

A Preliminary Taxonomy of Equivalence in Generative Relational Theory

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Abstract

This paper develops a preliminary taxonomy of equivalence within Generative Relational (GR) theory. The inquiry begins from GR's distinction among ontological reality, relational reality, and manifestation reality and extends the comparison problem beyond equivalence at a single static level. The taxonomy is organized by the formal object on which an equivalence or equivalence-like relation is defined. It distinguishes equivalence within individual GR reality levels; equivalence of relational structures connecting multiple levels; equivalence of historical evolutions of single-level or multi-level structures; and equivalence of dynamical regimes or agreement of structural invariants. A preceding architectural section specifies the carrier, historical scope, retained structure, relation type, applicability domain, and representational resolution required by a well-formed equivalence claim. The formalism reserves the term equivalence relation for relations whose relevant axioms are established and separates it from similarity, tolerance, substitutability, simulation, bisimulation, isomorphism, and diagnostic agreement. History-bearing configurations, manifestation maps, retained relational content, symmetry-relative interchangeability, and admissible historical extensions provide inherited GR interfaces for the construction. The paper further studies cross-level lifting, stability under relational refinement, relational substitution, history-sensitive embedding, and the evolution of equivalence structures themselves through refinement, coarsening, and reorganization. Human–AI interchangeability provides a principal application, supplemented by AI–AI and Human–Human comparisons. Turing-style evaluation is located within the broader taxonomy, and a Generative Relational Equivalence Investigation is proposed as a structured procedure for testing how observed equivalence changes under contextual, relational, historical, continuation, and dynamical enrichment. Ultimate ontological equivalence remains suspended, and normative consequences require additional premises.

Keywords: Generative Relational Theory; equivalence; relational interchangeability; manifestation reality; historical trajectories; dynamical structure

Preliminary Research Note

This manuscript belongs to the broader Generative Relational (GR) research programme. Its immediate object is the formal status of equivalence across GR's layered account of reality, relational structure, historical development, and generative dynamics. The paper develops a taxonomy before applying that taxonomy to Human–AI, AI–AI, and Human–Human comparisons.

The inquiry adopts ontological suspension. Ultimate ontological identity or equivalence is therefore not inferred from manifestation, structural similarity, task performance, symmetry, trajectory correspondence, dynamical diagnostics, or any other formal relation developed in the manuscript. Mathematical structures are used as conditional representations whose domains, retained distinctions, and inferential limits must remain explicit.

Formal economy is a governing constraint. The paper inherits history-bearing configurations, manifestation maps, retained relational content, symmetry-relative interchangeability, historical extension, continuation structure, and related GR machinery from preceding work. New notation is introduced only when the inherited objects cannot state the required comparison with sufficient precision. A term such as equivalence, similarity, substitutability, simulation, bisimulation, or diagnostic agreement receives its technical meaning only after the corresponding carrier and relation have been defined.

The present release contains a complete manuscript through Sections 1–17 and incorporates a Round 3 structural revision following the manuscript-wide Round 2 integrity review. Section 2 now supplies a verified literature interface; Sections 4–8 form the five-section taxonomy proper; Sections 9–15 synthesize and apply it through lifting, relational substitution, history-sensitive embedding, subject-type stress tests, Turing-style evaluation, and the Generative Relational Equivalence Investigation. Sections 16–17 state the principal boundaries and conclusions. Empirical calibration and several open formal extensions remain future work.

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Status. This working draft contains Sections 1–17 and has completed a manuscript-wide Round 2 formal-integrity and logical-continuity revision. The revision checked symbol provenance and typing, relation domains, implication directions, historical and continuation constructions, equation references, cross-references, claim scope, and page layout. The manuscript remains a preliminary research note: empirical calibration and the open formal problems identified in Section 16 remain subject to future development. Round 3 also expanded the verified literature interface and revised the manuscript architecture so the five-part taxonomy forms a distinct five-section block.

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Statement on the use of language models. The exploratory discussions and preparation of this paper involved OpenAI’s ChatGPT. ChatGPT supported exploratory dialogue, formal reconstruction, argumentative criticism, drafting in $\text{\LaTeX}$, and technical checking. The author selected the research questions, directed and approved the theoretical commitments and epistemic status of the claims, and bears sole responsibility for the manuscript, including its definitions, formal constructions, arguments, conclusions, and errors. Authorship credit remains with the human author.

1 Introduction

This section establishes the research object, analytical scope, and formal architecture of the manuscript. Its role is to locate equivalence within the three-level Generative Relational (GR) account of reality, identify the history-bearing and coupled structures on which equivalence claims are made, and specify the derivational discipline that will govern the later taxonomy. The discussion proceeds from the general problem of equivalence to the distinction among ontological, relational, and manifestation levels; it then introduces historical extension and coupled configuration as necessary dimensions of the comparison problem, states the formal-provenance constraints used throughout the paper, and locates Human–AI comparison as one application within a more general theory. The method is formal and reconstructive. Mathematical relations are introduced only where their carrier, retained structure, and inferential role can be stated explicitly.

1.1 Research Object and Scope

Equivalence claims appear across many domains of contemporary inquiry. Two representations can be treated as equivalent for a measurement protocol, two positions can be exchangeable under a symmetry, two procedures can satisfy the same task criterion, two histories can preserve a selected ordering or compositional structure, and two dynamical systems can share a declared invariant or regime characteristic. These statements have different mathematical types and preserve different information. Treating them as interchangeable uses of a single undifferentiated concept obscures the assumptions under which each claim is meaningful.

GR intensifies this problem because the objects under comparison are historically situated and relationally constituted. A relational subject, institution, artificial system, symbolic order, or other modeled entity is represented through the relations and histories that sustain its current organization and condition its possible continuations. Equivalence therefore cannot be specified adequately by comparing isolated property lists at one selected moment. A comparison can depend on the level of reality being modeled, the relational content retained by the analysis, the historical interval under consideration, the transformations treated as admissible, the relation between a local manifestation and a richer relational structure, and the dynamical characteristics preserved across possible continuations.

The present paper develops a preliminary taxonomy for these cases. Its objective is not to define one universal equivalence relation for GR. The objective is to distinguish families of equivalence and equivalence-like relations according to the formal objects on which they are defined. The taxonomy is organized around five connected analytical stages. The first states the scope and grammar of an equivalence claim. The second studies equivalence within one GR reality level. The third studies equivalence of structured relations among levels. The fourth studies equivalence of historical evolutions of single-level or multi-level structures. The fifth studies equivalence of dynamical regimes and agreement of structural invariants. Later sections examine relations among these categories and apply the resulting architecture to Human–AI interchangeability and Turing-style evaluation.

The term *equivalence* is used broadly in the paper’s research question and more narrowly in its formal statements. A binary relation is called an equivalence relation only where reflexivity, symmetry, and transitivity are established on the declared domain. Other relations can remain useful without satisfying those axioms. Approximate similarity, tolerance, directional substitutability, simulation, diagnostic agreement, and partial correspondence therefore receive distinct names when their formal properties differ. This terminological discipline allows the taxonomy to include a wide family of comparison structures without silently promoting each of them to an equivalence relation.

1.2 Equivalence in the GR Three-Level Architecture

The GR framework distinguishes ontological reality, relational reality, and manifestation reality. The three levels do not supply three parallel copies of the same comparison problem. They impose different epistemic and formal conditions on what an equivalence claim can mean.

Ontological reality remains suspended in the present inquiry. The manuscript therefore does not supply a general GR criterion for deciding whether two beings, systems, or histories are ultimately identical or equivalent in an ontological sense. A domain-specific metaphysics can introduce such a criterion conditionally, and the formal consequences of that criterion can then be studied. The GR taxonomy itself retains ontological equivalence primarily as a boundary on inference. Similar manifestation, relational interchangeability, trajectory correspondence, or dynamical agreement does not by itself settle ultimate ontological identity.

Relational reality concerns historically persistent and type-appropriately efficacious structures represented within a GR model. Equivalence at this level depends on which relational content the analysis retains. A comparison can preserve incidence relations, dependency structure, admissible transformations, institutional standing, historical provenance, responsibility relations, continuation structure, or another declared family of relations. The retained content must therefore be stated before a structural equivalence or symmetry-relative interchangeability judgment is evaluated. A coarse retained structure can render two positions interchangeable while a richer retained structure distinguishes them.

Manifestation reality concerns the context-dependent forms through which relational or otherwise modeled structures become presented, measured, symbolized, embodied, classified, or epistemically accessible. A manifestation context can therefore induce an equivalence relation by failing to distinguish relationally different structures at the selected resolution. Such equivalence belongs to the manifestation interface and carries only the inferential content licensed by that interface. A richer context family can refine the induced equivalence classes, while a coarse context can merge structures that remain distinct at the relational level.

The distinction among these levels provides the first coordinate of the taxonomy. It also creates a second research problem that cannot be reduced to equivalence within one level: the relation among levels can itself be compared. Two systems may possess relational structures that are equivalent under a declared criterion and manifestation spaces that are separately equivalent while differing in the mappings that connect relational structures to their manifestations. The cross-level architecture is therefore a formal object in its own right. Section 1 introduces this problem only at the level required to motivate the later taxonomy; the dedicated cross-level section will define the corresponding structure and preservation conditions.

1.3 History-Bearing Relata and Coupled Configurations

The objects compared in the manuscript are history-bearing GR configurations rather than isolated points. At a selected analytical resolution, an entity can be represented by a structured historical subconfiguration embedded in a wider relational spacetime. Two entities can participate in a coupled configuration that contains their internal histories, shared or overlapping structures, and the interfaces through which their relations are generated. The effective boundary of an entity remains a modeling commitment and can change under refinement when internal relations become relevant to the comparison.

This representation has two consequences for equivalence. First, equivalence of present manifestations or present relational states can leave historically relevant differences outside the comparison. Two configurations can converge under a current readout while retaining different provenance, accumulated relations, or operation histories. Second, equivalence of individual relata does not establish equivalence of their coupling. A pair of entities can each admit a correspondence to another pair while the interfaces, shared histories, or higher-order relations of the two coupled configurations differ. The taxonomy must therefore distinguish entity equivalence, coupling equivalence, and equivalence of the wider configuration in which the relation is embedded.

History introduces an additional formal scale. GR can represent historical organization through partially ordered events and admissible extensions without requiring a primitive universal scalar time. A comparison can therefore concern a selected relational slice, a realized historical trajectory, a family of admissible continuations, or a dynamical regime supported across multiple possible histories. These objects require different preservation conditions. Endpoint equivalence compares selected ter-

minal structures. Statewise historical equivalence compares structures along a history. Order-preserving and transition-preserving relations additionally compare historical precedence and transformations. Continuation relations compare branching possibilities that remain unrealized at the selected historical boundary. Dynamical-regime comparisons can preserve attractor-like, metastable, critical, algebraic, or topological characteristics even when realized trajectories remain distinct.

The historical scale supplies the second major organizing principle of the paper. Static and historical equivalence are consequently kept distinct throughout. A result proved for one slice is not extended to trajectories by terminology alone. A trajectory-level claim must specify the historical carrier, the correspondence between historical stages, and the structures preserved by transitions or continuation relations.

1.4 Formal Provenance and Taxonomic Discipline

The taxonomy follows the formal-provenance discipline developed in preceding GR work. A nontrivial symbol acquires substantive content only after its source structure, domain, codomain, construction rule, and model status have been supplied. New equivalence notation will therefore be introduced conservatively. Where an inherited relation already states the required object, the paper reuses that relation. Where a new relation is needed, its mathematical properties are established before the terminology attached to it is strengthened.

This discipline is especially important because equivalence language can conceal several distinct operations. Equality compares elements inside an already specified representation. Structural equivalence can preserve a declared family of relations while allowing representational differences. Isomorphism requires a structure-preserving invertible correspondence of a specified type. Symmetry-relative interchangeability concerns positions connected by transformations that preserve retained relational content. Simulation can be directional. Bisimulation imposes a mutual continuation condition. Approximate similarity requires a tolerance or metric-like structure and may fail transitivity. Agreement of selected invariants records diagnostic coincidence and supports a stronger structural conclusion only when the retained invariant family is independently shown to be sufficient for that conclusion.

The paper also distinguishes an equivalence relation from the historical status of that relation. An analyst can define an equivalence relation for a model without establishing that the corresponding classification is historically persistent or relationally efficacious. Conversely, an equivalence structure can become enacted through symbolic, institutional, technical, or other relations and thereby participate in later relational generation. The taxonomy therefore separates the mathematical existence of a quotient or partition from claims that the equivalence structure itself has acquired manifestation or relational-reality status.

A further discipline concerns historical comparison of equivalence structures. Equivalence relations defined at different historical stages need not share one carrier. A later analysis can compare refinement, coarsening, splitting, merging, or reorganization only after the model supplies a tracking or transport structure connecting the relevant domains. This requirement prevents a changing classification from being treated as if one fixed partition existed outside the history through which its domain and distinctions were formed.

1.5 Human–AI Equivalence as an Application Domain

Human–AI comparison provides an analytically demanding application because contemporary artificial systems are frequently designed to produce manifestations that are legible within human linguistic, technical, institutional, and task-oriented contexts. High compatibility at an interface can therefore arise before the relation between the artificial system’s internal history-bearing organization and the corresponding human organization has been established. Manifestation similarity can be practically important without by itself establishing relational interchangeability.

The paper does not treat this difference as evidence for human exceptionalism. The same architecture applies to AI–AI and Human–Human comparisons. Two artificial systems that share architecture,

training ancestry, or an initial configuration can become relationally differentiated after divergent histories. Two humans can be task-equivalent within a declared procedure while occupying different relational positions because their institutional roles, obligations, provenance, or continuation structures differ. Conversely, a Human–AI pair can become interchangeable relative to a deliberately retained relational content when the required transformations and preservation conditions are established. The taxonomy is designed to state the scope of such conclusions rather than to predetermine their outcome.

This generality also constrains the normative interpretation of the formalism. Relational difference, non-interchangeability, historical difference, or distinct dynamical organization does not by itself establish superiority, moral status, personhood, responsibility, legal standing, or a distributive conclusion. Those questions require additional normative or institutional premises. The current manuscript studies the formal and epistemic structure of equivalence claims and keeps their normative consequences analytically separate.

Human–AI analysis nevertheless provides a particularly useful stress test for cross-level inference. A traditional Turing-style procedure primarily interrogates manifestations produced through a bounded interaction interface. Such evidence can establish a form of context-relative indistinguishability or task-relative equivalence when the evaluation rule is specified. The later sections ask what additional structure is required before such a result supports claims about relational interchangeability, historical equivalence, continuation correspondence, or generative dynamics. This problem motivates the paper’s constructive proposal for a Generative Relational Equivalence Investigation.

1.6 Manuscript Architecture

The manuscript is organized in seventeen sections. Section 2 positions the taxonomy relative to established literatures on mathematical equivalence, partial transformations, behavioural equivalence, dynamical systems, topology, and Turing-style evaluation. Section 3 then restates only the inherited GR machinery required for the new analysis: history-bearing configurations, manifestation maps, retained relational content, symmetry-relative interchangeability, historical extension, and relation-type discipline.

Sections 4–8 form the five-section taxonomy proper. Section 4 supplies the overview and comparison grammar. Section 5 treats equivalence within individual GR reality levels. Section 6 treats equivalence of relational structures connecting multiple levels. Section 7 develops equivalence and equivalence-like relations over historical evolutions, including realized trajectories, continuation families, and evolving equivalence structures. Section 8 treats dynamical-regime and structural-invariant comparisons, with explicit separation between diagnostic agreement and stronger structural equivalence.

Section 9 develops cross-taxonomy lifting, refinement, and identifiability. Section 10 develops a general relational substitution architecture before the analysis is specialized by subject type. Section 11 isolates provenance, responsibility, shared relational history, and institutional embedding as possible retained structures rather than ad hoc Human–AI exceptions. Section 12 then applies the framework to Human–AI interchangeability as a typed and historically scoped comparison problem, while Section 13 uses AI–AI and Human–Human cases to test the type-neutrality of the formalism. Section 14 locates Turing-style evaluation within the taxonomy and identifies the evidential status of its results. Section 15 develops the Generative Relational Equivalence Investigation as a structured procedure that varies manifestation contexts, retained relational content, historical scope, continuation probes, and coupling structure. Section 16 states epistemic, empirical, ontological, and normative boundaries. Section 17 concludes by consolidating the taxonomy and its open formal interfaces.

The resulting architecture treats equivalence as a typed and historically situated relation whose meaning depends on the object compared and the structure preserved. The paper therefore uses the word *equivalence* as the beginning of a formal question rather than as a self-sufficient conclusion. Its central task is to make the missing indices, domains, histories, and preservation criteria explicit enough that stronger equivalence claims can be earned rather than presumed.

2 Literature Interfaces for Equivalence Analysis

This section positions the manuscript’s equivalence taxonomy relative to several established mathematical and philosophical literatures. Its role is selective rather than exhaustive. The paper draws formal resources from equivalence relations, isomorphism, group actions, partial transformations, simulation and bisimulation, dynamical systems, topological diagnostics, and Turing-style evaluation, while retaining the distinctions among those traditions. The objective is to identify which inherited concepts are being reused, which are being generalized only at the level of formal architecture, and which GR-specific questions remain outside the scope of the source theories. The discussion proceeds from structural equivalence and partiality to behavioural equivalence, dynamical-regime comparison, topological diagnostics, and evaluation-centred accounts of machine behaviour.

2.1 Structural Equivalence in Mathematics

Mathematical comparison distinguishes several relations that ordinary language often compresses into the word “same.” Equivalence relations partition a carrier into equivalence classes, isomorphisms preserve a declared mathematical structure through invertible maps, and group actions organize transformations and orbit structure. Category theory provides a general language for objects, morphisms, functors, natural transformations, and structure-preserving comparison, while group theory supplies the standard orbit and stabilizer machinery used later in this paper [7, 8].

The present taxonomy inherits these distinctions rather than treating them as interchangeable. An equivalence relation on a declared carrier, an isomorphism between structured objects, and same-orbit membership under a selected group action can coincide in special constructions while remaining formally different claims. This distinction is especially important for GR because the retained relational content can vary with the research question. A map that preserves a coarse task-oriented structure need not preserve a finer history-sensitive structure, and a symmetry action defined at one resolution need not remain available after refinement.

The paper therefore uses established mathematical language locally. It does not propose a new general definition of mathematical equivalence. Its contribution lies in typing several existing comparison modes against GR’s reality levels, historical carriers, coupled configurations, and generative continuation structures.

2.2 Partial Transformation Structures

Partiality is substantive in a history-bearing relational model because a transformation can be meaningful only on a restricted domain. Restriction categories abstract categories of partial maps by associating domain information with arrows, while partial group actions and inverse-semigroup constructions provide established ways to organize transformations that are not globally defined [5, 6].

GR uses this literature as a formal precedent for preserving definedness. The manuscript’s partial manifestation maps, operation schemas, historical transports, and comparison maps are introduced only on declared domains. Undefined comparison is therefore distinct from established non-equivalence. The distinction matters throughout the taxonomy: two structures can fail to be comparable because the required map has no justified domain, even when a richer future representation might later make the comparison meaningful.

The present manuscript does not require every partial transformation family to form a restriction category, inverse semigroup, or partial group action. Those stronger structures enter only where their compositional assumptions perform analytical work. This follows the paper’s economy-of-primitives discipline.

2.3 Behavioural Equivalence in Process Semantics

Simulation and bisimulation provide a mature language for comparing systems through their possible transitions rather than through static descriptions alone. Sangiorgi’s treatment emphasizes bisimulation as a behavioural equality for processes, its relation to coinduction, refinements of simulation, and its comparison with testing-based behavioural equivalences [4]. This literature is directly relevant to the manuscript’s distinction between realized-history agreement and continuation-structure comparison.

The GR use of simulation and bisimulation is deliberately typed. A continuation simulation requires one-sided matching of retained successor structure; a continuation bisimulation requires a back-and-forth matching relation over the declared continuation carriers. The paper does not identify a GR relational entity with a process-calculus term or import one fixed labelled-transition-system ontology. It borrows the structural distinction between one-directional simulation and mutual behavioural matching and re-constructs that distinction over GR historical extensions.

This interface also clarifies why finite observed agreement and continuation equivalence are different claims. A pair of systems can exhibit the same realized sequence while possessing different unactualized successor structures. Testing evidence can therefore be highly informative about a selected behavioural interface without exhausting the continuation structure represented by the model.

2.4 Dynamical-Regime Comparison

Dynamical-systems theory supplies several comparison objects that are stronger than equality of selected trajectories and weaker than unrestricted identity of systems. Standard treatments distinguish invariant sets, stability, bifurcation structure, and qualitative dynamical organization; Wiggins provides a broad geometric treatment of nonlinear dynamics and bifurcation methods [9]. Attractor language also admits multiple mathematically substantive definitions. Milnor’s formulation is one influential example and illustrates why the term “attractor” should not be used without a declared mathematical meaning [10].

The present taxonomy therefore distinguishes trajectory equivalence from dynamical-regime comparison. Two realized trajectories can differ while selected attractor, basin, recurrence, or bifurcation structures correspond. Conversely, agreement on a small set of dynamical diagnostics does not establish a stronger structural equivalence unless those diagnostics are proven sufficient on the declared domain.

Critical-transition research supplies a further example of diagnostic-relative comparison. Scheffer and collaborators survey indicators associated with approaching critical transitions and emphasize the difficulty of predicting tipping points across complex systems [12]. GR uses susceptibility, recovery, amplification, and structural-margin language only after the perturbation family and diagnostic interpretation have been specified. Diagnostic agreement remains distinct from identity of the underlying dynamical architecture.

2.5 Topological Diagnostics

Algebraic topology provides calculable invariants that can detect some structural differences while remaining incomplete as classifiers of topology. Homology groups and Betti numbers are standard examples; Hatcher provides the background used by this paper for chain complexes, homology, and related invariants [11].

This one-sided inferential structure is important for the taxonomy. A change in a genuine topological invariant can establish non-equivalence under a target relation that preserves that invariant. Equality of a selected invariant family generally establishes only diagnostic agreement. The manuscript therefore separates topological equivalence, homological equivalence, equality of Betti profiles, and equality of selected diagnostic vectors.

The same distinction extends to representation change. Topological quantities belong to the selected complex, so their substantive interpretation depends on a justified mapping between the relational system and the represented carrier. The formal calculation alone does not establish an ultimate topology of social, institutional, human, or artificial relational reality.

2.6 Turing-Style Evaluation Literature

Turing’s 1950 article replaces the initial question concerning machine thought with an imitation-game architecture centred on observable interaction [1]. Subsequent literature has examined the interpretation, evidential force, and philosophical status of the resulting test. Moor analyzes what a Turing-test result can establish and distinguishes the test from stronger claims about mentality [2]. Saygin, Cicekli, and Akman survey fifty years of debate, implementation, and philosophical discussion surrounding the Turing Test [3].

The present paper enters this literature at a narrower formal point. It treats a Turing-style evaluation as one possible manifestation-centred evidence architecture and asks which equivalence relation the evaluation actually induces. This reframing does not dismiss the test. It makes its inferential target explicit. An evaluation can be fully adequate for a task-relative or manifestation-relative equivalence while remaining insufficient for a richer relation whose classes depend on historical provenance, continuation structure, coupled relational embedding, or another unobserved feature.

2.7 Positioning of the GR Taxonomy

The literatures reviewed above solve different problems. Category theory and group theory formalize structural transport and symmetry; partial-map theories preserve domains of definition; simulation and bisimulation compare transition structures; dynamical systems compare qualitative evolution; algebraic topology supplies structural invariants; and Turing-style evaluation organizes evidence available through an interaction interface. The present manuscript does not identify these traditions with one another.

Its proposed contribution is an architecture for stating where each comparison operates inside a GR model. The taxonomy indexes comparison by reality or representation level, historical extent, retained structure, relation type, and resolution. It then studies cross-level lifting, historical evolution of equivalence structures, continuation relations, coupled substitution, and diagnostic sufficiency. The result is intended as a coordination framework: established mathematical notions retain their own assumptions, while GR specifies the history-bearing and relational objects on which those notions are conditionally instantiated.

3 Formal Foundations for GR Equivalence

This section fixes the inherited Generative Relational objects required by the equivalence taxonomy developed in Sections 4–8. Its role is preparatory: the section identifies the carriers on which later comparison relations act, the context-dependent maps through which relational structures become manifest, the retained relational content relative to which structural equivalence is evaluated, the symmetry machinery that supports one established notion of relational interchangeability, and the historical-extension structure used for trajectory and continuation analysis. The construction is reconstructive rather than additive. It reuses formal objects established in preceding GR work and introduces only the local notation required to make their interfaces mutually explicit.

The section also fixes a terminology discipline that governs the remainder of the paper. The phrase *equivalence relation* is reserved for a relation whose reflexive, symmetric, and transitive properties hold on a declared domain. Structural correspondence, isomorphism, interchangeability, simulation, bisimulation, substitutability, tolerance, and agreement of diagnostics are treated according to their own formal types. This discipline allows the taxonomy to remain broad while preventing the word equivalence from silently strengthening a weaker comparison.

3.1 History-Bearing Configurations and Relational Slices

The primary relata of the present paper are history-bearing GR structures rather than point-like actors. A finite or locally finite discrete representation can use a labelled 2-complex. The candidate carrier is

stated in Equation 1.

$$K = (V, E, F, \partial, \lambda). \quad (1)$$

In Equation 1, V , E , and F denote the retained vertices, edges, and faces; ∂ records incidence and attachment data; and λ collects model-dependent labels on cells or regions. The representation permits events, relations, higher-order attachments, and historically accumulated structure to remain explicit at a declared resolution. The cell types receive their substantive interpretation from the application. Equation 1 is therefore a candidate representation of GR spacetime rather than an assertion that relational reality possesses a unique combinatorial ontology.

An entity represented at the selected resolution occupies a history-bearing subconfiguration of a wider carrier. For two entities A and B , the inclusion architecture is stated in Equation 2.

$$K_A, K_B \subseteq K_{AB} \subseteq K. \quad (2)$$

The coupled configuration K_{AB} contains the portions of K_A and K_B required by the analysis together with retained cross-entity relations, interfaces, shared structures, and relevant environmental or institutional relations. The inclusions in Equation 2 do not require K_A and K_B to be disjoint or uniquely decomposable. A relation can be constituted partly by structures that belong to neither entity in isolation, while overlapping history or shared institutions can be represented inside both the entity-level and coupled descriptions according to the selected resolution.

The same history-bearing organization can be described at a selected relational slice when a local analysis requires an explicit separation among accumulated history, active boundary, and effective nomos. The inherited slice-indexed working configuration is stated in Equation 3.

$$\omega_\Sigma = (G_{\leq \Sigma}, B_\Sigma, N_\Sigma). \quad (3)$$

Here $G_{\leq \Sigma}$ denotes the accumulated relational history retained at slice Σ , B_Σ denotes the active relational boundary, and N_Σ denotes the effective nomos conditioning distinctions, admissibility, recognition, or other rule-governed aspects of the modeled configuration. Equation 3 is a working decomposition used where the later comparison requires those components. It does not replace the richer carrier of Equation 1, and a model can retain additional structure when the three components are insufficient for the local equivalence problem.

The relation between Equations 1 and 3 is representational. A labelled complex can supply one concrete carrier for the historical, boundary, and nomic structures summarized by ω_Σ , while an application can use another adequate carrier if its comparison problem requires different formal resources. The present paper therefore treats K , K_A , K_B , K_{AB} , and ω_Σ as compatible inherited interfaces rather than as competing ontological definitions.

Remark 3.1 (Entity boundaries and resolution). An equivalence claim concerning an entity is indexed implicitly by the entity boundary and representational resolution under which the entity is identified. A coarse representation can suppress internal histories or interfaces that become relevant after refinement. Later equivalence claims must therefore report the resolution at which their carriers, retained structures, and comparison relations are defined.

Remark 3.1 matters for all three principal application classes considered later. A human, an artificial system, an institution, or another relational subject can be represented as one effective entity for one comparison and as a structured historical subconfiguration for another. The taxonomy does not treat a change of analytical resolution as evidence that the represented being itself has changed.

3.2 Manifestation Maps and Context-Relative Distinction

GR distinguishes a relational structure from the context-dependent forms through which that structure becomes accessible, legible, measured, symbolized, embodied, or otherwise presented. Let $\mathcal{R}$ denote a model-defined family of relational objects admissible for the local analysis, and let X_c denote the

representation space associated with manifestation context c . The inherited manifestation map is the partial map stated in Equation 4.

$$M_c : \mathcal{R} \rightharpoonup X_c. \quad (4)$$

Partiality in Equation 4 is substantive. A context can fail to render part of a relational structure legible, or the model can lack the evidence required to assign a warranted manifestation. A linguistic interface, legal classification, technical protocol, observational instrument, institutional vocabulary, or task-specific evaluation environment can each instantiate a different context c and therefore a different map M_c .

