transport of evaluations can supply a separate state-transfer map, use a Heisenberg-oriented algebra map with reversed source-target convention, or work at a W^* level where a predual dynamics has been specified. The present paper keeps these conventions explicit so that an algebraic arrow is not silently reinterpreted as a unique physical time evolution.

For an endomorphic family $\alpha_t : \mathfrak{A} \rightarrow \mathfrak{A}$, the same equation permits the familiar dual evaluation update $\omega_t = \omega_0 \circ \alpha_t$ under the adopted convention. This specialization is useful for calculational examples, while the typed inter-algebra form remains available when the represented spacetime or local carrier changes.

7.4 GNS-Derived Operator Implementations

A Hilbert-space operator enters only after an algebra and an evaluation state have been chosen. Section 2.5 reviewed the GNS construction used here [3]. Let $\alpha : \mathfrak{A} \rightarrow \mathfrak{B}$ be a *-homomorphism. Choose a state ω on $\mathfrak{A}$ and a state ν on $\mathfrak{B}$ satisfying the compatibility condition stated in Equation 66.

$$\omega = \nu \circ \alpha. \quad (66)$$

Equation 66 allows the algebra map to be represented between the two GNS constructions without selecting an independent universal Hilbert space.

Let $(\pi_{\omega}, \mathcal{H}_{\omega}, \Omega_{\omega})$ and $(\pi_{\nu}, \mathcal{H}_{\nu}, \Omega_{\nu})$ be the corresponding GNS triples. On the dense cyclic subspace generated by $\mathfrak{A}$, define the map in Equation 67.

$$V_{\alpha}[a]_{\omega} := [\alpha(a)]_{\nu}. \quad (67)$$

Equation 67 preserves the GNS norm because α is multiplicative and $*$ -preserving and because Equation 66 holds. It therefore extends to an isometry from $\mathcal{H}_\omega$ into $\mathcal{H}_\nu$. The induced map intertwines the represented algebras according to Equation 68.

$$V_\alpha \pi_\omega(a) = \pi_\nu(\alpha(a)) V_\alpha, \quad a \in \mathfrak{A}. \quad (68)$$

Equation 68 makes precise the sense in which an operator can represent algebraic spacetime evolution. The operator V_α is derived from the algebra map and compatible evaluations. Its existence does not identify GR power with an operator.

A useful specialization occurs when $\mathfrak{A} = \mathfrak{B}$, α is a $*$ -automorphism, and ω is invariant under α . In that case V_α is onto and hence unitary. Writing it as U_α , the represented automorphism satisfies Equation 69.

$$U_\alpha \pi_\omega(a) U_\alpha^* = \pi_\omega(\alpha(a)). \quad (69)$$

Equation 69 is therefore a derived reversible regime. General GR dynamics can fall outside it through irreversibility, changing algebras, changing evaluation contexts, or absence of an invariant state.

Completely positive maps require a different implementation theory because multiplicativity, used in Equation 67, is absent. The present formalism therefore treats a completely positive algebra map itself as the primary dynamical object unless an additional dilation or representation theorem has been selected for the application. This prevents the notation of a single operator from concealing assumptions that belong to a particular realization.

7.5 W^* -Level Normal Dynamics

A W^* -level formulation becomes useful when weak limits, normal states, representation-dependent closures, or predual dynamics are required. For a selected GNS representation, the associated von Neumann algebra is defined in Equation 70.

$$\mathfrak{M}_\omega := \pi_\omega(\mathfrak{A})''. \quad (70)$$

Equation 70 is a representation-dependent completion, consistent with the background architecture reviewed in Section 2.5 [3]. The C^* -level quasi-local algebra and the W^* -level closure therefore perform different roles: the former assembles compatible local algebraic structure, while the latter supplies analytic structure tied to a chosen representation.

Let $\bar{\alpha} : \mathfrak{M} \rightarrow \mathfrak{N}$ be a normal bounded map. Its action on normal linear functionals is represented by the preadjoint in Equation 71.

$$\bar{\alpha}_*(\rho) := \rho \circ \bar{\alpha}, \quad \rho \in \mathfrak{N}_*. \quad (71)$$

Equation 71 maps the predual of the target algebra into the predual of the source algebra. When the model adopts a Heisenberg-oriented evolution map, the same duality can be oriented so that normal states evolve forward under the predual. The convention must be recorded because the algebraic and state pictures reverse arrows.

The W^* level also provides a natural setting for representation-dependent limiting behavior. Distinct evaluation contexts can lead to different von Neumann closures, and potentially to inequivalent representations. The paper treats such differences as formal possibilities that may become useful for classifying emergent regimes. A claim that a concrete GR system realizes distinct operator-algebraic sectors requires an application-specific construction and evidence beyond the abstract availability of the mathematics.

7.6 Generators and Spectral Objects

Section 6.7 introduced generators at the level of parameterized dynamical maps. An operator-algebraic specialization can sharpen that construction. Let $\{\alpha_s\}$ be a differentiable family of endomorphisms on a common algebra $\mathfrak{A}$ with $\alpha_0 = \text{id}$. For elements in the domain on which the limit exists, the algebraic generator is defined in Equation 72.

$$\delta(a) := \left. \frac{d}{ds} \alpha_s(a) \right|_{s=0}. \quad (72)$$

Equation 72 is the algebra-level realization of the generator concept. Its domain and analytic properties depend on the chosen family. An automorphism group can lead to a derivation-type generator, while a completely positive semigroup can lead to a dissipative generator of a different class. The distinction follows the map class rather than the substantive label of power.

When the family is represented by operators V_s that, after any required declared identifications, belong to one common Banach space of operators (for example $\mathcal{B}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$), differentiability can be defined and an additional represented generator can be introduced on its differentiability domain. A family on genuinely varying Hilbert spaces requires a separately specified trivialization or transport before the derivative below is meaningful. The definition is stated in Equation 73.

$$\widehat{G} := \left. \frac{d}{ds} V_s \right|_{s=0}. \quad (73)$$

Equation 73 supplies a calculational operator only after the implementation family exists. Self-adjointness, skew-adjointness, boundedness, and spectral properties require additional assumptions and are not inferred from the GR power interpretation.

Several distinct spectra can then appear, and they should remain conceptually separated. An element $a \in \mathfrak{A}$ has its C^* -algebraic spectrum; a bounded dynamical map can have a spectrum as an operator on the Banach space underlying $\mathfrak{A}$; and a represented generator $\widehat{G}$ can have a Hilbert-space operator spectrum when its analytic conditions are satisfied. Table 8 records these levels.

Table 8 prevents the later criticality analysis from treating every eigenvalue-like quantity as the same object. A spectral diagnostic becomes meaningful only after its algebra, map, representation, parameterization, and empirical interpretation have been fixed. The later near-criticality analysis will use this hierarchy when discussing near-critical configurations.

7.7 Finite Matrix Specializations

Finite-dimensional matrix algebras provide a direct route from the abstract construction to executable calculations. Let a finite region R be represented by $\mathfrak{A}_K(R) = M_n(\mathbb{C})$. In a reversible specialization, choose a unitary matrix $U_R[p_R]$ and define the local algebra map by Equation 74.

$$\alpha_R[p_R](A) := U_R[p_R]^* A U_R[p_R], \quad A \in M_n(\mathbb{C}). \quad (74)$$

Equation 74 is a finite $*$ -automorphism that can be composed exactly along the ordered histories of Section 6.4. The matrix realization permits explicit checks of multiplicativity, involution preservation, order sensitivity, spectra, and represented cycle residuals.

Irreversible finite models can replace Equation 74 with a declared positive, completely positive, stochastic, or more general linear superoperator, depending on the interpretation attached to the algebra. The map class should be tested directly rather than inferred from the application label. A topology-changing model can likewise replace one matrix algebra by another and record the inter-algebra map together with the structural transformation of the underlying complex.

The operator-algebraic finite verification workflow is summarized in Table 9.

Table 9 separates logical verification from empirical validation. Passing the listed checks shows that a finite realization executes coherently under its declared assumptions. Empirical adequacy requires an additional mapping from observed or reconstructed relational data to the algebraic objects and parameters.

7.8 Representation Dependence and Covariant Diagnostics

The algebra-first architecture permits more than one representation of the same abstract local or quasi-local algebra. Suppose two GNS or other faithful representations are related by a unitary equivalence $W : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. Representation equivalence is expressed by Equation 75.

$$W \pi_1(a) W^* = \pi_2(a), \quad a \in \mathfrak{A}. \quad (75)$$

Table 8: Spectral objects available at different levels of the operator-algebraic construction.

Spectral object	Required structure	Potential formal use
$\text{Spec}_{\mathfrak{A}}(a)$	Element of a unital Banach or C^* algebra	Algebraic classification of a selected local or quasi-local quantity
$\text{Spec}(\alpha)$	Bounded linear map on a common Banach space	Decay, persistence, or amplification modes in a fixed-domain specialization
$\text{Spec}(\widehat{G})$	Densely defined represented generator with suitable analytic control	Mode structure and later criticality diagnostics in a selected representation
Spectral gap or near-zero mode	Ordered or parameterized family with a declared spectral comparison	Candidate transition or susceptibility diagnostic after application-specific justification

Table 9: Executable checks for a finite operator-algebraic GR realization.

Check	Computation	Failure indicated
Typing	Verify source and target dimensions or algebra identifiers	Undefined composition or unrecorded carrier change
Regional covariance	Evaluate the two routes in Equation 63	Local dynamics fails to lift coherently
Map-class test	Check multiplicativity, involution, positivity, or selected norm bounds	Claimed algebraic specialization is unsupported
Evaluation pullback	Test positivity and normalization after Equation 65	Selected state/evaluation class is not preserved
GNS intertwinement	Evaluate Equation 68 on a generating set	Implementing operator is inconsistent with the algebra map
Spectral calculation	Compute the declared algebraic, map, or represented spectrum	Candidate systemic diagnostic changes or becomes ill conditioned
Representation covariance	Conjugate by an admitted representation equivalence and recompute diagnostics	Reported quantity depends on arbitrary representational choice

Equation 75 supplies the transformation rule under which a representation-covariant diagnostic should retain its declared meaning. For example, spectra of unitarily conjugate bounded operators agree, while coordinate entries of a matrix representation generally change. A reported system characteristic should therefore be tied either to the abstract algebra or to quantities stable under the admitted representation equivalence.

Inequivalent representations require a different strategy. The abstract algebra remains the common comparison layer when available, while representation-specific quantities retain their local interpretation. This possibility is one reason the paper avoids defining the GR system through one universal Hilbert space. Algebraic quantum field theory and quantum statistical mechanics provide established settings in which inequivalent representations can occur [6, 3]; the present formalism uses that mathematical possibility as a warning against premature identification of one representation with the relational system itself.

The same discipline applies to coarse-graining. A finite matrix representation can be useful for computation even when a finer or quasi-local model has a different representation dimension. Comparisons across resolutions should therefore use the restriction, coarse-graining, or inter-algebra maps established by the model and should distinguish invariant structure from resolution-dependent coordinates.

7.9 Interface for Historical Dynamical Regimes

The operator-algebraic layer now supplies a family of calculational objects that can be reused under different assumptions about historical change. Table 10 summarizes the objects passed to the following history-preserving and history-reconfiguring sections.

Table 10 fixes the division of labor for the two historical-dynamics regimes. The history-preserving regime will impose the practical restriction that the already-instantiated historical subcomplex remains fixed while future continuations and their algebraic dynamics change. The history-reconfiguring regime will release that restriction and permit transformations of the represented historical complex itself, including cases detected by changed incidence or topological data.

The resulting hierarchy now reaches a calculational operator level without making that level foundational. GR power remains the substantive relational object. Field values provide compatible local formal data. Dynamical maps specify how those data act. Algebraic realizations preserve a declared amount of local structure and can lift to quasi-local maps when compatibility permits. States and W^* closures enter after an evaluation context has been selected. GNS implementations and represented generators supply additional computational tools under further assumptions. This layered construction allows later claims about path dependence, historical persistence, topology change, cycles, emergence, and criticality to identify exactly which algebraic and representational commitments they require.

Table 10: Operator-algebraic objects supplied to the historical-dynamics layers.

Object	Current formal role	Use in subsequent sections
$\alpha_R[p_R]$	Typed local algebraic dynamics	Separate fixed-carrier evolution from changes of regional algebra
$\alpha_K[\mathbf{P}_{\mathcal{P}}]$	Compatible quasi-local lift when it exists	Track system-level evolution without a primitive global state vector
$\alpha^* / \bar{\alpha}_*$	Dual evaluation transport	Compare evaluation change under retained or reconstructed history
V_α or U_α	Derived operator implementation under representation assumptions	Enable matrix, norm, and spectral calculations
δ and $\hat{G}$	Algebraic and represented generators	Support local stability and later criticality analysis
Inter-algebra morphism	Evolution between distinct algebraic carriers	Represent historical or topological reconfiguration when K and K' differ

8 History-Preserving Dynamical Regimes

This section develops the fixed-past regime introduced kinematically in Section 3.4 and anticipated by the operator-algebraic interface in Section 7.9. Its role is to identify the additional constraints under which a GR power field can transform admissible continuations while a selected realized historical subcomplex remains structurally anchored. The objective is practical and formal: many applications require a stable record of instantiated history even when future possibilities, admissible operations, local algebras outside the anchor, or evaluation weights change substantially. The construction proceeds from an anchored extension relation to anchor-preserving local and algebraic dynamics, branching continuation deformation, ordered composition, relative topological diagnostics, graduated preservation conditions, and finite consistency checks. The fixed-past regime is treated throughout as a modeling restriction. Section 9 will release the structural anchor and permit transformations of represented history itself.

8.1 Historical Anchor and Regime Definition

Let K^- denote the realized historical complex selected at a given modeling stage. Section 3.4 represents an admissible continuation by an extension pair (K_i, ι_i) with an embedding $\iota_i : K^- \hookrightarrow K_i$. The fixed-past regime requires transformations between admissible extensions to preserve this embedded anchor.

Definition 8.1 (Anchor-preserving historical transformation). Let (K_i, ι_i) and (K_j, ι_j) be admissible extensions of K^- . A transformation $f_{ij} : K_i \rightarrow K_j$ is anchor preserving when it respects the incidence, labels, and other historical structure declared fixed by the selected regime and satisfies Equation 76.

$$f_{ij} \circ \iota_i = \iota_j. \quad (76)$$

Equation 76 states that the two routes from the historical anchor into the target extension agree. It does not require f_{ij} to be an isomorphism on the full extension. Cells, labels, admissible transitions, weights, and topology outside the embedded image of K^- can change. The definition also permits the two extensions to have different sizes and different continuation structures.

The set of extension pairs together with anchor-preserving transformations can be regarded as an anchored extension system over K^- . Category-theoretic language is available when identity and composition laws are required explicitly, although the present section uses only the resulting compositional structure. The fixed-past regime therefore needs no primitive scalar time. Historical succession can continue to be

indexed by the extension relation of Equation 17 or by another application-specific ordering compatible with the anchor.

Proposition 8.2 (Closure of anchor preservation under composition). *Let $f_{ij} : K_i \rightarrow K_j$ and $f_{jk} : K_j \rightarrow K_k$ satisfy Equation 76. Whenever the composite $f_{jk} \circ f_{ij}$ is defined, the composite also preserves the historical anchor.*

Proof. The defining relations give $f_{ij} \circ \iota_i = \iota_j$ and $f_{jk} \circ \iota_j = \iota_k$. Associativity then yields $(f_{jk} \circ f_{ij}) \circ \iota_i = f_{jk} \circ \iota_j = \iota_k$. $\square$

Proposition 8.2 makes the regime stable under multi-step evolution. Consequently, a sequence of power-conditioned transformations can remain history preserving while its continuation structure changes at successive steps.

8.2 Anchored Continuation Dynamics

The fixed-past restriction applies to the realized anchor, not to the continuation family as a whole. Let $\text{Ext}_0(K^-)$ denote the baseline family of admissible extensions under a declared reference model. A GR power-field configuration can change which extensions remain admissible or how they are related while retaining the same historical anchor. The power-conditioned continuation family is written in Equation 77.

$$\text{Ext}_{\mathcal{P}}(K^-) = \{(K_i, \iota_i) \mid \iota_i : K^- \hookrightarrow K_i \text{ is an admissible extension embedding under } \mathcal{P}\}. \quad (77)$$

Equation 77 permits three logically distinct changes. First, a previously admissible extension can be removed. Second, a previously unavailable extension can become admissible. Third, the extension set can remain unchanged while the transitions, order relations, or evaluation weights among retained extensions change. These effects should be recorded separately in applications because equal extension counts can conceal different branch structures.

For a finite model, it is useful to represent the anchored extension system as a rooted directed graph $\mathcal{G}^-$. The root represents K^- , vertices represent equivalence classes of admissible extensions, and directed edges represent selected elementary anchor-preserving transformations. A power-conditioned field changes the graph to $\mathcal{G}_{\mathcal{P}}^-$ by modifying vertices, edges, or edge data while keeping the root fixed. Figure 5 illustrates this distinction schematically.

Figure 5 emphasizes that historical preservation places no requirement of dynamical weakness on the continuation sector. The future branch structure can be substantially reorganized while the realized root remains fixed.

8.3 Local Dynamics Across the Historical Interface

Local field action can overlap the historical anchor without modifying the anchor itself. This distinction is important because the constitutive support of a power relation can lie partly in realized history even when the selected dynamical regime allows change only in its continuations. Let $R_i \subseteq K_i$ and $R_j \subseteq K_j$ be corresponding regions under an anchor-preserving transformation, and define the historical overlap regions by $S_i = R_i \cap \iota_i(K^-)$ and $S_j = R_j \cap \iota_j(K^-)$. On the retained historical content, Equation 76 induces a canonical identification $h_{ij} : S_i \rightarrow S_j$ through the common anchor whenever the relevant restrictions of ι_i and ι_j are injective.

A local dynamical map $\Phi_{R_i}[p_{R_i}]$ is history preserving when its induced structural action on S_i agrees with this canonical anchor identification. Outside S_i , the map can alter continuation data, admissible operations, interface structure, and algebraic carriers according to the rules established in Sections 6 and 7.

The local distinction can be expressed through restriction. Let $\rho_{R_i S_i}^{\text{hist}}$ denote restriction of the structural part of the local dynamics to the historical overlap. Exact local history preservation is stated in Equation 78.

$$\rho_{R_i S_i}^{\text{hist}}(\Phi_{R_i}[p_{R_i}]) = h_{ij} : S_i \longrightarrow S_j. \quad (78)$$

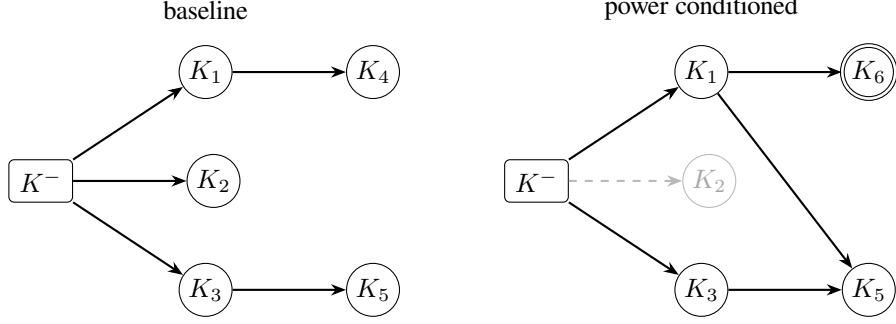


Figure 5: Schematic anchored continuation systems. The realized historical root K^- is retained in both models, while power-conditioned dynamics can suppress a branch, add a continuation, or alter transition structure among later extensions.

Equation 78 reduces to an identity condition only after S_i and S_j have been identified with the same copy of the anchor. It is stronger than requiring the power field itself to vanish on S_i . A nonzero field datum can encode historical dependence, accumulated institutions, inherited constraints, or other past-supported conditions that shape future dynamics. The fixed-past restriction concerns the induced update of the retained historical structure, not the location from which causal or generative influence is represented as arising.

This distinction also prevents a misleading support criterion. The constitutive-support family $\mathcal{S}_{\mathcal{P}}$ from Section 4 can intersect K^- , and the representational field support from Section 5 can likewise contain regions in K^- . History preservation requires that the transformation induced on the anchored structure satisfy the relevant invariance condition. It does not require historical regions to be excluded from the field description.

8.4 Algebraic Preservation of the Historical Anchor

The algebra-first architecture supplies a precise version of fixed-past dynamics even when the total source and target algebras differ. Let $\mathfrak{A}_{K^-}$ denote the quasi-local algebra of the historical anchor when the required completion exists, and let $j_i : \mathfrak{A}_{K^-} \rightarrow \mathfrak{A}_{K_i}$ and $j_j : \mathfrak{A}_{K^-} \rightarrow \mathfrak{A}_{K_j}$ be the embeddings induced by ι_i and ι_j . An algebraic transformation $\alpha_{ij} : \mathfrak{A}_{K_i} \rightarrow \mathfrak{A}_{K_j}$ preserves the historical anchor when Equation 79 holds.

$$\alpha_{ij} \circ j_i = j_j. \quad (79)$$

Equation 79 is the algebraic counterpart of Equation 76. It allows α_{ij} to act nontrivially on elements whose support extends beyond the historical subalgebra and permits $\mathfrak{A}_{K_i}$ and $\mathfrak{A}_{K_j}$ to differ. A fixed universal algebra is therefore unnecessary.

Proposition 8.3 (Composition of anchor-preserving algebra maps). *Suppose $\alpha_{ij} : \mathfrak{A}_{K_i} \rightarrow \mathfrak{A}_{K_j}$ and $\alpha_{jk} : \mathfrak{A}_{K_j} \rightarrow \mathfrak{A}_{K_k}$ each satisfy Equation 79. Then the composite $\alpha_{jk} \circ \alpha_{ij}$ also preserves the historical anchor.*

Proof. Using Equation 79 twice gives $(\alpha_{jk} \circ \alpha_{ij}) \circ j_i = \alpha_{jk} \circ j_j = j_k$. $\square$

Proposition 8.3 permits a history-preserving path to be represented by a sequence of inter-algebra morphisms rather than by repeated endomorphisms of one fixed algebra. This is useful when future extensions gain or lose relations, observables, or admissible operations while the historical anchor remains algebraically identifiable.

A finite model can quantify failure of Equation 79. Let $\mathcal{G}^-$ be a finite generating set for the chosen representation of $\mathfrak{A}_{K^-}$, and let $\|\cdot\|$ be a declared norm on the target algebra. The historical algebra defect is defined in Equation 80.

$$\varepsilon_{ij}^- := \max_{a \in \mathcal{G}^-} \|\alpha_{ij}(j_i(a)) - j_j(a)\|. \quad (80)$$

Equation 80 gives $\varepsilon_{ij}^- = 0$ for exact algebraic anchor preservation on the selected generating set. A nonzero value can indicate numerical error, coarse-graining mismatch, an intentionally approximate preservation regime, or a transition toward the historical-reconfiguration regime developed in Section 9. Interpretation depends on the source of the discrepancy.

8.5 Branching Continuations and Ordered Action

Fixing the historical anchor does not remove the order sensitivity developed in Section 6.5. Two power-conditioned transformations can preserve K^- individually and still generate different later extensions when their order is reversed. Let Θ_p and Θ_q be anchor-preserving transformations whose two compositions are well typed on a common initial extension (K_i, ι_i) . The resulting extensions are written in Equation 81.

$$(K_{pq}, \iota_{pq}) := \Theta_q \circ \Theta_p(K_i, \iota_i), \quad (K_{qp}, \iota_{qp}) := \Theta_p \circ \Theta_q(K_i, \iota_i). \quad (81)$$

Equation 81 preserves the same historical anchor in both branches while allowing K_{pq} and K_{qp} to differ outside that anchor. The source of noncommutativity can therefore lie entirely in future admissibility, interface formation, algebraic extension, or continuation topology. A fixed past does not imply path independence.

History-conditioned dynamics deepen this distinction. The local map at a later interface can depend on the realized path prefix through Equation 56, while the anchor itself remains unchanged. The historical anchor then functions as retained common structure, whereas the realized continuation prefix supplies additional memory. Institutional commitments, accumulated coalition structure, path-dependent legal effects, and other mechanisms can be represented at this level without rewriting the embedded past.

When a scalar evolution parameter is justified, a semigroup or cocycle specialization can be imposed on the continuation sector. The anchor-preservation condition then acts as a boundary constraint on the dynamical family. The general formalism retains composable extension maps because branching histories and changing future algebras need not admit one common one-parameter realization.

8.6 Relative Topology of Future Reconfiguration

The fixed-past regime permits topological change outside the historical anchor. Absolute homology of the full continuation can therefore change even when K^- remains unchanged. Relative homology provides a natural diagnostic of this situation because it records topological structure in an extension relative to the retained subcomplex [10].

For an anchored continuation (K_i, ι_i) , identify K^- with its embedded image when no ambiguity arises. The relative chain complex is formed by quotienting chains of K_i by chains supported entirely in the anchor. Its homology is denoted $H_q(K_i, K^-)$. The corresponding relative Betti number, when the coefficient system makes the rank well defined, is stated in Equation 82.

$$\beta_q^{\text{rel}}(K_i \mid K^-) := \text{rank } H_q(K_i, K^-). \quad (82)$$

Equation 82 measures a selected aspect of topological structure contributed beyond the retained anchor. If two power-conditioned continuations satisfy $\beta_q^{\text{rel}}(K_i \mid K^-) \neq \beta_q^{\text{rel}}(K_j \mid K^-)$ for some q , their future relational topology differs relative to the same realized history. This change does not by itself imply that the topology of K^- has changed.

The distinction between absolute and relative topology is summarized in Table 11.

Table 11 prevents a common ambiguity. A change in $H_q(K_i)$ can occur because the continuation sector changes while the historical anchor remains intact. Section 9 reserves the stronger expression *historical topology change* for transformations that alter the represented topology assigned to the realized history itself.

Relative topology also provides a calculational interface for later systemic analysis. Cycles can appear, disappear, or become homologous to different structures outside the anchor. Their interaction with power-conditioned operator composition will be developed in Section 10, where cycle observables and system-level characteristics are examined directly.

Table 11: Topological diagnostics inside the fixed-past regime.

Diagnostic	Structure examined	Interpretation in the fixed-past regime
$H_q(K^-)$	Realized historical anchor	Retained historical topology under the structural fixed-past assumption
$H_q(K_i)$	Entire admissible continuation	Global topology can change as future structure is added, removed, or reorganized
$H_q(K_i, K^-)$	Extension relative to the anchor	Topological contribution and organization beyond the retained past
Induced map on $H_q(K^-)$	Historical component of an anchor-preserving transformation	Identity or declared canonical identification when historical topology is fixed

8.7 Evaluation Stability and Preservation Levels

A structurally fixed past does not automatically freeze every statement, label, or evaluation associated with that past. The formalism therefore distinguishes several preservation levels. The minimal fixed-past regime used in this section preserves the declared historical carrier and its selected incidence structure. Additional assumptions can preserve labels, algebraic embeddings, or state-like evaluations.

Let ω_i and ω_j be evaluations on $\mathfrak{A}_{K_i}$ and $\mathfrak{A}_{K_j}$. Evaluation preservation on the anchor requires a common historical evaluation ω^- such that Equation 83 holds.

$$\omega_i \circ j_i = \omega^- = \omega_j \circ j_j. \quad (83)$$

Equation 83 is an additional modeling constraint. A model can preserve K^- and Equation 79 while allowing later evaluative frameworks to assign different meanings, weights, or salience to historical algebraic elements. Such retrospective re-evaluation remains distinct from changing the historical carrier itself.

Table 12 keeps the regime adaptable to empirical practice. An archival reconstruction may fix documentary events while allowing interpretive weights to change. A legal model may retain selected precedents and institutional records while changing their prospective effect. A computational toy model can impose the full package for simplicity. These are different formal commitments even when each is informally described as keeping the past fixed.

8.8 Finite Verification of Fixed-Past Dynamics

Finite anchored models permit direct checks of whether a claimed history-preserving transformation satisfies its own assumptions. Table 13 summarizes a minimal verification workflow.

The checks in Table 13 distinguish several outcomes that can otherwise be conflated. A model can pass anchor preservation while showing strong branch deformation and relative topological change. It can preserve historical incidence while failing evaluation preservation. It can also appear history preserving at one coarse resolution while a finer representation exposes a transformation inside the previously aggregated historical region. The regime should therefore be reported together with its preservation level and resolution.

These diagnostics establish logical and computational consistency under the selected representation. They do not establish that the empirical past is complete, uncontested, or uniquely reconstructible. Observational uncertainty, missing records, contested labels, and later reinterpretation belong to the empirical and epistemic layer of an application.

8.9 Interface for Historical Reconfiguration

The fixed-past regime supplies a precise baseline against which stronger historical transformations can be defined. Its characteristic constraints are the retained anchor of Equation 76, the algebraic commuting con-

Table 12: Graduated preservation conditions available within a fixed-past model.

Preservation level	Formal condition	What may still change
Structural anchor	Equation 76	Future cells, relations, branch structure, and topology outside K^-
Label retention	Structural anchor plus preservation of declared historical labels	Future labels and later contextual relations
Algebraic anchor	Equation 79	Algebraic structure outside the embedded historical subalgebra
Evaluation retention	Equation 83	Evaluations of future-supported or mixed-support quantities
Full selected fixed-past package	All preservation conditions declared by the application	Continuation dynamics compatible with the retained package

Table 13: Executable checks for a finite history-preserving GR realization.

Check	Computation	Failure indicated
Anchor commutation	Evaluate Equation 76 on historical cells or generators	Structural fixed-past assumption is violated
Historical algebra defect	Compute ε_{ij}^- from Equation 80	Algebraic anchor is only approximate or has changed
Branch comparison	Compare vertices, edges, or admissibility data of $\mathcal{G}^-$ and $\mathcal{G}_P^-$	Power-conditioned continuation structure has changed
Order sensitivity	Compare the two composites in Equation 81	Future dynamics depends on intervention order
Relative topology	Compute $H_q(K_i, K^-)$ or β_q^{rel} across retained extensions	Future topology relative to the same past has changed
Evaluation restriction	Test Equation 83 when evaluation retention is claimed	Historical interpretation or weighting is not fixed at the declared level
Resolution check	Repeat anchor and relative-topology diagnostics under an admitted refinement/coarse-graining map	Reported fixed-past effect depends on unexamined resolution choice

dition of Equation 79, and any additional preservation levels declared through Table 12. Within those constraints, the continuation family can branch differently, operator composition can remain noncommutative, local or quasi-local algebras outside the anchor can change, and relative topology can undergo substantial reconfiguration.

Section 9 will relax the structural condition that a common realized complex K^- embeds into every transformed history in a way that preserves the declared historical structure. The stronger regime will therefore require maps between different historical carriers, criteria for distinguishing relabeling from incidence change, and topological diagnostics applied directly to the represented past. The boundary between the two regimes is formal rather than rhetorical: future topology can change inside the present section, while historical topology change begins when the transformation no longer preserves the selected anchor structure itself.

This distinction also preserves the paper’s ontological restraint. Treating K^- as fixed stabilizes the domain against which future effects are calculated, while Section 9 releases that restriction. The two regimes specify which represented historical structures may change; neither by itself establishes that ultimate physical history is mutable.

9 Historical Reconfiguration and Topological Change

This section develops the stronger historical regime that follows once the anchor-preservation condition of Section 8 is released. Its role is to specify how a GR power-conditioned transformation may act on the represented historical carrier itself, rather than only on continuations attached to a retained carrier. The objective is to distinguish several mathematically different operations that are often compressed into the phrase *changing the past*: relabeling a retained complex, changing its incidence structure, changing the algebra attached to historical regions, and changing topological properties of the represented historical complex. The construction proceeds through typed reconfiguration sequences, algebraic transport between successive historical carriers, homological diagnostics, and finite verification criteria. The argument remains representational. A transformation of the modeled historical complex records a change in the structure through which history is represented or made relationally operative; stronger ontological readings require additional assumptions that are stated separately.

9.1 Historical Reconfiguration Regime

Let K_0^- denote the historical carrier used before a reconfiguration and let K_m^- denote the carrier used after it. Section 8 required a common anchor to remain embedded with selected structure preserved. The present regime permits the historical carrier to pass through a finite sequence of typed transformations. The sequence is stated in Equation 84.

$$\mathfrak{R}_{\mathcal{P}}^- := \left(K_0^- \xrightarrow{\rho_1} K_1^- \xrightarrow{\rho_2} \dots \xrightarrow{\rho_m} K_m^- \right). \quad (84)$$

Each ρ_i in Equation 84 is an admitted transformation of the chosen complex representation. It may preserve the underlying cell structure, alter labels, attach or delete cells, identify previously distinct cells, split cells, alter incidence relations, or combine several elementary operations. The maps need not be invertible. Their analytical meaning therefore differs from a symmetry action on one fixed carrier.

Definition 9.1 (Historical reconfiguration). A *historical reconfiguration* is a declared sequence $\mathfrak{R}_{\mathcal{P}}^-$ of admissible transformations whose source and target are historical carriers and for which at least one preservation condition defining the selected fixed-past regime is released. The reconfiguration is *structural* when the selected incidence data of the historical carrier change and *algebraic* when the associated historical algebra or its admissible embeddings change.

The definition allows a transformation to be historically reconfiguring even when the source and target complexes have the same homology. Topology change is therefore a stronger classification applied after the reconfiguration has been specified. This order prevents a combinatorial change from being called topological merely because the cell presentation differs.

A one-step reconfiguration can be written as $\rho_{\mathcal{P}}^- : K^- \rightarrow K'^-$ when a single map carries the intended structure. Equation 84 is more general because complex transformations may be easier to describe through elementary moves, and because intermediate carriers provide locations at which algebraic or topological changes can be checked.

The fixed-past regime is recovered as a special case. If every ρ_i restricts to the declared historical identity or canonical identification and preserves the selected historical package, then Equation 84 reduces to an anchor-preserving evolution of the type developed in Section 8. The present section therefore enlarges the admissible transformation class without changing the earlier regime's internal logic.

9.2 Historical Identity and Comparison Data

Historical reconfiguration requires a criterion for deciding which structures in K_0^- and K_m^- are being compared. A literal equality of cell labels is too restrictive because harmless relabeling or refinement can change the presentation while retaining the represented structure. A comparison based only on homology

is too coarse because distinct complexes can share the same homology. The formalism therefore separates *comparison data* from *invariants*.

Let $\mathcal{C}_{0m}^-$ denote the declared comparison data between K_0^- and K_m^- . Depending on the application, $\mathcal{C}_{0m}^-$ may contain a cell correspondence, an embedding into a common refinement, a quotient map, a zigzag of refinement/coarse-graining maps, or another explicitly specified relation that allows selected historical structures to be matched. The comparison data determine which differences count as representation changes and which count as substantive changes at the selected resolution.

A useful baseline is a representation-equivalence relation $\simeq_{\text{rep}}$ chosen for the application. Equation 85 states the corresponding classification.

$$K_0^- \simeq_{\text{rep}} K_m^- \iff \text{the declared comparison data identify the selected represented structure.} \quad (85)$$

Equation 85 is intentionally parameterized by the declared comparison data. In one model, subdivision of a face may be treated as representationally equivalent. In another, the new subdivision may carry additional historical meaning and therefore count as a substantive reconfiguration. The equivalence level should be reported together with any claim of historical change.

The distinction yields three useful layers. A *presentation change* alters labels, indexing, or admitted cellulation while remaining equivalent under $\simeq_{\text{rep}}$. A *structural reconfiguration* fails the selected representation equivalence because incidence or other declared structure has changed. A *topological change* is established only at the topological level supported by an explicit equivalence failure or by a diagnostic whose change is sufficient to establish a topological difference.

Table 14 summarizes these layers.

Table 14 also records an asymmetry in evidential strength. A change in a genuine invariant can establish a topological difference. Equality of a finite list of invariants generally supplies only compatibility with preservation. This limitation will govern the diagnostics below.

9.3 Algebraic Transport Across Reconfigured Histories

The algebra-first architecture permits historical reconfiguration without requiring the old and new historical carriers to inhabit one fixed Hilbert space. Associate each K_i^- in Equation 84 with an algebra $\mathfrak{A}_i^- := \mathfrak{A}_{K_i^-}$. A reconfiguration step ρ_i is accompanied, where the model supplies one, by a typed algebraic map stated in Equation 86.

$$\alpha_i^- : \mathfrak{A}_{i-1}^- \longrightarrow \mathfrak{A}_i^-. \quad (86)$$

The maps in Equation 86 need not be automorphisms. Cell deletion or quotienting can remove distinctions represented in the source algebra. Cell attachment can enlarge the codomain. Identification can map several source-supported quantities into one target-supported quantity. Open-system reduction can require a positive or completely positive map under additional assumptions. The admissible map class should therefore be chosen from the substantive model rather than imposed by the word *operator*.

When the maps are composable, the historical algebraic reconfiguration is the ordered composite in Equation 87.

$$A_{\mathcal{P}}^- := \alpha_m^- \circ \alpha_{m-1}^- \circ \cdots \circ \alpha_1^- : \mathfrak{A}_0^- \longrightarrow \mathfrak{A}_m^-. \quad (87)$$

Equation 87 preserves the sequence sensitivity developed in Section 6. Two historical reconstructions containing the same elementary operations can yield different target algebras or different composites when their ordering changes. Historical reconfiguration can therefore remain noncommutative even before cycle-level systemic effects are introduced.

A comparison with Section 8 makes the regime boundary explicit. Under algebraic anchor preservation, source and target historical algebras are connected by commuting embeddings of one retained historical algebra. In the present regime, the source historical algebra itself may be transformed. A finite model can measure the departure from an available reference identification. Suppose $J_{0m} : \mathfrak{A}_0^- \rightarrow \mathfrak{A}_m^-$ is the comparison map supplied by $\mathcal{C}_{0m}^-$. For a finite test set $\mathcal{T}^- \subset \mathfrak{A}_0^-$, define the algebraic reconfiguration defect by Equation 88.

$$\varepsilon_{\text{reconf}}^- := \max_{a \in \mathcal{T}^-} \|A_{\mathcal{P}}^-(a) - J_{0m}(a)\|. \quad (88)$$

Table 14: Levels of historical reconfiguration in the GR spacetime representation.

Level	Typical formal change	Classification requirement
Presentation change	Relabeling, reindexing, admitted refinement, or equivalent cellulation	Source and target remain equivalent under the declared $\simeq_{\text{rep}}$
Structural reconfiguration	Changed incidence, attachment, identification, deletion, splitting, or relational interface	Declared represented structure differs under $\mathcal{C}_{0m}^-$
Algebraic reconfiguration	Changed local algebra, embedding system, admissible operation set, or inter-algebra map	Algebraic comparison fails the selected preservation condition
Homology-detected topology change	$H_q(K_0^-) \not\cong H_q(K_m^-)$ for at least one tested degree	Changed homology supplies a sufficient diagnostic of topological difference
Stronger topological classification	Homeomorphism, homotopy-equivalence, or another declared topological criterion fails	Criterion and proof method must be stated explicitly

Equation 88 is meaningful only when the norm and comparison map are justified. A zero defect on the test set supports agreement with the selected reference identification; a nonzero value records an algebraically detected historical change at the tested resolution. It does not by itself classify the change as topological.

Evaluations can also be transported when the required maps exist. If ω_m^- is an evaluation on $\mathfrak{A}_m^-$, the pullback $\omega_m^- \circ A_{\overline{p}}$ defines an evaluation-like functional on $\mathfrak{A}_0^-$ under the conditions developed in Section 7. This permits explicit comparison between a historical quantity evaluated before and after reconfiguration while keeping the algebraic and evaluative layers distinct.

9.4 Elementary Structural Transformations

A finite 2-complex permits the historical reconfiguration sequence to be decomposed into elementary structural operations. This decomposition supplies a practical route to calculation because each operation has a localized effect on cell counts, boundary matrices, local algebras, and admissible continuation interfaces.

The principal elementary classes used here are cell attachment, cell deletion, identification, splitting, and incidence rewiring. An attachment creates a new cell together with its declared boundary relation. A deletion removes a cell subject to the consistency conditions required for the remaining structure to be a valid complex. An identification maps previously distinct cells or subcomplexes into a common target. A splitting replaces one represented unit with several units and new incidence data. Incidence rewiring changes attachment relations while retaining a selected set of cells.

These operations should not be assigned a fixed substantive meaning across applications. For example, adding an edge could represent recognition of an archival linkage, restoration of an institutional channel, or a stronger claim that a historical relation itself belongs to the modeled past. The mathematical operation is reusable; the interpretation depends on the declared epistemic and ontological level.