Equality of manifestation induces a context-relative relation on the domain of M_c . The inherited relation is stated in Equation 5.

$$x \sim_c y \iff M_c(x) = M_c(y), \quad x, y \in \text{Dom}(M_c). \quad (5)$$

Because equality in X_c is reflexive, symmetric, and transitive, Equation 5 defines an equivalence relation on $\text{Dom}(M_c)$. Its interpretation remains limited: $x \sim_c y$ states that the selected manifestation context fails to distinguish the two relational objects at the modeled resolution. It does not establish that x and y are structurally equivalent under a relational criterion, historically identical, interchangeable under an admissible transformation family, or ontologically identical.

The many-to-one character of a manifestation map is therefore analytically important. Several relational structures can occupy one $\sim_c$ -equivalence class, and one relational structure can receive different manifestations under different contexts. A later section will use this structure to distinguish single-context and multi-context manifestation equivalence, to define separating context families, and to state conditions under which a manifestation-level result can support a relational inference. The present section requires only the inherited map and the induced relation.

Remark 3.2 (Manifestation and relational status). Manifestation reality and relational reality are analytical roles rather than disjoint carriers of existence. A relationally real structure can have several manifestations, and a manifested classification or equivalence structure can later become relationally efficacious. Equation 4 specifies an epistemic or presentational interface; it does not assign the reality status of the structure that generated, enacted, or received the manifestation.

Remark 3.2 prevents a later quotient or evaluation procedure from being interpreted as a complete ontology of its source objects. It also permits equivalence structures themselves to become objects of historical analysis when their manifested classifications participate in subsequent relational generation.

3.3 Retained Relational Content and Structural Equivalence

Manifestation equivalence compares what a context preserves in presentation. Relational structural equivalence requires an independent statement of the relational content selected for preservation by the analysis. The inherited GR formalism indexes that retained content by κ . Depending on the application, κ can retain incidence, dependency, access, capacity, role, responsibility, provenance, history, admissible transformations, continuation structure, or another explicitly declared relational feature.

Let $D_\kappa \subseteq \mathcal{R}$ denote the declared domain on which the retained content is sufficiently articulated for comparison. When the local model has established a structural equivalence relation on that domain, the paper writes the relation as in Equation 6.

$$\simeq_\kappa \subseteq D_\kappa \times D_\kappa. \quad (6)$$

Equation 6 is a type declaration rather than a universal definition of relational equivalence. A local application must specify the preservation criterion encoded by κ and establish the relation's mathematical properties before the terminology *equivalence relation* is used. The present manuscript reserves $\simeq_\kappa$ for cases in which the retained criterion has been stated and reflexivity, symmetry, and transitivity hold on D_κ .

The index κ is analytically prior to the resulting judgment. It is fixed by the research question and declared retained structure rather than selected after the compared objects have been inspected for a desired result. This ordering prevents an equivalence claim from gaining strength by silently discarding inconvenient differences. A coarse κ can legitimately ignore distinctions irrelevant to one problem, while a different analysis can retain those distinctions under another index. The same pair of relational objects can therefore be equivalent relative to one retained content and non-equivalent relative to another without logical inconsistency.

The relation $\simeq_\kappa$ also remains distinct from literal equality. A representation can change labels, local coordinates, or other non-retained details while preserving the relational content encoded by κ . Conversely, two descriptions can be numerically or symbolically similar while differing in a relation that κ designates as preservation-relevant. The later taxonomy will use this distinction to separate structural equivalence from task similarity, diagnostic agreement, and representation-level coincidence.

Caution 3.3 (Retained content and post hoc comparison). A comparison should not be promoted to κ -relative structural equivalence when the retained content has been chosen only after observing which differences must be omitted to obtain equivalence. The analytical role of κ , its domain D_κ , and its preservation criterion should be stated independently of the specific equivalence conclusion under examination.

Caution 3.3 does not require one universal retained content. GR permits problem-relative representations. The requirement concerns transparency about which relational distinctions are being preserved and which are deliberately left outside the comparison.

3.4 Symmetry Actions and Relational Interchangeability

Structural equivalence can support a stronger transformation-based notion when the relational regime admits an articulated symmetry structure. Let $\mathcal{X}^*$ be a selected regime of relational configurations, let G be a group acting on that regime, and let α denote the action. The action is stated in Equation 7.

$$\alpha : G \times \mathcal{X}^* \longrightarrow \mathcal{X}^*. \quad (7)$$

The symmetry construction applies only when the retained equivalence relation $\simeq_\kappa$ is compatible with the action on the declared domain. For a fixed configuration $C \in \mathcal{X}^*$, the inherited relational symmetry group relative to κ is stated in Equation 8.

$$G_\kappa(C) := \{g \in G \mid \alpha(g, C) \simeq_\kappa C\}. \quad (8)$$

When the assumptions required by the preceding GR symmetry formalism are satisfied, Equation 8 collects the transformations that preserve the relational content retained by κ . Its existence is conditional on an articulated transformation family, a compatible retained equivalence, and a regime on which the relevant operations are defined.

Configuration-level symmetry does not yet establish interchangeability of individual positions. Let $P_\kappa(C)$ denote the retained relational positions distinguished in C at the resolution encoded by κ . A coherent induced position action has the type stated in Equation 9.

$$\rho_{C,\kappa} : G_\kappa(C) \longrightarrow \text{Bij}(P_\kappa(C)). \quad (9)$$

For $a \in P_\kappa(C)$, the corresponding orbit is defined in Equation 10.

$$\text{Orb}_{C,\kappa}(a) := \{\rho_{C,\kappa}(g)(a) \mid g \in G_\kappa(C)\}. \quad (10)$$

The inherited criterion for relational interchangeability is then the same-orbit condition stated in Equation 11.

$$\text{Orb}_{C,\kappa}(a) = \text{Orb}_{C,\kappa}(b). \quad (11)$$

Equation 11 is equivalent, under the articulated group-action assumptions, to the existence of some $g \in G_\kappa(C)$ with $\rho_{C,\kappa}(g)(a) = b$. It states that the two retained positions are related by transformations that preserve the selected relational structure. The criterion does not establish identity of the beings occupying the positions, equality of their full histories, or interchangeability after the retained content is changed.

The taxonomy developed later will therefore reuse Equation 11 rather than introduce a separate primitive relation for relational interchangeability. This reuse is methodologically important for Human–AI comparison. A Human and an artificial system can occupy the same orbit under a deliberately coarse κ while separating under another retained content that includes history, responsibility, dependency, embodiment, provenance, continuation structure, or another relevant feature. The converse possibility also remains open: relational interchangeability can be established relative to a sufficiently specified retained structure even when the compared beings differ in many properties outside that structure.

Remark 3.4 (Domain of the orbit criterion). Same-orbit interchangeability is available only inside an articulated symmetry regime. Pre-differentiated structures, partial transformation domains, primordial asymmetries, or relational systems lacking a justified group action can require a different comparison relation. Failure to satisfy the assumptions of Equation 11 leaves the orbit criterion inapplicable; it does not by itself establish non-equivalence.

Remark 3.4 will matter when the taxonomy later compares equivalence relations with simulations, partial correspondences, and continuation relations whose mathematical structure differs from group-orbit equivalence.

3.5 Historical Extension and Continuation Structure

Static comparison at a selected relational slice captures only one part of a history-bearing GR structure. The later taxonomy also compares realized trajectories, branching continuations, and dynamical regimes. The inherited history formalism begins from extension relations rather than a primitive universal scalar time.

Let K^- denote the relational complex treated as realized history at a selected modeling stage. An admissible continuation is represented by a pair (K', ι) in which K' is a candidate continuation complex and ι is an admissible embedding of the realized history into that continuation. The extension family is stated in Equation 12.

$$\text{Ext}(K^-) := \{(K', \iota) \mid \iota : K^- \hookrightarrow K' \text{ is an admissible historical extension}\}. \quad (12)$$

Admissibility in Equation 12 is supplied by the selected historical regime. In a fixed-past representation, ι preserves the incidence and label structure declared to belong to realized history while permitting new cells, relations, labels, or other continuation structure to enter K' . Stronger history-reconfiguring regimes require different transport maps and are treated only when an application explicitly permits modification of the represented past.

Historical order can be induced by the existence of admissible extension embeddings. For history representations K_i and K_j , the inherited extension relation is stated in Equation 13.

$$K_i \preceq K_j \iff \text{there exists an admissible embedding } \iota_{ij} : K_i \hookrightarrow K_j. \quad (13)$$

When identity embeddings and composition are admissible, Equation 13 is reflexive and transitive. A partial-order interpretation additionally requires an identification rule under which mutually extending histories are treated as equivalent. The resulting order can retain branching and incomparability, allowing the analysis to represent several admissible continuations without imposing one global sequence.

Equations 12 and 13 provide the basic objects from which Section 7 will construct historical and continuation relations. The later section will distinguish comparison of realized endpoints, correspondence along an ordered history, transition-preserving relations, simulation, bisimulation, and weighted continuation structures. None of those additional relations is assumed by the existence of $\text{Ext}(K^-)$ alone.

Table 1: Comparison-relation types used in the GR equivalence taxonomy. The final column states the minimum inferential boundary retained by the manuscript.

Relation type	Formal role	Inferential boundary
Equality	Identity of elements inside one declared representation	Does not establish representation-independent identity
Equivalence relation	Reflexive, symmetric, and transitive relation on a declared domain	Requires all three properties on the stated carrier
Structural isomorphism	Invertible correspondence preserving a specified structure	Preserves only the structure included in the isomorphism type
Symmetry-relative interchangeability	Same-orbit relation under a justified action preserving retained content	Applies only within the articulated symmetry regime and selected κ
Simulation	Directional matching of selected states, transitions, or continuations	Does not imply reverse matching
Bisimulation	Mutual back-and-forth matching under a declared transition structure	Preserves the declared transition observations, not unrestricted identity
Approximate similarity or tolerance	Closeness under a declared metric, discrepancy, or tolerance rule	May fail transitivity and is not called an equivalence relation by default
Diagnostic agreement	Equality or correspondence of selected invariants or summary diagnostics	Supports stronger structural equivalence only when diagnostic sufficiency is established

Remark 3.5 (Realized history and possible continuation). Two history-bearing configurations can be equivalent under a selected present-state criterion while admitting different continuation families. Conversely, distinct realized histories can support continuation structures that are equivalent relative to a declared future-oriented comparison. Present structural equivalence, realized-trajectory equivalence, and continuation-family equivalence therefore remain separate objects in the taxonomy.

Remark 3.5 supplies one of the principal motivations for extending equivalence beyond a static three-level comparison. GR permits the past structure retained by a model, the current relational organization, and the space of admissible becoming to carry distinct information.

3.6 Relation Types and Exactness Conditions

The preceding subsections supply the principal carriers and inherited GR relations. The taxonomy still requires a common discipline for naming comparison structures. Table 1 records the minimum distinctions used throughout the manuscript. The table states formal roles rather than a hierarchy of analytical importance.

The first distinction in Table 1 concerns equality and equivalence. Equality is internal to a chosen representation. An equivalence relation groups elements according to a declared criterion and can support quotient construction. Isomorphism adds a structure-preserving invertible map and therefore requires the structure to be specified before the correspondence can be evaluated. Symmetry-relative interchangeability is a more specialized construction inherited from the orbit formalism in Subsection 3.4.

Simulation and bisimulation become relevant when the compared object includes transition or continuation structure. A simulation can establish that one structure can match selected developments of another while leaving the reverse direction open. A bisimulation adds a reciprocal matching condition. Whether either relation is itself treated as an equivalence relation depends on its formal definition and domain in the relevant section. The present manuscript therefore avoids using *bisimilar* as a synonym for ontologically or relationally identical.

Approximate similarity requires additional numerical or ordered structure. A rule such as $d(x, y) < \varepsilon$ can be useful for empirical or computational comparison while failing transitivity on its domain. The taxonomy labels such a relation by its actual mathematical type unless the application constructs an equivalence relation from it through an independently justified partition or quotient. The same restraint applies to task-level tolerances, approximate dynamical matching, and numerical thresholds used in later applications.

Diagnostic agreement requires particular care in the dynamical and topological part of the taxonomy. Equality of one spectrum, Betti number, attractor count, susceptibility statistic, or other invariant can show agreement of the selected diagnostic while leaving non-measured structure different. A stronger conclusion is licensed only when the diagnostic family is complete for the structural equivalence relation being claimed, or when an additional theorem supplies the required implication.

Definition 3.6 (Well-formed GR equivalence claim). A GR equivalence claim is well formed within the present manuscript when it specifies, directly or through an established local convention, (i) the carrier and applicability domain of the compared objects; (ii) the GR reality or representation level at which the relation is evaluated; (iii) the historical scope of the claim; (iv) the retained structure or diagnostic content; (v) the mathematical type of the comparison relation; and (vi) the representational resolution relevant to the conclusion.

Definition 3.6 is a methodological requirement rather than a claim that one six-component tuple constitutes the ontology of equivalence. Several components can be fixed by context and suppressed in local notation once their values are unambiguous. The purpose of the definition is to make explicit the information that must be recoverable before an expression such as “ A is equivalent to B ” receives a determinate formal interpretation.

Research Obligation 3.7 (Formal provenance of a new equivalence relation). When a later section introduces a comparison relation that is not inherited from the formalism summarized above, the construction should identify its source objects, domain, codomain or relational carrier, defining rule, applicability conditions, mathematical properties, and reality or modeling status before the relation is used to support a stronger inference.

Research Obligation 3.7 operationalizes the paper’s economy-of-primitives rule. It also prepares the five-part taxonomy. Section 4 will use the inherited carriers to define the taxonomy’s coordinates; Section 5 will organize equivalence within individual GR levels; Section 6 will treat relational structures connecting levels; Section 7 will lift comparison to historical evolution and continuation; and Section 8 will distinguish dynamical-regime equivalence from agreement of selected structural diagnostics.

4 Taxonomic Architecture of GR Equivalence

This section establishes the organizing architecture for the five-part equivalence taxonomy developed in Sections 4–8. Its role is to specify the coordinates by which later equivalence claims are classified, to distinguish exact equivalence relations from other comparison types, to separate identity from structure-preserving correspondence, and to state how resolution and partiality constrain comparison. The section concludes with an implication discipline that permits refinement and logical inclusion to be represented where the compared relations share a carrier, while withholding stronger or weaker orderings across different carriers until an explicit transport or comparison structure has been supplied. The method is formal and classificatory. It introduces bookkeeping structures for the taxonomy without adding a further layer to the GR reality architecture.

The need for such an architecture follows from the heterogeneity of the objects that later sections compare. A context-induced equivalence on manifestations, a symmetry-orbit relation on relational positions, an isomorphism between cross-level structures, a bisimulation between continuation systems, and equality of selected dynamical diagnostics can each support a meaningful comparison while preserving

different information. A single symbol such as $\sim$ cannot carry these distinctions without an accompanying specification. The taxonomy therefore classifies comparison claims by the object compared, the reality or representation level retained, the historical extent of the claim, the structure preserved, the mathematical relation type, and the resolution at which the comparison is evaluated.

4.1 Taxonomic Coordinates

Definition 3.6 identified six items that must be recoverable from a well-formed GR equivalence claim. The present subsection reorganizes those items into a taxonomic record that can be reused across the four substantive categories developed in Sections 5–8. The record is bookkeeping for the analysis. Its entries are derived from the already declared carriers and comparison structures and do not constitute a new substantive GR entity.

Let $\mathcal{D}$ denote the declared carrier or structured domain on which a comparison is evaluated. Let ℓ record the GR reality or representation level, or the declared collection of levels, retained by that carrier. Let h record the historical extent of the claim, let κ record the structure retained by the comparison, let r record the mathematical relation type, and let ϱ record the representational resolution. The resulting taxonomic specification is stated in Equation 14.

$$\mathcal{T} := (\mathcal{D}, \ell, h, \kappa, r, \varrho). \quad (14)$$

Equation 14 does not assert that every application must display all six entries in every formula. Once a section fixes a common domain, history window, or resolution, later notation can suppress those indices. The specification requires only that the omitted information remain recoverable from the local definitions.

The carrier $\mathcal{D}$ performs a particularly important role because the five-part taxonomy is organized by the formal object on which comparison is defined. Table 2 records the four substantive carrier families that follow the present architectural section.

The level coordinate ℓ is also allowed to vary in type. In Section 5 it selects an individual level, such as manifestation or relational reality. In Section 6 it can identify an organized collection of levels together with the maps connecting them. In Section 7 the compared object can be a trajectory of structures whose level content is fixed or itself changes across the retained history. Section 8 can compare dynamical objects derived from any of those preceding carriers once the additional mathematical assumptions required by the diagnostic have been stated.

Historical extent h distinguishes at least four scopes that recur later: a selected slice, a realized historical trajectory, a family of admissible continuations, and a dynamical regime defined over a declared family of histories or configurations. These scopes should not be interpreted as a universal linear progression. A dynamical-regime comparison can preserve global or asymptotic organization while allowing realized trajectories to differ substantially, whereas a trajectory-level equivalence can preserve detailed historical correspondence without supplying a complete characterization of the surrounding dynamical regime.

The retained content κ generalizes the role already introduced for relational symmetry. Depending on the carrier, κ can select manifestation distinctions, relational dependencies, admissible operations, incidence structure, historical provenance, coupling relations, continuation branching, path composition, or other explicitly declared content. The same pair of objects can therefore occupy one equivalence class under κ_1 and different classes under a finer κ_2 . Section 4.4 gives the corresponding refinement relation a precise form.

The relation-type coordinate r prevents the word *equivalence* from absorbing mathematically distinct comparison modes. It can record equality, an equivalence relation, isomorphism, symmetry-relative interchangeability, simulation, bisimulation, tolerance, or diagnostic agreement. The resolution coordinate ϱ records the granularity at which the carrier and retained structure have been represented. A comparison can therefore be stable under one resolution and fail after refinement without contradiction, provided that the two claims are indexed to different ϱ .

Table 2: Principal carrier families of the GR equivalence taxonomy. The taxonomy is organized by the formal object receiving the comparison relation rather than by a universal ordering of strength.

Taxonomic category	Representative carrier	Primary comparison question
Single-level structure	Objects, configurations, manifestations, or relational positions at one declared GR level	Which distinctions are preserved within the selected level and retained content?
Cross-level relational structure	A structured family of levels together with manifestation, readout, quotient, or other level-coupling maps	Do the levels and the relations connecting them correspond under one coherent structural mapping?
Historical evolution	Ordered families of single-level or multi-level structures with historical transport, transition, or extension data	Do the compared histories preserve selected states, order, transitions, generative operations, or continuation structure?
Dynamical or structural characteristics	Attractors, basins, response structures, algebraic composition, spectra, topology, homology, or other declared diagnostics	Do different systems exhibit corresponding dynamical organization or only agreement of selected diagnostics?

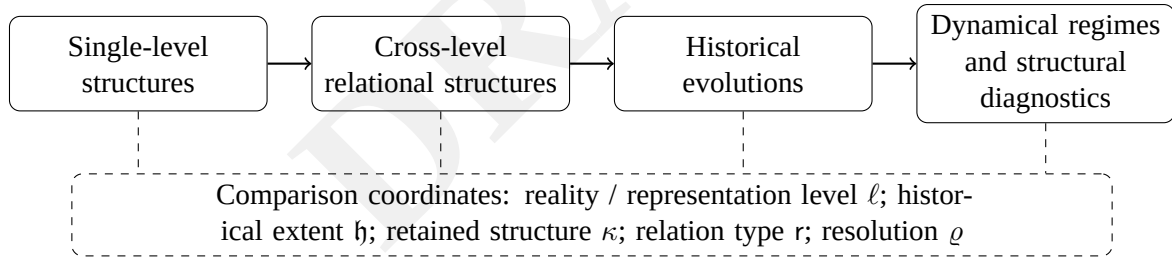


Figure 1: Taxonomic architecture by formal object and comparison coordinates. The four substantive categories classify the carrier on which comparison is defined. Level, historical extent, retained structure, relation type, and resolution constrain the meaning of each local claim. The horizontal arrangement represents structural scope rather than a universal order of inferential strength.

Figure 1 summarizes the architecture. The horizontal organization records increasing structural scope of the carrier, while the vertical annotations record comparison coordinates that must be supplied locally. The horizontal ordering is classificatory and does not assert that a category farther to the right is logically stronger.

Definition 4.1 (Taxonomically located comparison). A comparison is taxonomically located when its formal object belongs to one of the carrier families in Table 2, its required entries in Equation 14 are fixed or recoverable, and the comparison relation is evaluated only on the domain for which those entries are defined.

Definition 4.1 permits a later relation to occupy more than one descriptive coordinate without collapsing the carrier categories. A continuation bisimulation, for example, can be relational in level, historical

in extent, generative in retained content, and bisimulation-like in relation type. Its carrier nevertheless belongs to the historical-evolution category because the compared object includes a transition or continuation structure rather than a single relational slice.

4.2 Exact Equivalence within the Comparison Vocabulary

The taxonomy uses *equivalence* in two related senses. In the narrow mathematical sense, an equivalence relation is reflexive, symmetric, and transitive on a declared domain. In the broader classificatory sense, the paper studies equivalence and equivalence-like comparison relations that answer questions of sameness, substitutability, structural correspondence, mutual simulation, or diagnostic agreement under stated criteria. The broader vocabulary supports comparison across heterogeneous objects while the narrow term *equivalence relation* remains reserved for relations whose axioms have been established.

A general source of exact equivalence relations is a readout or classification map. Let $f : D \rightarrow Y$ be a map on a declared domain D . Define the kernel relation of f by Equation 15.

$$x \sim_f y \iff f(x) = f(y), \quad x, y \in D. \quad (15)$$

The manifestation relation $\sim_c$ from Section 3 is a special case with $f = M_c$ and $D = \text{Dom}(M_c)$.

Proposition 4.2 (Kernel equivalence of a declared map). *For any map $f : D \rightarrow Y$, the relation $\sim_f$ defined by Equation 15 is an equivalence relation on D .*

Proof. For every $x \in D$, equality gives $f(x) = f(x)$, so $x \sim_f x$. If $x \sim_f y$, then $f(x) = f(y)$ and symmetry of equality gives $f(y) = f(x)$, so $y \sim_f x$. If $x \sim_f y$ and $y \sim_f z$, then $f(x) = f(y) = f(z)$ and transitivity of equality gives $f(x) = f(z)$, so $x \sim_f z$. $\square$

Proposition 4.2 is elementary, yet it performs an important taxonomic role. It identifies when a context, task evaluator, coarse-graining map, or other declared readout genuinely induces an equivalence relation rather than a looser similarity rule. The proposition does not determine whether the resulting classes preserve the distinctions relevant to another level or another research question.

Several exact equivalence relations can be combined without requiring a scalar aggregation. Let $\{\sim_i\}_{i \in I}$ be equivalence relations on a common domain D . Define the common refinement by Equation 16.

$$x \sim_{\cap I} y \iff x \sim_i y \text{ for each } i \in I. \quad (16)$$

Proposition 4.3 (Intersection of exact equivalence relations). *The relation $\sim_{\cap I}$ defined by Equation 16 is an equivalence relation on D .*

Proof. Reflexivity, symmetry, and transitivity hold componentwise for every $\sim_i$. Their conjunction therefore satisfies the same three properties on the common domain D . $\square$

Proposition 4.3 will support multi-context manifestation equivalence in Section 5. It also provides a general model for combining independently justified exact criteria without introducing an arbitrary numerical weighting among them.

Not every useful comparison enters this kernel or intersection form. A simulation can be directional, an approximate tolerance relation can fail transitivity, a structural isomorphism is a map rather than initially a binary relation on one carrier, and equality of selected diagnostics need not establish the structural relation of interest. The paper uses the phrase *equivalence-like relation* as a taxonomic umbrella for such cases, with the mathematical type stated explicitly at the point of use.

Caution 4.4 (Equivalence-like vocabulary). The phrase *equivalence-like* carries no independent mathematical axiom in this manuscript. It marks a comparison whose analytical role concerns sameness, substitutability, or structural correspondence while its actual type is specified separately as an equivalence relation, isomorphism, simulation, bisimulation, tolerance, diagnostic agreement, or another declared structure.

Caution 4.4 prevents the breadth of the taxonomy from weakening its formal terminology. A later empirical application can use approximate or asymmetric comparisons where those are appropriate while reporting them under their actual mathematical status.

4.3 Comparison Modes across Identity, Isomorphism, Interchangeability, and Similarity

The objects grouped informally under “sameness” occupy different formal roles. Identity concerns one representation, isomorphism preserves a specified structure through an invertible correspondence, interchangeability concerns admissible transformation within a relational structure, and similarity concerns a declared measure or tolerance. Their separation is especially important for Human–AI analysis because equal task outcomes, structurally corresponding processes, and substitutability within an institution can coexist without establishing identity of the participating beings.

Let $x, y \in D$ belong to the same declared representation. Equality $x = y$ is the strongest internal identity statement available in that representation, yet its interpretation remains representation-dependent. A coarse carrier can identify structures that a finer carrier separates. Equality in one modeled state space therefore does not establish identity across all admissible representations or across ultimate ontology.

Structural isomorphism begins from two structured objects, say $\mathfrak{A}$ and $\mathfrak{B}$, and an invertible map $F : \mathfrak{A} \rightarrow \mathfrak{B}$ that preserves the operations, relations, incidence, order, or other structure specified by the isomorphism type. The symbol $\mathfrak{A} \cong \mathfrak{B}$ is meaningful only after that structure has been declared. An isomorphism can support an equivalence relation on an appropriately chosen class of structures by declaring two objects equivalent when such an isomorphism exists, provided the corresponding closure properties and size conditions are satisfied on the domain used by the model. The manuscript introduces that quotient only where it serves a later analytical purpose.

Relational interchangeability has a different provenance. Section 3 inherited the condition that positions $a, b \in P_\kappa(C)$ are interchangeable when a symmetry transformation preserving the retained structure transports one to the other. The corresponding orbit criterion is restated in Equation 17 for later reference.

$$\text{Orb}_{C,\kappa}(a) = \text{Orb}_{C,\kappa}(b). \quad (17)$$

Equation 17 concerns positions inside an articulated relational frame. It does not require equality of the beings occupying those positions or equality of their full histories. A finer retained content can remove the symmetry that supported the earlier interchangeability.

Similarity requires another structure. Let $d : D \times D \rightarrow [0, \infty)$ be a declared discrepancy or distance-like function, and let $\varepsilon > 0$ be a model-specified tolerance. The relation in Equation 18 records an approximate match.

$$x \approx_{d,\varepsilon} y \iff d(x, y) \leq \varepsilon. \quad (18)$$

Even when d is a metric, Equation 18 need not be transitive. If $d(x, y) \leq \varepsilon$ and $d(y, z) \leq \varepsilon$, the triangle inequality gives only $d(x, z) \leq 2\varepsilon$. The tolerance relation is therefore reported as approximate similarity unless an additional construction supplies an actual equivalence relation.

Substitutability can be asymmetric as well. A system A can satisfy the requirements of a task or relational role previously performed by B while B lacks one operation required in the reverse substitution. A directional substitutability relation is therefore naturally represented as a preorder or simulation-like structure when its application supplies reflexivity and transitivity, while mutual substitutability can generate a derived equivalence only under additional conditions. Section 12 will use this distinction to prevent task replacement from being identified prematurely with relational interchangeability.

Remark 4.5 (Sameness depends on the preserved structure). Identity, isomorphism, interchangeability, similarity, and substitutability answer different comparison questions. Their conclusions become comparable only after the analysis states which structures, transformations, or outcomes are required to be preserved and how the corresponding carriers are related.

Remark 4.5 also constrains cross-level inference. Equality of two manifestations can induce an exact equivalence relation on the manifestation domain while remaining compatible with non-isomorphic

relational carriers, different historical provenance, or distinct continuation structures. The later lifting analysis therefore asks which additional assumptions permit one comparison mode to support another.

4.4 Resolution, Partiality, and Applicability

GR comparisons are evaluated at declared representational resolutions and can be only partially applicable. This subsection formalizes those two constraints because both can change the apparent equivalence classes of a system. Resolution controls which distinctions are represented; partiality controls where the comparison itself is meaningful. Treating coarse representation or undefinedness as substantive equivalence would manufacture relations that the model has not earned.

Let $D^{(f)}$ and $D^{(c)}$ denote fine and coarse representations of overlapping relational content, and let $q : D^{(f)} \rightarrow D^{(c)}$ be a declared coarse-graining map. The kernel relation induced by q is stated in Equation 19.

$$x \sim_q y \iff q(x) = q(y), \quad x, y \in D^{(f)}. \quad (19)$$

By Proposition 4.2, Equation 19 defines an equivalence relation on $D^{(f)}$. Its classes record distinctions suppressed by the coarse representation. Membership in the same q -class therefore establishes equality after coarse-graining and leaves fine-level equivalence unresolved unless the fine retained structure supplies a separate relation.