For a finite historical complex K^- , let $n_q(K^-)$ denote the number of q -cells. The cell-count profile is stated in Equation 89.

$$\mathbf{n}(K^-) := (n_0(K^-), n_1(K^-), n_2(K^-)). \quad (89)$$

Changes in Equation 89 are combinatorial diagnostics. They can reveal attachment, deletion, splitting, or identification, while they cannot by themselves establish topology change because different cellulations of the same topology can have different cell counts.

The boundary matrices provide a more informative representation. Let $B_q(K^-)$ denote a matrix representation of ∂_q in chosen cellular bases. An elementary reconfiguration changes one or more of the matrices B_1 and B_2 . The condition $B_{q-1}B_q = 0$ must continue to hold after any transformation claimed to produce a valid chain complex. This supplies a direct finite consistency test before homology is computed.

A reconfiguration sequence can therefore be audited step by step. Each ρ_i changes a declared local region, updates the affected boundary matrices and algebraic assignments, and passes the chain-complex and typing conditions before the next step is composed. This operational form is especially useful for later computational experiments in which power-conditioned transformations are generated algorithmically.

9.5 Homological Diagnostics of Historical Topology

Homology supplies a calculable sufficient diagnostic for selected forms of historical topology change. For each historical carrier K_i^- , the homology groups $H_q(K_i^-)$ are computed from the boundary maps introduced in Section 2.9. A reconfiguration is homology-detected as topology changing when Equation 90 holds for at least one tested degree.

$$H_q(K_0^-) \not\cong H_q(K_m^-) \quad \text{for some tested } q. \quad (90)$$

Over coefficient fields for which ranks are defined, the corresponding Betti-number change is stated in Equation 91.

$$\Delta\beta_q^- := \beta_q(K_m^-) - \beta_q(K_0^-). \quad (91)$$

A nonzero value of Equation 91 establishes a change in the tested homological structure. For finite complexes, the Euler-characteristic difference supplies another diagnostic. It is defined in Equation 92.

$$\Delta\chi^- := \chi(K_m^-) - \chi(K_0^-). \quad (92)$$

A nonzero $\Delta\chi^-$ establishes that the source and target cannot be homeomorphic. The converse does not follow. Likewise, $\Delta\beta_q^- = 0$ for the tested degrees does not establish topological equivalence. These diagnostics are deliberately one-sided where mathematics requires one-sided inference.

For a chosen finite family of invariants $\mathcal{Q} = \{I_1, \dots, I_r\}$, define an invariant profile by Equation 93.

$$\mathfrak{I}_{\mathcal{Q}}(K^-) := (I_1(K^-), \dots, I_r(K^-)). \quad (93)$$

Equation 93 provides a reproducible comparison record. A profile change establishes a structural difference whenever the changed component is a genuine invariant under the equivalence being tested. Profile equality means only that the selected diagnostics did not detect a difference.

The induced action on homology supplies additional information when each ρ_i is a chain map or induces one after the declared comparison construction. The composite then yields maps of the form stated in Equation 94.

$$(\mathfrak{R}_{\mathcal{P}}^-)_{*,q} : H_q(K_0^-) \longrightarrow H_q(K_m^-). \quad (94)$$

Equation 94 distinguishes a mere comparison of Betti numbers from a transformation that tracks how source homology classes are mapped, annihilated, or identified in the target. Target classes outside the image of the induced map can then be reported separately as classes not inherited from the mapped source. When no natural chain map exists, the paper retains the invariant-profile comparison and does not invent an induced map.

9.6 Historical Reconfiguration and Continuation Structure

Changing the historical carrier also changes the domain from which future continuations are generated. Let $\text{Ext}(K_0^-)$ and $\text{Ext}(K_m^-)$ denote the continuation families associated with the source and target historical carriers. A historical reconfiguration therefore induces, when defined, a continuation transformation of the form stated in Equation 95.

$$\mathfrak{C}_{\mathcal{P}} : \text{Ext}(K_0^-) \longrightarrow \text{PowSet}(\text{Ext}(K_m^-)), \quad (95)$$

where $\text{PowSet}(S)$ denotes the power set of S . The notation avoids overloading $\mathcal{P}$, which elsewhere denotes substantive GR power.

Equation 95 is set-valued because one historical reconfiguration can change which continuations are admissible without determining a unique successor. The transformation may close branches, create branches, merge previously distinct branches, or change which local operations can be composed. Historical change can therefore alter the generative architecture of the future even when no scalar history weight has been assigned.

This mechanism makes the distinction between future reconfiguration and historical reconfiguration operational. In Section 8, the root K^- remains fixed while the branch structure changes. Here the root itself becomes K_m^- , and the continuation graph is rebuilt relative to the transformed historical carrier. The two processes can be composed, but they should be reported separately because they answer different counterfactual questions.

A useful diagnostic is the change in the admissible-operation family. Let $\text{AdmExt}(K^-)$ denote the declared family of admissible local transformations extending K^- . This notation is kept distinct from $\mathfrak{D}(K)$, which Section 4.4 reserves for deformation records. When the application supplies a correspondence from source operations to their admissible descendants after reconfiguration, represent it by the set-valued map in Equation 96.

$$\mathfrak{D}_{\mathcal{P}} : \text{AdmExt}(K_0^-) \longrightarrow \text{PowSet}(\text{AdmExt}(K_m^-)) . \quad (96)$$

Equation 96 is optional because a historical reconfiguration need not supply a canonical operation correspondence. When defined, it records a stronger effect than reweighting a fixed menu of futures: a source operation can disappear, acquire several admissible descendants, or be replaced by operations with different composability conditions. A transformed historical structure may therefore change the grammar of admissible continuation itself. This feature will later matter for system characteristics because cycles and invariants available after reconfiguration can differ from those available before it.

9.7 Interpretive Levels of Historical Change

The same formal map can support different substantive readings, and the paper keeps these readings separate. Table 15 distinguishes three levels that are sufficient for the present formal programme.

The epistemic level is compatible with an underlying past that remains fixed while the present representation changes. The relational-historical level concerns changes in which historical relations remain causally or normatively operative within the present relational system. The stronger ontological level assigns the formal topology change to the underlying past itself. The present paper develops mathematics capable of representing all three while adopting neither the second nor the third as an automatic consequence of the formalism.

This separation is important for power analysis. A government, institution, archive, platform, or community may exercise power over historical representation or recognized continuity without altering physical events that occurred. Such changes can still reshape present and future generative possibilities because the relationally operative past enters current institutions, legal structures, identities, expectations, and admissible actions. The formal relevance of historical reconfiguration therefore does not depend on endorsing retrocausal ontology.

The distinction also constrains language concerning topology. A change in $H_q(K^-)$ is a topology change of the *represented historical complex*. The stronger statement that physical history has changed requires a separately defended identification between the complex and ultimate historical ontology. Under GR ontological suspension, that identification remains open.

9.8 Finite Verification of Historical Reconfiguration

Finite models make the stronger regime directly auditable. Table 16 gives a verification workflow that separates combinatorial validity, algebraic consistency, topological diagnostics, and interpretive claims.

The finite workflow also suggests a minimal computational experiment. One can construct K_0^- , calculate its boundary matrices and invariant profile, apply a specified elementary reconfiguration sequence,

Table 15: Interpretive levels for historical reconfiguration.

Interpretive level	Object treated as changing	Representative application
Epistemic reconstruction	Accessible evidence, labels, inferred links, salience, or the current representation of prior relations	Archival revision, revised causal reconstruction, corrected attribution, new documentary evidence
Relational-historical reconfiguration	Historically operative links connecting prior events to present institutions, norms, memories, or recognized continuities	Changed precedent, institutional succession, collective-memory structure, restored or severed recognized lineage
Stronger ontological reading	Historical carrier is interpreted as representing change in the underlying past structure itself	Speculative extension requiring an ontology that permits such change

Table 16: Executable checks for a finite history-reconfiguring GR realization.

Check	Computation	Failure or result indicated
Transformation typing	Verify source and target of every ρ_i and α_i^-	Reconfiguration sequence is not composable at the declared level
Chain validity	Recompute boundary matrices and test $B_{q-1}B_q = 0$ after each structural step	Target fails to define the claimed chain complex
Representation comparison	Apply $\mathcal{C}_{0m}^-$ and the declared $\simeq_{\text{rep}}$	Distinguishes admitted presentation change from substantive represented change
Algebraic defect	Compute $\varepsilon_{\text{reconf}}^-$ when Equation 88 is defined	Historical algebra differs from the selected reference identification
Homology profile	Compute H_q , β_q , and selected invariant profile entries	Detects selected topological changes when an invariant differs
Induced-class tracking	Compute Equation 94 when a chain map exists	Shows which homology classes persist, merge, vanish, or map nontrivially
Continuation comparison	Compare $\text{Ext}(K_0^-)$, $\text{Ext}(K_m^-)$, and AdmExt	Historical reconfiguration changes the future generative architecture
Resolution robustness	Repeat diagnostics under admitted refinement/coarse-graining relations	Reported change depends on an unexamined representation scale

construct K_m^- , recalculate the same quantities, and compare the induced continuation structure. Algebraic maps can be evaluated in parallel on finite generating sets. This produces a reproducible record of what changed and at which formal level.

A topology-changing result should therefore be reported with at least four pieces of information: the source and target historical carriers, the reconfiguration sequence or comparison data, the invariant or equivalence criterion that detects the change, and the interpretive level assigned to the transformation. Omitting any of these items makes it difficult to distinguish topology change from a presentation change or from an unsupported ontological reading.

The checks remain formal and computational. They do not resolve empirical uncertainty concerning which historical relations should be represented, whether archival evidence is complete, or whether one interpretation of historical continuity is normatively preferable. Those questions enter through data construction, domain interpretation, and later empirical validation.

9.9 Interface for Cycles and System Characteristics

Historical reconfiguration completes the paper's distinction among three increasingly strong transformation levels. A power-conditioned model can reweight or transform dynamics on a fixed carrier, can alter future continuations while retaining the historical anchor, or can transform the historical carrier itself. The strongest level can be further classified by whether the change is representational, structural, algebraic, or topological according to the diagnostics developed in this section.

The next section shifts from transformation classes to compositional system structure. Once local fields and operator-algebraic maps act across a history-bearing complex, closed paths and repeated compositions can carry information that is absent from isolated local transformations. Historical reconfiguration matters directly for that analysis because changing K^- can create or destroy cycles, alter which cycle composites are defined, change homology classes, and modify the algebra in which residual transformations are evaluated.

Section 10 will therefore examine cycles, invariant and approximately invariant quantities, and emergent system characteristics. The present section supplies its topological boundary conditions: cycle-level conclusions are always relative to a represented carrier and its admitted equivalences, and a system characteristic that appears stable under local perturbation can still change when a power-conditioned transformation moves the historical configuration into a different structural or topological class.

10 Emergent System Structure under Cyclic Composition

This section develops the first explicitly system-level consequences of the local and operator-algebraic dynamics constructed in Sections 5–9. Its role is to determine when repeated composition around closed relational structures yields information that is not recoverable from isolated local transformations alone. The objective is to define cycle return maps, residual and curvature-like diagnostics, representation-stable invariants, cycle-generated transformation structures, and a restricted formal criterion for emergent system characteristics. The discussion proceeds from typed closed compositions to base-point and representation invariance, then to local face cycles, global topological cycle classes, diagnostic-relative emergence, and finite executable examples. The method is formal and computational. The term *emergent* is used relative to a declared family of local diagnostics and does not establish an ontological doctrine of emergence.

10.1 Closed Compositional Structures

Equation 51 becomes a cycle-level object only after the terminal carrier can be compared with the initial carrier. Consider a composable path $c = (e_1, \dots, e_n)$ from a retained region R_0 to a retained region R_n . A *closure identification* is a declared comparison map $\kappa_c : \text{Poss}_{K_n}(R_n) \rightarrow \text{Poss}_{K_0}(R_0)$ whose domain contains the retained image of the path composite. The corresponding cycle return map is defined in Equation 97.

$$\mathcal{C}_c := \kappa_c \circ \Phi_{e_n} \circ \dots \circ \Phi_{e_1}. \quad (97)$$

Equation 97 reduces to the ordinary closed composite when the terminal and initial carriers are literally identical and κ_c is the identity. Retaining κ_c explicitly is important for GR because a relational cycle can return to an equivalent region through a changed local representation, a refinement/coarse-graining identification, or an inter-algebra correspondence rather than through literal equality of carriers.

The algebraic specialization follows the same logic. Suppose each interface has an algebraic component $\alpha_{e_i} : \mathfrak{A}_i \rightarrow \mathfrak{A}_{i+1}$ and the closure comparison is an algebra map $\varkappa_c : \mathfrak{A}_{n+1} \rightarrow \mathfrak{A}_1$. The algebraic cycle return map is stated in Equation 98.

$$\mathcal{A}_c := \varkappa_c \circ \alpha_{e_n} \circ \dots \circ \alpha_{e_1} : \mathfrak{A}_1 \longrightarrow \mathfrak{A}_1. \quad (98)$$

Equation 98 gives an endomorphism of the selected base algebra even when the intermediate algebras differ. A cycle therefore does not require every local transformation to act on one universal algebra. It requires only typed composability and a declared closure comparison at the level where a return diagnostic is to be evaluated.

Definition 10.1 (GR cycle return structure). A GR cycle return structure consists of a composable closed relational path, its local dynamical or algebraic maps, and a declared closure identification that places the terminal retained structure in comparison with the initial retained structure. The resulting map $\mathcal{C}_c$ or $\mathcal{A}_c$ is the cycle return map at the selected formal level.

Definition 10.1 separates three possible failures. A path can fail to compose, a terminal carrier can lack a justified closure identification, or the resulting return map can be well defined and substantively non-trivial. Only the third situation supports a cycle residual. This distinction prevents missing representational infrastructure from being misreported as a systemic effect.

10.2 Cycle Residuals and Return Equivalence

A cycle becomes systemically informative when its return map fails to reproduce the retained starting structure under a declared equivalence. Let $\equiv_c$ denote the comparison adopted for the base possibility structure. The cycle is *return-trivial* on a retained domain D_c when $\mathcal{C}_c(x) \equiv_c x$ for every $x \in D_c$. The residual set is defined in Equation 99.

$$\Omega_c := \{x \in D_c \mid \mathcal{C}_c(x) \not\equiv_c x\}. \quad (99)$$

Equation 99 remains meaningful for nonlinear, partially defined, or carrier-changing histories because it requires no subtraction. A nonempty Ω_c records a residual effect after completing the declared relational cycle.

When a distance d_c is defined on comparable outputs, the pointwise cycle residual admits the quantitative specialization in Equation 100.

$$\varepsilon_c(x) := d_c(\mathcal{C}_c(x), x), \quad x \in D_c. \quad (100)$$

A finite calculation can aggregate Equation 100 by a supremum, mean, quantile, or another declared functional over a retained sample. The chosen aggregation becomes part of the model and should not be interpreted as a primitive scalar magnitude of power.

At the algebraic level, the fixed set of one cycle is defined in Equation 101.

$$\text{FixSet}(\mathcal{A}_c) := \{a \in \mathfrak{A}_1 \mid \mathcal{A}_c(a) = a\}. \quad (101)$$

Equation 101 records algebraic content preserved by the complete cycle even when individual steps act nontrivially. For a general linear, positive, or completely positive return map, this object is only a fixed set (and, when linear, a fixed linear subspace). When $\mathcal{A}_c$ is a unital $*$ -endomorphism, the fixed set is closed under the algebraic operations and will be denoted $\text{FixAlg}(\mathcal{A}_c)$. In a social or political application, the meaning of a preserved algebra element depends on the interpretation assigned to the algebra; the formal condition itself makes no claim that the element is a conserved physical quantity.

The paper uses the term *holonomy-like residual* only for return maps of the form in Equation 97 or Equation 98 when the required transport and closure structure has been defined. This terminology marks a structural analogy with closed-path transport. It does not identify GR power with a physical gauge connection.

10.3 Base-Point and Representation Invariance

A candidate cycle characteristic should not depend on an arbitrary choice of starting point when the formal conditions make different starting points equivalent. Consider an invertible cycle on one common carrier with maps $T_1, \dots, T_n$. The return map based before T_1 is $C_1 = T_n \cdots T_1$. The same oriented cycle based after T_1 has return map $C_2 = T_1 T_n \cdots T_2$.

Proposition 10.2 (Base-point conjugacy for invertible cycle maps). *If every map in a finite cycle is invertible on the retained common carrier, then return maps obtained by cyclically shifting the base point are conjugate. In particular, the one-step shift satisfies Equation 102.*

$$C_2 = T_1 C_1 T_1^{-1}. \quad (102)$$

Proof. Substituting $C_1 = T_n \cdots T_2 T_1$ gives $T_1 C_1 T_1^{-1} = T_1 T_n \cdots T_2$, which is the return map of the same oriented cycle after shifting the base point across T_1 . $\square$

Proposition 10.2 gives a concrete criterion for admissible cycle invariants. Similarity-invariant quantities are unchanged by the base-point shift when they are defined. In finite-dimensional linear realizations these can include the characteristic polynomial, spectrum with algebraic multiplicity, trace, and determinant. For bounded operators or algebra automorphisms, the available invariant family depends on the analytic setting. The paper therefore records the object and the equivalence under which each invariant is claimed.

Local representation changes yield a parallel result. Suppose the field and dynamical carriers transform through invertible maps g_R satisfying the representation-covariance conditions of Sections 5 and 6. For a cycle based at R_0 , the transformed return map has the covariance law in Equation 103.

$$C'_c = g_{R_0} \circ C_c \circ g_{R_0}^{-1}. \quad (103)$$

Equation 103 follows because the intermediate coordinate maps cancel pairwise in the ordered composite. Conjugacy-invariant cycle diagnostics therefore survive admitted local reparameterizations. A quantity that changes under this transformation may still be computationally useful, but it should be identified as representation dependent.

Table 17 separates several invariant levels used later in the section.

10.4 Face Cycles and Curvature-Like Residuals

A two-dimensional complex distinguishes cycles that bound retained faces from cycles whose global status depends on the topology of the carrier. Let f be a retained oriented face with boundary path ∂f . When the local maps along ∂f admit a closure identification, the face return map is $C_{\partial f}$ from Equation 97. Let $\equiv_f$ denote the return-equivalence relation $\equiv_c$ of Section 10.2 specialized to the retained base carrier of the face cycle. The field-dynamical realization is *cycle-flat on f* when Equation 104 holds on the retained domain.

$$C_{\partial f} \equiv_f \text{id}. \quad (104)$$

Equation 104 supplies a formal local flatness criterion. A nontrivial face residual is called *curvature-like* because it measures failure of local closed composition to return the retained structure to its initial equivalence class. The analogy remains limited: a conventional curvature two-form requires additional differential-geometric structure that the present discrete GR construction does not assume.

In a differentiable common-domain specialization, a small closed commutator loop makes the relation between sequence sensitivity and a curvature-like residual explicit. Let A and B be bounded generators on a common representation space and define the return operator in Equation 105.

$$C_\epsilon(A, B) := e^{-\epsilon B} e^{-\epsilon A} e^{\epsilon B} e^{\epsilon A}. \quad (105)$$

A second-order expansion gives Equation 106.

$$C_\epsilon(A, B) = I + \epsilon^2[B, A] + O(\epsilon^3). \quad (106)$$

Equation 106 shows that the leading residual of this small cycle is controlled by the same order-sensitive commutator structure developed in Section 6.5. The result applies only in the bounded differentiable specialization used for the expansion. It does not imply that every GR cycle possesses a Lie-algebraic curvature representation.

Local flatness also need not settle global cycle structure. A complex can admit non-boundary cycles even when every selected face boundary is return-trivial. In such a case, a global closed path can carry a nontrivial return map because the path belongs to a different topological class or because the adopted transport structure is globally nontrivial. This distinction motivates the separate treatment of homology classes below.

Table 17: Invariant levels for cyclic GR dynamics.

Invariant level	Required structure	Stable information
Return-equivalence class	Declared $\equiv_c$ on the base carrier	Whether the complete cycle is trivial or residual on retained content
Conjugacy class	Invertible cycle maps and admitted changes of base point or coordinates	Cycle action modulo re-expression at the base region
Finite spectral data	Finite-dimensional linear or matrix realization	Characteristic polynomial, eigenvalue multiset, trace, determinant where defined
Fixed subalgebra	Algebra endomorphism or automorphism	Algebraic content preserved by the complete cycle
Homology class	Chain-complex structure on the carrier	Topological cycle class modulo boundaries [10]

10.5 Cycle-Generated Transformation Structure

A system usually contains many cycles. Their joint structure can be represented without collapsing them into a single scalar. Let $\mathcal{C}(K)$ be a selected family of admissible closed paths based, after declared identifications, on a common retained algebra $\mathfrak{A}$. The cycle-generated transformation family is defined in Equation 107.

$$\mathfrak{G}_{\text{cyc}}(K) := \langle \mathcal{A}_c \mid c \in \mathcal{C}(K) \rangle_{\circ}, \quad (107)$$

where $\langle \cdot \rangle_{\circ}$ denotes closure under the declared finite compositions that remain well typed. Equation 107 gives a transformation monoid or semigroup in the general noninvertible case and a group when every retained cycle return map is invertible and inverses are admitted.

The algebraic content fixed by every selected cycle is stated in Equation 108.

$$\mathfrak{F}^{\text{cyc}} := \bigcap_{c \in \mathcal{C}(K)} \text{FixSet}(\mathcal{A}_c). \quad (108)$$

Equation 108 defines a system-level common fixed structure relative to the selected cycle family. In the general map class it is a set, or a linear subspace when all return maps are linear on one common vector space. If every retained $\mathcal{A}_c$ is a unital $*$ -endomorphism of the same algebra, then $\mathfrak{F}^{\text{cyc}}$ is a fixed subalgebra. It can be strictly smaller than the fixed structure of any one local map because an element must survive every retained closed composition. In an empirical application, membership in $\mathfrak{F}^{\text{cyc}}$ can be estimated only after the local algebras, cycle family, and transformation maps have been operationalized.

The cycle-generated family also records interactions among cycles. Two cycle return maps can fail to commute even when each is individually well defined. In a common linear realization, the cycle-level commutator $[\mathcal{A}_{c_1}, \mathcal{A}_{c_2}]$ then measures order sensitivity between repeated systemic circuits rather than between two isolated local interventions. Such a quantity can reveal mesoscopic organization that is absent from pairwise edge statistics.

10.6 Diagnostic-Relative Emergence

The language of emergence requires a precise comparison class. Let X denote a finite GR realization consisting of a carrier, field data, and dynamical maps. Let $\mathcal{L}_r(X)$ be a declared collection of local diagnostics computed from regions no larger than a chosen resolution or locality scale r . Two realizations are locally diagnostic-equivalent when Equation 109 holds.

$$X \equiv_{\text{loc}}^{(r)} Y \iff \mathcal{L}_r(X) = \mathcal{L}_r(Y) \text{ under the declared regional correspondence.} \quad (109)$$

Equation 109 intentionally depends on the chosen diagnostic family. Matching edge spectra, matching local norms, matching degree profiles, or matching local response records are different equivalence relations.

Definition 10.3 (Diagnostic-relative emergent characteristic). Let S be a system characteristic defined on a class of GR realizations. The characteristic S is emergent relative to the local diagnostic family $\mathcal{L}_r$ when there exist realizations X and Y such that $X \equiv_{\text{loc}}^{(r)} Y$ while $S(X) \neq S(Y)$ under the declared comparison of system characteristics.

Definition 10.3 establishes local underdetermination, not metaphysical irreducibility. A richer diagnostic family can remove an emergence result obtained under a coarser family. The definition therefore makes the scale, local information set, and system characteristic part of every emergence claim.

Several system characteristics developed in the present paper can be tested in this way. Examples include the conjugacy or spectral profile of cycle return maps, the common cycle-fixed structure $\mathfrak{F}^{\text{cyc}}$, homology groups or Betti numbers of the carrier, and later multi-field or criticality diagnostics. Each requires its own representation and comparison conditions.

A finite matrix example demonstrates the logic. Consider two three-interface realizations on $M_2(\mathbb{C})$ in which the local algebra maps are inner automorphisms of the form in Equation 74. Define the matrices in Equation 110.

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (110)$$

In realization X_A , take the ordered edge representatives (X, X, I) . In realization X_B , take (X, Z, I) . The corresponding inner automorphisms Ad_X and Ad_Z have the same superoperator spectrum $\{1, 1, -1, -1\}$, while the identity edge agrees exactly. If the local diagnostic family records only these edge-map spectra, then $X_A \equiv_{\text{loc}}^{(r)} X_B$ for that diagnostic choice. The cycle representatives differ as stated in Equation 111.

$$U_A = X X I = I, \quad U_B = X Z I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (111)$$

The algebraic cycle maps are $\mathcal{A}_{c,A} = \text{Ad}_{U_A}$ and $\mathcal{A}_{c,B} = \text{Ad}_{U_B}$, using the convention of Equation 74. Their superoperator spectra are stated in Equation 112.

$$\text{Spec}(\mathcal{A}_{c,A}) = \{1, 1, 1, 1\}, \quad \text{Spec}(\mathcal{A}_{c,B}) = \{1, 1, -1, -1\}. \quad (112)$$

Equation 112 distinguishes the two realizations even though the selected local algebra-map spectra do not. This is a concrete instance of Definition 10.3. It does not show that the global difference is unrecoverable from every possible local data set; it shows that it is unrecoverable from the one explicitly retained in the example.

10.7 Topological Cycle Classes and Global Residuals

The carrier topology supplies a second route to system-level structure. For a finite cellular or simplicial realization, one-dimensional cycles lie in the kernel of the boundary map ∂_1 , while boundaries of two-dimensional cells lie in the image of ∂_2 . The first homology group is therefore $H_1(K) = \ker \partial_1 / \text{im } \partial_2$ under the standard chain-complex construction [10]. A nonzero class records a cycle that is not homologous to a retained two-boundary within the chosen complex.

Cycle dynamics need not factor through homology automatically. Two homologous paths can have different return maps if the local dynamics depend on geometric detail, history, labels, or other information discarded by the homology class. When the model intends a topological cycle observable, an additional invariance condition is required. Let $[c]$ denote the homology class of a cycle. A cycle characteristic I_c factors through $H_1(K)$ when Equation 113 holds.

$$[c_1] = [c_2] \implies I_{c_1} = I_{c_2}. \quad (113)$$

Equation 113 is a testable model requirement. If it fails, the characteristic remains a path- or geometry-sensitive observable rather than a topological invariant.

Historical reconfiguration from Section 9 can change both sides of the cycle analysis. A transformation $K \rightarrow K'$ can create or remove homology classes, and an accompanying inter-algebra map can change the return dynamics assigned to surviving cycles. When the structural transformation is a chain map, the induced homology map developed in Section 9.5 can be used to compare selected cycle classes before and after reconfiguration. When no chain map exists, cycle correspondence must be supplied separately and the lack of a canonical comparison should remain explicit.

This distinction also clarifies the relation between local curvature-like residuals and global topology. A model may have nontrivial residuals on face boundaries, indicating local compositional curvature-like structure. It may also have return-trivial face boundaries while retaining a nontrivial return map on a non-boundary homology class. The second pattern is a global residual relative to the selected transport and topology. Neither pattern alone determines the ontology of power; both are calculable properties of the formal representation.

10.8 Finite Verification of Cyclic System Structure

Finite realizations permit direct verification of the system-level constructions. Table 18 gives a workflow that separates topological, dynamical, algebraic, and emergence claims.

The finite workflow supports several kinds of calculation without requiring a universal phase space. Boundary matrices determine selected topological cycle classes. Local dynamical or algebraic maps determine ordered cycle composites. Matrix or superoperator realizations supply spectra, fixed spaces, and commutators. Comparison across locally diagnostic-equivalent systems tests whether a proposed system characteristic contains information absent from the retained local summaries.

Formal success at this level remains distinct from empirical validation. A computationally nontrivial cycle residual can be substantively meaningful only if the regions, interfaces, local maps, closure identifications, and selected diagnostics correspond to defensible features of the domain being studied. The formalism supplies consistency tests and calculational objects; it does not select the empirical encoding by itself.

10.9 Interface for Multiple Power Fields

The present section has treated the local dynamics as already containing the power-conditioned field data relevant to each interface. The next step is to relax that single-field presentation. A region can be simultaneously affected by several GR power constructions with distinct constitutive supports, field carriers, and dynamical realizations. Their combined action need not be additive, and the ordering of their induced maps can alter both cycle residuals and the cycle-generated system algebra.

Section 11 will therefore examine multiple power fields and their compositional dynamics. The cycle architecture developed here supplies the systemic diagnostics required for that analysis. Multi-field interaction can reinforce or cancel selected local effects, alter the conjugacy class of a cycle return map, change the common fixed system structure $\mathfrak{F}^{\text{cyc}}$, create or remove admissible cycles through structural transformation, and modify which local diagnostic families are sufficient to recover global behavior.

The main formal conclusion of this section is accordingly restricted. System-level characteristics become calculable once local power-conditioned transformations are assembled into typed closed compositions and evaluated through representation-stable or explicitly representation-dependent diagnostics. Some such characteristics can be emergent relative to a declared local information set, and some can be tied to global topological classes when the required invariance conditions hold. These results deepen the field-theoretic programme without turning compositional or topological structure into an ontological definition of power.

Table 18: Executable checks for cyclic and emergent system structure in a finite GR realization.

Check	Computation	Result or failure indicated
Cycle typing	Verify source, target, and closure identification for each selected cycle	Distinguishes valid cycle return maps from unclosed or noncomposable paths
Return residual	Evaluate Equation 99 or Equation 100	Detects a nontrivial complete-cycle effect on retained content
Base-point covariance	Recompute after cyclically shifting the starting point	Tests the conjugacy relation in Proposition 10.2
Representation covariance	Apply admitted local reparameterizations and Equation 103	Separates substantive invariants from coordinate-dependent diagnostics
Face residual	Evaluate $\mathcal{C}_{\partial f}$ for retained faces	Detects local curvature-like compositional structure
Cycle-fixed structure	Intersect fixed sets or solve $\mathcal{A}_c(a) = a$ on a finite basis	Identifies content preserved across the selected cycle family
Homology profile	Compute boundary matrices, H_1 , and β_1	Separates boundary cycles from nontrivial topological cycle classes
Emergence test	Hold $\mathcal{L}_r$ fixed while comparing S across realizations	Tests diagnostic-relative local underdetermination under Definition 10.3
Resolution check	Repeat under admitted refinement/coarse-graining	Shows whether the systemic diagnostic is stable at the declared representation scale

11 Compositional Dynamics of Multiple GR Power Fields

This section extends the field and operator-algebraic construction from a single GR power-field representation to families of simultaneously retained power constructions. Its role is to determine which aspects of multi-field behavior follow from ordered composition of already-defined local maps and which require additional joint interaction laws. The objective is to define regional overlap structure, typed multi-field compositions, factorization and interaction residuals, common invariant structures, field-coloured cycles, and finite diagnostics that remain calculable without introducing a primitive global state space. The discussion proceeds from indexed field families to overlap geometry, ordered and joint dynamics, representation-dependent interaction measures, common fixed structures, mixed cycle residuals, and higher-order overlap patterns. The method is formal and computational. Terms such as *reinforcement*, *cancellation*, and *interaction* are used only relative to explicitly declared maps or diagnostics.

11.1 Indexed GR Power-Field Families

Let $\mathcal{I}$ be a finite or locally finite index set of GR power constructions retained in one modeled spacetime K . For each $a \in \mathcal{I}$, let $\mathbf{P}^{(a)}$ denote a field realization of the form developed in Section 5. On every region R belonging to the retained regional domain $\mathcal{R}_a$, the field supplies a local datum $p_R^{(a)} \in \mathcal{V}_a(R)$ and a local dynamical realization $\Phi_R^{(a)} := \Phi_R[p_R^{(a)}]$. The multi-field family is stated in Equation 114.

$$\mathbf{P}^{\text{multi}} := \left\{ \mathbf{P}^{(a)} \mid a \in \mathcal{I} \right\}. \quad (114)$$

Equation 114 organizes distinct field carriers under a shared index family. Distinct GR power constructions can use different local value spaces, different constitutive supports, different effect families, and different

dynamical laws. A common vector-space structure, common algebra, or common scalar magnitude is therefore an additional specialization.

For a retained region R , the locally represented index family is given in Equation 115.

$$\mathcal{I}_R := \{a \in \mathcal{I} \mid R \in \mathcal{R}_a\}. \quad (115)$$

Equation 115 records where more than one field description is available. When the representation-level supports of Equation 34 are defined, a narrower active family can be obtained by additionally requiring $R \in \text{Supp}_{\text{rep}}(\mathbf{P}^{(a)})$. The distinction matters because coexistence of two field descriptions on a region does not by itself imply that both have non-neutral data there.

The substantive GR power objects indexed by a also remain distinct from the field family in Equation 114. The multi-field construction supplies simultaneous formal representations while preserving the plurality of the substantive power relations or structures.

11.2 Regional Overlap and Interaction Domains

Multi-field composition can occur only where the required local data and maps are jointly defined or where a declared transport connects their domains. For a finite nonempty subset $J \subseteq \mathcal{I}$, define the regional overlap family in Equation 116.

$$\mathcal{O}_J := \bigcap_{a \in J} \mathcal{R}_a. \quad (116)$$

Equation 116 is representational. A region in $\mathcal{O}_J$ supports the retained descriptions for every field in J , while substantive interaction still requires a dynamical rule linking those descriptions.

Several overlap notions can be retained simultaneously. The constitutive overlap of two substantive power constructions is $\mathcal{S}_{\mathcal{P}^{(a)}} \cap \mathcal{S}_{\mathcal{P}^{(b)}}$ when the support families are comparable. Their effect overlap is $\mathcal{E}_{\mathcal{P}^{(a)}}^\Delta \cap \mathcal{E}_{\mathcal{P}^{(b)}}^\Delta$. Their representational overlap is the intersection of their field domains or, when defined, their representation-level supports. Equality among these overlaps requires a separate model argument.

Higher-order overlap is relevant because a three-field interaction can be defined on a region even when pairwise summaries lose information about the joint configuration. The family $\{\mathcal{O}_J\}_{J \subseteq \mathcal{I}}$ therefore supplies a first combinatorial description of where multi-field dynamics can be posed. Section 11.8 will turn this family into a finite overlap complex without assuming that such a complex is topologically equivalent to the underlying GR spacetime.

11.3 Ordered Multi-Field Composition

The weakest multi-field construction uses only the single-field maps already available. Let R be a retained region and let $\mathbf{a} = (a_1, \dots, a_m)$ be an ordered word with each $a_i \in \mathcal{I}_R$. When the successive source-target types are compatible, the ordered regional composite is defined in Equation 117.

$$\Phi_R^{\mathbf{a}} := \Phi_R^{(a_m)} \circ \dots \circ \Phi_R^{(a_1)}. \quad (117)$$

Equation 117 uses map composition directly and therefore remains meaningful when the different $p_R^{(a)}$ live in heterogeneous carriers. The construction also preserves the historical lesson of Section 6: an earlier field-induced transformation can change the domain on which a later field-induced map is defined.

For two fields a and b , order sensitivity can be measured whenever the two composites are comparable. Let d_{dyn} be a declared discrepancy on the retained map class. The two-field order residual is given in Equation 118.

$$\Delta_R^{a,b} := d_{\text{dyn}}\left(\Phi_R^{(b)} \circ \Phi_R^{(a)}, \Phi_R^{(a)} \circ \Phi_R^{(b)}\right). \quad (118)$$

Equation 118 is representation dependent unless the discrepancy itself is invariant under the admitted reparameterizations. A nonzero value establishes sequence sensitivity within the chosen realization. It does not by itself establish an intrinsic scalar amount of interaction between the substantive GR power relations.

When the maps are linear endomorphisms on a common algebraic carrier, Equation 118 can be supplemented by the commutator $[\alpha_R^{(b)}, \alpha_R^{(a)}]$. In a differentiable common-domain regime, Section 6.5 showed that the leading discrepancy between opposite orderings of small transformations is controlled by the corresponding generator commutator. Multi-field noncommutativity is therefore one calculable mechanism of interaction, while the general typed composition in Equation 117 remains available when commutators are undefined.

11.4 Joint Dynamical Laws and Factorization

Ordered composition assumes that each single-field map remains meaningful when applied in sequence. A stronger model can posit a local law that depends jointly on several field data before any factorization into single-field maps is chosen. For a finite set $J \subseteq \mathcal{I}_R$, define a joint realization map as stated in Equation 119.

$$\mathcal{D}_R^J : \prod_{a \in J} \mathcal{V}_a(R) \longrightarrow \text{Dyn}_K(R), \quad (p_R^{(a)})_{a \in J} \longmapsto \Psi_R[J]. \quad (119)$$

Equation 119 permits context dependence that is absent from independent single-field laws. For example, the local rule associated with field a can be modified by the simultaneous presence of field b , or the pair can activate an admissible transformation unavailable under either field separately.

A joint law is *sequentially factorable* on the retained domain when some ordering of the single-field maps reproduces it under the adopted map equivalence. For a finite J , let $\text{Perm}(J)$ denote the finite set of admissible permutations of the retained field labels; application-specific typing constraints may remove permutations whose ordered composites are undefined. Given a discrepancy d_{dyn} , define the factorization defect in Equation 120.

$$\varepsilon_R^{\text{fact}}(J) := \inf_{\pi \in \text{Perm}(J)} d_{\text{dyn}}(\Psi_R[J], \Phi_R^{\pi(J)}), \quad (120)$$

where $\Phi_R^{\pi(J)}$ denotes the ordered composite generated by the permutation π through Equation 117. Equation 120 is meaningful only when the compared maps have compatible types and the discrepancy is defined. A positive defect identifies a joint effect that the retained single-field maps cannot reproduce through any tested ordering.

Definition 11.1 (Representation-level multi-field interaction). For a declared joint law $\Psi_R[J]$, single-field dynamical family $\{\Phi_R^{(a)}\}_{a \in J}$, comparison class, and discrepancy d_{dyn} , the family exhibits a representation-level multi-field interaction on R when $\varepsilon_R^{\text{fact}}(J) > 0$ or when no admissible sequential factorization has the required source-target type.

Definition 11.1 is intentionally conditional on the retained law and factorization class. Enriching the single-field state of information or enlarging the admissible map family can change the result. The definition therefore identifies a model-relative failure of the specified decomposition.

A differentiable common-carrier specialization makes nonadditivity separately calculable. Suppose $\mathcal{L}_R^{(a)}$ is the generator associated with field a and $\mathcal{L}_R^J$ is the generator of the joint law. The interaction-generator remainder is defined in Equation 121.

$$\Delta \mathcal{L}_R[J] := \mathcal{L}_R^J - \sum_{a \in J} \mathcal{L}_R^{(a)}. \quad (121)$$

Equation 121 separates two effects that are easily conflated. A vanishing $\Delta \mathcal{L}_R[J]$ means that the joint infinitesimal generator is additive under the stated assumptions. Finite sequential dynamics can still be order sensitive when the generators fail to commute. Conversely, a nonzero $\Delta \mathcal{L}_R[J]$ records local nonadditivity even before order effects from exponentiation or repeated composition are considered.

11.5 Observable-Relative Interaction Diagnostics

Terms such as reinforcement and cancellation require a declared observable or response functional. Let $\mathfrak{A}_R$ be a common retained algebra, let ω be a normalized evaluation when such an evaluation is part of the model, and let $x = x^* \in \mathfrak{A}_R$ be a retained self-adjoint observable in a C^* -algebraic specialization. For an algebraic map $\alpha : \mathfrak{A}_R \rightarrow \mathfrak{A}_R$, define the response in Equation 122.

$$\delta_{\omega,x}(\alpha) := \omega(\alpha(x)) - \omega(x). \quad (122)$$

Equation 122 produces a scalar only after the evaluation and observable have been selected. It is therefore a diagnostic of one represented consequence under the selected evaluation and observable.