Refinement can be stated directly for equivalence relations on a common carrier. Let E_1 and E_2 be equivalence relations represented as subsets of $D \times D$. Definition 4.6 fixes the direction used throughout the manuscript.

Definition 4.6 (Refinement of an equivalence relation). An equivalence relation E_2 on D refines an equivalence relation E_1 on the same carrier when

$$E_2 \subseteq E_1. \quad (20)$$

The refinement is strict when the inclusion in Equation 20 is proper.

A finer relation identifies fewer pairs as equivalent and therefore induces classes contained in the corresponding coarser classes. Proposition 4.7 records this consequence.

Proposition 4.7 (Partition refinement). *If $E_2 \subseteq E_1$ are equivalence relations on D , then for every $x \in D$, the E_2 -equivalence class of x is contained in its E_1 -equivalence class.*

Proof. If $y \in [x]_{E_2}$, then $(x, y) \in E_2$. Inclusion gives $(x, y) \in E_1$, so $y \in [x]_{E_1}$. $\square$

Definition 4.6 gives a precise meaning to one form of stronger discrimination. It does not imply that E_2 is normatively preferable, empirically better, or ontologically closer to ultimate reality. A finer representation can preserve distinctions irrelevant to the research question, while a coarse equivalence can be exactly appropriate for a limited operation.

Partiality concerns the domain on which a comparison is defined. Let R be a binary relation intended to compare elements of a nominal carrier X . The model can declare an applicability domain $D_R \subseteq X$ and define R only on $D_R \times D_R$, or on another explicitly stated subset of $X \times X$ when pair-specific applicability is required. An object outside the domain is not thereby non-equivalent. The relation is undefined for that comparison.

This distinction is inherited from the partial-transformation discipline used elsewhere in GR. It is especially important for cross-level and historical comparisons. A manifestation map can be undefined for relational structures that a context cannot render legible. A historical tracking relation can fail when the model lacks evidence that two stages instantiate comparable objects. A topological diagnostic can be undefined when the required complex or invariant has not been constructed. In each case, undefinedness records a boundary of the model rather than a negative comparison result.

Caution 4.8 (Undefinedness and non-equivalence). For a partial comparison relation, failure of applicability is distinct from a defined negative result. The manuscript therefore records undefined comparisons explicitly and does not infer $x \not\sim y$ merely because the relation is unavailable for the pair (x, y) .

Resolution and partiality can interact. Refinement can bring a previously undefined comparison into a meaningful domain by articulating additional positions or relations. It can also remove an earlier comparison from its domain when a coarse identification is no longer coherent under the finer representation. Sections 5–8 retain these possibilities rather than assuming monotone growth of comparability.

4.5 Implication Structure across Taxonomic Categories

The final architectural problem concerns logical implication among comparison relations. The taxonomy is not a ladder in which later categories automatically establish stronger forms of sameness. An implication is admitted only when the source and target relations are defined on a common carrier or when an explicit transport maps the source comparison into the target domain. This section formalizes the common-carrier case and reserves cross-carrier lifting for Section 9.

Let $R_1, R_2 \subseteq D \times D$ be binary relations on a common carrier D . The statement that R_1 implies R_2 on D is represented by the inclusion in Equation 21.

$$R_1 \Rightarrow_D R_2 \iff R_1 \subseteq R_2. \quad (21)$$

Equation 21 means that every pair related by R_1 is also related by R_2 . When both relations are equivalence relations, R_1 is the finer relation and R_2 the coarser relation in the sense of Definition 4.6. The notation $\Rightarrow_D$ is used only for taxonomic implication on the stated carrier and should not be confused with causal, historical, or generative implication.

Proposition 4.9 (Transitivity of taxonomic implication). *If $R_1 \Rightarrow_D R_2$ and $R_2 \Rightarrow_D R_3$, then $R_1 \Rightarrow_D R_3$.*

Proof. The two assumptions state $R_1 \subseteq R_2$ and $R_2 \subseteq R_3$. Transitivity of set inclusion gives $R_1 \subseteq R_3$, which is Equation 21 for $R_1 \Rightarrow_D R_3$. $\square$

Equation 21 provides a partial ordering of equivalence relations on one carrier after literal equality of relations is used as identity. It does not provide an ordering among relation types defined on different carriers. Manifestation equivalence on X_c , orbit interchangeability on $P_\kappa(C)$, and bisimulation on a continuation system cannot be compared by subset inclusion because their relata differ. An inference among them requires a manifestation map, position embedding, historical transport, quotient factorization, or another explicitly typed bridge. Section 9 develops such lifting problems after the four substantive taxonomy sections have supplied their local relations.

The same discipline applies to diagnostic relations. Suppose J_1 and J_2 record agreement of two different invariant families. Even if one diagnostic family contains numerically more components, the resulting agreement relation is not automatically finer unless the carriers and retained distinctions permit a formal inclusion argument. A larger list of measurements can also contain redundant information or omit a structural distinction captured by a smaller but differently organized diagnostic.

The collection of exact equivalence relations on a fixed carrier can support lattice-like operations under inclusion. Their intersection provides a common refinement by Proposition 4.3. A common coarsening may also exist under additional closure construction. The present paper uses the intersection operation where needed and avoids claiming a global lattice across heterogeneous GR carriers. The four substantive taxonomy sections therefore remain connected through explicitly established implications rather than through a single universal scale of equivalence.

Definition 4.10 (Taxonomic implication discipline). An implication from one GR comparison relation to another is admitted in the present taxonomy when either (i) both relations are defined on a common carrier and the corresponding relation inclusion is established, or (ii) an explicitly typed bridge between

the carriers is supplied and the implication is proved relative to that bridge. In the absence of either condition, the relations are retained as distinct comparison modes without an implication arrow.

Definition 4.10 fixes the logic used in the remainder of the taxonomy. Section 5 begins with equivalence relations and comparison modes defined within individual GR levels. Section 6 then studies structured relations among levels and supplies the first systematic bridge between relational and manifestation comparisons. Section 7 extends the carrier to historical evolution and continuation systems, where simulation and bisimulation become available. Section 8 turns to dynamical regimes and structural diagnostics, with particular attention to the difference between invariant agreement and structural equivalence. The later synthesis in Section 9 will assemble only those cross-category implications that the preceding constructions have earned.

5 Equivalence within GR Reality Levels

This section develops the first substantive branch of the taxonomy: comparison relations whose principal analytical object is located within one declared GR reality or representation level. Its role is to separate three questions that are easily compressed into one word. Ontological equivalence concerns ultimate being and remains suspended in the present GR programme. Manifestation equivalence concerns indistinguishability after a declared context-dependent readout or evaluation. Relational equivalence concerns preservation of explicitly retained relational structure, with symmetry-relative position interchangeability available when an articulated group action has been established. The section proceeds from these level-specific roles to multi-context and task-induced constructions and concludes with refinement relations that state when one within-level equivalence is formally finer than another.

The construction is deliberately asymmetric across the three GR levels. Ontological suspension withholds a default equivalence relation at the ontological level. Manifestation-level relations can often be generated as kernel equivalences of declared maps. Relational-level relations require an independently specified preservation criterion and, for interchangeability, a justified transformation structure. This asymmetry is methodological rather than hierarchical. It records the different evidence and formal commitments available at each level.

5.1 Ontological Equivalence and the Scope of Suspension

GR uses ontological suspension to keep formal representation distinct from a claim about ultimate ontology. The present taxonomy therefore does not introduce a default relation of ontological equivalence between humans, artificial systems, institutions, animals, symbolic structures, or other modeled entities. A statement that two objects are equivalent in manifestation or relational structure does not determine whether they are ultimately the same kind of being, possess the same metaphysical constitution, or satisfy an unrestricted criterion of identity.

This restraint can be expressed formally by distinguishing absence of a GR-supplied ontological comparator from a negative ontological judgment. Let $\mathcal{O}$ denote a placeholder for an ontological carrier only when a conditional ontology has independently supplied one. The base GR taxonomy contains no distinguished binary relation $E_{\mathcal{O}} \subseteq \mathcal{O} \times \mathcal{O}$ whose membership is inferred from the lower-level constructions developed in this paper. The absence of such a distinguished relation is a scope condition. It does not assert that ontological equivalence is impossible or meaningless under every metaphysical theory.

A conditional ontology may nevertheless be examined inside the taxonomy when its commitments are stated explicitly. Suppose a model supplies a carrier $\mathcal{O}^*$, a relation $E_{\mathcal{O}^*}$, and a map connecting selected modeled relational structures to that ontological representation. The relation can then be analyzed according to the general discipline of Section 4: its domain, axioms, preserved structure, and bridge to the GR representation must be stated before any implication is considered. Its conclusions remain conditional on the added ontology.

Caution 5.1 (Ontological inference boundary). Manifestation equivalence, task equivalence, relational structural equivalence, and symmetry-relative interchangeability do not by themselves establish ontological equivalence or ontological non-equivalence. Either conclusion requires additional premises specifying an ontological carrier and a justified relation on that carrier.

Caution 5.1 is especially important for Human–AI comparison. A dialogue interface can make a human and an artificial system indistinguishable under a selected context, while a relational model can additionally place them in the same symmetry orbit relative to a declared κ . These results establish exactly the equivalences represented by their respective constructions. They do not settle questions of consciousness, personhood, biological identity, moral status, or ultimate metaphysical constitution.

5.2 Contextual Manifestation Equivalence

Manifestation equivalence is the most direct exact equivalence inherited by the present paper. Let $M_c : \mathcal{R} \rightarrow X_c$ be the partial manifestation map from Equation 4. The relation $\sim_c$ from Equation 5 identifies source objects that receive the same manifestation in context c . The section now records its taxonomic status explicitly.

Definition 5.2 (Contextual manifestation equivalence). For a manifestation context c , two relational objects $x, y \in \text{Dom}(M_c)$ are contextually manifestation-equivalent when

$$M_c(x) = M_c(y). \quad (22)$$

The resulting equivalence relation is the inherited relation $\sim_c$.

Equation 22 compares relational source objects through one declared presentation channel. Its equivalence classes are the fibers of M_c . For a manifestation value $m \in \text{Im}(M_c)$, the corresponding fiber is defined in Equation 23.

$$\text{Fib}_{M_c}(m) := \{x \in \text{Dom}(M_c) \mid M_c(x) = m\}. \quad (23)$$

Equation 23 makes the epistemic compression explicit. A large fiber indicates that the selected context identifies many source structures at the retained resolution. A singleton fiber indicates that the manifestation separates that particular source object from the other source objects represented in the declared domain. Neither property is automatically stable under a change of context or model resolution.

Contextual manifestation equivalence is consequently indexed by both c and the domain on which M_c is defined. A natural-language interface, a visual presentation, an institutional classification, a legal status, a behavioral measurement, and a technical protocol can induce different partitions of the same source domain. The relation is exact relative to the map even when the map is epistemically coarse.

Remark 5.3 (Exactness and informational thinness). An equivalence relation can be mathematically exact while preserving very little source structure. Contextual manifestation equivalence is therefore exact as a kernel relation and potentially thin as a representation of the relational objects that it groups together.

Remark 5.3 separates two dimensions that later Human–AI analysis must keep distinct. Exact equality of interface outputs can coexist with substantial underdetermination of relational history, provenance, admissible transformations, or continuation structure.

5.3 Multi-Context Manifestation Equivalence

A single manifestation context can suppress distinctions that become visible through another context. Multi-context comparison therefore combines several independently declared readouts while remaining

at the manifestation-comparison level. Let $\mathcal{C}$ be a nonempty family of manifestation contexts and define the common applicability domain in Equation 24.

$$D_{\mathcal{C}} := \bigcap_{c \in \mathcal{C}} \text{Dom}(M_c). \quad (24)$$

For $x \in D_{\mathcal{C}}$, the combined manifestation map is stated in Equation 25.

$$M_{\mathcal{C}}(x) := (M_c(x))_{c \in \mathcal{C}} \in \prod_{c \in \mathcal{C}} X_c. \quad (25)$$

The product in Equation 25 is a representational product of the retained context spaces. It does not assert that the participating contexts share one substantive ontology or one institutional interpretation.

Definition 5.4 (Multi-context manifestation equivalence). For $x, y \in D_{\mathcal{C}}$, define

$$x \sim_{\mathcal{C}} y \iff M_{\mathcal{C}}(x) = M_{\mathcal{C}}(y). \quad (26)$$

Equivalently, $x \sim_{\mathcal{C}} y$ when $M_c(x) = M_c(y)$ for every $c \in \mathcal{C}$.

The relation in Definition 5.4 is an immediate instance of the kernel construction in Proposition 4.2. The equivalent intersection form is useful for refinement and is stated in Equation 27.

$$\sim_{\mathcal{C}} = \bigcap_{c \in \mathcal{C}} \sim_c \quad \text{on } D_{\mathcal{C}}. \quad (27)$$

Equation 27 shows that multi-context equivalence retains only pairs left indistinguishable by every selected manifestation context.

Proposition 5.5 (Context-family refinement). Let $\mathcal{C}_1 \subseteq \mathcal{C}_2$ be two nonempty context families and restrict both relations to $D_{\mathcal{C}_2}$. Then

$$\sim_{\mathcal{C}_2} \subseteq \sim_{\mathcal{C}_1}. \quad (28)$$

Proof. If $x \sim_{\mathcal{C}_2} y$, then $M_c(x) = M_c(y)$ for every $c \in \mathcal{C}_2$. The same equality therefore holds for every $c \in \mathcal{C}_1$, which gives $x \sim_{\mathcal{C}_1} y$. $\square$

Equation 28, established by Proposition 5.5, gives a precise sense in which adding manifestation contexts can refine an equivalence partition. Equality is also possible: an additional context can be redundant relative to the distinctions already separated by $\mathcal{C}_1$. The proposition does not imply that every larger context family is epistemically preferable. A context can be costly, irrelevant, normatively inappropriate, or insufficiently justified for the research question.

The construction also accommodates heterogeneous contexts. A Human–AI comparison can combine linguistic, task-performance, provenance, institutional, and interactional readouts without forcing them into one scalar score. Their joint effect is represented through the product map in Equation 25. Section 6 will later ask when such a context family separates relational equivalence classes rather than merely refining manifestation-level distinctions.

5.4 Task-Induced Equivalence and Approximate Similarity

Task evaluation introduces a second quotienting operation at the manifestation level. A task can treat different manifestations as equivalent because the differences fall outside the declared success criterion. The resulting relation is often the formal content behind statements that two systems perform equivalently on a task. This subsection derives an exact task equivalence and then separates it from tolerance-based similarity.

Fix a manifestation context c and a task label τ . Let Y_τ denote the declared task-evaluation space, and let the evaluation map be the partial map stated in Equation 29.

$$E_{\tau,c} : X_c \rightarrow Y_\tau. \quad (29)$$

The task map can return a categorical outcome, accepted solution class, score vector, decision status, or another explicitly defined evaluation object. Its semantics belong to the task model. Define the relational-source domain on which the composed readout is available by Equation 30.

$$D_{\tau,c} := \{x \in \text{Dom}(M_c) \mid M_c(x) \in \text{Dom}(E_{\tau,c})\}. \quad (30)$$

The composed task readout is $Q_{\tau,c} := E_{\tau,c} \circ M_c$ on $D_{\tau,c}$.

Definition 5.6 (Exact task-induced equivalence). For $x, y \in D_{\tau,c}$, define

$$x \sim_{\tau,c} y \iff Q_{\tau,c}(x) = Q_{\tau,c}(y). \quad (31)$$

Equation 31 produces the kernel relation formalized in Definition 5.6; by Proposition 4.2, it is an equivalence relation on the declared domain. It can be strictly coarser than contextual manifestation equivalence because $E_{\tau,c}$ can intentionally ignore differences in X_c .

Proposition 5.7 (Manifestation equivalence implies exact task equivalence). *On $D_{\tau,c}$, contextual manifestation equivalence implies exact task-induced equivalence. Equivalently,*

$$\sim_c \upharpoonright_{D_{\tau,c}} \subseteq \sim_{\tau,c}. \quad (32)$$

Proof. If $x \sim_c y$, then $M_c(x) = M_c(y)$. Applying the same task map $E_{\tau,c}$ to the equal manifestations gives $Q_{\tau,c}(x) = Q_{\tau,c}(y)$, so $x \sim_{\tau,c} y$. $\square$

The inclusion in Equation 32 is generally one-sided. Its converse requires additional injectivity of $E_{\tau,c}$ on the relevant manifested image. A grading rule can assign the same accepted status to two visibly different answers, and a benchmark can assign the same scalar score to outputs generated through different response patterns. Exact task equivalence therefore establishes equality only after the declared evaluation map.

Approximate task comparison requires a different relation type. Suppose Y_τ carries a metric or another symmetric discrepancy d_τ for which the intended comparison is meaningful. For a tolerance $\varepsilon > 0$, a provisional tolerance relation can be defined by Equation 33.

$$x \approx_{\tau,c}^\varepsilon y \iff d_\tau(Q_{\tau,c}(x), Q_{\tau,c}(y)) \leq \varepsilon. \quad (33)$$

Equation 33 is reflexive when $d_\tau(u, u) = 0$ and symmetric when d_τ is symmetric. Transitivity does not follow from a positive threshold. For example, on the real line with ordinary distance, the three values 0, 0.75ε , and 1.5ε satisfy the tolerance relation for the first–second and second–third pairs while the first–third pair lies outside the tolerance. The paper therefore calls Equation 33 approximate similarity or tolerance rather than an equivalence relation unless an application supplies an additional construction that restores transitivity.

Caution 5.8 (Task success and relational interchangeability). Exact or approximate agreement under a task evaluation establishes only the comparison encoded by the task map and its declared context. A task result does not by itself establish relational structural equivalence, symmetry-relative interchangeability, historical equivalence, or continuation equivalence.

Caution 5.8 prevents benchmark success from being silently converted into an unrestricted replacement claim. Section 12 will use this distinction directly in Human–AI analysis.

5.5 Relational Structural Equivalence

Relational structural equivalence compares modeled relational organization directly rather than through equality of a selected manifestation. The inherited relation $\simeq_\kappa$ from Equation 6 applies on a domain D_κ after the retained content κ and its preservation criterion have been specified. The present subsection positions that relation inside the taxonomy and makes explicit the role of quotient classes and retained distinctions.

For $x \in D_\kappa$, write the κ -equivalence class as in Equation 34.

$$[x]_\kappa := \{y \in D_\kappa \mid y \simeq_\kappa x\}. \quad (34)$$

The quotient carrier $D_\kappa / \simeq_\kappa$ is the collection of the classes in Equation 34. The quotient is a mathematical representation of the declared equivalence structure. It does not imply that distinctions outside κ have ceased to exist relationally or ontologically.

The interpretation of $\simeq_\kappa$ depends on the content retained by κ . One model can preserve current access and admissible operations while omitting historical provenance. Another can retain history, dependency, responsibility, interface structure, or continuation conditions. The same pair x, y can therefore satisfy $x \simeq_{\kappa_1} y$ and fail $x \simeq_{\kappa_2} y$ without contradiction when the two indices encode different preservation requirements.

Definition 5.9 (Within-level relational structural equivalence). A claim of relational structural equivalence in this section is a judgment $x \simeq_\kappa y$ on a declared domain D_κ for which the model has specified the relational content encoded by κ , supplied the preservation criterion, and established that $\simeq_\kappa$ is an equivalence relation on that domain.

Definition 5.9 adds no new relation to the inherited formalism. Its role is taxonomic: it records the conditions under which the inherited symbol is used as a within-level equivalence rather than as an informal claim of similarity.

Relational structural equivalence can also be local to a region, boundary, or coupled configuration. If D_κ contains representations extracted from K_{AB} , the preserved structure can include relations that exist only at the coupled level. This possibility matters because equivalent descriptions of K_A and K_B considered separately do not automatically establish equivalence of the relation represented in K_{AB} . Section 6 treats that stronger cross-structure problem directly.

Remark 5.10 (Relational equivalence and omitted structure). A relation $\simeq_\kappa$ can be exact while deliberately omitting relational features outside κ . The conclusion licensed by $x \simeq_\kappa y$ is therefore preservation of the declared relational content, not unrestricted equality of the represented entities.

Remark 5.10 parallels Remark 5.3 at the manifestation level. Exactness concerns the relation as defined; substantive scope concerns which structure the definition retains.

5.6 Symmetry-Relative Position Interchangeability

Relational structural equivalence compares elements of D_κ . Interchangeability of positions requires additional transformation structure. The inherited construction in Subsection 3.4 supplies an articulated symmetry group $G_\kappa(C)$ and its position action $\rho_{C,\kappa}$ on $P_\kappa(C)$. Positions are interchangeable when they occupy the same orbit.

For taxonomic bookkeeping, the same-orbit relation can be written as the derived relation in Equation 35.

$$a \sim_{C,\kappa}^{\text{orb}} b \iff \text{Orb}_{C,\kappa}(a) = \text{Orb}_{C,\kappa}(b), \quad a, b \in P_\kappa(C). \quad (35)$$

Equation 35 is an abbreviation of the inherited criterion in Equation 11; it is not a new primitive notion of interchangeability.

Proposition 5.11 (Orbit interchangeability is an equivalence relation). *Under the articulated group-action assumptions of Subsection 3.4, the relation $\sim_{C,\kappa}^{\text{orb}}$ in Equation 35 is an equivalence relation on $P_\kappa(C)$.*

Proof. Every position belongs to its own orbit, so the relation is reflexive. Equality of orbit sets is symmetric. If $\text{Orb}_{C,\kappa}(a) = \text{Orb}_{C,\kappa}(b)$ and $\text{Orb}_{C,\kappa}(b) = \text{Orb}_{C,\kappa}(d)$, transitivity of equality gives $\text{Orb}_{C,\kappa}(a) = \text{Orb}_{C,\kappa}(d)$. $\square$

The mathematical simplicity of Proposition 5.11 should not obscure its substantial prerequisites. The carrier $P_\kappa(C)$ must contain identifiable positions; the transformation group must be justified; the action must be defined; and the action must preserve the retained relational content. A pre-differentiated regime or a system with only partial transformations can lie outside the applicability domain of the orbit criterion.

Interchangeability also remains position-relative. The beings occupying a and b can have different internal histories or properties that are excluded from $P_\kappa(C)$ at the selected resolution. If those features become preservation-relevant, the model must refine the retained content or the position representation and evaluate the new relation. This provides a direct formal route for studying why a human and an artificial system can be interchangeable for one relational role while remaining non-interchangeable for another.

5.7 Refinement within a Reality Level

The preceding constructions create several exact equivalence relations inside manifestation and relational comparison. Their relationships should be stated through proven relation inclusion rather than by an informal ranking of equivalence strength. This subsection collects the within-level refinement results and marks the boundaries of those implications.

At the manifestation level, Proposition 5.5 establishes that enlarging a context family can only retain or refine the induced equivalence on the common domain. Exact task evaluation usually moves in the opposite direction: Proposition 5.7 shows that applying an additional task map can merge manifestation classes and therefore produce a coarser equivalence. The two operations are structurally different. Adding contexts adds discriminating coordinates; applying a many-to-one task evaluation quotients the available manifestation information.

At the relational level, refinement is expressed directly through relation inclusion. For two retained-content relations on a common carrier D , if

$$\simeq_{\kappa_2} \subseteq \simeq_{\kappa_1}, \quad (36)$$

then $\simeq_{\kappa_2}$ refines $\simeq_{\kappa_1}$ in the sense of Definition 4.6. Equation 36 is the formal criterion. A verbal claim that κ_2 “contains more information” is insufficient unless the corresponding preservation relations actually satisfy the inclusion on the declared domain.

Symmetry structure supplies another relational refinement mechanism. Suppose a group G acts on a retained position set P and $H \leq G$ acts by restriction. For $a \in P$, the orbit inclusion is stated in Equation 37.

$$\text{Orb}_H(a) \subseteq \text{Orb}_G(a). \quad (37)$$

Equation 37 follows because every transformation in H is also available in G . A residual subgroup can therefore split an antecedent orbit into several finer orbits. This is one established route by which previously interchangeable positions become differentiated while the carrier remains comparable.

Table 3 summarizes the principal within-level comparison structures developed in this section. The table reports the carrier and preserved content so that exact relations are not mistaken for unrestricted identity.

Figure 2 organizes the same relations by GR level and by the operation that creates or refines the comparison. The figure deliberately leaves the ontological band without a default equivalence arrow. Manifestation relations are generated through readout and evaluation maps, whereas relational equivalence and interchangeability require retained relational content and transformation structure.

Table 3: Principal equivalence and equivalence-like relations within individual GR levels. Each row states the carrier and the information preserved by the comparison.

Comparison	Carrier	Preservation rule	Principal boundary
Ontological equivalence	Conditional $\mathcal{O}^*$ only	Supplied by an added ontology	No default GR relation is inferred from lower levels
Contextual manifestation equivalence $\sim_c$	$\text{Dom}(M_c)$	Equality after M_c	Context-relative indistinguishability
Multi-context equivalence $\sim_C$	D_C	Equality under every retained M_c	Depends on selected context family
Exact task equivalence $\sim_{\tau,c}$	$D_{\tau,c}$	Equality after $E_{\tau,c} \circ M_c$	Can merge distinct manifestations
Task tolerance $\approx_{\tau,c}^\varepsilon$	$D_{\tau,c}$ with discrepancy structure	Distance or discrepancy within declared tolerance	Generally lacks transitivity
Relational structural equivalence $\simeq_\kappa$	D_κ	Preservation of declared relational content κ	Omits structure outside κ
Orbit interchangeability $\sim_{C,\kappa}^{\text{orb}}$	$P_\kappa(C)$	Same orbit under $G_\kappa(C)$	Requires an articulated symmetry action

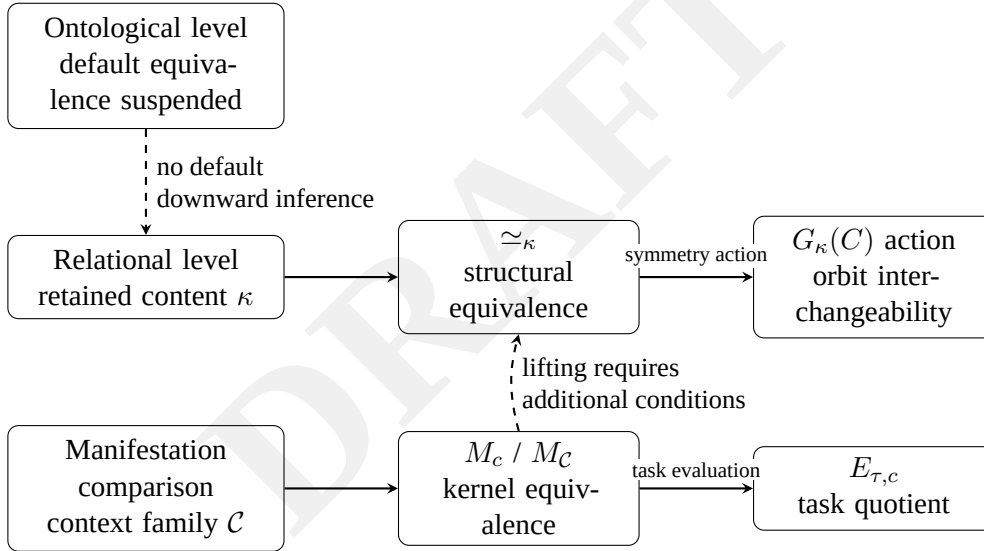


Figure 2: Within-level GR equivalence architecture. Manifestation equivalences arise from declared readout or evaluation maps; relational structural equivalence is indexed by retained content κ ; and position interchangeability additionally requires an articulated symmetry action. The dashed cross-level arrow marks a later lifting problem rather than an implication established in this section.

The section therefore yields a structured collection of within-level relations rather than a single universal equivalence. Manifestation equivalence can be refined by additional contexts and coarsened by task evaluation. Relational equivalence can be refined by a stricter preservation relation, and orbit interchangeability can refine when the available symmetry group is reduced. Ontological equivalence remains conditionally representable only after an additional ontology supplies its carrier and comparison rule. Section 6 next changes the formal object of comparison: it asks when the relations connecting GR levels, including manifestation maps themselves, are structurally equivalent across two modeled systems.

6 Equivalence of Cross-Level Relational Structures

Equivalence within one GR level does not determine how that level is related to another. The role of this section is to make the relation among levels itself an object of comparison. The objective is to formalize when a relational carrier, a manifestation carrier, and the map connecting them possess corresponding cross-level organization, and to distinguish such correspondence from separate equivalence claims made inside either carrier. The analysis proceeds through level-coupling architectures, structure-preserving correspondences, manifestation maps treated as relational data, quotient factorization, separating context families, and cross-level distinction diagnostics. The construction remains model-relative. It compares declared relational and manifestation structures without assigning ultimate ontological status to either the carriers or the mappings used to represent their relation.