Let $\alpha^{(a)}$ and $\alpha^{(b)}$ be the algebraic components of the corresponding single-field dynamics, and let $\alpha^{a,b}$ be the algebraic component of the declared joint law $\Psi_R[\{a, b\}]$, all represented as comparable endomaps of $\mathfrak{A}_R$ for this specialization. The interaction-response residual is given in Equation 123.

$$\Sigma_{\omega,x}^{a,b} := \delta_{\omega,x}(\alpha^{a,b}) - \delta_{\omega,x}(\alpha^{(a)}) - \delta_{\omega,x}(\alpha^{(b)}). \quad (123)$$

A positive or negative $\Sigma_{\omega,x}^{a,b}$ can be described as reinforcing or offsetting only relative to the orientation and interpretation of x . Another observable, evaluation, or representation can yield a different sign. The diagnostic therefore supports local empirical interpretation without restoring a primitive total magnitude of power.

Order effects can likewise be projected onto one observable. Define the observable order spread in Equation 124.

$$\Omega_{\omega,x}^{a,b} := \left| \delta_{\omega,x}(\alpha^{(b)} \circ \alpha^{(a)}) - \delta_{\omega,x}(\alpha^{(a)} \circ \alpha^{(b)}) \right|. \quad (124)$$

Equation 124 is directly estimable in a finite calibrated model. It also illustrates why an interaction can remain substantively important even when an aggregate scalar response is small: two opposite orderings can have equal magnitudes and different directions or can cancel on one observable while remaining distinct on another.

11.6 Common Invariants and Generated Transformation Structure

A family of fields can constrain a system through the transformation structure generated collectively by their local maps. Suppose first that every retained algebraic map $\alpha_R^{(a)}$ is a unital $*$ -endomorphism of a common unital $*$ -algebra $\mathfrak{A}_R$. The multi-field transformation family is defined in Equation 125.

$$\mathfrak{G}_{\text{multi}}(R) := \left\langle \alpha_R^{(a)} \mid a \in \mathcal{I}_R \right\rangle_{\circ}. \quad (125)$$

Equation 125 is a semigroup, or a monoid when the identity is included, and becomes a group when all retained maps are invertible and inverses are admitted.

Because this subsection has assumed unital $*$ -endomorphisms of one common algebra, the algebraic content fixed by every retained field map is given in Equation 126. For weaker nonmultiplicative map classes, the same intersection is only a common fixed set unless closure under the algebraic operations is proved separately.

$$\mathfrak{A}_R^{\text{multi}} := \bigcap_{a \in \mathcal{I}_R} \text{FixAlg}(\alpha_R^{(a)}). \quad (126)$$

Every element of Equation 126 is also fixed by every finite composition generated by Equation 125. The common fixed algebra can therefore be much smaller than the fixed algebra associated with any one field. This provides a calculable form of collective constraint without requiring the field values themselves to be added.

A finite inner-automorphism specialization gives a useful interpretation. Let $\mathfrak{A}_R = M_n(\mathbb{C})$ and let $\alpha_R^{(a)} = \text{Ad}_{U_a}$ with invertible unitary U_a . Then Equation 127 holds.

$$\mathfrak{A}_R^{\text{multi}} = \{x \in M_n(\mathbb{C}) \mid U_a x = x U_a \text{ for every } a \in \mathcal{I}_R\}. \quad (127)$$

Equation 127 identifies the common fixed algebra with the commutant of the retained implementing family. Its dimension can be computed directly in a finite matrix realization. A decrease in this dimension after adding another field records loss of jointly preserved algebraic content under the chosen representation.

When the fields act between different algebras, Equation 126 need not be defined globally. The appropriate replacement is a typed diagram of inter-algebra maps together with invariant subobjects or equalizers defined only where the source-target structure permits them. The algebra-first formulation therefore retains multi-field comparison without requiring an artificial common carrier.

11.7 Field-Coloured Cycles and Collective Residuals

Section 10 associated return maps with closed compositional paths. Multiple power fields add a second layer of information: each interface can be acted on by a selected field or joint field law. Let $c = (e_1, \dots, e_m)$ be a retained closed path and let $\mathbf{a} = (a_1, \dots, a_m)$ assign a field index to each step. The field-coloured cycle map is stated in Equation 128.

$$\mathcal{C}_{c,\mathbf{a}} := \kappa_c \circ \Phi_{e_m}^{(a_m)} \circ \dots \circ \Phi_{e_1}^{(a_1)}, \quad (128)$$

where κ_c is the closure identification defined for the retained cycle. Equation 128 shows that the same geometric cycle can support different systemic dynamics under different field assignments.

A collective residual can appear even when simple single-field backtracking cycles are trivial. Let U and V be invertible finite representatives of two fields. The individual backtracking products $U^{-1}U$ and $V^{-1}V$ are identities, while the mixed four-step cycle gives the group-commutator representative in Equation 129.

$$C(U, V) := V^{-1}U^{-1}VU. \quad (129)$$

Equation 129 equals the identity when the two representatives commute and can remain nontrivial even though each single-field backtracking cycle is trivial. The residual is therefore genuinely compositional relative to the selected local maps.

When the selected field actions possess algebraic components $\alpha_{e_i}^{(a_i)}$ and the cycle has an algebraic closure map κ_c , the corresponding field-coloured algebraic return map is derived from Equation 98 by retaining the field labels:

$$\mathcal{A}_{c,\mathbf{a}} := \kappa_c \circ \alpha_{e_m}^{(a_m)} \circ \dots \circ \alpha_{e_1}^{(a_1)}. \quad (130)$$

The cycle-generated structure of Section 10.5 can now be refined by field words. Let $\mathcal{C}_{\text{col}}(K)$ be a retained family of admissible field-coloured cycles for which Equation 130 is well typed. The generated family is stated in Equation 131.

$$\mathfrak{G}_{\text{col}}(K) := \langle \mathcal{A}_{c,\mathbf{a}} \mid (c, \mathbf{a}) \in \mathcal{C}_{\text{col}}(K) \rangle_{\circ}. \quad (131)$$

Equation 131 retains both geometry and field assignment. Two systems with the same uncoloured cycle structure can therefore differ at the multi-field dynamical level.

11.8 Multi-Field Overlap Complex

Higher-order overlap can be summarized by a finite combinatorial object. Assume for this subsection that $\mathcal{I}$ is finite and that each field has a retained representational domain $\mathcal{R}_a$. Define the *multi-field overlap complex* $\mathcal{N}_{\mathbf{P}}$ to have one vertex for each $a \in \mathcal{I}$ and a simplex on a finite nonempty set $J \subseteq \mathcal{I}$ whenever Equation 132 holds.

$$J \in \mathcal{N}_{\mathbf{P}} \iff \mathcal{O}_J \neq \emptyset. \quad (132)$$

Because nonempty common overlap for J implies nonempty common overlap for every nonempty subset of J , Equation 132 is downward closed and therefore defines an abstract simplicial complex on the retained index set. It records pairwise and higher-order coexistence of field representations. A triangle in $\mathcal{N}_{\mathbf{P}}$ means that three fields share at least one retained region; three pairwise edges without the filled triangle mean that every pair overlaps somewhere while no retained region supports all three simultaneously. Figure 6 displays this distinction.

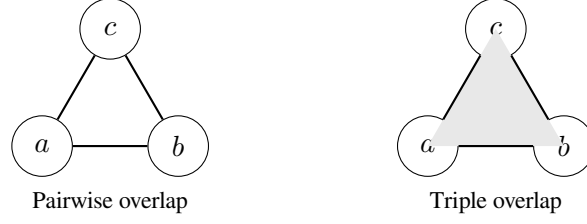


Figure 6: Two overlap-complex patterns for three retained GR power fields. The left complex records three pairwise overlaps without a common three-field region. The filled simplex on the right records at least one retained region belonging to all three field domains.

Within the present formalism, $\mathcal{N}_{\mathbf{P}}$ is defined solely as a calculable summary of higher-order interaction opportunity. A claim of topological equivalence with the underlying GR spacetime or with the union of field supports would require additional assumptions on the regional cover. Its incidence structure, connected components, cycles, and homology can be used as diagnostics of how field domains overlap at the chosen resolution.

A structural transformation can change $\mathcal{N}_{\mathbf{P}}$ even when each individual field law is retained. For example, a historical or future topology change can create a new region in which three formerly pairwise fields coexist, filling a previously absent simplex. Conversely, loss of a shared region can remove a higher-order interaction domain without eliminating any individual field. Multi-field composition therefore has a topological aspect at the level of interaction opportunity as well as at the level of the underlying GR carrier.

11.9 Finite Two-Field Specialization

A two-dimensional matrix realization demonstrates that trivial single-field backtracking can coexist with a nontrivial mixed cycle. Let σ_x and σ_z be the matrices in Equation 133.

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (133)$$

Define the unitary representatives in Equation 134.

$$U := \frac{1}{\sqrt{2}}(I - i\sigma_x), \quad V := \frac{1}{\sqrt{2}}(I - i\sigma_z). \quad (134)$$

Each field has a trivial two-step backtracking cycle because $U^{-1}U = V^{-1}V = I$. Their mixed cycle from Equation 129 is stated explicitly in Equation 135.

$$C(U, V) = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \end{pmatrix}. \quad (135)$$

Equation 135 has trace 1 and determinant 1, hence eigenvalues $e^{i\pi/3}$ and $e^{-i\pi/3}$. The corresponding inner automorphism $\text{Ad}_{C(U,V)}$ therefore has a nontrivial action even though the two selected single-field backtracking cycles are identities. This finite example isolates a collective cycle residual generated by the relative ordering of two field representations.

The same matrices also illustrate common invariant reduction. The fixed algebra of Ad_U contains matrices commuting with U , and the fixed algebra of Ad_V contains matrices commuting with V . Because σ_x and σ_z generate noncommuting directions in $M_2(\mathbb{C})$, the matrices commuting with both U and V are the scalar multiples of I . Thus the common fixed algebra in Equation 127 is one-dimensional even though each individual fixed algebra is larger. The example has the epistemic status of a finite operator-algebraic specialization under the paper's ontological suspension.

11.10 Finite Verification of Multi-Field Dynamics

Finite realizations permit direct checks of the distinctions introduced in this section. Table 19 summarizes a workflow that separates overlap, composition, interaction, invariant, and cycle claims.

These checks support logical and computational verification without presupposing that every field can be expressed in one common coordinate system. Heterogeneous carriers can be retained through typed compositions, while common-carrier specializations allow matrix norms, commutators, spectra, fixed spaces, and response functionals to be computed directly. A failure of one check should be interpreted locally. For example, a nonzero factorization defect supports only a model-relative failure of the tested decompositions, and a disconnected overlap complex can change under a different regionalization.

Empirical use requires an additional mapping from observed institutions, relations, actions, or historical structures to the retained fields and maps. The formalism does not decide that encoding. Its contribution is to distinguish mechanisms that would otherwise be collapsed into one phrase such as “combined power”: coexistence, ordered action, noncommutative order effects, a joint non-factorizable law, observable-relative reinforcement or offset, common invariant reduction, and collective cycle residuals.

11.11 Interface for Balance Dynamics

The multi-field construction changes the formal question posed by balance analysis. Balance can be represented at the level of several GR power fields whose constitutive supports, effect domains, local maps, and cycle contributions partially overlap, rather than being reduced to equality between scalar magnitudes assigned to actors. Counteraction can then occur through composition, modification of admissible transformations, common invariant structure, or reconfiguration of the interaction-overlap complex.

The distinction between sequential composition and joint interaction is especially important for international applications. Two countervailing power structures can produce small net change in one observable while remaining strongly noncommutative, while another pair can share an approximately additive infinitesimal generator and still generate a nontrivial mixed cycle over repeated interaction. A balance diagnosis therefore requires a declared level of analysis: local response, ordered map composition, joint factorization, cycle residual, fixed structure, or system topology.

Section 12 will use this architecture to develop a more detailed balance dynamics in international relations. Its task is to determine what balance means once multiple power fields are represented as calculable local and systemic transformations, and to identify which balancing claims can be formulated as counter-deformation, invariant retention, residual suppression, or structural reconfiguration within the present field-theoretic framework.

Table 19: Executable checks for multiple GR power fields in a finite realization.

Check	Computation	Result or failure indicated
Regional overlap	Compute $\mathcal{O}_J$ for retained field families	Identifies where joint dynamics can be posed at the selected resolution
Ordered typing	Verify each source-target chain in Equation 117	Separates valid sequences from forced common-domain assumptions
Order residual	Evaluate Equation 118 or a commutator in a common linear realization	Detects sequence sensitivity of retained field actions
Factorization defect	Evaluate Equation 120 for all tested permutations	Detects a joint law not reproduced by the retained single-field sequences
Interaction generator	Evaluate Equation 121 when differentiable generators exist	Separates infinitesimal nonadditivity from finite-order noncommutativity
Observable response	Evaluate Equations 123 and 124	Tests reinforcement, offset, and order effects relative to a declared observable
Common fixed structure	Solve Equation 126 on a finite basis	Identifies algebraic content preserved under every retained field map
Field-coloured cycle	Compute Equation 128 for selected field words	Detects collective residuals dependent on both geometry and field assignment
Overlap complex	Build $\mathcal{N}_{\mathbf{p}}$ from Equation 132	Records pairwise and higher-order interaction opportunity at the chosen scale
Resolution check	Repeat overlap and interaction diagnostics after admitted refinement/coarse-graining	Tests whether multi-field conclusions persist across representation scales

12 Balance Dynamics in International Relations

This section applies the multi-field formalism to balance dynamics in international relations. Its role is to move beyond the earlier GR reconstruction of balance as a general counter-deformation and to identify consequences that become visible only after local field realizations, ordered composition, cycle residuals, and higher-order overlap have been made explicit. The objective is therefore formal and comparative. The section defines a balance corridor relative to a declared international-political model, derives typed counter-deformation criteria, distinguishes first-order compensation from higher-order compositional residuals, introduces order-robustness and cycle-balance diagnostics, and shows how balancing can reconfigure the overlap structure among several GR power fields. The resulting construction remains a structural formalization rather than an empirical law of international politics.

The disciplinary interface requires restraint because balance-of-power theory contains several distinct mechanisms. Structural realism treats the distribution of capabilities and recurrent balancing under anarchy as central systemic objects [12]. Balance-of-threat theory shifts the relevant comparison toward perceived threat and alliance response [13]. Reviews of the balance-of-power tradition also distinguish automatic, semi-automatic, and more deliberately managed versions of balancing and emphasize persistent disagreement over the concept's operation and empirical scope [14]. The present section uses these theories as comparison points. It does not identify their concepts with the GR constructions developed below.

12.1 Selection Interfaces for International-Power Modeling

A field-theoretic representation requires a rule for deciding which relational structures enter the retained international-political model. Capability and threat assessments can serve this role without being identified with power itself. Let K^{IR} be a history-bearing GR spacetime representation of the international configuration. Let c denote a declared empirical context, such as a period, geographical scope, evidential source set, or decision setting, so that every context-indexed readout below has an explicit interpretation. A context-dependent capability readout is written in Equation 136.

$$r_c^{\text{cap}} : K^{\text{IR}} \longrightarrow \mathcal{X}_c^{\text{cap}}. \quad (136)$$

Equation 136 may retain military, economic, technological, demographic, institutional, or other variables selected by the application. Its codomain can be multidimensional. A scalar aggregation requires an additional model-specific rule.

A threat-oriented interface adds relational and interpretive information. Equation 137 represents this step abstractly.

$$r_c^{\text{thr}} : (K^{\text{IR}}, r_c^{\text{cap}}, \mathcal{H}_c, \mathcal{I}_c) \longrightarrow \mathcal{X}_c^{\text{thr}}, \quad (137)$$

where $\mathcal{H}_c$ records the historical information retained by the model and $\mathcal{I}_c$ records additional interpreted relations such as proximity, declared commitments, or intention-sensitive indicators where the empirical design supports them. Equation 137 is a selection and interpretation interface. It does not imply that threat constitutes a primitive GR field variable.

The retained power-field family is consequently written as $\mathbf{P}^{\text{IR}} = \{\mathbf{P}^{(a)}\}_{a \in \mathcal{I}}$, where the index set $\mathcal{I}$ is determined by the declared modeling problem rather than by a universal list of states or capabilities. Capability and threat readouts can influence which fields are retained, which regions are compared, and which perturbations are treated as substantively relevant. The field-theoretic construction then begins from the relational deformations associated with those retained objects.

Table 20 states the interpretive boundary used throughout the section. In particular, the GR construction does not require equal capability readouts for a balance condition and does not infer balance from a single alliance relation.

12.2 Counter-Deformation Relative to a Balance Corridor

A balance condition requires a reference class against which a deformation can be assessed. Let R be a retained international-political region for which the relevant local maps have a common comparison domain. Let $\text{Dyn}_{K^{\text{IR}}}(R)$ be the typed admissible map class inherited from Section 6.1, restricted here to a common comparison domain, and equip the retained comparison class with a distance or discrepancy functional d_R when such a quantity is available. The *balance corridor* $\mathfrak{B}_R$ is a nonempty model-declared subset of $\text{Dyn}_{K^{\text{IR}}}(R)$ containing the transformations regarded as balance-compatible for the local problem.

The distance from an admissible map Θ to the corridor is stated in Equation 138.

$$\delta_R^{\text{bal}}(\Theta) := \inf_{B \in \mathfrak{B}_R} d_R(\Theta, B). \quad (138)$$

Equation 138 makes the balance criterion explicit. The corridor can be a singleton containing a baseline identity map in a highly simplified model, a family of transformations preserving declared invariants, or a broader region satisfying empirical or normative-independent stability criteria. The present formalism does not determine the corridor independently of the research design.

Let a denote a focal field and let J be a finite family of responding fields. Let $\Pi(J) \subseteq \text{Perm}(J)$ denote the orderings for which the retained response maps compose with the required source-target types. In a common-carrier specialization, write $\Phi_R^{(a)}$ for the focal map. For an admissible ordering $\pi \in \Pi(J)$, define the ordered response word by $\Phi_R^{J, \pi} := \Phi_R^{\pi(J)}$ using Equation 117. The combined transformation is defined in Equation 139.

$$\Theta_R^{(a; J, \pi)} := \Phi_R^{J, \pi} \circ \Phi_R^{(a)}. \quad (139)$$

Table 20: International-relations concepts and the limited formal roles assigned to them in the present GR construction. The correspondences are modeling interfaces rather than theoretical equivalences.

IR concept	GR formal interface	Analytical use in this section
Capability distribution	Context-dependent readout r_c^{cap}	Selects coarse observable information without defining power as a scalar possession
Threat assessment	Context-dependent readout r_c^{thr}	Selects relationally relevant response targets and comparison domains
Balancing action	Typed counter-deformation map or composition word	Tests whether a response moves a declared dynamical object toward a balance corridor
Alliance or coordinated response	Joint or ordered multi-field dynamics	Allows factorization, sequence, and noncommutative effects to be examined
System balance	Membership in, or proximity to, a declared balance corridor together with systemic diagnostics	Separates local compensation from order robustness, cycle residuals, and overlap structure

For typed inter-algebra dynamics, Equation 139 is replaced by the corresponding composable diagram and the balance corridor is defined in the appropriate source-target hom-set or comparison class.

The balancing gain associated with the response word is stated in Equation 140.

$$G_R^{\text{bal}}(a; J, \pi) := \delta_R^{\text{bal}}\left(\Phi_R^{(a)}\right) - \delta_R^{\text{bal}}\left(\Theta_R^{(a; J, \pi)}\right). \quad (140)$$

A positive value in Equation 140 means that the declared response moves the retained dynamical object closer to the balance corridor under the selected metric. A zero or negative value records absence of improvement under that specification. The sign therefore depends on the declared corridor, distance, region, and ordering.

Definition 12.1 (GR counter-balancing response). For a declared region R , balance corridor $\mathfrak{B}_R$, discrepancy d_R , focal field a , response family J , and ordering π , the response is *counter-balancing relative to the specification* when $G_R^{\text{bal}}(a; J, \pi) > 0$.

Definition 12.1 makes balance explicitly model-relative. It also permits overcompensation: a response can reduce the initial deviation while still overshooting into another part of the dynamical space. A corridor containing several admissible configurations handles such cases more naturally than equality to one scalar benchmark.

Figure 7 illustrates this geometry schematically. The same focal deformation is followed by two orderings of the same response family. One ordering enters the declared balance corridor while another terminates outside it.

12.3 Order Sensitivity of Balancing Sequences

The multi-field formalism makes the order of balancing operations an explicit object of analysis. Using the admissible ordering set $\Pi(J)$ defined above, the order spread of the balance defect is defined in Equation 141.

$$\Omega_R^{\text{bal}}(a; J) := \sup_{\pi, \pi' \in \Pi(J)} \left| \delta_R^{\text{bal}}\left(\Theta_R^{(a; J, \pi)}\right) - \delta_R^{\text{bal}}\left(\Theta_R^{(a; J, \pi')}\right) \right|. \quad (141)$$

Equation 141 vanishes when the selected corridor-distance diagnostic is invariant across admissible response orderings. A small value does not establish that the underlying maps commute because distinct transformations can have equal distances to the corridor.

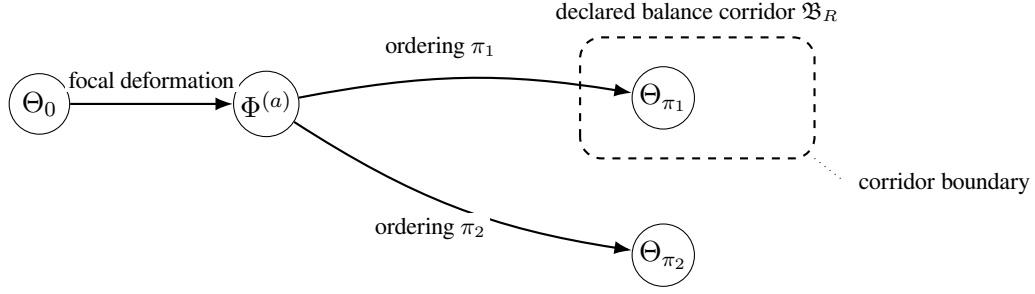


Figure 7: Schematic balance corridor and order-sensitive counter-deformation. The geometry is illustrative; the actual dynamical space and discrepancy are application-specific.

A stronger worst-order diagnostic is stated in Equation 142.

$$D_R^{\text{rob}}(a; J) := \sup_{\pi \in \Pi(J)} \delta_R^{\text{bal}} \left(\Theta_R^{(a; J, \pi)} \right). \quad (142)$$

Equation 142 is useful when the ordering of diplomatic, military, economic, institutional, or other responses cannot be controlled precisely. A response family can have a favorable best ordering while remaining fragile under plausible alternatives.

The distinction between first-order compensation and higher-order residuals becomes explicit in a differentiable finite specialization. To identify the second-order coefficient rather than hide unknown terms inside $O(\epsilon^2)$, assume for this illustrative regime that the three local maps are generated by bounded matrices or bounded operators through exponentials, as stated in Equation 143.

$$\Phi_\epsilon^{(a)} = e^{\epsilon A}, \quad \Phi_\epsilon^{(b)} = e^{\epsilon B}, \quad \Phi_\epsilon^{(c)} = e^{\epsilon C}. \quad (143)$$

Assume that the linearized deformations satisfy the first-order compensation condition in Equation 144.

$$A + B + C = 0. \quad (144)$$

The ordered product with a acting first, followed by b and then c , has the expansion stated in Equation 145.

$$\Phi_\epsilon^{(c)} \Phi_\epsilon^{(b)} \Phi_\epsilon^{(a)} = I + \epsilon^2 \mathcal{R}_{cba}^{(2)} + O(\epsilon^3), \quad \mathcal{R}_{cba}^{(2)} := \frac{1}{2}(A^2 + B^2 + C^2) + BA + CA + CB. \quad (145)$$

Under Equation 144, the coefficient can equivalently be written as $\mathcal{R}_{cba}^{(2)} = \frac{1}{2}([B, A] + [C, A] + [C, B])$. Equation 145 therefore shows that cancellation of the first-order generators can leave a generally nonzero, composition-dependent second-order residual.

Proposition 12.2 (First-order balance with order-sensitive residual). *Under Equations 143 and 144, interchanging the order of the two responding maps changes the total transformation at second order according to Equation 146.*

$$\Phi_\epsilon^{(c)} \Phi_\epsilon^{(b)} \Phi_\epsilon^{(a)} - \Phi_\epsilon^{(b)} \Phi_\epsilon^{(c)} \Phi_\epsilon^{(a)} = \epsilon^2 [C, B] + O(\epsilon^3). \quad (146)$$

Proposition 12.2 provides a concrete distinction unavailable to scalar equalization. A response can satisfy a linearized balance condition while leaving a finite compositional residual whose magnitude and direction depend on ordering. In the international-political application, the proposition does not assert that actual policy sequences obey a matrix expansion. It identifies a calculable mechanism available to a calibrated finite model.

12.4 Systemic Balance under Cyclic Composition

Pairwise or local counter-deformation does not by itself establish a systemic balance condition. A multi-actor configuration can contain closed sequences whose cumulative effect remains nontrivial even when selected bilateral diagnostics are small. Let $(c, \mathbf{a})$ be a field-coloured closed cycle retained from Section 11.7. When the cycle closes on a common carrier, define the cycle-balance residual in Equation 147.

$$\eta_{c,\mathbf{a}}^{\text{bal}} := d_c(\mathcal{C}_{c,\mathbf{a}}, \text{id}), \quad (147)$$

where d_c is a declared discrepancy on the return-map space. A corresponding distance to a cycle-level balance corridor can be used when identity return is too restrictive.

Equation 147 yields a direct systemic diagnostic. Consider a triangle of actors or subsystems in which each selected bilateral response appears locally counter-balancing. Noncommutative or history-conditioned composition can still produce $\eta_{c,\mathbf{a}}^{\text{bal}} > 0$. The residual records a circulatory or compositional asymmetry relative to the chosen cycle representation. It does not imply a unique substantive interpretation such as domination, instability, or conflict.

The common fixed structure developed in Section 11.6 gives a complementary invariant diagnostic. Let $\mathfrak{G}_R^{\text{bal}}$ be the model-declared subfamily of the multi-field transformation structure $\mathfrak{G}_{\text{multi}}(R)$ from Section 11 generated under composition by the focal maps and the counter-deformation maps retained for the balance diagnosis. When these maps are unital $*$ -endomorphisms of one common regional algebra $\mathfrak{A}_R$, Equation 148 defines their jointly preserved algebraic content.

$$\mathfrak{A}_R^{\text{bal}} := \bigcap_{\Theta \in \mathfrak{G}_R^{\text{bal}}} \text{FixAlg}(\Theta). \quad (148)$$

For weaker map classes, the same intersection is interpreted only as a common fixed set or subspace, according to the available linear structure. Changes in $\dim \mathfrak{A}_R^{\text{bal}}$ in a finite endomorphic model can record changes in jointly preserved algebraic content even when scalar balance indicators remain similar.

The cycle and invariant diagnostics therefore separate two questions. Equation 147 asks whether a closed compositional process returns approximately to its declared reference. Equation 148 asks which algebraic content remains invariant under the generated balance dynamics. Neither quantity alone defines political stability.

12.5 Overlap Structure of Higher-Order Alignment

Balancing can change the architecture through which several power fields coexist. Let $\mathcal{N}_{\mathbf{P}}$ be the multi-field overlap complex from Section 11.8. A balancing episode can alter representational domains, coordination interfaces, or admissible joint laws, producing a new complex $\mathcal{N}'_{\mathbf{P}}$. The change can be summarized through ordinary topological diagnostics of the finite overlap complex.

For each retained homological degree q , define the overlap-topology change in Equation 149.

$$\Delta\beta_q^{\text{ov}} := \beta_q(\mathcal{N}'_{\mathbf{P}}) - \beta_q(\mathcal{N}_{\mathbf{P}}). \quad (149)$$

Equation 149 does not measure military or political strength. It records a change in the combinatorial topology of retained field overlap. For example, three pairwise overlaps can become one genuine triple-overlap simplex when a common coordination region becomes available. Conversely, loss of a higher-order overlap can fragment a previously shared response domain while preserving all three actor labels.

This perspective gives alliances, institutions, mediation channels, and coordinated responses a possible formal role beyond pairwise edge weights. A new arrangement can change which collections of fields admit a joint regional law. The resulting systemic effect can therefore appear as a change in higher-order overlap even when the local field magnitudes or capability readouts change little.

The overlap complex remains representation-dependent. A different empirical threshold for what counts as a shared field domain can alter its simplices. Persistent or multiscale constructions may later reduce sensitivity to one threshold, but the present paper retains the simpler finite object and requires the thresholding rule to be reported.

12.6 Robustness Conditions for Balance Classification

The preceding diagnostics identify several distinct ways in which a configuration can appear balanced under one coarse description while remaining dynamically unbalanced under another. Table 21 summarizes these mechanisms.

The table supports a broader distinction between *local balance*, *compositional balance*, and *structural balance* within the present formalism. These labels denote analytical levels rather than a new taxonomy of realist theories. Local balance concerns a declared region or bilateral comparison. Compositional balance concerns ordered multi-field and cycle behavior. Structural balance concerns invariant and overlap-level properties across the retained system.

A robust balance claim should therefore state its level. In a finite implementation, one may require a tuple of tolerances as in Equation 150.

$$\mathfrak{T}_{\text{bal}} := (\tau_{\text{loc}}, \tau_{\text{ord}}, \tau_{\text{cyc}}, \tau_{\text{ov}}), \quad (150)$$

where the components specify admissible local corridor distance, order spread, cycle residual, and overlap-structure change for the empirical problem. The tuple does not create a universal balance index. It records the declared tolerances required for reproducible classification.

12.7 Finite Verification of Balance Dynamics

The balance construction is intended to support executable checks in finite models. Table 22 lists a minimal verification programme.

A finite empirical application would require an additional calibration layer connecting observations to the local algebras, field realizations, dynamical maps, and discrepancy functions. The present section supplies the formal objects that such a calibration would need to estimate. It does not claim that currently available international-political data uniquely determine those objects.

12.8 Interface for Near-Critical Power Configurations

Balance becomes especially interesting when small changes in field composition or ordering produce large changes in the diagnostics developed above. A configuration can lie close to the declared balance corridor while exhibiting high order sensitivity, rapidly changing cycle residuals, shrinking invariant structure, or topological change in the overlap complex. Such a configuration differs from a deeply stable equilibrium in which nearby perturbations decay with little effect on the retained system characteristics.

The Criticality Hypothesis of International Politics proposes, conditionally, that some international relational systems may exhibit regimes combining persistence with substantial susceptibility and reconfigurability [7]. The field-theoretic balance construction provides several candidate quantities through which that conjecture can be sharpened: derivatives of corridor distance with respect to field perturbations, sensitivity of cycle residuals, spectral changes in represented generators, changes in common invariant structures, and topology changes in the overlap complex.

The next section treats balance and criticality as distinct dynamical properties. It asks when a balance-compatible multi-field configuration approaches a regime in which small perturbations can reorganize compositional, spectral, or topological characteristics. The balance diagnostics developed in the present section provide the baseline against which that susceptibility can be defined.

Table 21: Formal failure modes for balance diagnostics in the GR field-theoretic representation. Each row describes a mismatch between a coarse balance indicator and a more demanding compositional test.

Formal pattern	Diagnostic signature	Interpretive caution
Scalar compensation with compositional residual	First-order or aggregate response is small while Equation 145 remains nonzero	Aggregate equality can conceal sequence-sensitive residual structure
Order-fragile response	Large Ω_R^{bal} or D_R^{rob}	Favorable balance depends strongly on the response sequence
Local balance with cycle residual	Small local defects coexist with $\eta_{c,a}^{\text{bal}} > 0$	Pairwise compensation does not establish systemic return
Overlap lock-in or fragmentation	Significant $\Delta\beta_q^{\text{ov}}$ or simplex changes in $\mathcal{N}_{\mathbf{P}}$	Balance responses can reorganize coordination structure independently of capability equality
Invariant contraction	Decrease in the retained common fixed structure	The jointly preserved content of the system can shrink while conventional balance indicators remain stable

Table 22: Finite checks for the balance-dynamics construction. The tests concern internal model behavior and do not by themselves establish empirical validity.

Check	Computation	Interpretation
Corridor distance	Evaluate $\delta_R^{\text{bal}}(\Theta)$ before and after a response	Tests whether the declared response reduces the selected balance defect
Ordering test	Enumerate admissible $\pi \in \Pi(J)$ and compute Ω_R^{bal} and D_R^{rob}	Detects sequence dependence and worst-order fragility
First-order cancellation	Verify $A + B + C = 0$ and compute the second-order residual	Separates linearized compensation from higher-order compositional effects
Cycle test	Compute $\mathcal{C}_{c,a}$ and $\eta_{c,a}^{\text{bal}}$ on retained closed paths	Detects systemic residuals invisible to selected pairwise checks
Invariant test	Compute the common fixed algebra or another declared invariant family	Measures jointly preserved structure under balance dynamics
Overlap test	Compare simplices, Betti numbers, and selected connectivity measures of $\mathcal{N}_{\mathbf{P}}$ and $\mathcal{N}'_{\mathbf{P}}$	Detects higher-order reconfiguration of field coexistence
Resolution test	Repeat the diagnostics under declared coarse-graining or threshold changes	Identifies representation-sensitive balance classifications

13 Near-Critical Power Configurations

This section develops a model-relative account of near-criticality for the GR power-field formalism. Its role is to connect the balance and compositional constructions of Sections 12 and 11 to explicit diagnostics of susceptibility, recovery, residual amplification, and structural transition. The objective is formal rather than predictive. The section defines perturbation families and finite-scale susceptibilities, introduces spectral recovery and resolvent diagnostics for finite operator realizations, extends susceptibility analysis to order and cycle observables, defines structural margins for discrete invariant and topology changes, and combines these objects into a declared criticality specification. A finite specialization then shows how a configuration can remain compatible with a selected balance criterion while becoming increasingly sensitive to small

perturbations. The Criticality Hypothesis of International Politics enters as a conditional application of this machinery rather than as an empirical premise of the formalism [7].

Critical transitions are studied in several areas of complex-systems research through mechanisms such as slowing recovery, increasing variance or autocorrelation, and abrupt regime change under parameter variation [8]. Those mechanisms motivate several diagnostics below, while the present construction does not assume that one warning signal, one bifurcation class, or one physical model transfers universally to international or social systems. Within the present paper, *near-criticality* therefore refers to a declared relation among a GR configuration, an admissible perturbation family, and a set of formal diagnostics whose thresholds and interpretation are fixed by the modeling problem.

13.1 Criticality as a Model-Relative Dynamical Property

Let $\mathcal{X}_{\text{GR}}$ denote a class of admissible finite or quasi-local GR model configurations. An element $X \in \mathcal{X}_{\text{GR}}$ may contain a history-bearing complex, a family of local algebras, one or more GR power-field realizations, their dynamical maps, and any derived overlap or cycle structures required by the local analysis. The notation X packages these objects for perturbation analysis. It does not assert that they form a complete ontological state of the underlying social or political world.

For each X , let Δ_X be a declared family of admissible perturbations. A perturbation $\delta \in \Delta_X$ has a model-specific magnitude $\|\delta\|_X \geq 0$ and produces a perturbed configuration $X[\delta]$ whenever the perturbation is admissible. The family may contain changes in local field values, interaction parameters, ordering constraints, algebraic maps, edge or cell weights, overlap thresholds, or other retained model quantities. Different empirical applications can therefore use different perturbation geometries.

A criticality claim requires at least three declared ingredients: the configuration family, the admissible perturbations, and the diagnostics used to assess response. This dependence is summarized by the criticality specification in Equation 151.

$$\mathfrak{S}_{\text{crit}} := (\mathcal{X}_{\text{GR}}, \{\Delta_X\}_{X \in \mathcal{X}_{\text{GR}}}, \mathfrak{Q}_{\text{crit}}, \mathfrak{T}_{\text{crit}}), \quad (151)$$

where $\mathfrak{Q}_{\text{crit}}$ is the retained family of continuous and structural diagnostics and $\mathfrak{T}_{\text{crit}}$ contains their declared thresholds, scales, and decision rules. Equation 151 makes the classification reproducible relative to the model. A change in $\mathfrak{S}_{\text{crit}}$ can change the classification.

This formulation separates three analytical objects. A *critical set* is a subset of the declared model space at which a selected regularity, stability, or structural condition changes. A *near-critical configuration* lies within a declared perturbation scale of such behavior according to $\mathfrak{S}_{\text{crit}}$. A *critical transition* is a transformation between configurations that crosses the selected condition. These notions remain mechanism-specific. Spectral softening, abrupt topology change, and strong cycle amplification can therefore represent different routes to near-critical behavior.

13.2 Perturbation Families and Finite-Scale Susceptibility

A continuous diagnostic should record how strongly an observable changes under admissible perturbations. Let $O : \mathcal{X}_{\text{GR}} \rightarrow \mathcal{Y}_O$ be a declared diagnostic with discrepancy d_O on its codomain. The finite-scale susceptibility of O at X is defined in Equation 152.

$$\chi_O(X; \varepsilon) := \sup_{\substack{\delta \in \Delta_X \\ 0 < \|\delta\|_X \leq \varepsilon}} \frac{d_O(O(X[\delta]), O(X))}{\|\delta\|_X}. \quad (152)$$

Equation 152 is defined when the admissible perturbation set in the supremum is nonempty and the displayed discrepancies are finite; otherwise the susceptibility is left undefined for that scale. It is defined without assuming differentiability. In a finite smooth specialization, if O is Fréchet differentiable with respect to the chosen perturbation coordinates, the small-scale limit is bounded by the induced derivative norm. The finite-scale definition remains useful when the model contains thresholds, piecewise maps, changing overlap complexes, or other nonsmooth structures.

The balance formalism supplies several immediate choices for O . The local corridor distance δ_R^{bal} , the order spread Ω_R^{bal} , the robust balance defect D_R^{rob} , and a cycle residual $\eta_{c,a}^{\text{bal}}$ can each be inserted into Equation 152. A configuration can therefore remain close to a balance corridor while the derivative or finite-scale response of one of these diagnostics grows substantially.

For a vector of retained diagnostics $\mathbf{O} = (O_1, \dots, O_m)$, the present paper avoids collapsing the components into one universal scalar. Instead, the model records the susceptibility family in Equation 153.

$$\chi(X; \varepsilon) := \left\{ \chi_{O_j}(X; \varepsilon) \right\}_{j=1}^m. \quad (153)$$

Equation 153 preserves the interpretation and units of each diagnostic. A scalar aggregation can be introduced in a specific application after normalization and weighting rules have been justified.

Table 23 is a menu rather than a checklist that every application must satisfy. A study should retain only diagnostics supported by its formal and empirical specification.

13.3 Spectral Recovery and Amplification

Finite operator realizations permit a more specific recovery analysis. Let L_X be a finite-dimensional generator obtained from the local or quasi-local dynamics at configuration X . Suppose the application identifies a neutral subspace $\mathcal{N}_X$ that is excluded from recovery assessment, for example a stationary direction preserved by construction. Let $\sigma_{\text{rec}}(L_X)$ denote the remaining spectrum relevant to return toward the declared reference regime. The spectral recovery gap is defined in Equation 154.

$$g_{\text{rec}}(X) := - \sup_{\lambda \in \sigma_{\text{rec}}(L_X)} \text{Re } \lambda, \quad (154)$$

provided the retained recovery spectrum is nonempty and lies in the open left half-plane. Equation 154 is positive for the corresponding linearly restoring regime. A sequence X_n with $g_{\text{rec}}(X_n) \rightarrow 0^+$ contains an increasingly slow linear recovery mode under the selected representation. This provides one formal analogue of critical slowing down, while the inference remains conditional on the adequacy of the local linearization and the selected generator [8].

Eigenvalues alone can understate finite-time sensitivity when the generator is strongly non-normal. A complementary quantity is the resolvent amplification in Equation 155.

$$\mathcal{R}_X(z) := \|(zI - L_X)^{-1}\|, \quad z \notin \sigma(L_X). \quad (155)$$

Equation 155 measures the response of the linear resolvent at a selected complex parameter z . An application can summarize the relevant region through Equation 156.

$$\mathcal{A}_\Omega(X) := \sup_{z \in \Omega} \mathcal{R}_X(z), \quad (156)$$

where Ω is a declared subset of the resolvent set $\rho(L_X) = \mathbb{C} \setminus \sigma(L_X)$ chosen for the local problem. Equations 155 and 156 are diagnostic constructions. They do not imply that the full GR system has a single linear generator.

The spectral and resolvent diagnostics serve different purposes. The recovery gap tracks the slowest retained asymptotic linear mode under the stated assumptions. Resolvent amplification can detect large forced or transient responses associated with the represented operator even when the eigenvalues remain inside a nominally restoring region. A near-criticality specification may retain either quantity, both quantities, or neither, depending on the model.