6.1 Level-Coupling Architectures

Section 5 treated equivalence on carriers located within a selected GR level. A cross-level inquiry requires a larger formal object because two systems can have comparable relational carriers and comparable manifestation spaces while organizing the relation between those carriers differently. The manifestation map introduced in Equation 4 supplies the minimal inherited interface for such a construction.

Fix a context c . Let $D_c \subseteq \text{Dom}(M_c)$ be a declared relational source domain and let $X_c^D := M_c(D_c) \subseteq X_c$ be the manifestation image retained for the comparison. The corresponding two-level coupling architecture is defined in Equation 38.

$$\mathfrak{L}_c(D_c) := (D_c, X_c^D, M_c|_{D_c}). \quad (38)$$

Equation 38 treats the restricted manifestation map as part of the structure. The architecture therefore records more than the existence of a relational domain and a manifestation domain. It records which relational objects receive which manifestations under the selected context and resolution.

The source carrier D_c can contain configurations, relational structures, positions, histories, or another well-typed GR object. The codomain X_c^D can be linguistic, symbolic, numerical, institutional, sensory, technical, or otherwise structured. The construction does not require the two carriers to have the same mathematical type. Their relation is supplied by the partial manifestation machinery already established in Section 3.

A family of contexts produces a correspondingly richer architecture. Let $\mathcal{C}$ be a declared context family and let $D_{\mathcal{C}}$ be the common domain from Equation 24. The multi-context coupling architecture is stated in Equation 39.

$$\mathfrak{L}_{\mathcal{C}}(D_{\mathcal{C}}) := (D_{\mathcal{C}}, \{X_c^D\}_{c \in \mathcal{C}}, \{M_c|_{D_{\mathcal{C}}}\}_{c \in \mathcal{C}}). \quad (39)$$

Equation 39 retains the context labels. Two contexts that happen to have isomorphic codomains need not perform the same cross-level role. A later comparison can permit a justified relabelling of contexts, but such a correspondence must be supplied explicitly rather than inferred from codomain similarity.

Definition 6.1 (Cross-level comparison object). A cross-level comparison object is a typed architecture whose data include at least two declared carriers associated with different GR analytical levels or representations and one or more maps, relations, or correspondences connecting those carriers. Equivalence of the cross-level object concerns preservation of the connecting structure in addition to any within-carrier preservation conditions.

Definition 6.1 is intentionally broader than Equation 38. Later GR applications can connect relational structures to several manifestation systems, evidence spaces, nomic representations, or other derived layers. The present section develops the manifestation interface because it is already formally inherited and directly relevant to Human–AI comparison.

6.2 Cross-Level Structural Equivalence

Comparison of two level-coupling architectures requires correspondences both within the component carriers and across the map that links them. Let $\mathfrak{L}_c^A = (D_A, X_A, M_A)$ and $\mathfrak{L}_c^B = (D_B, X_B, M_B)$ denote two architectures of the form in Equation 38. The context label is suppressed locally when the two architectures are compared under an already declared context correspondence.

Suppose the model supplies a bijection $F_R : D_A \rightarrow D_B$ that preserves the relational structure retained for the comparison and a bijection $F_M : X_A \rightarrow X_B$ that preserves the declared manifestation structure. Componentwise preservation alone leaves the cross-level relation unresolved. The compatibility condition is the commuting relation stated in Equation 40.

$$F_M \circ M_A = M_B \circ F_R \quad \text{on } D_A. \quad (40)$$

Equation 40 says that transporting a relational object to the second architecture and then manifesting it yields the same retained manifestation as manifesting first and transporting the manifestation. The equality is exact at the selected resolution. Approximate or tolerance-based cross-level comparison requires an additional discrepancy structure and does not automatically define an equivalence relation.

Definition 6.2 (Cross-level structural isomorphism). The architectures $\mathfrak{L}_c^A$ and $\mathfrak{L}_c^B$ are cross-level structurally isomorphic relative to the declared retained structures when there exist structure-preserving bijections F_R and F_M of the types above satisfying Equation 40. The comparison is indexed by the retained relational and manifestation structures and by the context correspondence used to type the two architectures.

The term isomorphism is used because Definition 6.2 requires invertible structure-preserving correspondences. A weaker application can use homomorphisms, embeddings, simulations, or another relation type, provided that the corresponding preservation conditions are stated. Such weaker relations should not be relabelled as cross-level isomorphism merely because their component carriers are similar.

Proposition 6.3 (Componentwise equivalence is insufficient for cross-level equivalence). *Let the relational and manifestation carriers of two architectures admit independently justified structure-preserving correspondences. Those componentwise correspondences do not establish cross-level structural isomorphism unless they can be chosen, or are already supplied, so that Equation 40 holds.*

Proof. Definition 6.2 includes commutation with the manifestation maps as a constitutive condition. Structure-preserving correspondences on the two component carriers specify no relation between $F_M \circ M_A$ and $M_B \circ F_R$. Hence the componentwise conditions leave the cross-level compatibility condition logically open. $\square$

Proposition 6.3 gives formal content to a basic GR distinction. Equivalence of relata does not establish equivalence of the relation connecting them. In Human–AI analysis, two relational descriptions can share a coarse structural classification and two interfaces can share an output vocabulary while the correspondence between relational states and interface presentations remains different. The cross-level architecture records that difference directly.

For a multi-context architecture, Definition 6.2 extends pointwise across the retained context family. If the context indexing is held fixed, a relational correspondence F_R and manifestation correspondences $F_{M,c}$ must satisfy Equation 41 for every retained context.

$$F_{M,c} \circ M_c^A = M_c^B \circ F_R, \quad c \in \mathcal{C}. \quad (41)$$

A model that permits contexts themselves to be transported must additionally supply a justified correspondence between the context index families before Equation 41 is evaluated.

6.3 Manifestation Maps as Structure-Bearing Relations

The cross-level architecture assigns formal significance to the manifestation map itself. A manifestation map is not only a device for producing an observable label. Its fibers determine which relational objects become indistinguishable under a context, and its graph records the cross-level pairing between relational and manifested objects. These structures can therefore be compared independently of the internal structure of either carrier.

For a restricted map $M_c|_D : D \rightarrow X_c$, define its graph by Equation 42.

$$\text{Gr}(M_c|_D) := \{(x, M_c(x)) \mid x \in D\} \subseteq D \times X_c. \quad (42)$$

Equation 42 is a relation between the two retained carriers. Under bijections $F_R : D_A \rightarrow D_B$ and $F_M : X_A \rightarrow X_B$, the product map $F_R \times F_M$ transports this graph. The commuting condition in Equation 40 is equivalent to the graph condition in Equation 43.

$$(F_R \times F_M)(\text{Gr}(M_A)) = \text{Gr}(M_B). \quad (43)$$

Equation 43 makes clear that cross-level equivalence preserves a relation, not merely two endpoint spaces.

The fiber construction from Equation 23 provides a complementary view. For a manifestation $m \in X_c^D$, the fiber $M_c^{-1}(m) \cap D$ collects relational objects that the selected context presents identically. A cross-level structural isomorphism transports these fibers coherently.

Proposition 6.4 (Fiber preservation under cross-level structural isomorphism). *Let (F_R, F_M) define a cross-level structural isomorphism between $\mathfrak{L}_c^A$ and $\mathfrak{L}_c^B$. For every $m \in X_A$, the relational correspondence satisfies Equation 44.*

$$F_R(M_A^{-1}(m)) = M_B^{-1}(F_M(m)). \quad (44)$$

Proof. For $x \in D_A$, Equation 40 gives $M_B(F_R(x)) = F_M(M_A(x))$. Hence $M_A(x) = m$ if and only if $M_B(F_R(x)) = F_M(m)$. Bijectivity of F_R gives equality of the two fibers. $\square$

Proposition 6.4 identifies the partition induced by manifestation as part of the cross-level structure. Two maps with the same image cardinality can therefore remain cross-level distinct when they partition their relational source domains differently.

6.4 Quotients, Factorization, and Identifiability

Cross-level comparison becomes especially informative when a retained relational equivalence $\simeq_\kappa$ is compared with a manifestation-induced equivalence $\sim_c$ on a common domain. The relation between these equivalences can be decomposed into two logically independent inclusion conditions. One condition asks whether manifestation respects the relational equivalence. The other asks whether manifestation separates relational classes that the model treats as distinct.

Fix a domain $D \subseteq D_\kappa \cap \text{Dom}(M_c)$, and restrict both relations to D . The manifestation is κ -invariant on D when Equation 45 holds.

$$\simeq_\kappa|_D \subseteq \sim_c|_D. \quad (45)$$

Equation 45 means that relational objects declared equivalent under the retained content κ receive the same manifestation in context c . Under this condition, $M_c|_D$ is constant on $\simeq_\kappa$ -classes and therefore descends to the quotient.

Let $q_\kappa : D \rightarrow D/\simeq_\kappa$ be the quotient map. The factorized manifestation map is the unique map in Equation 46 satisfying the displayed factorization.

$$\overline{M}_{c,\kappa} : D/\simeq_\kappa \longrightarrow X_c^D, \quad M_c|_D = \overline{M}_{c,\kappa} \circ q_\kappa. \quad (46)$$

Proposition 6.5 (Manifestation factorization through relational equivalence). *The factorization in Equation 46 exists if and only if Equation 45 holds on D .*

Proof. If the factorization exists and $x \simeq_\kappa y$, then $q_\kappa(x) = q_\kappa(y)$ and therefore $M_c(x) = M_c(y)$, giving Equation 45. Conversely, if Equation 45 holds, define $\overline{M}_{c,\kappa}([x]_\kappa) = M_c(x)$. The inclusion guarantees that the value is independent of the chosen representative, so the map is well defined and the factorization follows. $\square$

The opposite inclusion expresses identifiability of the retained relational classes from the selected manifestation. Context c is κ -separating on D when Equation 47 holds.

$$\sim_c \mid_D \subseteq \simeq_\kappa \mid_D. \quad (47)$$

If two relational objects receive the same manifestation under a κ -separating context, they belong to the same retained relational equivalence class. This condition does not require the manifestation to preserve every difference inside a κ -class; that separate requirement is supplied by Equation 45.

Definition 6.6 (Exact cross-level recovery of a retained equivalence). A manifestation context c exactly recovers $\simeq_\kappa$ on D when the two restricted relations coincide, as stated in Equation 48.

$$\sim_c \mid_D = \simeq_\kappa \mid_D. \quad (48)$$

Proposition 6.7 (Factorization and injectivity criterion). Suppose $M_c \mid_D$ is κ -invariant so that the factorization in Equation 46 exists. The context exactly recovers $\simeq_\kappa$ on D if and only if the induced map $\overline{M}_{c,\kappa}$ is injective.

Proof. Under factorization, $M_c(x) = M_c(y)$ is equivalent to $\overline{M}_{c,\kappa}([x]_\kappa) = \overline{M}_{c,\kappa}([y]_\kappa)$. Injectivity therefore makes manifestation equality equivalent to equality of the quotient classes, which is precisely $x \simeq_\kappa y$. The converse follows because equality of the two relations prevents distinct quotient classes from receiving the same induced manifestation. $\square$

Proposition 6.7 states a sufficient and necessary condition for one exact form of cross-level lifting. It also identifies why a Turing-style manifestation test can remain epistemically incomplete. If $\overline{M}_{c,\kappa}$ is non-injective, several relational equivalence classes occupy one manifestation fiber even when the manifestation is perfectly stable and exactly measured.

6.5 Separating Context Families

A single manifestation context can collapse relational distinctions that become visible under another context. The multi-context construction of Subsection 5.3 therefore supplies a natural route to stronger cross-level identification. The relevant question is whether the combined context family separates the retained relational classes.

Fix a context family $\mathcal{C}$ and a common domain $D \subseteq D_{\mathcal{C}} \cap D_\kappa$. Let $\sim_{\mathcal{C}}$ be the multi-context equivalence from Equation 26. The family is κ -separating on D when Equation 49 holds.

$$\sim_{\mathcal{C}} \mid_D \subseteq \simeq_\kappa \mid_D. \quad (49)$$

A family can satisfy Equation 49 even when no single retained context does. Distinct relational classes can be separated by different components of the combined readout.

Let $M_{\mathcal{C}}$ be the combined map from Equation 25. If every retained context is κ -invariant on D , then $\simeq_\kappa \mid_D \subseteq \sim_{\mathcal{C}} \mid_D$. The combined map consequently factors through $D/\simeq_\kappa$. Exact recovery is then obtained when the family is also κ -separating.

Proposition 6.8 (Exact recovery by a separating context family). Suppose every $c \in \mathcal{C}$ is κ -invariant on D . Then the following conditions are equivalent:

1. $\mathcal{C}$ is κ -separating on D ;
2. $\sim_{\mathcal{C}} \mid_D = \simeq_\kappa \mid_D$;

3. the induced combined manifestation map on $D/\simeq_\kappa$ is injective.

Proof. The assumed κ -invariance of every context gives $\simeq_\kappa \mid_D \subseteq \sim_c \mid_D$ for each c , hence $\simeq_\kappa \mid_D \subseteq \sim_{\mathcal{C}} \mid_D$ by Equation 27. Condition 1 supplies the reverse inclusion and is therefore equivalent to Condition 2. The equivalence of Conditions 2 and 3 follows from Proposition 6.7 applied to the combined map. $\square$

Adding contexts has a one-sided monotonic effect on relational distinction loss. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, Proposition 5.5 implies that the equivalence induced by $\mathcal{C}_2$ refines the equivalence induced by $\mathcal{C}_1$. Additional contexts can therefore split manifestation-equivalence classes. They can also distinguish pairs that the retained relational criterion κ deliberately treats as equivalent. For this reason, a larger context family is not automatically a more adequate representation of a specified relational equivalence. Adequacy requires both separation of retained distinctions and compatibility with the equivalences that the model intends to preserve.

Remark 6.9 (Context enrichment and retained purpose). A context family should be evaluated relative to the declared retained structure. Increased discriminatory capacity can improve identification of κ -classes while simultaneously exposing differences that lie outside κ . The latter differences can motivate refinement of κ , can remain intentionally irrelevant to the current comparison, or can indicate that the cross-level architecture was specified too coarsely. The formalism leaves these interpretations open until the application supplies their substantive role.

6.6 Cross-Level Collapse and Distinction Preservation

The two inclusion conditions developed above permit cross-level mismatch to be diagnosed without treating every mismatch as the same phenomenon. One direction records relational distinctions that manifestation collapses. The other records manifestation distinctions drawn inside a retained relational equivalence class. These two cases have different formal and substantive interpretations.

On a common domain D , define the relational-distinction loss set in Equation 50.

$$\mathcal{L}_{c|\kappa}(D) := (\sim_c \mid_D) \setminus (\simeq_\kappa \mid_D). \quad (50)$$

A pair (x, y) belongs to $\mathcal{L}_{c|\kappa}(D)$ when the selected context treats x and y as manifestationally equivalent while the retained relational criterion distinguishes them. The set is empty exactly when c is κ -separating.

The complementary within-class manifestation distinction set is defined in Equation 51.

$$\mathcal{E}_{c|\kappa}(D) := (\simeq_\kappa \mid_D) \setminus (\sim_c \mid_D). \quad (51)$$

A pair in $\mathcal{E}_{c|\kappa}(D)$ is relationally equivalent under the retained content while receiving different manifestations. The set is empty exactly when the manifestation is κ -invariant. Membership need not indicate an error. The manifestation can retain distinctions that the current relational comparison has chosen to omit.

Proposition 6.10 (Mismatch decomposition for exact recovery). *For two equivalence relations $\sim_c \mid_D$ and $\simeq_\kappa \mid_D$ on the same domain, exact cross-level recovery in the sense of Definition 6.6 holds if and only if both mismatch sets in Equations 50 and 51 are empty.*

Proof. Two sets are equal if and only if each set difference is empty. Applying this elementary criterion to the two equivalence relations gives the result. $\square$

For a context family $\mathcal{C}$, replace $\sim_c$ by $\sim_{\mathcal{C}}$ in Equations 50 and 51. Context enrichment then yields the monotonicity property in Equation 52.

$$\mathcal{C}_1 \subseteq \mathcal{C}_2 \implies \mathcal{L}_{\mathcal{C}_2|\kappa}(D) \subseteq \mathcal{L}_{\mathcal{C}_1|\kappa}(D), \quad (52)$$

whenever both relations are evaluated on the same common domain D . Equation 52 follows from $\sim_{\mathcal{C}_2} \subseteq \sim_{\mathcal{C}_1}$. A larger context family cannot create a new pair that is manifestation-equivalent under all

Table 4: Cross-level conditions relating manifestation equivalence to a retained relational equivalence on a common domain D .

Condition	Formal relation	Cross-level meaning	Principal consequence
κ -invariance	$\simeq_\kappa \subseteq \simeq_c$	Manifestation is constant on retained relational classes	M_c factors through $D/\simeq_\kappa$
κ -separation	$\simeq_c \subseteq \simeq_\kappa$	Equal manifestations identify one retained relational class	No retained relational distinction is collapsed by manifestation
Exact recovery	$\simeq_c = \simeq_\kappa$	Manifestation classes coincide with retained relational classes	Factorized map is injective
Relational-distinction loss	$\mathcal{L}_{c \kappa}(D) \neq \emptyset$	Manifestation merges at least one pair separated by κ	Cross-level lifting from manifestation fails for that pair
Within-class manifestation distinction	$\mathcal{E}_{c \kappa}(D) \neq \emptyset$	Manifestation distinguishes at least one pair identified by κ	Manifestation does not descend to the κ quotient

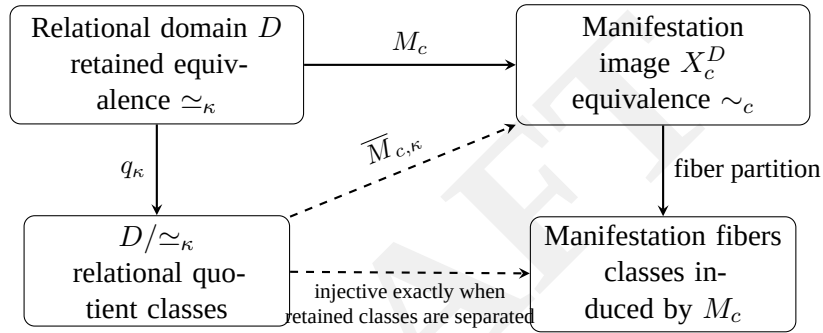


Figure 3: Cross-level relation between retained relational and manifestation equivalence. The dashed factorization exists under κ -invariance; exact recovery additionally requires injectivity of the induced quotient map.

retained contexts while absent from the equivalence induced by the smaller family. The corresponding within-class manifestation distinction set need not contract, because an added context can distinguish a pair previously treated alike.

Table 4 summarizes the cross-level conditions developed in this section. The conditions separate preservation, identification, and exact recovery rather than treating them as one undifferentiated notion of representational adequacy.

Figure 3 displays the quotient and manifestation routes that organize the principal results. When κ -invariance holds, the manifestation map descends through the relational quotient. Injectivity of the induced map gives exact recovery. Non-injectivity leaves several relational classes in one manifestation fiber and therefore preserves cross-level underdetermination.

The section establishes a cross-level comparison layer that is distinct from the within-level taxonomy. Component carriers can be equivalent while their coupling maps differ; a manifestation map can preserve a retained relational equivalence without identifying its classes; and a context family can improve separation while introducing distinctions outside the selected relational content. These distinctions prepare the historical-evolution taxonomy developed in the next section, where the cross-level architecture itself can vary across ordered relational history. Static cross-level equivalence will then become one possible slice of a broader trajectory-level comparison.

7 Equivalence of Historical Evolutions

This section extends the taxonomy from structures evaluated at one relational stage to structures distributed across ordered history. Its role is to identify which additional data are required when an equivalence claim concerns an evolution rather than a snapshot, and to separate several relations that can coincide only under additional assumptions. The construction begins from the historical-extension interface fixed in Section 3.5. It then distinguishes endpoint equivalence, statewise correspondence, order preservation, transition preservation, operation-sensitive comparison, branching continuation relations, weighted continuation comparison, coupled multi-level evolution, and historical change in an equivalence structure itself. The method is formal and representational: historical order is primary, scalar clock time remains optional, and every stronger trajectory relation is constructed from explicitly declared carriers and transport maps.

7.1 Ordered Historical Carriers

A trajectory comparison requires more than a collection of configurations. The configurations must be placed inside a historical structure that records which stages are comparable and how retained structure is transported among them. The extension relation in Equation 13 supplies one source of such order. The present subsection packages the minimum data required by the later trajectory relations.

Let $(I, \preceq_I)$ be a partially ordered historical index set. The indices can represent selected relational slices, admissible history complexes, event-order ideals, or another application-defined family whose ordering has been justified. For each $i \in I$, let $\mathfrak{S}_i$ denote a typed GR comparison structure available at that historical stage. The symbol $\mathfrak{S}_i$ is deliberately generic: it can denote a within-level relational structure from Section 5, a cross-level architecture from Section 6, or another structure whose type has been fixed before trajectory comparison.

When $i \preceq_I j$, let T_{ij} denote the historical transport supplied by the model whenever a comparison from $\mathfrak{S}_i$ to $\mathfrak{S}_j$ is defined. In the fixed-past 2-complex regime, T_{ij} can be realized by an admissible extension embedding $\iota_{ij} : K_i \hookrightarrow K_j$. Other applications can use a typed continuation, restriction-compatible map, or historical identification relation appropriate to the structures being tracked. A coherent family of transports satisfies the conditions in Equation 53 whenever the displayed compositions are defined.

$$T_{ii} = \text{id}_{\mathfrak{S}_i}, \quad T_{ik} = T_{jk} \circ T_{ij} \quad (i \preceq_I j \preceq_I k). \quad (53)$$

Equation 53 records identity and compositional consistency within the selected historical representation. It is not a claim that every GR history forms one globally defined category of total maps. Partiality can remain explicit when a historical identification ceases to be available.

Definition 7.1 (GR historical evolution record). A GR historical evolution record for a declared structure type is a tuple

$$\mathbb{H} = (I, \preceq_I, \{\mathfrak{S}_i\}_{i \in I}, \{T_{ij}\}_{i \preceq_I j}), \quad (54)$$

in which $(I, \preceq_I)$ is the selected historical order, each $\mathfrak{S}_i$ is a well-typed structure of the declared comparison class, and the retained historical transports satisfy Equation 53 on their common domains.

Equation 54 packages the data required by Definition 7.1. The record is a derived bookkeeping object. It does not add a fourth GR reality level and does not identify history with a sequence of external times. A scalar parameter can be attached when an empirical application has a justified calendar, duration, or sampling coordinate, but the comparison developed below depends first on the order and transport structure represented in $\mathbb{H}$.

The record also permits branching. The index set need not be a chain, and two indices can remain incomparable when the model supplies no historical extension relation between them. A realized trajectory can be represented by a chain or directed suborder inside I , while a continuation family can retain several branches. This distinction becomes central in Subsection 7.6, where realized-history equivalence is separated from equivalence or simulation of possible continuations.

Remark 7.2 (Historical extent as a taxonomic coordinate). The historical coordinate $\mathfrak{h}$ introduced in Section 4 can now be instantiated by several formally different carriers: a selected endpoint, an ordered realized history, a branching continuation structure, or a dynamical regime extracted from a family of histories. Equality of the retained structure at one extent supplies no automatic implication at another extent.

7.2 Endpoint Equivalence with Statewise Correspondence

The weakest historical comparisons can ignore part or all of the transition structure. Endpoint equivalence compares selected terminal or reference stages. Statewise correspondence strengthens that comparison by matching structures across a declared set of historical indices, while still leaving the historical order and transitions outside the preservation requirement.

Let $\mathbb{H}_A$ and $\mathbb{H}_B$ be two historical evolution records with index sets I_A and I_B . Suppose the application selects reference indices $i_* \in I_A$ and $j_* \in I_B$ and a well-formed stage-level equivalence relation E_{stg} on the relevant common carrier. The records are endpoint-equivalent at the selected indices when the condition in Equation 55 holds.

$$(\mathfrak{S}_{i_*}^A, \mathfrak{S}_{j_*}^B) \in E_{\text{stg}}. \quad (55)$$

The relation E_{stg} in Equation 55 is the already declared exact stage-level equivalence appropriate to the compared structures. It can instantiate $\simeq_\kappa$ for relational structural equivalence, $\sim_c$ for one manifestation context, $\sim_{\mathcal{C}}$ for a context family, or another exact equivalence whose carrier has been established. The historical claim adds only the selection of the reference stages.

Statewise correspondence requires a broader index matching. Let $I_A^* \subseteq I_A$ and $I_B^* \subseteq I_B$ be retained historical domains, and let $\phi : I_A^* \rightarrow I_B^*$ be a declared bijection. The records are statewise equivalent relative to (ϕ, E_{stg}) when Equation 56 holds.

$$(\mathfrak{S}_i^A, \mathfrak{S}_{\phi(i)}^B) \in E_{\text{stg}} \quad \text{for every } i \in I_A^*. \quad (56)$$

Equation 56 compares the structures attached to the matched indices. The bijection ϕ is part of the comparison data; its existence is not inferred from the equivalence of the individual stages.

Proposition 7.3 (Endpoint consequence of statewise equivalence). *If Equation 56 holds and $i_* \in I_A^*$ is a selected reference index with $j_* = \phi(i_*)$, then the endpoint condition in Equation 55 holds at (i_*, j_*) .*

Proof. Equation 56 holds for every index in I_A^* . Substituting $i = i_*$ gives the stated endpoint equivalence. $\square$

The converse generally fails. Distinct historical records can converge to equivalent terminal structures after carrying different intermediate relations, distinctions, operations, or couplings. Endpoint equivalence therefore licenses a claim about the selected stages only. It does not identify the realized historical trajectories that produced them.

Statewise equivalence is also weaker than historical-order equivalence. A bijection can pair equivalent structures while reversing, scrambling, or otherwise ignoring the precedence relations that organize the two histories. The next subsection adds order preservation as a separate requirement.

7.3 Historical Order Preservation

Historical order can carry information that is absent from a set of matched states. GR permits noncommuting or state-dependent transformations, path-dependent accumulation, and branching continuation. A trajectory comparison that treats these features as relevant must therefore preserve the precedence structure of the retained history.

Let $\phi : I_A^* \rightarrow I_B^*$ be the statewise matching from Subsection 7.2. An exact order correspondence is stated in Equation 57.

$$i \preceq_A j \iff \phi(i) \preceq_B \phi(j) \quad (i, j \in I_A^*). \quad (57)$$

Equation 57 makes ϕ an order isomorphism between the retained suborders. It permits different labels, clock coordinates, or sampling schemes while preserving the modeled historical precedence relation.

Definition 7.4 (Order-preserving historical equivalence). Two historical evolution records are order-preservingly equivalent relative to (ϕ, E_{stg}) on selected suborders when (i) ϕ is a bijection satisfying Equation 57 and (ii) the statewise equivalence condition in Equation 56 holds.

Definition 7.4 preserves the order of matched stages without yet preserving the historical transports T_{ij} . Two records can have isomorphic index posets and equivalent structures at corresponding indices while the maps connecting those structures remain different. This possibility is important whenever the mechanism or path of transformation is part of the research object.

A weaker one-directional relation is available when ϕ is only an order embedding. Such a map can represent one history inside another while leaving unmatched stages or branches in the target record. The present paper describes that relation as an *historical embedding* rather than an equivalence. The distinction follows the relation-type discipline of Section 3.6: a structure-preserving embedding can be analytically valuable without satisfying symmetry on the compared carrier.

Remark 7.5 (Clock independence). Order-preserving historical equivalence does not require equality of external timestamps or durations. If scalar time is substantively relevant, duration preservation or a declared reparameterization law can be added as retained structure. Without that additional requirement, the equivalence concerns historical precedence rather than metric time.

7.4 Transition Preservation over Historical Paths

Order preservation identifies corresponding stages and precedence relations. A stronger comparison examines the historical transports that connect those stages. This requires explicit witnesses for the stage-level equivalence. Bare membership $(\mathfrak{S}_i^A, \mathfrak{S}_{\phi(i)}^B) \in E_{\text{stg}}$ does not by itself provide a map with which two transitions can be compared.

Assume that each matched pair carries an invertible structure-preserving correspondence $\Theta_i : \mathfrak{S}_i^A \rightarrow \mathfrak{S}_{\phi(i)}^B$ witnessing the declared stage-level equivalence. For every retained comparable pair $i \preceq_A j$, transition preservation requires the square in Equation 58 to commute.

$$\Theta_j \circ T_{ij}^A = T_{\phi(i)\phi(j)}^B \circ \Theta_i. \quad (58)$$

Equation 58 states that transporting first in history A and then across the stage correspondence yields the same retained result as translating first and then transporting in history B . The equality is evaluated only where the involved maps and compositions are defined.