13.4 Compositional Susceptibility of Orders and Cycles

The field-theoretic formalism permits critical sensitivity to arise through composition even when individual local maps vary smoothly. Let $\Theta_R^\pi(X)$ be the ordered composite associated with an admissible ordering

Table 23: Candidate near-criticality diagnostics available from the preceding GR construction. Each diagnostic requires a declared perturbation family and interpretation.

Diagnostic object	Finite computation	Near-critical indication within a declared model
Balance susceptibility	Apply Equation 152 to δ_R^{bal} or D_R^{rob}	Small perturbations strongly change proximity to the balance corridor
Order susceptibility	Apply Equation 152 to Ω_R^{bal}	Small perturbations strongly alter sequence dependence
Cycle susceptibility	Apply Equation 152 to a declared cycle-map discrepancy	Closed compositional effects become highly perturbation-sensitive
Recovery gap	Compute a declared spectral recovery gap of a finite generator	Restoring modes soften under the selected linearization
Resolvent amplification	Compute $\ (zI - L)^{-1}\ $ on a declared spectral region	Small forcing can be strongly amplified despite modest eigenvalue movement
Structural margin	Compute the smallest perturbation that changes a selected invariant profile	Discrete structural change is accessible under small perturbations
Overlap-topology margin	Repeat the overlap-complex construction under perturbed inputs	Higher-order field coexistence can reconfigure under small perturbations

π , and let d_{dyn} be the declared dynamical discrepancy. The ordering susceptibility at scale ε is defined in Equation 157.

$$\chi_{\text{ord}}(X; \varepsilon) := \sup_{\substack{\delta \in \Delta_X \\ 0 < \|\delta\|_X \leq \varepsilon}} \frac{\sup_{\pi, \pi'} \left| d_{\text{dyn}} \left(\Theta_R^\pi(X[\delta]), \Theta_R^{\pi'}(X[\delta]) \right) - d_{\text{dyn}} \left(\Theta_R^\pi(X), \Theta_R^{\pi'}(X) \right) \right|}{\|\delta\|_X}. \quad (157)$$

Equation 157 measures sensitivity of sequence dependence itself. A configuration may have a modest current order spread while a small field perturbation makes that spread increase sharply.

Closed cycles provide a second compositional channel. For a retained cycle c with cycle map $\mathcal{C}_c(X)$, let $\mathfrak{E}_c$ be a declared nonempty reference class of return maps and let d_c^{map} be a discrepancy on the corresponding map space. Define the map-level cycle residual by $\eta_c(X) := \inf_{E \in \mathfrak{E}_c} d_c^{\text{map}}(\mathcal{C}_c(X), E)$. This derives the scalar used below from the more primitive return map and declared comparison structure. The cycle susceptibility is stated in Equation 158.

$$\chi_c(X; \varepsilon) := \sup_{\substack{\delta \in \Delta_X \\ 0 < \|\delta\|_X \leq \varepsilon}} \frac{|\eta_c(X[\delta]) - \eta_c(X)|}{\|\delta\|_X}. \quad (158)$$

A large value of Equation 158 means that a small perturbation can strongly alter the residual generated by a closed compositional process. This is a system-level susceptibility because η_c depends on the ordered product around the cycle rather than on one local edge in isolation.

In a differentiable small-parameter regime, commutator growth can supply one mechanism for increasing compositional susceptibility. If local generators $A(\mu)$ and $B(\mu)$ vary with a parameter μ , the leading small-cycle residual depends on $[B(\mu), A(\mu)]$ by the result of Section 10. A rapid change in the norm or spectrum of that commutator can therefore alter cycle sensitivity before any selected local scalar readout changes comparably.

13.5 Structural Margins and Topological Transition

Continuous susceptibility does not capture discrete changes in topology, invariant structure, or admissible relational architecture. Let I be a declared structural diagnostic taking values in a discrete or equivalence-class codomain. Examples include a selected Betti-vector profile, the isomorphism class of a finite overlap complex, the dimension of a common fixed algebra, or another invariant family established by the model. The structural margin associated with I is defined in Equation 159.

$$m_I(X) := \inf \{ \|\delta\|_X : \delta \in \Delta_X, I(X[\delta]) \not\sim_I I(X) \}, \quad (159)$$

where $\sim_I$ is the equivalence relation declared for the diagnostic. If the set is empty, the convention $m_I(X) = +\infty$ is used. A small finite value of Equation 159 means that a small admissible perturbation can cross the selected structural boundary.

For the multi-field overlap complex, let $\mathcal{N}_{\mathbf{P}}(X)$ denote the overlap complex reconstructed from configuration X by Equation 132. For a declared maximum degree q , define the selected homological profile by $\beta^{\text{ov}}(X) := (\beta_0(\mathcal{N}_{\mathbf{P}}(X)), \dots, \beta_q(\mathcal{N}_{\mathbf{P}}(X)))$. The corresponding topological-detection margin is stated in Equation 160.

$$m_{\text{top}}^{\text{ov}}(X) := \inf \{ \|\delta\|_X : \beta^{\text{ov}}(X[\delta]) \neq \beta^{\text{ov}}(X) \}. \quad (160)$$

If the set in Equation 160 is empty, the convention $m_{\text{top}}^{\text{ov}}(X) = +\infty$ is used. Equation 160 detects access to a homology-changing overlap reconfiguration. Equality of the selected Betti vector across perturbations does not establish topological equivalence, so the quantity should be read as a detection margin relative to the chosen invariant profile.

A similar construction applies to the history-bearing complex itself, to relative homology in a fixed-past regime, or to the historical carrier in the reconfiguration regime of Section 9. The relevant margin must specify which carrier is permitted to change. This keeps future topology change, overlap topology change, and historical topology change formally distinct.

13.6 Balance-Compatible Near-Critical Regimes

Balance and criticality describe different properties of the represented system. A balance diagnostic evaluates position relative to a declared balance corridor. A criticality diagnostic evaluates sensitivity or structural proximity under a declared perturbation family. The two properties can therefore vary independently.

Let $D^{\text{bal}}(X)$ be a retained scalar balance defect, for example the robust defect of Equation 142. For the two-dimensional visualization only, let $Q_{\text{crit}}(X)$ be one declared scalar criticality diagnostic or a separately justified normalized aggregation of several diagnostics from Ω_{crit} . Choose thresholds τ_{bal} and τ_{crit} . The conceptual partition is shown in Figure 8.

Figure 8 clarifies the hypothesis examined in this paper. A configuration can remain inside or near a balance corridor while moving toward a regime with a small recovery gap, large order or cycle susceptibility, or a small structural margin. The phrase *near-critical balance* is used here only for this conjunction relative to the declared specification.

Definition 13.1 (Balance-compatible near-critical configuration). Let $\mathfrak{S}_{\text{crit}}$ be a declared criticality specification, whose decision component is $\mathfrak{T}_{\text{crit}}$, and let $\mathfrak{T}_{\text{bal}}$ be a declared balance tolerance. A configuration X is *balance-compatible and near-critical relative to the specification* when it satisfies $\mathfrak{T}_{\text{bal}}$ and the decision rule in $\mathfrak{T}_{\text{crit}}$ classifies the required retained criticality diagnostics as near-critical.

Definition 13.1 does not identify balance with criticality. It supplies a reproducible condition under which the two coexist. This distinction is relevant to the Criticality Hypothesis of International Politics because a system can display persistence and recurrent balancing while retaining substantial susceptibility to perturbation and reconfiguration [7]. The present formalism supplies candidate observables for testing such a hypothesis in a specified empirical model.

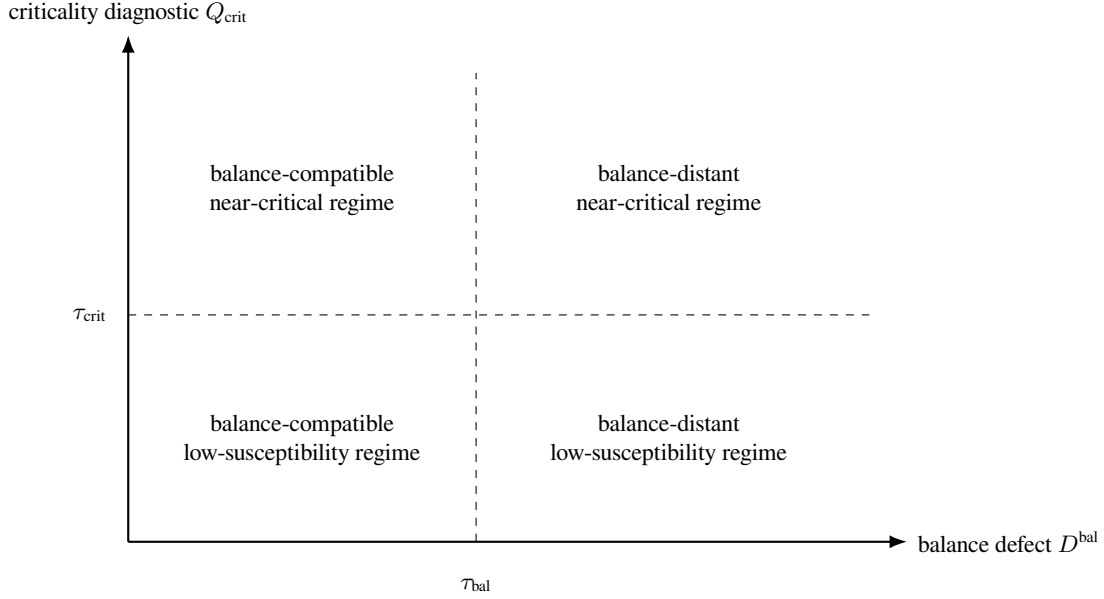


Figure 8: Conceptual separation of balance compatibility and near-criticality. The axes and thresholds are model-declared; the diagram does not define a universal phase space of international politics.

13.7 Finite Illustrative Specialization

A two-dimensional linear specialization illustrates how balance compatibility and perturbation sensitivity can separate. Let $\mu > 0$ and $\kappa \in \mathbb{R}$. Consider the finite generator in Equation 161.

$$L_\mu := \begin{pmatrix} -\mu & \kappa \\ 0 & -1 \end{pmatrix}. \quad (161)$$

Suppose the surrounding multi-field construction satisfies a declared balance criterion for every $\mu \in (0, \mu_0]$. This assumption isolates the criticality calculation from the balance classification. The eigenvalues of Equation 161 are $-\mu$ and -1 , so the recovery gap for $0 < \mu \leq 1$ is stated in Equation 162.

$$g_{\text{rec}}(\mu) = \mu. \quad (162)$$

Equation 162 approaches zero as $\mu \rightarrow 0^+$, producing a progressively slower restoring mode in the finite linearization.

The same specialization also shows forced amplification. For a constant perturbation $f = (0, 1)^\top$, the stationary linear response x_μ satisfying $L_\mu x_\mu + f = 0$ is stated in Equation 163.

$$x_\mu = -L_\mu^{-1} f = \begin{pmatrix} \kappa/\mu \\ 1 \end{pmatrix}. \quad (163)$$

For $\kappa \neq 0$, the first component in Equation 163 grows as μ^{-1} . Thus a family can remain balance-compatible under the selected balance rule while becoming increasingly sensitive to a fixed perturbation in the represented dynamics.

This example establishes a formal possibility rather than an empirical description of an international system. Its analytical role is to demonstrate that a balance criterion and a recovery or amplification criterion can be independent even in a minimal finite model. Empirical use would require a justified mapping from observed relations to L_μ , μ , κ , the perturbation f , and the balance corridor.

13.8 Finite Verification of Near-Critical Configurations

The near-criticality construction is designed for explicit checking in finite realizations. Table 24 gives a minimal verification programme.

Table 24: Finite checks for near-critical GR power configurations. The procedures test the declared formal model and do not by themselves validate an empirical criticality claim.

Check	Computation	Interpretation
Perturbation calibration	Specify Δ_X , $\ \delta\ _X$, and the scale ε	Makes susceptibility claims reproducible relative to the admitted perturbations
Continuous susceptibility	Compute Equation 152 for the retained diagnostics	Measures finite-scale response of balance, order, cycle, or other observables
Spectral recovery	Compute g_{rec} from the retained generator spectrum	Detects softening of selected restoring modes
Resolvent test	Evaluate Equation 156 on the declared region Ω	Detects represented amplification that may not be summarized by the spectral gap alone
Structural margin	Search for the smallest perturbation changing I under Equation 159	Estimates proximity to a selected discrete structural boundary
Overlap-topology test	Recompute $\mathcal{N}_{\mathbf{p}}$ and its selected invariant profile under perturbation	Detects higher-order coexistence reconfiguration
Balance-criticality separation	Compare the balance defect with the retained criticality diagnostics across the same perturbation family	Tests whether balance compatibility and near-criticality vary independently
Resolution test	Repeat the calculations under declared coarse-graining, threshold, or region changes	Identifies classifications that depend strongly on representation choices

A credible empirical study should report the perturbation family and thresholds before interpreting a high susceptibility as evidence of near-criticality. It should also compare multiple diagnostics when the hypothesized mechanism implies more than one observable consequence. The absence of one generic warning signal does not, within this framework, prove distance from every possible critical mechanism.

13.9 Interface for Scope and Formal Limits

The preceding construction converts the Criticality Hypothesis from a broad dynamical conjecture into a family of model-specific tests. It identifies continuous susceptibility, spectral recovery, compositional sensitivity, and structural margins as distinct formal routes to near-critical behavior. It also separates balance compatibility from criticality and allows a system to occupy any of the four conceptual regions shown in Figure 8.

Several limitations remain structurally important. The perturbation norm is model-dependent. A finite generator can describe only the retained local approximation. Betti-number changes detect selected topological differences without classifying every possible topology. Susceptibility can be large because of poor parametrization, ill-conditioned inference, or observational noise rather than a substantive critical mechanism. Threshold-based overlap complexes can change when the threshold changes. The present formalism therefore requires sensitivity analysis across reasonable representations before a criticality interpretation is advanced.

These limitations motivate the next section, which separates structural formalization, physical analogy, empirical parameterization, and ontological interpretation; it also states which claims in the present paper are definitions, derived propositions, computational possibilities, hypotheses, and open research obligations.

14 Epistemic Status of the Field-Theoretic Formalism

This section states the epistemic status, interpretive scope, and principal limits of the field-theoretic construction developed in Sections 2–13. Its role is methodological: the preceding sections introduced increasingly explicit formal objects, while the present section specifies what those objects can support as definitions, derivations, computational diagnostics, empirical models, and broader hypotheses. The discussion proceeds through four interfaces. It first separates structural formalization from physical analogy and ontological interpretation. It then identifies representation dependence and the role of coarse-graining, states the requirements for empirical parameterization and identifiability, and clarifies the scope of the historical, balance, and criticality applications. The section closes with research obligations that would be required before the formalism could support stronger empirical claims.

14.1 Formal Status of the Construction

The present paper is a formal research programme rather than a completed empirical theory of social or political power. Its principal objects are constructed to make relational claims explicit enough to inspect, compose, and calculate. A labelled 2-complex supplies one candidate representation of history-bearing relational structure; local and quasi-local algebras organize admissible quantities and transformations; GR power-field realizations encode region-indexed deformation data; dynamical maps represent local and compositional action; and homological, spectral, cyclic, and criticality diagnostics summarize selected structural properties of those representations.

These objects occupy different epistemic levels. Table 25 records the status assigned to the main constructions in the present draft.

Table 25 prevents a useful formal representation from acquiring a stronger status merely because it is mathematically explicit. For example, a proposition proved for compatible local algebra maps establishes a property of the formal system under its assumptions. It does not establish that an observed political system satisfies those assumptions. Similarly, a finite computation can verify the implementation of a diagnostic without validating the empirical interpretation assigned to that diagnostic.

14.2 Representation Layers and Interpretive Boundaries

The formalism uses several mathematical traditions because they solve different representational problems. Discrete complexes represent relational incidence and history. Operator algebras represent local quantities and transformations without requiring one primitive global state vector. Field language represents region-indexed deformation and compatibility. Homology detects selected global structural differences. Spectral and resolvent objects characterize finite or represented dynamical realizations. Each layer has an analytical role, and none of these roles by itself determines the ontology of the modeled domain.

A useful interpretive separation is therefore maintained among four levels:

1. the substantive GR claim about historically constituted relational power;
2. the selected mathematical representation of that claim;
3. the calculations or propositions internal to the selected representation; and
4. the empirical or ontological interpretation assigned to the resulting quantities.

The first level motivates the model. The second makes the model explicit. The third can be checked mathematically or computationally. The fourth requires evidence or philosophical argument beyond internal formal consistency.

The terms *field*, *curvature*, *holonomy*, *operator*, *spacetime*, and *spin-foam-inspired history* are used at the second and third levels unless a stronger claim is separately established. Their use records structural correspondences and calculational resources. The present paper therefore leaves open whether another

Table 25: Epistemic status of principal constructions in the present paper. The table distinguishes defined model objects and derived formal results from empirical claims that require additional evidence and calibration.

Construction	Status in this paper	What the construction supports	Additional requirement for stronger use
Labelled GR 2-complex	Structural representation proposal	Explicit history-bearing regions, interfaces, paths, cycles, and cell-level reconfiguration	Empirical rule connecting observed relations and events to cells and labels
Local and quasi-local algebras	Algebra-first modeling architecture	Typed local quantities, transformations, restriction maps, and open-system composition	Justification of algebra class, observables, norms, and empirical interpretation
GR power-field realization	Derived representation layer	Region-indexed, overlap-compatible representation of power-conditioned deformation	Calibration from substantive power relations to field carriers and local maps
Operator-algebraic dynamics	Derived specialization under stated assumptions	Composition, generators, spectra, GNS implementations, and finite matrix checks	Evidence that the chosen operator realization captures the relevant empirical dynamics
Cycle and curvature-like residuals	Defined formal diagnostics	Detection of residual effects produced by closed composition	Substantive interpretation and robustness across path, region, and representation choices
Homological and topological diagnostics	Sufficient structural diagnostics within the selected complex	Detection of selected connectivity, cycle, and topology changes	Stronger invariants or comparison tools when homology does not distinguish candidate structures
Balance corridor and balance defects	Application-specific formal specification	Reproducible classification of counter-deformation relative to declared tolerances	Empirical justification of capability, threat, field, discrepancy, and tolerance mappings
Near-criticality diagnostics	Model-relative hypothesis-testing framework	Susceptibility, recovery, amplification, and structural-margin calculations	Empirical perturbation model, threshold calibration, uncertainty analysis, and comparative validation

mathematical language could represent the same GR commitments more efficiently or with stronger empirical adequacy.

The algebra-first architecture has the same status. The decision to begin from local algebras and quasi-local assembly addresses open, heterogeneous, and potentially unbounded relational systems. It does not establish that social reality is fundamentally a C^* -algebra or a W^* -algebra. Particular applications may require only finite associative algebras, stochastic kernels, categories of transformations, graph operators, or other local structures. The C^* and W^* specializations become appropriate only when their additional algebraic and analytic assumptions are useful and substantively defensible.

14.3 Representation Dependence and Structural Robustness

Several formal quantities depend on modeling choices made before any calculation begins. These choices include region boundaries, cell decomposition, labels, local algebra classes, overlap thresholds, admissible perturbations, norms, discrepancy functions, continuation rules, and coarse-graining maps. A result that changes substantially under small reasonable changes in those choices should be interpreted as representation-sensitive.

Let $\mathfrak{R}$ be a declared family of admissible representation changes for a configuration X , and let Q be a diagnostic taking values in a metric space with distance d_Q . The representation spread of Q is defined in Equation 164.

$$\Delta_Q(X; \mathfrak{R}) := \sup_{\rho \in \mathfrak{R}} d_Q(Q(X), Q(\rho X)). \quad (164)$$

Equation 164 is a robustness diagnostic rather than a universal criterion. The family $\mathfrak{R}$ has to be chosen according to the modeling problem. It may contain alternative region partitions, plausible threshold changes, refinement or coarse-graining maps, alternative codings of the same empirical relation, or other transformations that preserve the substantive question being studied.

A small value of $\Delta_Q(X; \mathfrak{R})$ supports stability of the selected diagnostic across that declared representation family. A large value identifies a dependence that should remain visible in substantive interpretation. The quantity does not decide which representation is substantively preferable. That judgment depends on evidence, explanatory purpose, and the semantics of the modeled relations.

The same logic applies to topology. Betti numbers, Euler characteristics, and cycle classes are properties of the selected complex. A topological change in the model can be meaningful when the complex has a justified empirical interpretation and the change is robust across reasonable constructions. The mathematical fact that a complex changes topology does not, on its own, establish that the underlying social or political world possesses a topology in the same ontological sense.

14.4 Empirical Parameterization and Identifiability

Empirical use requires an additional interface that is intentionally absent from the abstract formalism. Let Y denote an observational record and let Θ collect a declared empirical specification, including variable definitions, region construction, coding rules, uncertainty treatment, and any estimation procedure. A calibration map assigns an empirical record to a GR model configuration as stated in Equation 165.

$$\mathcal{C}_\Theta : Y \mapsto X_\Theta \in \mathcal{X}_{\text{GR}}. \quad (165)$$

Equation 165 makes the empirical interface explicit. Different specifications can map the same record to different GR configurations, and incomplete observations can leave several configurations compatible with the same evidence.

Let $\mathcal{O}_\Theta$ be a declared observation map from a GR configuration to the empirical quantities that the model is expected to reproduce or predict. Two configurations are observationally equivalent relative to that map when they satisfy Equation 166.

$$X \sim_{\mathcal{O}_\Theta} X' \iff \mathcal{O}_\Theta(X) = \mathcal{O}_\Theta(X'). \quad (166)$$

Equation 166 shows why identifiability is separate from calculability. A cycle residual, field value, topology, or generator can be calculated exactly inside a fitted model while remaining weakly identified by the available observations. Empirical work should therefore report which latent structures are directly measured, which are inferred, which are fixed by modeling convention, and which remain observationally equivalent under the available data.

The empirical interface also introduces uncertainty that the current deterministic notation often suppresses. Measurement error, missing relations, uncertain event ordering, ambiguity in actor or region boundaries, and coding disagreement can propagate through field realization, operator composition, topology, balance, and criticality diagnostics. A mature empirical implementation would therefore attach uncertainty to $\mathcal{C}_\Theta$ and propagate it to the reported diagnostics rather than presenting one fitted complex as uniquely determined by the observations.

14.5 Formal Verification and Empirical Validation

The repeated finite checks in Sections 5–13 address internal formal behavior. Their evidential role can be organized into the layers shown in Table 26.

Table 26 preserves methodological pluralism. A qualitative historical analysis can be decisive for the meaning of a relation even when a formal diagnostic is exactly calculable. Conversely, computational checks can expose contradictions or path dependence that are difficult to detect in prose alone. The programme therefore treats formal, computational, quantitative, historical, and interpretive inquiry as complementary sources of constraint on different parts of the model.

14.6 Historical Reconfiguration under Ontological Suspension

Section 9 allows the represented historical carrier to change. That extension is deliberately broader than the fixed-past regime of Section 8, but its interpretation remains layered.

At an epistemic level, a changed historical complex can represent revision of accessible evidence, classification, narrative connection, or institutional memory. At a relational-historical level, it can represent a change in which past events, precedents, obligations, or identities remain connected to current relations and future possibilities. At a stronger ontological level, one could ask whether the underlying past itself is mutable. The present formalism supplies typed transformations that can represent each reading once the corresponding semantics are declared; it does not select the strongest reading by mathematical form alone.

The topology-changing diagnostics inherit this restraint. A change in $H_q(K^-)$ or in a stronger invariant establishes a change in the represented historical carrier under the selected model. It can support a substantive claim about institutional or relational history only after the cells, attachment maps, and comparison data have been given an empirically or interpretively justified meaning. Physical retrocausality is outside the evidential scope of the present paper.

14.7 Scope of the International-Relations Application

Sections 12 and 13 use international relations as one application domain. Their purpose is to demonstrate how the general formalism can express counter-deformation, order effects, higher-order alignment, cycle residuals, recovery gaps, susceptibility, and structural margins. These constructions do not replace the substantive theories from which empirical variables or mechanisms would be selected.

In particular, a GR balance corridor is a model-declared object. Its construction can use capability measures, threat perceptions, alliance relations, institutional constraints, or other empirically justified quantities, but the formalism does not determine which selection is appropriate for every international system. Likewise, a near-criticality classification is conditional on the perturbation family, diagnostic set, thresholds, and timescale. The calculations can make those assumptions explicit and test their consequences; historical and empirical research remains necessary to determine whether the assumptions describe a concrete case.

Table 26: Verification and validation layers for the GR power-field programme. Higher layers require additional evidence and are not implied by success at lower layers.

Layer	Primary question	Typical evidence
Typing and definition	Are domains, codomains, restrictions, and admissibility conditions well specified?	Symbolic inspection, proof obligations, automated type or schema checks
Formal consistency	Do propositions and compositions follow from the declared assumptions?	Mathematical derivation, counterexample search, algebraic verification
Computational reproducibility	Does a finite implementation reproduce the declared calculation?	Code, test fixtures, numerical replication, exact finite examples
Representation robustness	Does the result persist across reasonable resolutions and encodings?	Sensitivity analysis, alternative partitions, thresholds, coarse-graining, model comparison
Empirical identification	Do observations constrain the formal quantities being interpreted?	Measurement design, estimation, uncertainty analysis, identifiability assessment
Empirical validation	Does the calibrated model account for observations beyond the fitting choices?	Out-of-sample tests, historical comparison, cross-case analysis, prospective or quasi-experimental evidence where available
Substantive interpretation	Does the formal quantity support the political, social, institutional, or historical meaning assigned to it?	Domain scholarship, qualitative evidence, mechanism tracing, interpretive comparison

The criticality framework also avoids a universal prediction that balanced systems should approach transition. Sections 12 and 13 permit balance-compatible configurations with low susceptibility, balance-compatible configurations with high susceptibility, and corresponding balance-distant regimes. The empirical distribution of cases among these regimes remains an open question.

14.8 Normative Scope and Reflexive Systems

The formalism represents power-conditioned changes in generative possibilities, histories, and admissible transformations. Normative properties such as legitimacy, justice, responsibility, rights, welfare, or democratic authorization require additional evaluative structures. A transformation with a large field norm or strong cycle residual can be normatively beneficial, harmful, mixed, or indeterminate depending on the substantive context. The current paper therefore supplies no conversion rule from dynamical magnitude to normative value.

Social and political systems also contain reflexive agents who can respond to classifications, forecasts, institutions, and models. Such reflexivity can alter the empirical mapping $\mathcal{C}_\Theta$, the relevant perturbation family, or the local transformations themselves. Parameters estimated from one historical regime may consequently lose meaning after actors learn, adapt, contest categories, or redesign institutions. An empirical GR power-field model should therefore treat parameter stability as a hypothesis to be checked rather than as a standing assumption.

This reflexive character provides another reason for retaining local and revisable representations. A model can update regional algebras, field carriers, histories, and comparison maps as the substantive system changes. The benefit of that flexibility is accompanied by a risk of post hoc accommodation. Predeclared diagnostics, archived model versions, sensitivity analysis, and out-of-sample comparison can help distinguish genuine explanatory improvement from unconstrained reinterpretation.

14.9 Research Obligations for Empirical Development

The present formal programme creates several concrete obligations for subsequent work. They are stated as research obligations because satisfying them requires empirical, computational, or theoretical work beyond the first-draft formal construction.

Research Obligation 14.1 (Empirical grounding of local structure). An application should specify how observed actors, events, relations, institutions, and historical records determine regions, cells, labels, local algebras, and comparison maps. Competing plausible constructions should be compared where the mapping is underdetermined.

Research Obligation 14.2 (Identification of power-field quantities). An application should state which observations constrain local field carriers and dynamical maps, which quantities are latent, and which parameters remain observationally equivalent under the available evidence.

Research Obligation 14.3 (Representation and resolution robustness). Reported cycle, topology, balance, and criticality conclusions should be tested across a declared family of reasonable thresholds, partitions, coarse-grainings, and coding alternatives when those choices materially affect the result.

Research Obligation 14.4 (Uncertainty propagation). Empirical implementations should propagate measurement, coding, and estimation uncertainty into derived field, operator, topological, balance, and criticality diagnostics when those uncertainties can alter substantive classification.

Research Obligation 14.5 (Cross-method interpretation). Formal diagnostics should be interpreted together with the historical, institutional, legal, qualitative, and domain-specific evidence required to assign substantive meaning to the modeled relations and transformations.

Research Obligation 14.6 (Predictive and comparative discrimination). Where the formalism is used explanatorily or predictively, empirical studies should compare it with simpler and competing models and identify observations that could discriminate among their mechanisms rather than relying only on retrospective fit.

The obligations above are deliberately demanding. Their purpose is to preserve the distinction between a mathematically articulated research programme and an empirically established theory. They also provide a practical agenda for converting the present paper's finite consistency checks into cumulative empirical research.

14.10 Interface for the Conclusion

The field-theoretic formalism developed in this paper has therefore reached a specific level of completion. Its local, algebraic, compositional, topological, multi-field, balance, and criticality constructions can be defined and checked within finite or quasi-local models. Several propositions establish consistency properties under explicit assumptions, and the finite examples demonstrate that sequence-sensitive, cycle-level, multi-field, balance, and near-critical effects are calculable within those models.

The paper leaves a different set of questions open. Empirical work must determine how observed relations instantiate the formal objects, which representations remain robust across reasonable alternatives, which latent quantities are identifiable, and whether the resulting diagnostics improve explanation or prediction relative to simpler models. Philosophical work must continue to examine the interpretation of relational spacetime, historical reconfiguration, and ontological suspension. Normative work must supply evaluative structures when the consequences of power are assessed as legitimate, just, harmful, or desirable.

The concluding section can therefore summarize the paper without treating formal explicitness as empirical closure. The principal contribution is a calculable and revisable language for GR power dynamics in which local deformation, open-system algebraic evolution, history, cycles, multiple fields, structural change, balance, and near-criticality can be studied within one coherent architecture while their empirical and ontological status remains separately stated.

15 Conclusion

This section closes the paper by consolidating the formal programme developed in Sections 1–14. Its role is synthetic rather than generative: it restates the principal constructions, clarifies how the different scales of the formalism fit together, and identifies the remaining path from formal specification to empirical use. The discussion proceeds from the algebra-first representation of GR power to historical regimes, compositional system structure, international-power applications, and the verification obligations that remain after a mathematically explicit model has been constructed.

15.1 Algebra-First Synthesis of GR Power Dynamics

The paper began from a limitation in the abstract account of power developed in earlier GR work. A relational conception of power can identify historically constituted asymmetry, differential generative capacity, persistence, and landscape deformation without yet specifying how those effects are represented locally, how they compose across a history-bearing configuration, or how a finite model can be checked. The present paper addressed that gap by developing a field-theoretic formalism whose basic objects are local relational regions, local algebras, compatible field carriers, and typed transformations.

The resulting architecture does not require a primitive global phase space or one universal Hilbert space. Section 2 introduced the mathematical resources needed for that choice, while Section 3 represented GR spacetime provisionally as a labelled 2-complex with history-bearing subconfigurations, relational interfaces, and extension-based continuation structure. Sections 4 and 5 then separated the substantive GR power relation from its formal representations. A region-indexed deformation record is constructed before a field is admitted, and field status requires compatible local carriers, restriction maps, overlap coherence, and an admissible gluing structure.

This separation is central to the paper’s methodological contribution. Power is not identified with a field value, an operator, a scalar ranking, or a norm. Those objects become representation-dependent tools whose admissibility depends on the problem being modeled. Section 6 derived local dynamics from the field layer and showed how ordered transformations can become sequence-sensitive. Section 7 then derived algebraic and operator representations from those dynamics. States, GNS representations, Hilbert-space operators, and W^* -level normal maps enter only when the required assumptions are available. The resulting hierarchy allows finite matrix realizations for explicit calculation while retaining a quasi-local formulation for open or indefinitely extensible systems.

The formal gain is therefore twofold. First, the paper specifies domains, codomains, compatibility conditions, composition laws, and representation changes that were previously left at an abstract level. Second, those specifications generate finite checks. Restriction consistency, overlap defects, order sensitivity, factorization, fixed structures, cycle residuals, spectra, homology, and coarse-graining sensitivity can be calculated within a declared model. Calculability does not by itself establish empirical adequacy, yet it creates a more corrigible formal object than a purely metaphorical use of field or operator language.

15.2 Historical Dynamical Regimes

The paper also separates two historical regimes that had previously been easy to conflate. Section 8 formalized a history-preserving regime in which a realized historical anchor K^- remains fixed while continuation structures, future topology, admissibility, and evaluative consequences may change. Relative homology makes the distinction precise: the topology of an extension relative to the fixed anchor can change even when the historical carrier itself does not.

Section 9 then relaxed that restriction. Historical reconfiguration is represented through typed transformations between historical carriers and their associated algebras. This permits changes in incidence structure, attachment structure, algebraic organization, and homology without assuming that every such transformation is invertible or that source and target histories share one privileged state space. The paper therefore treats the fixed-past assumption as a useful modeling regime rather than an ontological commitment.

The interpretive status of the stronger regime remains deliberately layered. A reconfigured historical complex can represent changes in accessible evidence, institutional memory, recognized precedent, relational identity, or the effective connection between past events and present possibilities. A stronger ontological interpretation concerning the mutability of the underlying past requires evidence beyond the formalism. Section 14 therefore keeps the topology-changing machinery available while preserving ontological suspension.

This distinction matters for the broader GR programme because historical dependence is represented as more than a parameter attached to a present state. History enters through the structure of the relational carrier, its admissible extensions, and the maps that connect one historically constituted configuration to another. The formalism can therefore represent future reorganization relative to a retained past and reorganization of the represented past itself as different mathematical regimes with different verification obligations.

15.3 Compositional Emergence at System Scale

Sections 10 and 11 moved from local dynamics to system-level structure. Closed compositions generate cycle return maps whose residual effects can remain after a relational sequence returns to its initial carrier type. Under the stated invertibility assumptions, base-point changes act by conjugacy, which permits similarity-invariant cycle diagnostics. Small closed compositions also recover commutator-governed residuals in the differentiable finite regime. These constructions provide a disciplined route from local sequence sensitivity to curvature-like and holonomy-like diagnostics without asserting a literal physical gauge ontology.

The paper uses these cycle structures to formulate a restricted and testable sense of emergence. A system characteristic is called diagnostic-relative when it contains information absent from a declared family of local diagnostics. The finite examples show that systems with the same selected local map spectra can nevertheless possess different cycle spectra, and that individually trivial backtracking processes can participate in a nontrivial mixed-field cycle. These examples establish compositional information that cannot be recovered from the selected local summaries alone. They do not establish metaphysical irreducibility to every conceivable microscopic description.

Multiple power fields add a second layer of collective structure. Ordered coexistence, joint dynamics, and non-factorizable interaction are distinguished formally. The factorization defect tests whether a joint field-induced transformation can be reproduced by an ordering of single-field maps, while interaction generators separate infinitesimal nonadditivity from finite noncommutativity. The multi-field overlap complex records the coexistence pattern of field domains and makes higher-order alignment structure calculable without identifying that overlap complex with the topology of the underlying GR spacetime.

Taken together, these constructions shift the analysis of power away from isolated pairwise magnitudes toward compositional structure. A system may exhibit properties that depend on cycles, ordering, overlap, common fixed structures, and global topology even when individual local relations appear moderate. The resulting framework therefore provides formal objects for studying circulation, amplification, frustration, persistence, and collective reconfiguration as system-level consequences of relational composition.

15.4 International-Power Dynamics near Criticality

The international-relations application in Sections 12 and 13 demonstrates how the general machinery can be specialized without reducing the formalism to one disciplinary theory. Balance is represented relative to a declared selection interface and balance corridor. Counter-deformation, order spread, cycle residuals, common fixed structures, and overlap-complex changes then supply diagnostics for different senses in which several power structures may compensate, reorganize, or continue to generate systemic residuals.

The finite balance calculation shows why first-order compensation is insufficient as a complete systemic criterion. Local generators can cancel at first order while ordered finite transformations retain higher-order residuals, and reversing a response sequence can change those residuals through commutator terms. The

distinction among local compensation, compositional balance, and structural balance therefore follows from the formal construction rather than from a new classification imposed on realist theory.

The criticality section adds an independent axis. Near-criticality is defined relative to a declared specification containing perturbations, diagnostics, scales, and thresholds. Finite-scale susceptibility, spectral recovery gaps, resolvent amplification, order and cycle sensitivity, structural margins, and overlap-topology margins represent different possible mechanisms. The balance-compatible near-critical regime is consequently a conjunction of two separately evaluated properties. A configuration may satisfy a balance tolerance while becoming increasingly susceptible to perturbation, slow recovery, compositional amplification, or structural transition.

This separation gives the Criticality Hypothesis of International Politics a more demanding formal interface. An empirical application would have to identify which critical mechanism is proposed, map observations to the corresponding GR quantities, test representation robustness, and evaluate whether the retained diagnostics change as the hypothesis predicts. Balance, persistence, susceptibility, and transition can then be studied as related but distinct properties of an instantiated model.

15.5 Verification Path for Empirical Development

The paper's formal explicitness is intended to support verification rather than to substitute for evidence. Section 14 therefore placed the constructions inside a layered research programme. Typing and definitional coherence come first. Formal consistency, finite computational reproducibility, and representation robustness follow. Empirical identification and validation require additional observation models, coding decisions, uncertainty treatment, and substantive interpretation.

The calibration map introduced in Section 14 makes that empirical interface explicit. A dataset does not arrive already partitioned into GR regions, labelled cells, local algebras, power fields, admissible perturbations, or topology. Those objects have to be constructed through a declared empirical specification. Alternative reasonable specifications can produce different instantiated models, and observationally equivalent configurations can remain underdetermined by the available evidence. The representation-spread diagnostic provides one way to report how much a retained conclusion varies across an admitted family of encodings.

This requirement is especially important for the more visually compelling outputs of the formalism. A nonzero cycle residual, a topological transition, a closing spectral gap, or a large susceptibility is mathematically meaningful only relative to the specified model. Substantive interpretation requires a justified mapping between the formal object and the political, institutional, historical, or social process under study. Future empirical work should therefore combine computational verification with historical evidence, qualitative interpretation, uncertainty propagation, sensitivity analysis, and comparative or predictive testing where the research design permits them.

The same caution applies to normative interpretation. The formalism can represent differential generative capacity, constraint, access, persistence, option-space deformation, and historical reconfiguration. It does not derive legitimacy, justice, responsibility, or desirability from those quantities alone. Normative analysis requires additional premises and forms of evidence appropriate to the question being studied.

15.6 GR Power Formalism under Ontological Suspension

The paper has developed a preliminary field-theoretic formalism rather than a final ontology of power. The terms *spacetime*, *field*, *operator*, *curvature*, *topology*, and *criticality* name mathematical resources whose use is warranted only when the corresponding structures have been defined and their interpretive scope has been stated. Their successful use in a model does not establish that political or social reality possesses the same ontology as the physical theories from which some of the mathematical language is historically familiar.

Within that restraint, the paper advances a concrete programme. GR power can be represented locally without reducing it to a scalar possession. Local representations can be assembled into a field only when compatibility conditions are satisfied. Field-induced transformations can be composed along histories and

cycles, lifted to quasi-local algebraic dynamics, and represented by operators when additional assumptions permit. Historical dynamics can be studied both with a fixed anchor and with reconfigured historical carriers. Multiple fields can generate collective structures that are absent from selected local diagnostics. Balance and near-criticality can be assigned separate, computable criteria inside a declared international-relations model. Each step supplies explicit objects that can fail, disagree across representations, or be contradicted by empirical work.

That possibility of failure is part of the intended value of the formalism. A field representation can fail its gluing conditions. A proposed interaction can factorize. A cycle residual can vanish. A purported invariant can depend strongly on representation. A topological transition can disappear under reasonable reconstruction. A criticality diagnostic can remain bounded. An empirical calibration can leave the central quantities unidentified. These outcomes would narrow or redirect the theory rather than undermine the usefulness of formalization itself.

The present paper therefore closes with a methodological commitment shared by the broader GR programme: formal structure should increase the precision of revisable inquiry. Power is treated here as historically constituted, relationally distributed, compositionally dynamic, and potentially generative of system-level structure. The field-theoretic and operator-algebraic framework supplies one preliminary language for making those claims explicit enough to derive, calculate, compare, and eventually test. Its further development depends on mathematical refinement, computational implementation, empirical calibration, and sustained dialogue with the substantive disciplines in which particular forms of power are observed and interpreted.

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