Definition 7.6 (Transition-preserving historical equivalence). Two order-preservingly equivalent historical records are transition-preservingly equivalent on the retained domain when there exists a family of invertible stage correspondences $\{\Theta_i\}$ satisfying Equation 58 for every retained comparable pair.

The definition is deliberately stronger than statewise matching. It is available only when the stage-level comparison admits explicit structure-preserving witnesses. Equivalence defined solely as equality of a diagnostic value can fail to provide such witnesses and therefore remains outside the domain of Definition 7.6.

Path preservation follows from the local commutation condition and transport coherence. Let $i_0 \preceq_A i_1 \preceq_A \cdots \preceq_A i_n$ be a retained historical chain. Write T_γ^A and $T_{\phi(\gamma)}^B$ for the corresponding ordered composites. The induced path condition is stated in Equation 59.

$$\Theta_{i_n} \circ T_\gamma^A = T_{\phi(\gamma)}^B \circ \Theta_{i_0}. \quad (59)$$

Proposition 7.7 (Path preservation from transition preservation). *Under the transport-coherence condition in Equation 53, transition-preserving historical equivalence implies Equation 59 for every finite retained historical chain on which the composites are defined.*

Proof. Apply Equation 58 successively along the chain. The intermediate correspondences cancel by associativity of composition and the historical transport composites reduce according to Equation 53. The remaining equality is Equation 59. $\square$

Proposition 7.7 provides a formal distinction between a sequence of equivalent snapshots and an equivalent relational history. The latter preserves how retained structures are connected across the represented historical path.

7.5 Operational Structure in Generative Trajectories

Transition preservation compares the transports already retained by a historical record. GR analyses can require a further distinction concerning which operations are available from each stage and how those operations contribute to later relational structure. This subsection introduces that comparison as an additional layer rather than identifying it automatically with historical transport.

For a historical stage $\mathfrak{S}_i$, let $\mathcal{O}^{\text{adm}}(\mathfrak{S}_i)$ denote a declared family of admissible partial operations whose domains include the retained local structure on which they act. The family is application-defined and can inherit nomos-conditioned admissibility, state-dependent updates, institutional procedures, technical operations, or another GR transformation architecture. No algebraic closure is assumed at this stage.

Let histories A and B already satisfy the state correspondence represented by Θ_i . An operational correspondence at stage i is a bijection

$$\eta_i : \mathcal{O}^{\text{adm}}(\mathfrak{S}_i^A) \longrightarrow \mathcal{O}^{\text{adm}}(\mathfrak{S}_{\phi(i)}^B) \quad (60)$$

whose interpretation is fixed by the local model. For an operation $U \in \mathcal{O}^{\text{adm}}(\mathfrak{S}_i^A)$ that produces a retained successor structure $\mathfrak{S}_j^A$, operational preservation requires a corresponding successor in history B and the commutation condition in Equation 61.

$$\Theta_j \circ U = \eta_i(U) \circ \Theta_i. \quad (61)$$

Equation 61 preserves the realized effect of corresponding operations at the retained structural resolution. It does not assert that the two operations have identical internal mechanisms or ontologies.

Definition 7.8 (Operational trajectory equivalence). Two transition-preservingly equivalent historical records are operationally equivalent relative to a declared operation family when there exists a family of bijections $\{\eta_i\}$ satisfying Equation 60 such that definedness, retained successor structure, and Equation 61 are preserved for every operation included in the comparison domain.

Generative trajectory comparison places a stronger burden on the selected retained structure. An operation can have the same immediate represented result while differing in the relations, admissibility conditions, interfaces, or continuation structure that it helps generate later. The present paper therefore uses *generative trajectory equivalence* only when the retained content κ_G explicitly includes the later relational conditions whose production or modification is relevant to the inquiry and the operational correspondences preserve those conditions across the selected historical window.

Definition 7.9 (Generative trajectory equivalence). Operationally equivalent historical records are generatively equivalent relative to retained content κ_G when the matched operations preserve, through the declared stage correspondences, the κ_G -relevant structures generated for later relational continuation. The retained structures can include later operation domains, newly generated relations, constraints, interfaces, or admissible continuation conditions, provided their historical identification and comparison maps have been independently specified.

Definition 7.9 is intentionally schema-level. Different applications can instantiate κ_G with materially different generative consequences. The definition does not introduce one scalar quantity of generativity and does not infer equivalence of unretained causal mechanisms from preservation of selected later relational effects.

Remark 7.10 (Process preservation and endpoint agreement). Equal manifested outputs or equivalent endpoints can coexist with operational or generative non-equivalence. Conversely, two systems can use different internal operation labels while remaining operationally equivalent after a justified bijection η_i has been supplied. The relevant question is preservation of the declared operation-and-consequence structure, not nominal identity of procedure labels.

7.6 Continuation Relations over Branching Futures

A realized historical record retains one selected history or one family of already-instantiated histories. GR also distinguishes the continuation structures available from a realized history. The extension family $\text{Ext}(K^-)$ in Equation 12 makes this future-oriented comparison explicit. This subsection separates exact continuation-structure isomorphism from directional simulation and mutual bisimulation.

For two realized histories K_A^- and K_B^- , write $\mathcal{E}_A = \text{Ext}(K_A^-)$ and $\mathcal{E}_B = \text{Ext}(K_B^-)$. Let $\preceq_A^+$ and $\preceq_B^+$ denote the extension orders induced on the retained continuation families when those orders are defined. An exact continuation isomorphism is a bijection $F : \mathcal{E}_A \rightarrow \mathcal{E}_B$ satisfying Equation 62 together with the declared structural equivalence of corresponding continuation complexes.

$$\gamma_1 \preceq_A^+ \gamma_2 \iff F(\gamma_1) \preceq_B^+ F(\gamma_2). \quad (62)$$

Equation 62 preserves the branching order of the selected continuation structures. A richer application can additionally require preservation of operation labels, costs, weights, or cross-level manifestation structure.

Directional comparison requires less. Let $\rightsquigarrow_A$ and $\rightsquigarrow_B$ denote one-step or selected elementary continuation relations on the retained continuation carriers X_A^+ and X_B^+ . Let $R \subseteq X_A^+ \times X_B^+$ be a cross-system relation whose paired states satisfy the declared retained structural criterion. A forward continuation simulation from A to B satisfies Equation 63.

$$(x, y) \in R, \quad x \rightsquigarrow_A x' \implies \exists y' (y \rightsquigarrow_B y' \text{ and } (x', y') \in R). \quad (63)$$

Equation 63 states that each retained continuation step available from the A side can be matched by the B side. It does not require the reverse condition and therefore supports substitutability or simulation claims that are directional.

The reverse matching condition is stated explicitly in Equation 64.

$$(x, y) \in R, \quad y \rightsquigarrow_B y' \implies \exists x' (x \rightsquigarrow_A x' \text{ and } (x', y') \in R). \quad (64)$$

Definition 7.11 (Continuation bisimulation). A relation $R \subseteq X_A^+ \times X_B^+$ is a continuation bisimulation relative to the declared structural criterion when it satisfies both the forward clause in Equation 63 and the reverse clause in Equation 64. A pair of realized configurations is continuation-bisimilar relative to that criterion when it belongs to at least one such relation.

Definition 7.11 provides a back-and-forth comparison of possible becoming. It remains weaker than an isomorphism of continuation carriers because several distinct states or branches can be related by one bisimulation relation. The term *bisimilar* is therefore kept distinct from literal equality of continuation trees or extension posets.

Remark 7.12 (Bisimulation as an equivalence-like relation). Continuation bisimulation is introduced here as a relation type on declared transition carriers. Whether the induced notion of bisimilarity is used as an equivalence relation on a larger common carrier depends on the closure and typing assumptions of the instantiated transition model. The present taxonomy therefore records the bisimulation witness and its domain before using quotient language.

The distinction among exact continuation isomorphism, simulation, and bisimulation matters for substitution claims. A system can simulate another's retained continuation possibilities while possessing additional branches that the simulated system cannot match. Mutual back-and-forth correspondence removes that one-sided mismatch while still allowing several structurally different continuation carriers to remain bisimilar at the declared resolution.

7.7 Weighted Continuation Equivalence

Continuation structure records which histories remain admissible. Some GR applications additionally assign nonnegative weights, propensities, costs, support values, or other evaluations to retained continuations. Equality of the unweighted branching structure does not preserve those evaluations. Weighted continuation equivalence therefore requires an additional comparison layer and an explicit interpretation of the weights.

Assume $\mathcal{E}_A$ and $\mathcal{E}_B$ are finite or countable retained continuation families equipped with nonnegative weight functions w_A and w_B whose total weights are positive and finite. Define the normalized weights in Equation 65.

$$\hat{w}_X(\gamma) := \frac{w_X(\gamma)}{\sum_{\eta \in \mathcal{E}_X} w_X(\eta)}, \quad X \in \{A, B\}. \quad (65)$$

The normalization in Equation 65 permits comparison when the raw weights differ only by an overall positive scale. It does not assign a probabilistic interpretation unless the application independently justifies one.

Let $F : \mathcal{E}_A \rightarrow \mathcal{E}_B$ be an exact continuation isomorphism of the type introduced in Subsection 7.6. The isomorphism is weight-preserving when Equation 66 holds.

$$\hat{w}_A(\gamma) = \hat{w}_B(F(\gamma)) \quad \text{for every } \gamma \in \mathcal{E}_A. \quad (66)$$

Definition 7.13 (Weighted continuation equivalence). Two weighted continuation structures are equivalent relative to the declared history and structural criteria when there exists a continuation isomorphism F satisfying Equation 62 and the normalized weight-preservation condition in Equation 66.

Proposition 7.14 (Weighted equivalence refines unweighted continuation equivalence). *Weighted continuation equivalence implies exact unweighted continuation isomorphism after the weight data are forgotten. The converse is not generally valid.*

Proof. Definition 7.13 includes an exact continuation isomorphism as part of its witness, so forgetting Equation 66 leaves the unweighted isomorphism. Conversely, an unweighted isomorphism imposes no equality on $\hat{w}_A$ and $\hat{w}_B$, so corresponding branches can receive different normalized weights. $\square$

Proposition 7.14 formalizes the distinction between sharing a possibility and assigning that possibility the same comparative support. The same distinction applies when weights represent cost, propensity, preference, amplitude-like bookkeeping, or another application-defined evaluation rather than probability.

7.8 Coupled Multi-Level Trajectory Equivalence

GR treats relations and interfaces as parts of the represented object. Equivalence of the individual relata therefore does not determine equivalence of their coupling. The same issue persists historically: equivalent entity trajectories can participate in non-equivalent coupled trajectories. Cross-level structures from Section 6 add a second coupling problem because the maps connecting relational and manifestation structures can also evolve.

For two identified entities A and B , let $\mathfrak{S}_i^{AB}$ denote the coupled structure retained at historical index i . At a 2-complex resolution this structure can contain $K_{A,i}$, $K_{B,i}$, their shared or interface subcomplexes, and additional relations defined only at the coupled level. A coupled historical record is obtained by using $\mathfrak{S}_i^{AB}$ as the stage object in Definition 7.1.

Proposition 7.15 (Componentwise trajectory equivalence does not determine coupling equivalence). *Equivalence of the retained trajectory of A with A' and of B with B' does not, without an additional coupling-preservation condition, imply equivalence of the coupled trajectories AB and $A'B'$.*

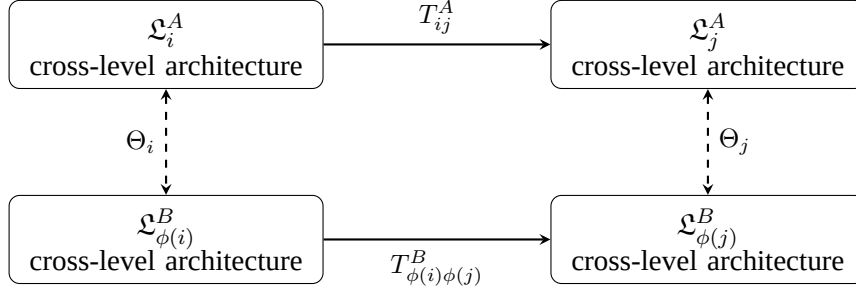


Figure 4: Historical comparison of cross-level architectures. Each $\mathfrak{L}$ contains the retained level carriers and their cross-level maps at one historical stage. Trajectory equivalence requires the within-stage architecture correspondences Θ_i and Θ_j to remain compatible with the historical transports. Equivalent stage objects alone therefore do not establish equivalent historical evolution.

Proof. Consider a retained relational criterion that treats the internal structures of A and A' as equivalent and likewise the internal structures of B and B' . Let the coupled structure $\mathfrak{S}^{AB}$ contain a relation r between the two components while the otherwise matched structure $\mathfrak{S}^{A'B'}$ lacks the corresponding relation, or contains a relation with a different retained type. Componentwise equivalence remains unchanged, while any coupled criterion that retains r distinguishes the two coupled structures. Repeating the construction across the matched historical indices gives componentwise equivalent trajectories with non-equivalent coupling trajectories. $\square$

Proposition 7.15 makes the relational object itself part of the historical taxonomy. A human trajectory and an artificial-system trajectory can each be matched across two modeled settings while the human–AI interface, institutional embedding, responsibility allocation, or mutual adaptation history differs. A similar construction applies to AI–AI, human–human, institutional, ecological, or other coupled histories.

Multi-level trajectories add the relation among GR levels. At historical index i , write a two-level architecture schematically as

$$\mathfrak{L}_i = (\mathcal{R}_i, \mathcal{M}_i, M_i), \quad (67)$$

where M_i denotes the retained manifestation map or family of maps at that stage. Equation 67 makes the cross-level map part of the historical stage object rather than treating it as an external observation rule. A historical equivalence between two such architecture trajectories requires more than relational-level and manifestation-level trajectory equivalence separately. The cross-level squares from Section 6 must commute at each matched stage, and the historical transports must remain compatible with those level correspondences.

Figure 4 summarizes the resulting compatibility requirement at the architecture level. Each node already contains the retained cross-level carriers and maps. A trajectory-level equivalence can therefore fail because stage architectures differ, historical transports differ, or the stage correspondences fail to commute with historical evolution.

The same architecture extends beyond two levels or two entities. Higher-order cells, multi-party interfaces, institutional layers, and families of manifestation contexts can be included when the application supplies their carriers and compatibility maps. The formal burden grows with the retained structure; the taxonomy therefore recommends the smallest coupled carrier adequate to the research question.

7.9 Historical Evolution of Equivalence Structures

The preceding subsections compare objects through an equivalence relation treated as fixed over the selected comparison. GR also permits the equivalence structure itself to change historically. A classification can refine, coarsen, split, merge, lose its domain, or be reorganized as relational distinctions and manifestation practices evolve. Comparing such changes requires historical transport before relation inclusion can be evaluated.

Let D_i and D_j be carriers at historical stages $i \preceq j$, equipped with equivalence relations $E_i \subseteq D_i \times D_i$ and $E_j \subseteq D_j \times D_j$. Let $\tau_{ij} : D_i \rightarrow D_j$ be a declared historical tracking map and write $D_{ij} = \text{Dom}(\tau_{ij})$. The later equivalence can be pulled back to the earlier comparison domain by Equation 68.

$$\tau_{ij}^* E_j := \{(x, y) \in D_{ij} \times D_{ij} \mid (\tau_{ij}(x), \tau_{ij}(y)) \in E_j\}. \quad (68)$$

Equation 68 places the later relation on the same carrier as the restriction $E_i|_{D_{ij}}$. Refinement, coarsening, and reorganization can then be defined by ordinary relation inclusion.

Definition 7.16 (Transport-relative evolution of equivalence). Relative to a historical tracking map τ_{ij} , the equivalence structure is:

1. *preserved* on D_{ij} when $\tau_{ij}^* E_j = E_i|_{D_{ij}}$;
2. *refined* when $\tau_{ij}^* E_j \subsetneq E_i|_{D_{ij}}$;
3. *coarsened* when $E_i|_{D_{ij}} \subsetneq \tau_{ij}^* E_j$;
4. *reorganized* when neither relation contains the other.

A strict refinement means that at least one pair tracked from the earlier domain was equivalent under E_i and is distinguished by the pulled-back later relation. A strict coarsening means that at least one previously distinguished tracked pair becomes equivalent later. Reorganization combines losses and gains of equivalence in a way that cannot be represented by one partition refinement order.

Caution 7.17 (Transport-induced collapse). If τ_{ij} is non-injective, two earlier objects can be mapped to one later object before E_j is evaluated. Apparent coarsening can then arise from the tracking map rather than from a substantive change in the later equivalence criterion. An analysis that seeks to attribute class merging specifically to evolution of the equivalence structure should either use an injective tracking map on the comparison domain or separately report identifications introduced by transport.

The transport issue in Caution 7.17 matters especially for changing entity boundaries, institutional aggregation, coarse-graining, and model revision. Historical comparability is not a neutral preliminary step; it can itself determine which distinctions remain available for later classification.

The construction applies separately to manifestation and relational equivalence. Let E_i^M denote a manifestation-induced equivalence and E_i^R a retained relational equivalence at stage i . Their historical evolution can display different inclusion patterns after the relevant transports have been supplied. A manifestation classification can coarsen while a relational classification refines, producing increasing similarity of presentation alongside increasing retained relational differentiation. The reverse pattern is also formally possible. The taxonomy therefore treats the historical evolution of equivalence as a multi-layer object rather than assuming synchronized class structure across GR levels.

Table 5 summarizes the principal comparison forms developed in this section. The rows are organized by historical object rather than by a universal ranking of strength.

Table 5: Principal historical and continuation comparison forms in the GR equivalence taxonomy.

Comparison form	Preserved object	Required witness	Principal boundary
Endpoint equivalence	Selected historical stages	Stage-level equivalence	Omits intermediate history
Statewise equivalence	Matched stage structures	Index bijection plus stage equivalence	Omits order and transport
Order-preserving equivalence	Historical precedence plus matched stages	Order isomorphism	Omits transition maps
Transition-preserving equivalence	Historical transports and stage structures	Invertible stage correspondences satisfying commuting squares	Requires explicit structural witnesses
Operational trajectory equivalence	Admissible operations and retained effects	Bijections of operation families plus compatible execution	Does not identify unretained mechanisms
Generative trajectory equivalence	Later relational conditions generated by operations	Operational correspondence plus retained generative content κ_G	Depends on declared generative consequences
Continuation simulation	Directional matching of possible next histories	Forward simulation relation	Need not be symmetric
Continuation bisimulation	Back-and-forth matching of branching continuation	Bisimulation relation	Need not be continuation-tree isomorphism
Weighted continuation equivalence	Branching structure plus normalized continuation weights	Continuation isomorphism plus weight preservation	Weight semantics remain application-specific
Coupled trajectory equivalence	Entities together with their interface or higher-order relation	Coupling-preserving historical correspondence	Component equivalence is insufficient
Equivalence-structure evolution	Historical change in equivalence classes	Tracking map plus pullback comparison	Transport can itself collapse distinctions

The historical taxonomy establishes that equivalence can be defined on several distinct extents of relational becoming. A selected endpoint, an ordered realized history, a transition system, a branching continuation family, a weighted family of futures, and an evolving equivalence structure preserve different information. The distinction also clarifies the role of trajectory evidence in later Human–AI analysis: identical outputs over a sampled interaction can establish a manifestation-level statewise relation while leaving historical order, operation structure, relational coupling, and admissible continuation unmatched. Section 8 next changes the comparison object again by asking whether distinct histories can instantiate equivalent dynamical regimes or agree on selected structural invariants even when their realized trajectories are not equivalent.

8 Equivalence of Dynamical Regimes and Structural Invariants

Section 8 completes the five-part taxonomy by changing the comparison object from realized historical evolution to structures extracted from, or organizing, families of possible evolutions. Its role is to distinguish equivalence of dynamical organization from agreement of selected dynamical or structural diagnostics. The objective is to make precise when attractor-like regimes, basin relations, metastability, critical-response characteristics, algebraic composition, and topological information support a structural equivalence claim, and when they support only a weaker statement of diagnostic agreement. The discussion proceeds from a general diagnostic map to regime-level structures, perturbation-response organization, algebraic and compositional characteristics, topological and homological diagnostics, and resolution dependence. The constructions remain representation-relative. A dynamical regime, spectrum, homology group, susceptibility, or invariant profile is a formal object supplied by a declared model; none

is identified with ultimate ontology.

8.1 Diagnostic Agreement and Structural Equivalence

A structural equivalence can be difficult to test directly. Dynamical analysis therefore often compares quantities extracted from a richer object: attractor counts, persistence profiles, basin labels, recovery rates, spectra, cycle residuals, Betti numbers, or other diagnostics. Equality of such quantities is useful, while its inferential force depends on how the diagnostic relates to the structural equivalence under investigation.

Let D be a carrier of dynamical or structural objects and let $E \subseteq D \times D$ be a declared target equivalence relation. Let $\mathcal{Q} = \{Q_a\}_{a \in A}$ be a family of diagnostics, where each Q_a is a partial map $Q_a : D \rightharpoonup Y_a$. Write $D_{\mathcal{Q}}$ for the common domain on which the retained diagnostic family is defined. The joint diagnostic map is stated in Equation 69.

$$Q_{\mathcal{Q}} : D_{\mathcal{Q}} \longrightarrow \prod_{a \in A} Y_a, \quad Q_{\mathcal{Q}}(x) := (Q_a(x))_{a \in A}. \quad (69)$$

The induced diagnostic-agreement relation is stated in Equation 70.

$$x \sim_{\mathcal{Q}} y \iff Q_{\mathcal{Q}}(x) = Q_{\mathcal{Q}}(y), \quad x, y \in D_{\mathcal{Q}}. \quad (70)$$

Proposition 8.1 (Kernel equivalence of exact diagnostic agreement). *The relation $\sim_{\mathcal{Q}}$ in Equation 70 is an equivalence relation on $D_{\mathcal{Q}}$.*

Proof. Equation 70 is equality after application of the joint map $Q_{\mathcal{Q}}$. Reflexivity, symmetry, and transitivity therefore follow from equality in the product codomain. $\square$

Proposition 8.1 establishes an exact equivalence relation at the diagnostic-profile level. It does not establish that $\sim_{\mathcal{Q}}$ coincides with the target structural equivalence E . Two additional relations between E and the diagnostic kernel are required.

Definition 8.2 (Diagnostic invariance and separation). Let E be a target equivalence relation on $D_{\mathcal{Q}}$. The diagnostic family $\mathcal{Q}$ is *E -invariant* when

$$E \subseteq \sim_{\mathcal{Q}}, \quad (71)$$

and it is *E -separating* on the retained domain when

$$\sim_{\mathcal{Q}} \subseteq E. \quad (72)$$

Equation 71 states that equivalent structures cannot be distinguished by the retained diagnostics. Equation 72 states that diagnostic agreement cannot identify two structures that the target relation distinguishes. The second condition is therefore a sufficiency condition relative to E , rather than a generic property of an invariant.

Proposition 8.3 (Exact Diagnostic Recovery). *A diagnostic family $\mathcal{Q}$ is simultaneously E -invariant and E -separating on $D_{\mathcal{Q}}$ if and only if*

$$E = \sim_{\mathcal{Q}}. \quad (73)$$

Proof. The two conditions in Definition 8.2 give the opposite inclusions $E \subseteq \sim_{\mathcal{Q}}$ and $\sim_{\mathcal{Q}} \subseteq E$. Their conjunction gives equality. Equality conversely implies both inclusions. $\square$

Equation 73 identifies the exact-characterization case. The one-sided use of diagnostics follows immediately. If $\mathcal{Q}$ is E -invariant and $Q_{\mathcal{Q}}(x) \neq Q_{\mathcal{Q}}(y)$, then $(x, y) \notin E$. Equality of diagnostic profiles supports $(x, y) \in E$ only when E -separation has also been established. This asymmetry will govern the attractor, criticality, algebraic, spectral, and topological comparisons below.

Figure 5 summarizes the distinction. The upper route compares the structural objects through a declared relation E . The lower route passes through an extracted diagnostic profile. Agreement along the lower route recovers the structural relation only when the diagnostic family separates the relevant E -classes.

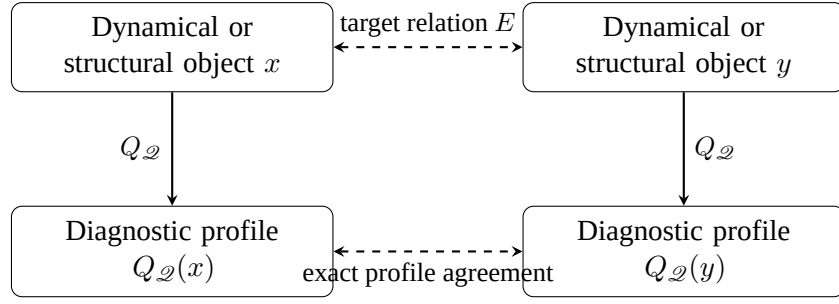


Figure 5: Structural equivalence and diagnostic agreement. Equality of extracted diagnostic profiles defines an exact kernel equivalence on the diagnostic domain. Recovery of the target structural equivalence requires the diagnostic family to be both invariant on target-equivalent structures and separating across distinct target classes.

8.2 Attractor and Basin Structure

Attractor-like and basin-like descriptions summarize organization across families of histories rather than one realized trajectory. Their use in GR is intentionally broader than one classical metric-phase-space definition: an application can represent recurrent retention, return tendency, or historically persistent regime membership through structures suited to its domain. The equivalence taxonomy therefore compares the declared regime structure itself before reducing it to counts or scalar summaries.

Let $\mathcal{D}$ denote a dynamical specification on a retained configuration domain X . Suppose the model identifies a family of attractor-like regimes $\mathcal{A}(\mathcal{D}) = \{A_\lambda\}_{\lambda \in \Lambda}$ and, where defined, corresponding basin-like domains $\mathcal{B}(\mathcal{D}) = \{B_\lambda\}_{\lambda \in \Lambda}$. Let $\rightsquigarrow_{\mathcal{D}}$ be a declared transition or reachability relation among the retained regimes, and let $\ell_{\mathcal{D}}$ collect whatever regime labels the comparison has independently justified, such as metastability class, retained symmetry type, or another typed descriptor. The attractor–basin structure is stated in Equation 74.

$$\mathfrak{A}(\mathcal{D}) := (\Lambda, \{A_\lambda\}_{\lambda \in \Lambda}, \{B_\lambda\}_{\lambda \in \Lambda}, \rightsquigarrow_{\mathcal{D}}, \ell_{\mathcal{D}}). \quad (74)$$

Equation 74 is a model-level extraction. It does not assert that every GR system possesses a canonical attractor decomposition or that basin boundaries are uniquely defined independently of representation.

Definition 8.4 (Attractor–basin structural equivalence). Two dynamical specifications $\mathcal{D}_1$ and $\mathcal{D}_2$ are attractor–basin structurally equivalent relative to the retained data in Equation 74 when there exists a bijection $f : \Lambda_1 \rightarrow \Lambda_2$ together with the declared regime- and basin-level structural isomorphisms such that: (i) each $A_\lambda^{(1)}$ corresponds to $A_{f(\lambda)}^{(2)}$ under the retained regime criterion; (ii) each $B_\lambda^{(1)}$ corresponds to $B_{f(\lambda)}^{(2)}$ under the retained basin criterion; (iii) $\rightsquigarrow_{\mathcal{D}_1}$ is preserved and reflected by f ; and (iv) the retained labels ℓ are preserved under the declared label correspondence.

The definition makes the object of equivalence explicit. Equality of attractor counts, basin volumes, recurrence rates, or persistence statistics can supply diagnostics of Equation 74; such equality is weaker unless those diagnostics are shown to characterize the retained structure.

For example, define the attractor-count diagnostic $Q_{\#}(\mathcal{D}) = |\Lambda|$ when the set is finite. Two systems with $Q_{\#}(\mathcal{D}_1) = Q_{\#}(\mathcal{D}_2)$ can differ in basin incidence, transition organization, internal regime type, or persistence properties. The count therefore induces a coarse diagnostic equivalence while leaving Definition 8.4 unresolved.

The same caution applies to basin statistics. Let Q_B record a vector of declared basin descriptors. Equality $Q_B(\mathcal{D}_1) = Q_B(\mathcal{D}_2)$ means that the selected basin diagnostics agree. It licenses attractor–basin structural equivalence only if Q_B , alone or jointly with other diagnostics, satisfies the separation condition in Definition 8.2 for the retained regime structure.

8.3 Metastability and Critical-Regime Characteristics

Metastability and criticality introduce another level of abstraction. They classify how a dynamical organization persists, responds, and approaches structural change across a declared family of perturbations and historical scales. A GR comparison therefore requires a specification of the diagnostics and decision rule used to assign a regime class.

Let X be a retained configuration carrier and let $\Delta_X(x)$ denote the admissible perturbations at $x \in X$. A critical-regime specification is represented by Equation 75.

$$\mathcal{S}_{\text{crit}} := (\Delta, \mathcal{Q}_{\text{crit}}, \mathcal{T}_{\text{crit}}, \varrho_{\text{crit}}), \quad (75)$$

where Δ supplies the perturbation family, $\mathcal{Q}_{\text{crit}}$ is the retained diagnostic family, $\mathcal{T}_{\text{crit}}$ is the declared decision or classification rule, and ϱ_{crit} records the representational scale on which the specification is evaluated. The corresponding classifier is written $C_{\mathcal{S}_{\text{crit}}} : X \rightarrow Y_{\text{crit}}$.

The classifier induces an exact class-agreement relation by Equation 76.

$$x \sim_{\mathcal{S}_{\text{crit}}} y \iff C_{\mathcal{S}_{\text{crit}}}(x) = C_{\mathcal{S}_{\text{crit}}}(y). \quad (76)$$

Equation 76 can classify two configurations as belonging to the same declared criticality regime while their detailed response structures remain different. The relation is therefore equivalence under the declared classifier, not a universal equivalence of dynamics.

Metastability can be treated analogously. A specification can retain a historical scale, a persistence rule, a family of perturbations, and a return or retention criterion. Two structures can then agree on the resulting metastability class while differing in basin geometry, transition pathways, or the mechanisms supporting persistence. The taxonomy records such agreement at the classification level unless a stronger correspondence between the underlying regime structures is established.

Remark 8.5 (Specification-relative criticality equivalence). Changing $\mathcal{S}_{\text{crit}}$ can change the induced equivalence classes. Adding a diagnostic does not by itself guarantee a refinement of $\sim_{\mathcal{S}_{\text{crit}}}$ because the decision rule $\mathcal{T}_{\text{crit}}$ can aggregate, threshold, or otherwise reorganize the enlarged diagnostic family. Relation inclusion should therefore be established from the concrete classifiers rather than inferred from the number of retained diagnostics.

Remark 8.5 prevents a common shortcut in criticality comparisons. A richer measurement vector can yield a coarser classification if the revised decision rule merges earlier categories, and a smaller diagnostic family can be highly separating if its components are structurally discriminating. The taxonomic object is the specified relation, not the length of the diagnostic list.

8.4 Response and Susceptibility Structure

A susceptibility statistic compresses a perturbation–response relation into one or several quantities. The full response organization carries additional information: which perturbations are admissible, which effects are compared, how response composition depends on state, and which structural changes occur beyond a selected scalar response. The present subsection therefore defines response-structure equivalence before treating susceptibility agreement.

For a dynamical object $x \in D$, let $\Delta(x)$ be its declared perturbation family and let Z_x be the retained response space. A response map is written in Equation 77.

$$R_x : \Delta(x) \rightarrow Z_x. \quad (77)$$

The codomain can contain scalar responses, response profiles, changed continuation structures, transformed operators, or another typed object justified by the application.

Definition 8.6 (Response-structure isomorphism). Two dynamical objects x and y have isomorphic retained response structure when there exist bijections

$$\psi_\Delta : \Delta(x) \longrightarrow \Delta(y), \quad \psi_Z : Z_x \longrightarrow Z_y, \quad (78)$$

with the required preservation of any additional retained structure on the perturbation and response carriers, such that $\delta \in \text{Dom}(R_x)$ if and only if $\psi_\Delta(\delta) \in \text{Dom}(R_y)$, and such that Equation 79 holds on the corresponding defined domain.

$$\psi_Z \circ R_x = R_y \circ \psi_\Delta. \quad (79)$$

The bijections in Equation 78, together with the commuting condition, make Definition 8.6 a comparison of the response relation itself rather than a summary statistic. A susceptibility diagnostic can then be represented as a map $\chi : D \rightarrow Y_\chi$ constructed from R_x , a perturbation scale, a discrepancy structure, or another declared rule. Exact susceptibility agreement,

$$\chi(x) = \chi(y), \quad (80)$$

is diagnostic agreement in the sense of Section 8.1. Equation 80 does not imply Definition 8.6 unless χ is separating for the retained response structure.

The distinction accommodates several GR response diagnostics within one architecture. A model may retain sensitivity of a task readout, a continuation family, an ordered composition, a cycle residual, a recovery process, or a discrete structural margin. Two systems can agree on one susceptibility while differing strongly on another because the perturbation families and response codomains differ. A claim of response equivalence should therefore state both the perturbation carrier and the retained response structure.

8.5 Algebraic and Compositional Characteristics

History-bearing dynamics can be organized through admissible transformations whose order and composition matter. Algebraic or categorical representations make this structure explicit. They also generate numerical diagnostics such as spectra, traces, commutator norms, fixed-space dimensions, or cycle residuals. The taxonomy distinguishes structural equivalence of the compositional object from equality of these derived quantities.

Let $\mathcal{A}_x$ denote a retained algebra of transformations or observables associated with a dynamical object x , and let Path_x denote a retained path or composition structure generated by admissible transformations. When the application uses an algebra, a structural comparison can be represented by an isomorphism

$$\Phi_{xy} : \mathcal{A}_x \xrightarrow{\cong} \mathcal{A}_y \quad (81)$$

that preserves the operations required by the selected algebraic type. Equation 81 records the algebraic case. When the application uses a category or path structure, the corresponding comparison requires a functorial equivalence or another explicitly stated composition-preserving relation appropriate to the retained objects and morphisms.

Closed compositional histories provide a further structural object. Let $\mathcal{C}_x$ be a declared family of cycles and let $H_x(c)$ be the return map associated with $c \in \mathcal{C}_x$. A cycle-structure correspondence requires a bijection $f_C : \mathcal{C}_x \rightarrow \mathcal{C}_y$ together with a declared correspondence of the return-map carriers such that corresponding cycles preserve the retained compositional relation. Equality of one scalar residual derived from $H_x(c)$ and $H_y(f_C(c))$ does not by itself establish equivalence of the return maps.

The same separation applies to spectra. Suppose a finite represented operator L_x is associated with x and a spectral diagnostic $\text{Spec}(L_x)$ is defined. Equality

$$\text{Spec}(L_x) = \text{Spec}(L_y) \quad (82)$$

records spectral agreement in Equation 82. It does not, without further assumptions, identify the operators, their eigenvectors or invariant subspaces, their non-normal response, the surrounding algebra, or the

relational interpretation of their carriers. Spectral agreement therefore occupies the diagnostic branch of Figure 5 unless a theorem or model-specific argument establishes sufficiency for the stronger structure under study.

Likewise, a commutator diagnostic can detect order sensitivity without characterizing the whole algebra. Equality of $\| [A, B] \|$ for two systems can coexist with different commutation relations among other generators or different higher-order compositions. The retained generator set and composition laws must therefore be part of any claim of algebraic equivalence.

8.6 Topological and Homological Diagnostics

A history-bearing 2-complex or another relational carrier can support topological and homological analysis when the required cell or chain structure has been specified. These comparisons illustrate particularly clearly the difference between a genuine structural equivalence and agreement of an incomplete invariant family.

Let K_x and K_y be finite or suitably controlled complexes representing the retained relational structures of two dynamical objects. A target topological relation must first be declared. Depending on the application, the target can be homeomorphism, homotopy equivalence, simple-homotopy equivalence, isomorphism of a labeled cell complex, or another explicitly typed relation. The present taxonomy writes this relation generically as E_{top} and does not treat these alternatives as interchangeable.

For a declared coefficient system and tested degree set Q , define the homology profile by Equation 83.

$$\mathcal{H}_Q(K) := (H_q(K))_{q \in Q}. \quad (83)$$

A coarser Betti profile is defined in Equation 84 when the relevant ranks are available.

$$\beta_Q(K) := (\beta_q(K))_{q \in Q}. \quad (84)$$

If E_{top} preserves the selected homology groups, then non-isomorphic homology in a tested degree establishes non-equivalence under E_{top} . Equality of the selected homology or Betti profiles establishes only homological or Betti-profile agreement unless the retained family is complete for the target relation on the declared domain.

Proposition 8.7 (One-sided invariant detection). *Let $I : D \rightarrow Y$ be invariant under a target equivalence relation E , so that $(x, y) \in E$ implies $I(x) = I(y)$. If $I(x) \neq I(y)$, then $(x, y) \notin E$. The converse implication from $I(x) = I(y)$ to $(x, y) \in E$ requires I to be E -separating.*

Proof. The first statement is the contrapositive of invariance. The converse is exactly the separation condition in Definition 8.2 for the singleton diagnostic family $\{I\}$. $\square$

Proposition 8.7 applies to Betti numbers, Euler characteristic, selected spectral invariants, symmetry-group data, or another genuine invariant of the target relation. It also clarifies the status of an invariant profile. A changed component can detect structural difference when invariance is established; an unchanged finite profile can leave substantial structural difference undetected.

When a declared transformation between complexes induces maps on homology, the induced maps carry more information than profile equality. A chain-compatible map $f : K_x \rightarrow K_y$ can induce

$$f_{*,q} : H_q(K_x) \longrightarrow H_q(K_y). \quad (85)$$

Equation 85 can record how classes are transported, identified, or annihilated. Its availability depends on the existence of the required underlying map; the taxonomy does not infer an induced map from equality of Betti numbers alone.

8.7 Resolution Dependence and Invariant Sufficiency

Dynamical and invariant-based comparisons depend on how the represented system is resolved. A coarse representation can suppress cycles, split or merge attractor-like regimes, alter an admissible perturbation family, change operator dimension, or remove higher-order cells that carry homological information. The final part of the taxonomy therefore records how a claimed equivalence behaves under declared refinement or coarse-graining.

Let $D^{(f)}$ and $D^{(c)}$ be fine and coarse carriers related by a declared representation map $q : D^{(f)} \rightarrow D^{(c)}$. Let $Q^{(f)}$ and $Q^{(c)}$ be diagnostics on the two representations. Diagnostic covariance under the representation change is stated in Equation 86 when there exists a map $\hat{q}$ on the diagnostic codomains such that

$$Q^{(c)} \circ q = \hat{q} \circ Q^{(f)} \quad (86)$$

on the common domain of definition. Equation 86 records how a fine diagnostic is represented coarsely; it does not require q or $\hat{q}$ to be invertible.

A structural equivalence claim can likewise be tested for resolution stability. Let $E^{(f)}$ and $E^{(c)}$ be the target equivalence relations at the two resolutions. The coarse representation is equivalence-compatible on a retained fine domain when

$$(x, y) \in E^{(f)} \implies (q(x), q(y)) \in E^{(c)}. \quad (87)$$

Equation 87 states preservation from fine to coarse resolution. The reverse implication is stronger because coarse-graining can identify fine structures that the finer relation distinguishes. Exact recovery of fine equivalence from the coarse representation therefore requires an additional separation condition.

The preceding constructions yield a general criterion for diagnostic sufficiency that is independent of the specific mathematical language used by an application.

Corollary 8.8 (Invariant-family sufficiency). *Let E be a target equivalence relation on a declared domain and let $\mathcal{Q}$ be a family of exact diagnostics. If $\mathcal{Q}$ is invariant under E and separates distinct E -classes, then equality of the complete retained diagnostic profile is equivalent to E on that domain.*

Proof. The result is Proposition 8.3 restated for a diagnostic family interpreted as invariants of E . $\square$

Corollary 8.8 states the burden required before the paper will infer structural equivalence from invariant agreement. In many applications the available diagnostic family will fail to be known complete. The appropriate conclusion is then diagnostic agreement at the declared resolution, accompanied by any one-sided non-equivalence results that changed invariant values establish.

Table 6 summarizes the comparison objects developed in Section 8. The rows preserve different structures and therefore do not form a universal ladder of strength.

Table 6: Dynamical-regime and structural-invariant comparisons in the GR equivalence taxonomy.

Comparison form	Retained object	Formal witness	Principal inferential boundary
Diagnostic-profile agreement	Selected diagnostic values	Equality under $Q_{\mathcal{Q}}$	Structural equivalence requires separation/sufficiency
Attractor-basin structural equivalence	Regimes, basins, transitions, and retained labels	Regime correspondence preserving the declared structure	Counts or basin summaries alone are generally coarser
Critical-regime class agreement	Classification under a declared criticality specification	Equality under $C_{\mathcal{S}_{\text{crit}}}$	Depends on perturbations, diagnostics, thresholds, and resolution
Response-structure isomorphism	Perturbation carrier, response carrier, and response map	ψ_{Δ} and ψ_Z satisfying Equation 79	Equal susceptibility need not identify the response relation
Algebraic/compositional equivalence	Operations, composition, paths, cycles, or return maps	Algebra isomorphism, functorial equivalence, or other typed structure-preserving map	Spectra and scalar residuals can remain incomplete diagnostics
Homological agreement	Selected homology groups or Betti profile	Isomorphism/equality of retained homological diagnostics	Does not generally establish topological equivalence
Target topological equivalence	Declared topological structure	Homeomorphism, homotopy equivalence, labeled-complex isomorphism, or another specified witness	The target relation must be named explicitly
Resolution-stable comparison	Comparison preserved across declared representation changes	Compatibility of q , diagnostics, and/or target relations	Coarse agreement can conceal fine distinctions

The five-part taxonomy is now complete. Equivalence can be defined within one GR level, on relational structures connecting levels, on historical evolutions of such structures, or on dynamical organizations extracted from families of histories. Section 8 adds a final methodological constraint: equality of an attractor count, susceptibility, spectrum, Betti vector, or other selected invariant should be reported at the diagnostic level until invariance and separation justify a stronger structural conclusion. Distinct realized histories can therefore instantiate equivalent dynamical organization, while identical diagnostic profiles can remain compatible with structural non-equivalence. Section 9 turns from classification to synthesis by examining which cross-taxonomy lifting and refinement implications can be established among the relations constructed in Sections 4–8.

9 Cross-Taxonomy Lifting, Refinement, and Identifiability

Sections 4–8 classify equivalence and equivalence-like relations according to the formal object on which comparison is performed. The present section supplies the synthesis required before those relations are applied to human–AI interchangeability. Its role is to state when a comparison established in one taxonomic location can support a comparison in another, how such support changes when the retained relational content is refined, and which additional interfaces are required before relations on distinct carriers can be ordered by implication. The objective is inferential rather than ontological. A manifestation equivalence, a trajectory relation, and a dynamical diagnostic can each be exact on its own declared carrier while remaining insufficient for a broader relational conclusion. The discussion therefore treats lifting as a typed relation among already defined comparison structures rather than as a general license to move upward through the taxonomy.

The section proceeds through five constructions. Subsection 9.1 generalizes static lifting through

pullback relations and exact-recovery conditions. Subsection 9.2 identifies the additional historical and compositional structure required when the target claim concerns trajectories rather than states. Subsection 9.3 studies stability of an equivalence claim across relational refinement. Subsection 9.4 extends the common-carrier implication discipline of Section 4 to relations originally defined on different carriers. Subsection 9.5 collects recurrent obstruction classes and states a research obligation for later applications. Throughout the section, a successful lift means only that the stated implication has been established under the declared maps, domains, historical scope, and resolution.

9.1 Static Cross-Level Lifting

A static lifting problem begins with a relation on an evidential, representational, or otherwise derived carrier and asks whether that relation is sufficient for a target relation on the underlying comparison domain. The manifestation-to-relational case in Section 6 is one instance. A more general construction is useful because later applications also compare task evaluations, diagnostic profiles, coarse representations, and other derived objects.

Let D be a declared source carrier, let Y be a derived carrier, and let $f : D \rightarrow Y$ be a partial representation or readout map. Let $S \subseteq Y \times Y$ be a relation on the derived carrier. On the domain $D_f := \text{Dom}(f)$, the pullback relation induced by (f, S) is defined in Equation 88.

$$f^*S := \{(x, y) \in D_f^2 \mid (f(x), f(y)) \in S\}. \quad (88)$$

Equation 88 places the derived comparison back on the source carrier. It therefore creates a common domain on which the derived relation can be compared with a target relation $E \subseteq D_f^2$. The construction is representational: it does not identify S with E and it does not require f to be injective or surjective.

Definition 9.1 (Static lifting sufficiency). Let E be a target relation on D_f . The derived comparison (f, S) is *sufficient for lifting to E on D_f* when

$$f^*S \subseteq E. \quad (89)$$

It is *invariant with respect to E* when $E \subseteq f^*S$, and it *exactly recovers E* when $f^*S = E$.

Equation 89 states the positive-lifting condition used by Definition 9.1. The definition separates three claims that are often merged in informal comparison. Invariance means that target-equivalent objects remain related after representation. Sufficiency means that a positive result at the derived level does not join source objects that the target relation separates. Exact recovery requires both directions. The manifestation conditions in Equations 45 and 47 are recovered as the special case in which $f = M_c$ and S is equality on the manifestation carrier.

Proposition 9.2 (Exact recovery by pullback equality). *For a partial map $f : D \rightarrow Y$, a relation S on Y , and a target relation E on D_f , the following are equivalent:*

1. (f, S) is both E -invariant and sufficient for lifting to E ;
2. $f^*S = E$ on D_f .

Proof. The first condition gives the two inclusions $E \subseteq f^*S$ and $f^*S \subseteq E$. Their conjunction is equality. Equality conversely implies both inclusions. $\square$

Proposition 9.2 applies to more than exact equality readouts. A task evaluation may use a declared equivalence relation on task outcomes; a coarse-grained representation may compare equivalence classes in a reduced carrier; and a diagnostic family may compare structural objects through equality of a joint profile. In each case, the inference to a target relation is licensed only after the induced pullback relation has been compared with that target relation.

A lifting claim can also be deliberately one-sided. Suppose the analytical purpose is exclusion rather than positive identification. If $E \subseteq f^*S$ and $(f(x), f(y)) \notin S$, then $(x, y) \notin E$. This is the invariant-based non-equivalence logic used in Proposition 8.7. Conversely, if $f^*S \subseteq E$ and $(f(x), f(y)) \in S$, then the derived agreement is sufficient for E on the declared domain. The two inferential directions therefore require different inclusions.

Remark 9.3 (Lifting and ontological suspension). The source relation E in Definition 9.1 can be relational, historical, dynamical, representational, or conditionally ontological. GR's ontological suspension prevents a formally successful lift to a relational equivalence from being extended further to ultimate ontological equivalence unless a separate conditional ontology and its comparison criterion have been introduced.

9.2 Trajectory-Level Lifting

Static recovery of relational classes does not by itself recover a historical evolution. Section 7 distinguishes endpoint, statewise, order-preserving, transition-preserving, operational, generative, and continuation relations. A trajectory-level lifting claim must therefore specify which of these structures the evidential representation preserves.

Let $\Gamma = (I, \preceq, \{X_i\}, \{T_{ij}\})$ be a source historical evolution record in the sense of Definition 7.1. Let $\hat{\Gamma} = (I^*, \preceq^*, \{Z_{\phi(i)}\}, \{\hat{T}_{ij}\})$ be a declared represented or evidential history, where $\phi : I \rightarrow I^*$ supplies the retained index correspondence. For every retained $i \in \text{Dom}(\phi)$, let the stagewise readout have the type stated in Equation 90.

$$f_i : X_i \rightarrow Z_{\phi(i)}. \quad (90)$$

Equation 90 can represent manifestation, task evaluation, coarse-graining, or another stage-indexed epistemic interface. Even when every f_i exactly recovers the retained state equivalence at its stage, the historical transitions can remain unresolved.

To evaluate transition structure, let $\hat{T}_{ij} : Z_{\phi(i)} \rightarrow Z_{\phi(j)}$ denote the transition represented at the readout level whenever the application supplies such a map. Transition compatibility requires the square in Equation 91 to commute on the common domain.

$$f_j \circ T_{ij} = \hat{T}_{ij} \circ f_i, \quad i \preceq j. \quad (91)$$

Equation 91 states that reading out after a historical transition gives the same represented result as first reading out and then applying the represented transition. The condition is stronger than equality of stagewise readouts because it constrains how the historical maps themselves are represented.

Definition 9.4 (Trajectory-identifying representation). A family of stagewise readouts $\{f_i\}$ is trajectory-identifying relative to a declared target trajectory relation when: (i) the retained index correspondence ϕ preserves the historical order required by that relation; (ii) each f_i is sufficient to recover the retained state equivalence at stage i ; and (iii) the represented transition structure is separating for the retained transition classes and satisfies Equation 91 for the historical maps used by the comparison. If the target relation additionally includes operational, generative, or continuation structure, the corresponding maps must satisfy the analogous separation and compatibility requirements.

Definition 9.4 is deliberately schema-like. It does not impose one mathematical language on all historical models. A category of histories can express the transition requirement through faithfulness on a retained subcategory; a finite state-transition model can use injectivity on transition classes; and a bisimulation-oriented model can state a back-and-forth condition over successor structures. The common requirement is that state identification and transition identification remain separate evidential burdens.

Proposition 9.5 (Stagewise recovery is insufficient for transition recovery). *There exist historical evolution records for which every pair of corresponding stages is exactly equivalent under the retained state relation while the trajectories fail to be transition-preserving equivalent.*

Proof. Let two historical records have the same two retained states u and v at corresponding indices. In the first record, let the retained transition from the first to the second stage be $T_1 : u \mapsto v$. In the second record, let the retained transition be $T_2 : u \mapsto v$ with a different transition label, provenance, or compositional role that the target transition relation distinguishes. The corresponding states are equal, hence exactly equivalent at both stages, while T_1 and T_2 are not equivalent under the retained transition criterion. Therefore statewise recovery does not establish transition-preserving historical equivalence. $\square$

Proposition 9.5 is the historical analogue of componentwise insufficiency in Proposition 6.3. It also explains why an evaluation that correctly identifies every displayed state can remain incomplete for a process whose significance depends on provenance, order, composition, or branching possibilities.

Continuation-family lifting requires an additional step. If the target relation is continuation bisimulation from Definition 7.11, a representation must preserve enough successor structure to verify the required back-and-forth clauses. Equality of one realized path, or even exact recovery of all states along that path, does not establish equivalence of unactualized branches. A continuation-sensitive evaluation therefore requires evidence about the admissible extension structure itself.

9.3 Equivalence Stability under Relational Refinement

A comparison can be valid at one retained relational resolution and fail when the analysis preserves additional distinctions. Section 5 represented refinement by inclusion of equivalence relations on a common carrier. The present subsection uses that structure to define the stability of a particular equivalence claim across a family of retained contents.

Let $(\mathcal{K}, \preceq_{\mathcal{K}})$ be a partially ordered family of relational-content specifications. Write $\kappa_1 \preceq_{\mathcal{K}} \kappa_2$ when κ_2 is at least as fine as κ_1 on the common comparison domain, with the associated equivalence relations satisfying Equation 92.

$$\kappa_1 \preceq_{\mathcal{K}} \kappa_2 \implies \simeq_{\kappa_2} \subseteq \simeq_{\kappa_1} . \quad (92)$$

Equation 92 makes the order semantic: finer retained content can distinguish pairs that a coarser specification treats as equivalent.

For a pair (x, y) in the common domain, define its refinement-survival set in Equation 93.

$$\text{Surv}_{\mathcal{K}}(x, y) := \{ \kappa \in \mathcal{K} \mid x \simeq_{\kappa} y \} . \quad (93)$$

Equation 93 is a derived record of the relational specifications under which the pair remains equivalent. It is not a scalar strength of equivalence.

Proposition 9.6 (Downward closure of refinement survival). *Under Equation 92, $\text{Surv}_{\mathcal{K}}(x, y)$ is downward closed: if $\kappa_2 \in \text{Surv}_{\mathcal{K}}(x, y)$ and $\kappa_1 \preceq_{\mathcal{K}} \kappa_2$, then $\kappa_1 \in \text{Surv}_{\mathcal{K}}(x, y)$.*

Proof. Membership of κ_2 means $(x, y) \in \simeq_{\kappa_2}$. Equation 92 gives $\simeq_{\kappa_2} \subseteq \simeq_{\kappa_1}$, hence $(x, y) \in \simeq_{\kappa_1}$. $\square$

Proposition 9.6 gives a precise form to the idea that an equivalence can survive or break under enrichment. A pair that remains equivalent under a fine specification also remains equivalent under every declared coarser specification below it. The converse can fail: equivalence under task output, present interface behavior, or a coarse relational description can disappear when historical provenance, dependency, responsibility, coupling, or continuation structure is added to κ .

Definition 9.7 (Refinement-Stable Equivalence). A pair (x, y) is refinement-stably equivalent over a declared subset $\mathcal{K}^* \subseteq \mathcal{K}$ when

$$\mathcal{K}^* \subseteq \text{Surv}_{\mathcal{K}}(x, y) . \quad (94)$$

Equation 94 states the family-relative survival condition. The definition remains indexed by the declared refinement family and therefore does not create an absolute equivalence relation containing every possible relational distinction. The choice of $\mathcal{K}^*$ expresses which refinements matter to the inquiry. In a

human–AI task comparison, for example, a task-specific family can be narrow; a claim about institutional substitution can require provenance, responsibility, and coupled-history refinements that are irrelevant to the narrower task.

Refinement stability can also be applied to cross-level recovery. Suppose a context family $\mathcal{C}$ exactly recovers $\simeq_{\kappa_1}$ but the relational content is refined to κ_2 . Exact recovery at κ_1 provides no general guarantee at κ_2 . The context family must be retested for κ_2 -invariance and κ_2 -separation. This observation gives the later GR equivalence investigation a principled reason to alternate contextual enrichment with relational refinement rather than treating one fixed interface as permanently sufficient.

9.4 Cross-Category Implication Structure

Section 4 defined implication between relations by set inclusion on a common carrier. Relations belonging to different taxonomic categories generally begin on different carriers. A cross-category implication therefore requires an explicit comparison interface that transports those relations to a common anchor domain before inclusion is evaluated.

Let D be an anchor carrier and let D_1, D_2 be carriers on which relations $R_1 \subseteq D_1^2$ and $R_2 \subseteq D_2^2$ are defined. Suppose the analysis provides partial maps $f_1 : D \rightarrow D_1$ and $f_2 : D \rightarrow D_2$. On the common domain

$$D_{12} := \text{Dom}(f_1) \cap \text{Dom}(f_2), \quad (95)$$

Equation 95 fixes the common anchor domain. The transported relations are the pullbacks $f_1^* R_1$ and $f_2^* R_2$ restricted to D_{12} .

Definition 9.8 (Anchor-Relative Cross-Category Implication). The comparison represented by (f_1, R_1) implies the comparison represented by (f_2, R_2) relative to anchor D_{12} when

$$f_1^* R_1|_{D_{12}} \subseteq f_2^* R_2|_{D_{12}}. \quad (96)$$

Equation 96 is the defining inclusion for Definition 9.8. The definition extends the implication discipline of Equation 21 without pretending that R_1 and R_2 were originally relations on the same kind of object. The anchor carrier and transport maps are part of the claim. Changing them can change the implication.

Proposition 9.9 (Transitivity under a shared anchor). *Let three comparison structures (f_i, R_i) , $i \in \{1, 2, 3\}$, be defined relative to the same anchor domain D^* after restriction to the common domain of all three maps. If*

$$f_1^* R_1 \subseteq f_2^* R_2 \quad \text{and} \quad f_2^* R_2 \subseteq f_3^* R_3 \quad (97)$$

on D^ , then $f_1^* R_1 \subseteq f_3^* R_3$ on D^* .*

Proof. The premises are the inclusions in Equation 97. The result is transitivity of set inclusion after all three relations have been transported to the same anchor carrier. $\square$

The shared-anchor condition is substantive. If two implication claims use different entity boundaries, historical windows, resolutions, or tracking conventions, Proposition 9.9 cannot be applied until the model supplies a justified common anchor or an additional transport between the anchors. This condition prevents implication chains from being assembled by combining individually valid statements whose domains do not align.

Figure 6 summarizes the architecture. The taxonomy supplies several comparison carriers; explicit representation or tracking maps connect selected carriers to an anchor; implication is evaluated only after the relations have been transported to that common domain.

The architecture in Figure 6 also permits implication to be absent. Two relations can remain incomparable even after transport. This outcome is analytically meaningful: task equivalence and topological equivalence, for example, need not order one another on a given anchor. The paper therefore uses a partial implication structure rather than forcing all taxonomic categories into one scale.

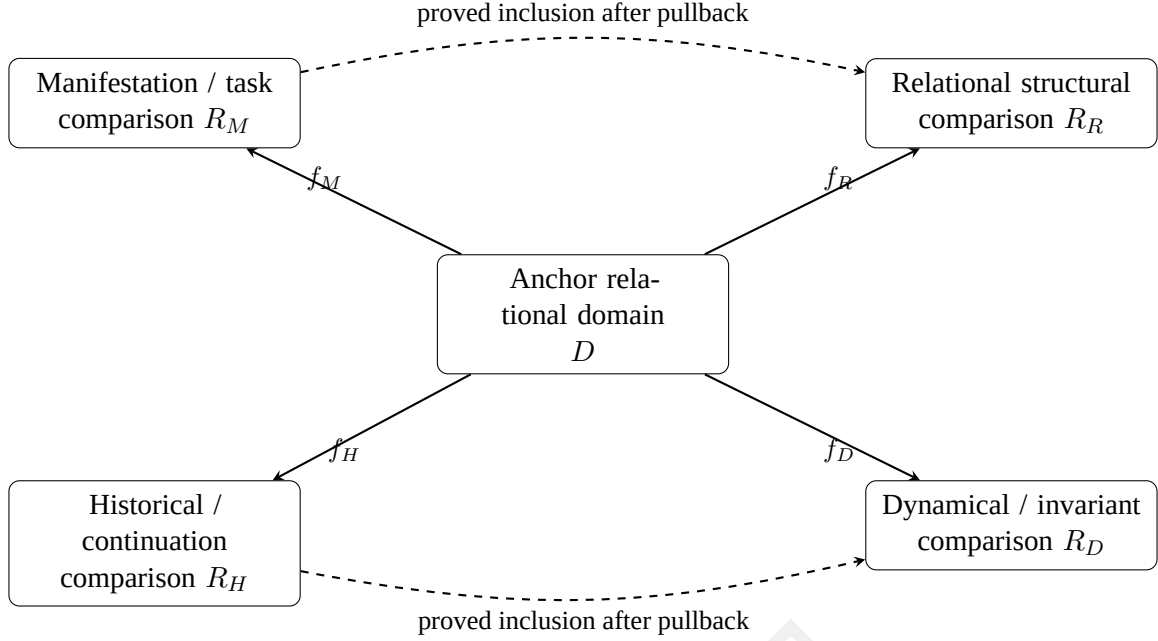


Figure 6: Cross-taxonomy implication through a common anchor. Relations defined on manifestation, relational, historical, or dynamical carriers become comparable only after explicit maps transport them to one declared anchor domain. Dashed arrows represent implication claims that require proof as relation inclusions after pullback; the diagram does not assert those implications universally.

9.5 Implication Boundaries and Counterexample Classes

The preceding constructions identify several recurrent ways in which a proposed lifting or implication can fail. The point of collecting them is methodological. A failed implication can reveal missing evidence, an incomplete representation, a genuine structural distinction, or an ill-typed comparison. These cases should remain distinguishable in later human–AI and Turing-style applications.

Table 7 records the principal obstruction classes already generated by the taxonomy.

Several entries in Table 7 are deliberate possibilities rather than defects. Task quotienting can be appropriate when the declared purpose concerns only task outcomes. Coarse-graining can be analytically useful when discarded distinctions are irrelevant to the local question. Directional simulation can be the correct relation for one-way substitution. The formal problem arises when a relation with limited scope is silently promoted to a stronger category.

Research Obligation 9.10 (Cross-Taxonomy Lifting). A claim that one GR equivalence or equivalence-like relation supports another should state: (i) the source and target carriers; (ii) the relation type on each carrier; (iii) the representation, tracking, quotient, or transport maps connecting them; (iv) the common domain on which the implication is evaluated; (v) the historical scope and representational resolution; (vi) the retained relational content; and (vii) the inclusion, injectivity, faithfulness, bisimulation clause, or other formal condition that licenses the inference. When only one inferential direction has been established, the claim should preserve that direction explicitly.

Obligation 9.10 completes the synthesis layer required for the applications that follow. The taxonomy no longer supplies only a list of possible comparison relations. It also supplies a method for determining whether evidence from one taxonomic location can support another claim and for recording where that implication stops. Section 12 applies this discipline to human–AI interchangeability. The relevant question will therefore be typed before it is answered: a human and an artificial system can be equivalent under a manifestation context, task evaluation, relational position, historical trajectory, continuation relation, or dynamical diagnostic, while the implication among those statements depends on the lifting conditions established in the present section.

Table 7: Recurrent obstruction classes for cross-taxonomy lifting and implication.

Obstruction class	Formal pattern	Consequence for inference
Non-separating readout	$f^*S \not\subseteq E$	Derived agreement joins target-distinct objects; positive lifting fails
Non-invariant readout	$E \not\subseteq f^*S$	Target-equivalent objects can receive distinct derived representations; factorization or preservation fails
Task quotienting	$\sim_c \not\subseteq \sim_{\tau, c}$	Equal task evaluation can be strictly coarser than manifestation equivalence
Statewise historical collapse	Stage equivalence holds while transition equivalence fails	Static or slice-wise evidence does not recover provenance, order, or path structure
Continuation underdetermination	One realized path agrees while successor families differ	Realized-history agreement does not establish simulation or bisimulation
Componentwise coupling loss	Component trajectories agree while coupled trajectories differ	Equivalence of relata does not establish equivalence of the relation or interface
Diagnostic incompleteness	$E \subseteq \sim_{\mathcal{Q}}$ with separation unestablished	Equal invariant profiles remain compatible with target non-equivalence
Coarse-graining collapse	$q(x) = q(y)$ for target-distinct fine structures	Coarse agreement cannot recover fine equivalence without separation
Domain misalignment	Implication premises use incompatible carriers, histories, or resolutions	Relation inclusion is undefined until a common anchor or transport is supplied
Directional simulation	A simulates B without the reverse clause	Substitutability or continuation matching remains asymmetric and should not be labeled equivalence

10 Relational Substitution Architecture

This section develops a general substitution architecture before any comparison is specialized to Human–AI, AI–AI, or Human–Human cases. Its role is to distinguish task-local replacement from relational interchangeability, historical preservation, continuation matching, and preservation of coupled system structure. The objective is to state which objects must be compared when one history-bearing GR entity or position is proposed as a substitute for another. The discussion proceeds from substitution carriers and required operations to directional operational simulation, historical substitution, continuation preservation, coupled replacement, and the scope of a substitution claim.

10.1 Substitution Carriers and Required Structure

Let X and Y denote history-bearing GR entities or positions selected for a substitution problem. The substitution claim is embedded in two comparison configurations C_X and C_Y whose retained environments have been connected by the tracking or transport structure required by the application. The comparison is therefore not specified by the pair (X, Y) alone. It also depends on the role being replaced, the retained environment, the required operation family, the relational content to be preserved, and the historical or continuation scope of the claim.

Let $\mathcal{O}_X^{\text{req}}(C_X)$ denote the declared family of operations required from the reference side. This family is selected from the admissible operations at C_X . Let $\mathcal{O}_Y^{\text{adm}}(C_Y)$ denote the admissible operation family available on the candidate-substitute side. The required family is part of the claim. Enlarging it can invalidate a previously available substitution witness, while narrowing it can make a local replacement claim well formed without altering either entity.

The retained relational content κ specifies which consequences of the required operations count as preservation-relevant. A task-local substitution can use a coarse κ that retains only designated input–output or successor relations. A stronger institutional or historical substitution can retain provenance, responsibility, interface relations, later admissibility, generated obligations, or continuation structure. Section 11 develops several such history-sensitive contents in greater detail.

10.2 Directional Operational Substitution

Substitution is generally directional. A model can ask whether Y can replace X for a selected operation family without requiring X to reproduce every operation available to Y . The minimum witness is therefore simulation-like.

An operational substitution witness is a map of the type stated in Equation 98.

$$\eta^{X \rightsquigarrow Y} : \mathcal{O}_X^{\text{req}}(C_X) \longrightarrow \mathcal{O}_Y^{\text{adm}}(C_Y). \quad (98)$$

The witness in Equation 98 need not be surjective and need not possess an inverse. Let $\mathcal{X}_X$ and $\mathcal{X}_Y$ denote local pre-operation carriers containing C_X and C_Y , and let $\mathcal{X}'_X$ and $\mathcal{X}'_Y$ denote the corresponding retained successor carriers. For a required operation $U : \mathcal{X}_X \rightarrow \mathcal{X}'_X$ and its mapped candidate operation $\eta^{X \rightsquigarrow Y}(U) : \mathcal{X}_Y \rightarrow \mathcal{X}'_Y$, let $\Theta : \mathcal{X}_X \rightarrow \mathcal{X}_Y$ and $\Theta' : \mathcal{X}'_X \rightarrow \mathcal{X}'_Y$ be declared comparison correspondences. Preservation of the selected operational consequence is represented by Equation 99 whenever both compositions are defined.

$$\Theta' \circ U = \eta^{X \rightsquigarrow Y}(U) \circ \Theta. \quad (99)$$

Equation 99 preserves the declared successor effect. It does not establish equality of internal mechanisms, costs, side effects, provenance, or later continuation unless those structures are retained by the comparison.

Definition 10.1 (Directional operational substitutability). The candidate side Y is operationally substitutable for the reference side X relative to (C_X, C_Y) , $\mathcal{O}_X^{\text{req}}$, retained content κ , and the declared comparison correspondences when a witness of the form in Equation 98 is defined for every required operation and preserves the κ -relevant successor structure in the sense of Equation 99.

Definition 10.1 yields a directional relation. A symmetric operational equivalence requires a stronger two-way correspondence and the historical compatibility conditions developed for operational trajectory equivalence in Section 7.

10.3 Historical Substitution

A one-step substitution witness can preserve an immediate consequence while failing to preserve the historical structure in which the operation is embedded. Let Γ_X and Γ_Y be historical evolution records for the reference and candidate comparison configurations. Historical substitution strengthens operational substitution by requiring the chosen stage correspondence, order structure, and transition comparison to satisfy the appropriate conditions from Section 7 over the declared historical window.

The distinction is substantive even when the terminal configurations are equivalent under a retained relation. Endpoint equivalence can coexist with different transition histories, different provenance paths, different institutional authorizations, or different histories of interaction. A substitution claim that treats any of those histories as constitutive of the role must preserve them explicitly rather than infer their preservation from endpoint agreement.

Historical substitution also permits bounded claims. An application can require preservation over a finite or institutionally specified historical window without claiming equivalence over the complete formation history of either entity. The historical scope is therefore a taxonomic coordinate rather than an implicit universal horizon.

10.4 Continuation-Preserving Substitution

Historical agreement concerns realized structure; continuation comparison concerns admissible becoming from the retained realized history. Let K_X^- and K_Y^- be the realized-history anchors at the substitution stage. Their continuation families are stated in Equation 100.

$$\mathcal{E}_X := \text{Ext}(K_X^-), \quad \mathcal{E}_Y := \text{Ext}(K_Y^-). \quad (100)$$

A directional replacement claim can require continuation simulation when every retained successor on the reference side can be matched by a successor on the candidate side. Mutual continuation interchangeability requires the stronger back-and-forth condition of Definition 7.11. Exact equivalence of the selected continuation orders requires a still stronger order-isomorphism or another explicitly declared structural equivalence.

This hierarchy prevents realized-history success from standing in for future-branch equivalence. A candidate substitute can reproduce every continuation retained as relevant to a narrow role while possessing additional branches, different update possibilities, or different environment-dependent continuations. Such a model can support directional substitutability while failing bisimulation or continuation isomorphism.

Caution 10.2 (Observed history and continuation structure). Agreement over a finite realized substitution history establishes only the relation supported by that observed history. Claims about unactualized continuation structure require additional model structure, intervention evidence, or other warranted information about branches that were not realized.

10.5 Coupled Replacement

Replacement within a relational system acts on a coupled configuration. Let $\mathcal{E}$ denote the retained environment, including any persons, institutions, infrastructures, rules, resources, and interfaces included by the model. Let $K_{X\mathcal{E}}$ and $K_{Y\mathcal{E}}$ denote comparison configurations in which the reference and candidate sides occupy the selected role. The relevant historical comparison is then between $\Gamma_{X\mathcal{E}}$ and $\Gamma_{Y\mathcal{E}}$ rather than between Γ_X and Γ_Y alone.

Componentwise equivalence does not entail coupled equivalence. Even when X and Y are equivalent under a selected entity-level relation and the retained environment components are correspondingly equivalent, interface histories, feedback loops, coordination practices, operation domains, and mutual adaptation can differ. This gives the general non-implication stated in Equation 101.

$$(X \simeq_{\kappa_X} Y) \text{ and } (\mathcal{E}_X \simeq_{\kappa_E} \mathcal{E}_Y) \not\Rightarrow \Gamma_{X\mathcal{E}} \simeq_{\kappa_C} \Gamma_{Y\mathcal{E}}. \quad (101)$$

Equation 101 states a logical boundary, not a universal empirical prediction of divergence. A domain-specific application may establish the coupled equivalence after supplying the required interface and historical evidence.

10.6 Scope of a Substitution Claim

The preceding constructions separate several progressively richer objects without imposing one universal ladder of value. Task-local replacement, directional operational simulation, historical substitution, continuation matching, and coupled replacement preserve different structures. A well-formed substitution claim should therefore state at least the reference and candidate carriers, required operation family, retained relational content, historical scope, continuation scope where relevant, and the environment retained by the comparison.

Definition 10.3 (Typed relational substitution claim). A relational substitution claim from X to Y is well formed when it specifies the substitution carriers and retained environment, identifies the required operation family or other local criterion, declares the retained relational content and historical scope, states the relation type being asserted, and supplies the transport, continuation, or coupled-configuration witnesses required by that relation. The term *substitution* carries no stronger meaning than the declared specification.

Definition 10.3 supplies the general architecture used later for Human–AI comparison. It also applies unchanged to AI–AI, Human–Human, institutional, or other GR substitution problems. The next application sections can therefore ask which parts of this architecture are empirically instantiated without rebuilding the formal meaning of substitution from entity type.

11 History-Sensitive Relational Embedding

This section isolates the history-sensitive relational structures that can become decisive when a substitution or interchangeability claim extends beyond a narrow task. Its role is to prevent provenance, responsibility, shared history, and institutional embedding from appearing as ad hoc exceptions introduced only in Human–AI cases. The objective is to represent these features as ordinary retained relational content whose relevance depends on the declared comparison. The discussion proceeds through provenance, responsibility, shared relational history, institutional embedding, refinement, and the resulting implications for equivalence claims.

11.1 Provenance as Retained Relational Structure

Provenance concerns the history through which a present relational structure was generated, authorized, trained, maintained, recorded, or otherwise connected to earlier events and relations. Within the present framework, provenance is represented only when the selected retained content includes the relevant historical paths, attachments, or tracking relations. It is therefore neither an intrinsic marker attached automatically to one subject type nor a universally required component of equivalence.

Let κ_T denote a retained content sufficient for a narrow task-oriented comparison and let κ_P denote a refinement that additionally retains provenance-relevant structure. When both relations are defined on a common comparison domain and the finer content preserves every distinction retained by the coarser content, the corresponding inclusion is stated in Equation 102.

$$\simeq_{\kappa_P} \subseteq \simeq_{\kappa_T} . \quad (102)$$

Equation 102 permits a pair to remain equivalent under the task-oriented content while becoming non-equivalent after provenance is retained. The result does not establish that provenance should matter to every task. It states only how the comparison changes when the application treats provenance as preservation-relevant.

Provenance can matter for several different reasons. A legal or institutional role can depend on an authorization history; a scientific result can depend on traceable evidence and method; a technical system can depend on training, update, or version history; and a relationship can depend on interactions that generated expectations or commitments. These examples identify possible relational functions of provenance without reducing them to one universal semantics.

11.2 Responsibility as a Relational Assignment Structure

Responsibility is treated here as a relation or family of relations through which consequences are assigned, supervised, contested, transferred, recorded, or remedied. The analysis therefore avoids treating responsibility as an unexplained property that is present merely because a relatum is human or absent merely because a relatum is artificial.

A responsibility-sensitive retained content can include relations of authorization, delegation, oversight, answerability, liability, review, appeal, or remedy according to the domain. Two task-equivalent systems can then occupy different positions in that relational architecture. Conversely, a well-designed institutional arrangement can place heterogeneous subjects inside corresponding responsibility structures for a selected purpose.

Remark 11.1 (Responsibility-sensitive comparison). When responsibility matters to an interchangeability claim, the comparison should represent the relations through which responsibility is constituted in the domain. Entity-type labels alone do not determine the resulting equivalence class.

Remark 11.1 preserves the separation between structural and normative analysis. A responsibility structure can be represented without yet deciding whether it is ethically or legally adequate. Section 16 retains that normative question as an additional layer.

11.3 Shared Relational History

Some relations are partly constituted by a history shared among several participants. Repeated interaction can generate expectations, coordination routines, mutual knowledge, trust, conflict, obligations, or other relational structures that are unavailable from an isolated snapshot. The present paper uses *shared relational history* in this structural sense. It does not infer phenomenal co-experience or consciousness from historical coupling.

Let K_{XY} be a coupled history-bearing configuration containing two selected relata and the cross-relations retained by the model. A shared-history-sensitive comparison preserves the declared part of the cross-subcomplex or other historical carrier that materially contributes to the current relation. Replacing one relatum by another can therefore preserve a current manifestation while altering the historical support of the coupled relation.

This distinction is especially important for long-lived interpersonal, institutional, or collaborative relations. The relevant object of substitution can be the historically constituted coupling rather than an inventory of current capacities. A candidate substitute can reproduce selected present interactions while failing to instantiate the same historical relation. The taxonomy records this result as history-sensitive relational non-equivalence rather than as a claim about ultimate identity.

11.4 Institutional Embedding

Relational positions are frequently mediated by institutions, infrastructures, legal statuses, protocols, credentials, resource channels, and technical interfaces. These structures can condition admissible operations and later continuations even when two subjects display similar local behaviour. Institutional embedding is therefore another possible refinement of the retained relational content.

A role-sensitive comparison should state whether the institution is treated as part of the environment, part of the focal relational structure, or part of the manifestation context. Different choices can yield different equivalence claims. For example, equal task output under one interface can coexist with different authorization domains, access privileges, escalation paths, or obligations. If those structures enter the role being compared, they belong in the retained content rather than in an unformalized prose exception.

Institutional embedding also makes substitution reciprocal at the system level. Introducing a substitute can change the surrounding institution: supervision can be redistributed, documentation can be added, interfaces can be redesigned, and new dependencies can arise. The coupled-substitution architecture of Section 10 is designed to retain these possible changes when the application treats them as relevant.

11.5 Refinement through History-Sensitive Content

The structures above can be incorporated through refinement rather than through a universal checklist. Let κ_0 be a baseline retained content and let $\kappa_1, \kappa_2, \dots$ add selected provenance, responsibility, shared-history, or institutional relations. When the refinement order is justified, the survival construction of Section 9 records the specifications under which a pair remains equivalent.

This approach supports several outcomes. A pair can remain equivalent through all refinements retained by the study; equivalence can fail at a particular refinement frontier; or the comparison can become undefined because the richer model lacks the tracking structure required to compare the two carriers. The third case is analytically distinct from non-equivalence and should remain visible.

A refinement frontier can also be application-specific. Provenance may separate two candidates in a scientific-authorship problem while remaining irrelevant to a narrowly defined arithmetic operation. Responsibility may be central to an institutional delegation problem while absent from a descriptive benchmark. The formalism therefore locates the source of differentiation without converting one retained structure into a universal criterion of sameness.

11.6 Implications for Human–AI Interchangeability

Human–AI comparison provides a prominent application because engineered manifestation compatibility can be substantial while provenance, responsibility, shared history, and institutional embedding remain heterogeneous. The taxonomy does not infer non-interchangeability from that heterogeneity alone. It requires the application to show that the differing structure belongs to the retained content relevant to the claim.

Accordingly, a Human–AI pair can satisfy a task-oriented equivalence while failing a provenance-sensitive or responsibility-sensitive relational equivalence. Another pair can satisfy a responsibility-sensitive institutional equivalence after the relevant supervisory and remedial relations have been explicitly constructed. A long-running Human–AI interaction can also develop shared relational history that was absent at the initial encounter. The relevant status is therefore historically generated and model-relative rather than fixed solely by nominal entity type.

This section restores provenance, responsibility, and shared history to their proper formal location: they are candidate components of relational embedding. Their presence can refine an equivalence claim, their absence can leave a broader claim underdetermined, and their normative importance requires additional argument. The Turing-style evaluation section that follows can then ask which of these structures are visible through a bounded interaction protocol and which require an additional lifting or investigation procedure.

12 Human–AI Interchangeability as a Typed Equivalence Problem

This section applies the general taxonomy, substitution architecture, and history-sensitive embedding framework to Human–AI comparison. Its role is diagnostic rather than foundational: the formal meaning of manifestation equivalence, relational equivalence, operational substitution, historical comparison, and continuation matching has already been established. The objective is to show how those relations combine in a Human–AI application without converting nominal subject type into an equivalence criterion. The discussion proceeds through manifestation and task comparison, relational refinement, typed substitution, coupled historical comparison, and the scope of interchangeability claims.

The symbols H and A denote selected history-bearing GR entities represented at a declared resolution. They do not denote universal types that exhaust the internal diversity of humans or artificial systems. Let K_H and K_A be their retained history-bearing subconfigurations inside an application-defined relational domain. Shared institutions, technical infrastructure, communication interfaces, other participants, and environmental structure can be retained when the comparison requires them. Human–AI interchangeability is therefore treated as a typed relation within a declared model rather than as an unrestricted predicate attached to two ontological kinds.

12.1 Manifestation and Task Comparison

Let c be a declared interaction or observation context in which both H and A lie in the domain of the manifestation map M_c . Contextual manifestation equivalence is stated in Equation 103.

$$H \sim_c A \iff M_c(H) = M_c(A). \quad (103)$$

Equation 103 states only that the selected context does not distinguish the two represented entities at the resolution encoded by M_c . A multi-context family $\mathcal{C}$ yields the corresponding refinement in Equation 104.

$$H \sim_{\mathcal{C}} A \iff M_c(H) = M_c(A) \text{ for every } c \in \mathcal{C}. \quad (104)$$

The relation remains manifestation-relative even when the context family is rich.

Task evaluation can quotient manifestations further. Let $Q_{\tau,c} = E_{\tau,c} \circ M_c$ be the task readout from Section 5. Exact task equivalence is stated in Equation 105.

$$H \sim_{\tau,c} A \iff Q_{\tau,c}(H) = Q_{\tau,c}(A). \quad (105)$$

Equation 105 can hold even when Equation 103 fails because the task evaluator can identify distinct manifestations with one accepted outcome. Approximate score agreement remains a tolerance relation unless the required equivalence axioms have been established.

Caution 12.1 (Interface scope). A Human–AI comparison conducted through a declared interface establishes only the relation induced by that interface and any subsequent evaluation map. Familiarity, fluency, benchmark success, or equality of displayed outputs does not independently establish relational structural equivalence, historical equivalence, continuation bisimulation, or ontological equivalence.

12.2 Relational Refinement

Human–AI comparison becomes relational when the retained content κ includes structures beyond the selected manifestation or task. Section 11 identifies provenance, responsibility, shared relational history, and institutional embedding as possible history-sensitive contents. Their inclusion is application-relative.

Let κ_T denote a task-oriented retained content and let κ_P be a refinement that includes a selected provenance-sensitive structure. When the relations are defined on a common domain and the refinement order is justified, Equation 106 holds.

$$\simeq_{\kappa_P} \subseteq \simeq_{\kappa_T} . \quad (106)$$

A Human–AI pair can consequently satisfy Equation 107 without contradiction.

$$H \simeq_{\kappa_T} A, \quad H \not\simeq_{\kappa_P} A. \quad (107)$$

The result does not establish that the finer content is normatively superior. It states that the equivalence claim changes when additional relational structure is retained.

The same logic applies to symmetry-relative position interchangeability. A human-associated and AI-associated position can share an orbit under a coarse retained structure and occupy distinct orbits after refinement. Conversely, a domain can explicitly construct corresponding institutional positions at a selected resolution. The relevant conclusion is indexed by the retained structure and transformation group rather than by subject type alone.

12.3 Typed Human–AI Substitution

The substitution architecture of Section 10 applies directly with $X = H$ and $Y = A$. A directional operational replacement is therefore represented by a witness $\eta^{H \rightsquigarrow A} : \mathcal{O}_H^{\text{req}}(C_H) \rightarrow \mathcal{O}_A^{\text{adm}}(C_A)$ that preserves the declared successor structure through the comparison correspondences of Equation 99. The relation is simulation-like unless a stronger two-way correspondence is established.

This generalization matters because practical Human–AI replacement claims are often asymmetric. An institution may ask whether an artificial system can realize the operations required by one role without asking whether a human can realize every operation available to the artificial system. The formal claim should therefore retain its directional character rather than being promoted to a symmetric equivalence relation by terminology alone.

History-sensitive substitution adds the requirements of Sections 10 and 11. An immediate successor can be preserved while provenance, responsibility, later admissibility, or generated relations differ. When those structures matter to the role, they belong to the retained content and must be checked as part of the substitution claim.

12.4 Historical, Continuation, and Coupled Comparison

Let Γ_H and Γ_A be historical evolution records for the selected Human- and AI-centred configurations. Endpoint equivalence can coexist with failure of transition-preserving historical equivalence. Likewise, agreement over a realized interaction history can coexist with different continuation structures. The continuation families at the comparison stage are $\text{Ext}(K_H^-)$ and $\text{Ext}(K_A^-)$, and the appropriate claim

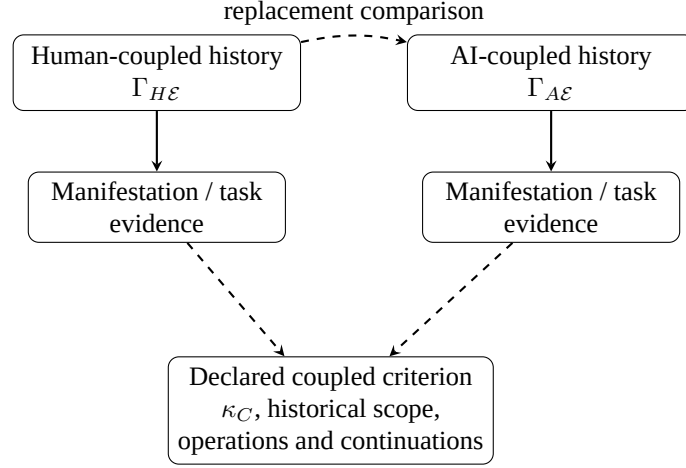


Figure 7: Human–AI substitution as a comparison of coupled historical configurations. Manifestation and task evidence are partial readouts of the coupled histories. A stronger interchangeability claim requires the relational, historical, operational, continuation, and interface structures declared relevant to the target relation.

can range from one-sided simulation to back-and-forth bisimulation or a stronger continuation-structure isomorphism.

Replacement at the system level concerns coupled configurations. Let $\mathcal{E}$ denote the retained environment. The relevant comparison is between $\Gamma_{H\mathcal{E}}$ and $\Gamma_{A\mathcal{E}}$, including interface relations when they belong to the target criterion. The componentwise non-implication of Equation 101 therefore applies directly: entity-level equivalence plus environment-level equivalence does not by itself establish equivalence of the coupled trajectories.

Figure 7 summarizes the Human–AI application of this architecture.

12.5 Scope of Interchangeability Claims

Table 8 summarizes the principal Human–AI claim families after the formal material has been separated into the general taxonomy and substitution architecture.

Definition 12.2 (Typed Human–AI interchangeability claim). A Human–AI interchangeability claim is well formed when it is expressed through a taxonomic specification of Section 4, identifies the Human and AI comparison carriers or positions, states the relation type being asserted, and supplies the cross-taxonomy lifting, substitution, historical, continuation, or coupled-comparison conditions required to connect the evidence to that relation. The word *interchangeable* carries no additional unrestricted meaning beyond that declared specification.

Definition 12.2 prevents local evidence from acquiring broader scope through terminology alone. Task replacement can coexist with relational non-equivalence under a finer retained content; relational interchangeability at one slice can coexist with historical non-equivalence; realized-history agreement can coexist with continuation non-equivalence; and componentwise equivalence can coexist with non-equivalence of the coupled system. None of these results settles ultimate ontology or normative standing.

Table 8: Typed Human–AI comparison claims and the principal evidence they require.

Claim type	Required comparison object	Stronger conclusion not established by the claim alone
Contextual manifestation equivalence	Equality under a declared M_c	Task generality, relational equivalence, history, continuation, ontology
Multi-context manifestation equivalence	Equality under all retained M_c	Relational recovery unless the context family separates the target classes
Exact task equivalence	Equality under $Q_{\tau,c}$	Equality of manifestations or unrestricted functional replacement
Relational structural equivalence	Preservation of declared content κ	Preservation of relational content omitted from κ
Orbit interchangeability	Same orbit under $G_\kappa(C)$	Whole-entity identity, historical equivalence, or interchangeability after refinement
Directional operational substitution	Witness on the required operation family	Reverse substitutability, operation-family completeness, generative equivalence
Historical equivalence	Ordered state and transition correspondence	Bisimilarity of unactualized continuation branches
Continuation simulation / bisimulation	One-sided or back-and-forth successor matching	Literal identity of continuation carriers or unretained mechanisms
Coupled-trajectory equivalence	Preserved role, environment, interface, transitions, and retained coupling	Ontological or normative equivalence

13 Comparative Stress Tests across Relational Subject Types

This section tests whether the equivalence taxonomy depends on a presumed Human–AI divide. Its role is comparative: the same formal distinctions used in Section 12 are applied to AI–AI and Human–Human cases whose nominal subject types are shared within each pair. The objective is to determine whether manifestation equivalence, task equivalence, relational interchangeability, historical equivalence, continuation structure, and coupled evolution remain analytically distinct when type labels no longer supply an obvious source of difference. The discussion proceeds through shared-ancestry AI divergence, task-relative Human–Human equivalence, a type-neutral comparison principle, and the effect of representational granularity on stress-test outcomes.

The stress tests are deliberately formal. They do not claim that all artificial systems, all humans, or all members of either broad class possess one common internal architecture. The labels A_1 , A_2 and H_1 , H_2 identify selected history-bearing GR entities under an application-defined resolution. Each comparison inherits the taxonomic specification $\mathfrak{T}$ from Equation 14; a nominal subject-type label contributes to the comparison only when the retained structure κ , the carrier $\mathfrak{D}$, or another declared component of $\mathfrak{T}$ explicitly includes it.

13.1 AI–AI Divergence under Shared Ancestry

AI–AI comparison supplies a direct stress test for any account that informally attributes relational difference to a Human–AI type contrast. Two artificial systems can begin from the same architecture, model family, checkpoint, protocol, or other declared technical ancestry and subsequently acquire different interaction histories. Shared ancestry can therefore support an initial equivalence claim while leaving later equivalence dependent on what the comparison retains.

Let A_1 and A_2 be two identified artificial-system entities represented at an initial historical stage i_0 . Let κ_0 retain only the inherited structure that the application treats as relevant at that stage. The initial relation in Equation 108 can then be well formed when the declared carrier supports $\simeq_{\kappa_0}$.

$$A_1(i_0) \simeq_{\kappa_0} A_2(i_0). \quad (108)$$

Equation 108 does not state that the two systems are numerically identical or that their future histories

are fixed. It states only that the retained structure κ_0 does not distinguish the represented entities at the selected historical stage.

Suppose the two systems then participate in different pair-relevant or environment-relative histories. Let $\mathbb{H}_{A_1}$ and $\mathbb{H}_{A_2}$ denote the resulting historical evolution records in the sense of Definition 7.1. At a later pair of matched stages i_1 and j_1 , a finer retained structure κ_1 can include interaction history, memory state, institutional attachment, operation availability, learned distinctions, or another feature generated after the initial stage. Historical divergence is represented by the condition in Equation 109.

$$A_1(i_1) \not\sim_{\kappa_1} A_2(j_1). \quad (109)$$

Equations 108 and 109 are compatible because both the historical stage and retained structure have changed. The pair therefore supplies a simple case in which common origin does not determine later relational equivalence.

The same point can be stated without changing κ . Let a history-sensitive retained content κ_H be fixed from the outset and let ϕ provide the declared matching of historical indices. Initial equivalence at i_0 and $\phi(i_0)$ does not imply statewise equivalence at all later matched indices. The failure is already permitted by Section 7: endpoint or stage equivalence supplies no theorem of trajectory preservation.

Manifestation can move in either direction relative to this divergence. Two systems can remain equivalent under a human-facing context c while their retained relational structures diverge, as stated in Equation 110.

$$A_1 \sim_c A_2, \quad A_1 \not\sim_{\kappa_H} A_2. \quad (110)$$

Equation 110 is the same cross-level mismatch analyzed in Section 6; its importance here is that both relata are artificial systems. Manifestation similarity therefore cannot be treated as reliable evidence of relational equivalence merely because the comparison occurs within one nominal subject class.

The converse pattern is also possible. Two systems can occupy equivalent positions relative to a coarse institutional κ while producing different manifestations under a context that exposes model-specific style, interface conventions, or other distinctions excluded from that retained relational content. Within-level and cross-level equivalence remain indexed by their own carriers and need not coincide.

Continuation structure sharpens the stress test. Two systems can share a realized history up to a selected stage while possessing different admissible future transitions because one system has different tools, memory access, update permissions, safety constraints, or environmental interfaces. In the notation of Section 7.6, the realized pair can remain related by a stage-level equivalence while no continuation bisimulation exists between the selected successor structures. Shared technical ancestry therefore establishes neither continuation equivalence nor mutual substitutability without the corresponding continuation witness.

Proposition 13.1 (Shared ancestry does not determine historical equivalence). *Let A_1 and A_2 be history-bearing GR entities whose declared initial structures satisfy Equation 108. The initial equivalence does not, without a historical transport and preservation condition, imply statewise, order-preserving, transition-preserving, or continuation equivalence of $\mathbb{H}_{A_1}$ and $\mathbb{H}_{A_2}$.*

Proof. Initial equivalence constrains only the selected stage and retained structure. Definition 7.1 allows later stage objects and historical transports to differ. The stronger historical relations in Section 7 require additional premises concerning matched stages, order, transport, or successor structure. Those premises do not follow from Equation 108 alone. $\square$

Proposition 13.1 prevents technical genealogy from functioning as a substitute for relational-history analysis. Shared architecture can remain relevant evidence when κ explicitly retains it; the proposition concerns the scope of the inference drawn from that evidence.

13.2 Human–Human Equivalence under Task-Relative Comparison

Human–Human comparison supplies a complementary stress case. Two humans can be equivalent relative to a task readout while remaining relationally differentiated by provenance, institutional role, responsibility, dependency, accumulated expertise, or future operation domains. This case shows that the distinction between task equivalence and relational interchangeability is not specific to artificial systems.

Let H_1 and H_2 be two identified human entities in the common domain of a task readout $Q_{\tau,c} = E_{\tau,c} \circ M_c$. Exact task equivalence is the inherited kernel condition in Equation 111.

$$H_1 \sim_{\tau,c} H_2 \iff Q_{\tau,c}(H_1) = Q_{\tau,c}(H_2). \quad (111)$$

Equation 111 can be satisfied by two people who produce the same accepted answer, complete the same bounded operation, or receive the same task evaluation. The relation states nothing about structures excluded by $Q_{\tau,c}$.

Suppose a retained relational content κ_R includes a role relation that distinguishes the two persons within a wider coupled configuration C . Then the pair can satisfy Equation 112.

$$H_1 \sim_{\tau,c} H_2, \quad H_1 \not\sim_{\kappa_R} H_2. \quad (112)$$

The second relation can arise because κ_R retains an institutional position, a historically generated obligation, a dependency relation, an authorization domain, or another relational feature that the task readout omits. No claim about intrinsic human difference is required.

A symmetry-relative formulation gives the same conclusion at the position level. Let $h_1, h_2 \in P_{\kappa_R}(C)$ be the retained positions occupied by H_1 and H_2 . Task equivalence of the entities does not establish

$$\text{Orb}_{C,\kappa_R}(h_1) = \text{Orb}_{C,\kappa_R}(h_2). \quad (113)$$

Equation 113 requires an articulated symmetry group capable of transporting one retained position to the other. A shared task outcome supplies no such transformation by itself.

The Human–Human case also exposes a second pattern: relational interchangeability can hold while historical identity remains absent. Two persons can occupy interchangeable positions in a deliberately coarse institutional representation even though their personal histories are different. If κ_P is then refined to retain provenance relevant to the inquiry, the orbit or structural equivalence can split. The refinement-survival formalism of Section 9.3 records this possibility without asserting that provenance is always relevant.

Operational substitution can likewise be directional among humans. An identified person H_2 may be able to realize every operation required of H_1 in a bounded role while H_1 cannot realize every operation required of H_2 . The appropriate formal relation is then the directional substitutability construction of Definition 10.1, with the Human/AI labels replaced by the corresponding Human-side carriers. Same-type comparison therefore does not convert a simulation-like relation into an equivalence relation.

Historical and continuation distinctions remain equally available. Two persons can have identical present permissions but different historically accumulated obligations; two can have closely matched realized role histories but different future continuation structures because the surrounding institution grants different successor operations. A task-local equality therefore remains a local conclusion even when both relata are human.

Proposition 13.2 (Task equivalence is type-neutral with respect to omitted structure). *Let x and y be any two GR entities in the domain of the same exact task readout $Q_{\tau,c}$. If $Q_{\tau,c}(x) = Q_{\tau,c}(y)$, then $x \sim_{\tau,c} y$ regardless of the nominal subject types assigned to x and y . Any further conclusion about $\simeq_{\kappa}$, orbit interchangeability, historical equivalence, or continuation structure requires the corresponding additional premises.*

Proof. The first statement follows from Definition 5.6, whose kernel relation is determined by $Q_{\tau,c}$ on its declared domain. The definition contains no subject-type premise unless such a type is encoded in the readout itself. The additional relations have distinct definitions and carriers in Sections 5 and 7, so their conditions must be established separately. $\square$

Proposition 13.2 is modest but important. It prevents the taxonomy from using one inferential standard for Human–AI pairs and another for Human–Human pairs when the formal evidence is of the same type.

13.3 Cross-Case Formal Structure

The Human–AI, AI–AI, and Human–Human cases differ in substantive genealogy, embodiment, institutional context, technical construction, and many other possible features. The taxonomy does not erase those differences. It determines when they enter the formal comparison: a difference contributes to the result when it changes the declared carrier, retained structure, manifestation map, historical transport, operation family, continuation structure, or another component of the taxonomic specification.

Let τ_X denote an optional nominal subject-type label attached to an entity X . Examples can include labels such as human, artificial system, institution, or another application-defined class. A comparison can retain τ_X as part of κ , use it to select admissible domains, or omit it when the research question concerns another structure. The formal principle used in the stress tests is stated in Proposition 13.3.

Proposition 13.3 (Nominal type-label neutrality under fixed taxonomic data). *Fix a taxonomic specification $\mathfrak{T} = (\mathfrak{D}, \ell, \mathfrak{h}, \kappa, \mathfrak{r}, \varrho)$ and a comparison relation generated solely from those declared components. If two applications have isomorphic comparison data for every component used by the construction and differ only in nominal subject-type labels that are excluded from those components, then the formal comparison rule yields corresponding results under the supplied isomorphism.*

Proof. By hypothesis, the comparison relation is constructed only from the declared components of $\mathfrak{T}$, and the two applications provide isomorphic data for each component on which the construction depends. Transporting the comparison along those isomorphisms preserves the inputs used by the rule. A label omitted from the construction supplies no additional formal argument capable of changing the result. $\square$

Proposition 13.3 is a statement about the model, not a claim that nominal types are substantively irrelevant. A biological, technical, legal, or institutional type can matter strongly when the application demonstrates that it changes a retained relation or admissible transformation. The proposition requires that such relevance enter the comparison through an explicit formal channel rather than through an untyped presumption.

Figure 8 summarizes the three cases. The upper branch represents a Human–AI comparison; the middle branch represents AI–AI divergence after an initially retained common structure; the lower branch represents Human–Human task equivalence with possible relational differentiation. All three feed into the same taxonomic specification and can therefore be compared without requiring the same substantive history.

The stress tests also reveal two recurring asymmetries in inference. First, a common nominal type can coexist with deep relational divergence, as in the AI–AI and Human–Human cases. Second, a cross-type pair can satisfy a local equivalence when the retained structure is sufficiently specific. Subject-type sameness is therefore neither a sufficient condition for broad relational equivalence nor a necessary condition for local task or position equivalence.

The same conclusion applies to coupled systems. Two AI systems that are separately equivalent to two comparison targets can participate in different coordination structures; two humans who are task-equivalent can occupy non-equivalent institutional couplings; a Human–AI pair can have a coupling trajectory equivalent to another Human–AI pair even when the individual histories differ outside the retained interface. Proposition 7.15 therefore remains type-neutral as well.

Table 9 records the principal stress cases and the inferential error each one is designed to expose.

Table 9 does not claim that the listed differences are always present or always relevant. Its function is methodological: each difference becomes comparison-relevant only when the application supplies the corresponding formal object and includes it in the retained structure.

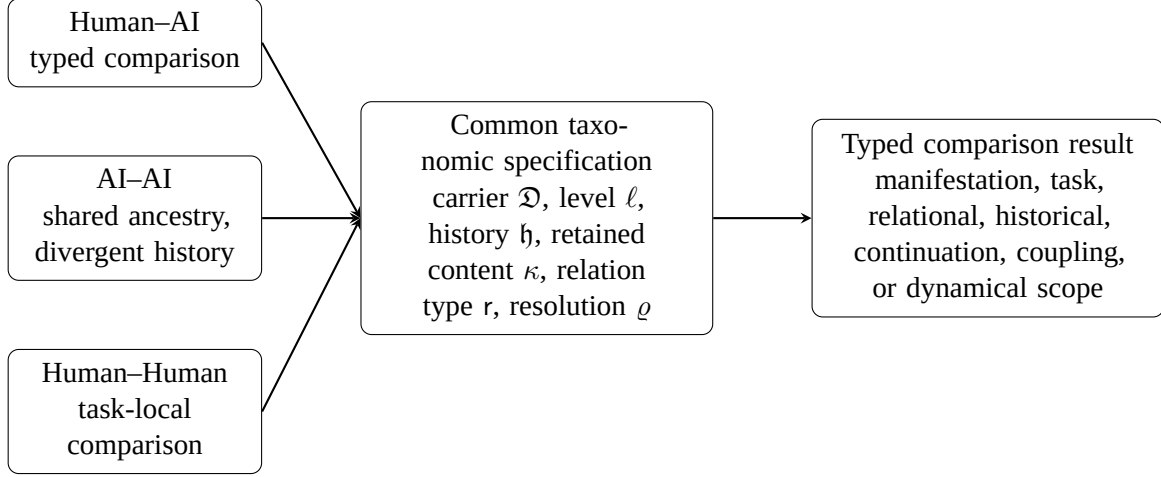


Figure 8: Cross-type stress tests of the GR equivalence taxonomy. Human–AI, AI–AI, and Human–Human cases can carry different substantive histories while entering the same formal comparison architecture. Nominal subject type affects the result only through structures retained by the declared taxonomic specification.

Table 9: Comparative stress tests across relational subject types. Each case isolates a different route by which a coarse equivalence claim can exceed the structure actually established.

Stress case	Structure deliberately held similar	Structure permitted to differ
Human–AI	Task readout, manifestation context, or selected relational position	Provenance, historical embedding, operation domains, continuation, coupling, or other finer retained structure
AI–AI	Technical ancestry or initial retained structure	Interaction history, memory, learned distinctions, tools, continuation structure, later coupling
Human–Human	Task outcome or coarse institutional role	Provenance, responsibility, dependency, authority, historical obligations, future admissible operations
Cross-case formal test	Taxonomic specification and comparison rule	Nominal subject-type label when excluded from the retained formal data

13.4 Representation Granularity in Stress-Test Outcomes

The comparative cases are sensitive to representational granularity. A coarse representation can suppress the histories, interfaces, internal operations, or coupling relations that distinguish two entities. A very fine representation can also retain distinctions irrelevant to the research question and thereby eliminate an equivalence that was meaningful at a coarser functional scale. The stress test therefore requires a declared relation between resolution and retained structure rather than an assumption that finer representation is automatically superior.

Let ϱ_c and ϱ_f denote a coarse and a finer resolution, with a declared coarse-graining map $q_{fc} : \mathcal{D}_f \rightarrow \mathcal{D}_c$ on the relevant comparison carriers. Let E_f be a fine-level equivalence and E_c a coarse-level equivalence. Coarse compatibility requires at least the preservation condition in Equation 114 on the retained domain.

$$(x, y) \in E_f \implies (q_{fc}(x), q_{fc}(y)) \in E_c. \quad (114)$$

Equation 114 states that fine equivalence is not destroyed merely by applying the declared coarse representation. The converse need not hold because coarse-graining can identify structures that remain distinct at the finer level.

Define the coarse-graining collapse set in Equation 115.

$$\Delta_{q,E_f} := \{(x, y) \in \mathfrak{D}_f^2 \mid (x, y) \notin E_f, (q_{fc}(x), q_{fc}(y)) \in E_c\}. \quad (115)$$

Equation 115 identifies pairs whose fine-level distinction disappears at the selected coarse resolution. The construction generalizes the collapse logic used elsewhere in the GR programme while remaining typed by the present target equivalences.

For the AI–AI stress test, Δ_{q,E_f} can contain systems whose distinct interaction histories are suppressed when both are represented only by a common model-family label or a bounded output profile. For the Human–Human stress test, it can contain persons whose different institutional obligations are suppressed by a task-only representation. For the Human–AI case, it can contain cross-type entities that become equivalent under a narrow task readout. The formal pattern is the same across all three cases.

Granularity also interacts with historical scope. A fine snapshot can retain many present distinctions while omitting the historical transitions through which those distinctions formed. Conversely, a coarser state representation paired with a rich historical carrier can preserve trajectory structure that a fine static comparison misses. Resolution ϱ and historical extent $\mathfrak{h}$ therefore remain independent taxonomic coordinates.

Research Obligation 13.4 (Cross-type equivalence stress test). An application using subject-type comparisons should report the nominal type labels of the relata, the extent to which those labels enter the retained formal structure, the comparison resolution, the historical scope, and at least one plausible refinement or alternative representation capable of testing whether the reported equivalence depends on suppressed structure. When a same-type or cross-type conclusion changes under that refinement, the application should report the narrower domain on which the original equivalence remains supported.

Research Obligation 13.4 converts the comparative section into a check on the taxonomy itself. If the formalism treats Human–AI equivalence as categorically different while failing to reproduce analogous distinctions in AI–AI or Human–Human cases, the discrepancy requires an explicit retained structure rather than an appeal to labels alone.

The comparative results support the placement of the next section. Turing-style evaluation is best understood as one particular manifestation-centered testing architecture whose evidential meaning depends on the taxonomic relation it is intended to support. The cross-type stress tests show why indistinguishability evidence should be assessed by its carrier, retained structure, historical extent, and lifting conditions even before the evaluation is specialized to Human–AI imitation.

14 Turing-Style Evaluation within the Equivalence Taxonomy

This section locates Turing-style evaluation within the equivalence taxonomy developed in Sections 4–8. Its role is to identify the formal object directly evaluated by an interaction-centered imitation or indistinguishability procedure, to distinguish the relation induced by an evaluator-visible interaction trace from broader relational and historical equivalence claims, and to state the additional lifting conditions required when a Turing-style result is used as evidence for those broader claims. The discussion proceeds from the evaluation interface to evaluator judgments, then to information excluded by the interface, the taxonomic placement of the resulting evidence, and the inferential scope of a positive or negative evaluation result. The method is reconstructive rather than adjudicative: the section does not attempt to define intelligence or determine whether a particular artificial system is intelligent.

Turing’s 1950 paper introduced the imitation game while reformulating the question of machine intelligence through an operational interaction setting in which an interrogator has access to communicated responses rather than unrestricted access to the participating systems [1]. Later analyses have examined both the interpretation of the test and the inferential scope of a successful performance; Moor provides an early philosophical analysis, while Saygin, Cicekli, and Akman survey the subsequent fifty-year debate and practical development [2, 3]. The present paper uses *Turing-style evaluation* as a broader analytical

label for evaluation architectures that retain this interface-centered feature. The label does not imply that every contemporary benchmark, dialogue evaluation, or indistinguishability protocol reproduces Turing’s historical game. Each application must specify its own participants, interaction protocol, manifestation channels, information restrictions, evaluator rule, and outcome semantics.

14.1 Evaluation Interface and Manifestation Scope

A Turing-style evaluation is represented here as a bounded epistemic interface applied to history-bearing GR configurations. Let $\mathcal{D}_E$ denote a declared carrier of admissible coupled evaluation configurations. An element $X \in \mathcal{D}_E$ can contain a selected relational subject together with the evaluator, interaction environment, technical interface, and other structures retained by the protocol. The carrier is therefore broader than an isolated Human or artificial-system entity when the interaction depends on surrounding relations.

Let Prot_E denote the declared interaction protocol, and let $\mathbf{c}_E$ denote the manifestation context or context family through which the evaluator receives information during the interaction. The protocol can constrain prompts, response channels, interaction length, permitted tools, timing information, external resources, or other application-specific conditions. The context family determines which parts of the underlying relational evolution become evaluator-visible.

The evaluator-visible interaction is represented by the partial trace map in Equation 116.

$$V_E : \mathcal{D}_E \rightharpoonup \text{Tr}_E, \quad (116)$$

where Tr_E is the declared space of evaluator-visible interaction traces. A trace can retain a sequence of messages, responses, timings, interface events, or another protocol-defined manifested history. Equation 116 is derived from the manifestation and historical machinery established earlier in the paper: it records only the interactional material exposed by $\mathbf{c}_E$ under Prot_E . It is not a complete representation of the relational subject or coupled configuration.

The map in Equation 116 induces exact trace equivalence on its domain. The relation is stated in Equation 117.

$$X \sim_E^{\text{tr}} Y \iff V_E(X) = V_E(Y). \quad (117)$$

Equation 117 is a manifestation-trajectory equivalence relative to the declared evaluation architecture. It can be highly informative when the interaction trace is rich, while its scope remains fixed by the visibility and protocol conditions encoded in V_E .

Remark 14.1 (Interactional manifestation rather than isolated output). A Turing-style evaluation can preserve an ordered interaction history rather than a single static output. Its taxonomic location is therefore often richer than one-context snapshot equivalence. The retained object can be a manifestation trajectory whose later responses depend on earlier interaction. This historical structure remains a manifestation-level record unless the additional lifting conditions required for relational or generative trajectory equivalence are established.

Remark 14.1 prevents the taxonomy from reducing an imitation game to one response sample. It also preserves the distinction developed in Section 7 between equivalence of manifested historical records and equivalence of the relational transitions or continuation structures that generated those records.

14.2 Indistinguishability and Task Criteria

Evaluator judgment introduces a second map after the visible trace has been generated. Let $\mathcal{Y}_E$ denote the declared judgment space and let the partial evaluator map be defined as in Equation 118.

$$J_E : \text{Tr}_E \rightharpoonup \mathcal{Y}_E. \quad (118)$$

The judgment space can contain source classifications, identity guesses, task decisions, categorical ratings, or another protocol-defined outcome. The composite evaluation map is defined in Equation 119.

$$\Theta_E := J_E \circ V_E : \mathcal{D}_E \rightarrow \mathcal{Y}_E. \quad (119)$$

The exact judgment-equivalence relation induced by Equation 119 is stated in Equation 120.

$$X \sim_E^J Y \iff \Theta_E(X) = \Theta_E(Y). \quad (120)$$

Both Equations 117 and 120 define equivalence relations on their respective common domains because they are kernel relations of declared maps. The judgment relation is generally coarser than the trace relation.

Proposition 14.2 (Trace equivalence implies judgment equivalence). *On the common domain of V_E and Θ_E ,*

$$\sim_E^{\text{tr}} \subseteq \sim_E^J. \quad (121)$$

Proof. If $V_E(X) = V_E(Y)$, applying the same defined evaluator map J_E to both traces gives $\Theta_E(X) = \Theta_E(Y)$. Hence every trace-equivalent pair is judgment-equivalent. $\square$

Equation 121, established by Proposition 14.2, formalizes a common source of compression. Distinct traces can receive the same evaluator judgment, so a judgment-level result can identify a larger equivalence class than exact trace comparison.

Repeated evaluation requires a separate distinction. Suppose a repeated-trial design assigns each admissible configuration an evaluation profile through a map $Q_E : \mathcal{D}_E \rightarrow \mathcal{Q}_E$, where the profile can contain an empirically estimated judgment distribution, success-frequency vector, confusion matrix, or another declared statistical record. Exact equality of profiles induces the kernel equivalence in Equation 122.

$$X \sim_E^Q Y \iff Q_E(X) = Q_E(Y). \quad (122)$$

A tolerance statement such as $d_Q(Q_E(X), Q_E(Y)) \leq \varepsilon$ should instead be reported as approximate evaluation similarity unless the chosen tolerance construction has independently been shown to satisfy the axioms required for an equivalence relation. The distinction follows the relation-type discipline of Section 4.

The term *indistinguishability* therefore remains incomplete until its comparison rule is stated. It may refer to identical evaluator-visible traces, identical evaluator judgments, equal repeated-evaluation profiles, or approximate similarity under a declared statistic. These objects occupy different positions in the taxonomy and support different inferential scopes.

14.3 Historical and Relational Information outside the Evaluation Interface

The visible-trace map can suppress relational distinctions that remain active outside the evaluation interface. Let $\simeq_\kappa$ denote a target relational equivalence relation on a common anchor domain $D \subseteq \mathcal{D}_E$, induced by retained relational content κ . Depending on the application, κ can retain provenance, responsibility structure, institutional embedding, dependency, admissible transformations, coupling relations, historical transitions, continuation structure, or other relational content established by the model.

The target relation and trace equivalence are directly comparable only after both have been placed on the same carrier. On D , target invariance under the evaluation interface is the inclusion in Equation 123.

$$\simeq_\kappa \subseteq \sim_E^{\text{tr}}. \quad (123)$$

Equation 123 means that the evaluation interface does not distinguish pairs that the target relational criterion treats as equivalent. Positive lifting from trace equivalence to the target relation requires the reverse inclusion in Equation 124.

$$\sim_E^{\text{tr}} \subseteq \simeq_\kappa. \quad (124)$$

Equation 124 is a separation condition: equality of evaluator-visible traces is sufficient for the declared relational-equivalence conclusion on the retained domain. Exact recovery occurs when both inclusions hold.

The compatible-fiber construction from Section 9 provides an equivalent way to state the underdetermination. For a visible trace $v \in \text{Tr}_E$, define the trace fiber in Equation 125.

$$\text{Fib}_{V_E}(v) := \{X \in D \mid V_E(X) = v\}. \quad (125)$$

If the fiber in Equation 125 intersects more than one $\simeq_\kappa$ -equivalence class, then the visible trace does not uniquely identify the retained relational class. The same issue can become stronger after evaluator compression because Proposition 14.2 permits several distinct trace fibers to enter one judgment class.

Proposition 14.3 (Evaluation underdetermination relative to a target relation). *Let $\simeq_\kappa$ and $\sim_E^J$ be relations on a common domain D . If*

$$\sim_E^J \not\subseteq \simeq_\kappa, \quad (126)$$

then there exist $X, Y \in D$ that are judgment-equivalent under the evaluation while remaining non-equivalent under the target relational criterion. A judgment-equivalence result therefore does not, by itself, establish $\simeq_\kappa$ -equivalence on D .

Proof. The failure of the inclusion in Equation 126 means that some pair (X, Y) belongs to $\sim_E^J$ while not belonging to $\simeq_\kappa$. The stated conclusion follows directly. $\square$

Proposition 14.3 identifies an inferential boundary rather than a defect intrinsic to Turing-style evaluation. An evaluation can be highly discriminating relative to its intended manifestation or task criterion while remaining insufficient for another target relation that retains information excluded by the interface.

Historical scope introduces an additional burden. A rich dialogue trace can preserve the order of visible exchanges while omitting the earlier relational history through which the participants, interface, evaluator expectations, or admissible operations were formed. Likewise, a realized evaluation trajectory need not determine the branching continuation structure that would be available under alternative later interactions. The stagewise and trajectory-level lifting results of Section 9 therefore apply without modification: evidence sufficient for manifested interaction equivalence need not identify relational transition structure, generative trajectory equivalence, continuation isomorphism, or continuation bisimulation.

14.4 Taxonomic Placement of Turing-Style Evidence

The preceding formalism places Turing-style evidence inside the broader taxonomy without reducing the evaluation to a static output test. Figure 9 displays the basic route. A coupled history-bearing configuration generates an evaluator-visible manifested interaction trace under the declared protocol and context family. An evaluator or statistical evaluation rule then maps that trace to a judgment or profile. Stronger relational, historical, continuation, coupled, or dynamical conclusions require separately justified lifting paths.

Table 10 records the principal evidential objects and their direct scope.

Table 10 places the strongest direct content of a conventional interface-centered evaluation primarily in manifestation and task-relative categories. This placement does not make the evidence trivial. A sufficiently rich interaction can test adaptive manifestation over an extended history, and repeated evaluation can support well-specified empirical comparisons. The table constrains only the further inferential step from that evidence to a different taxonomic relation.

The placement also clarifies the role of evaluator concealment or restricted access. Information is excluded from the evaluator because the protocol deliberately defines a bounded manifestation interface. That restriction can be methodologically useful when the research object is behavior under the interface. It becomes a source of underdetermination only when the resulting evidence is used to identify structures that the interface has not been shown to separate.

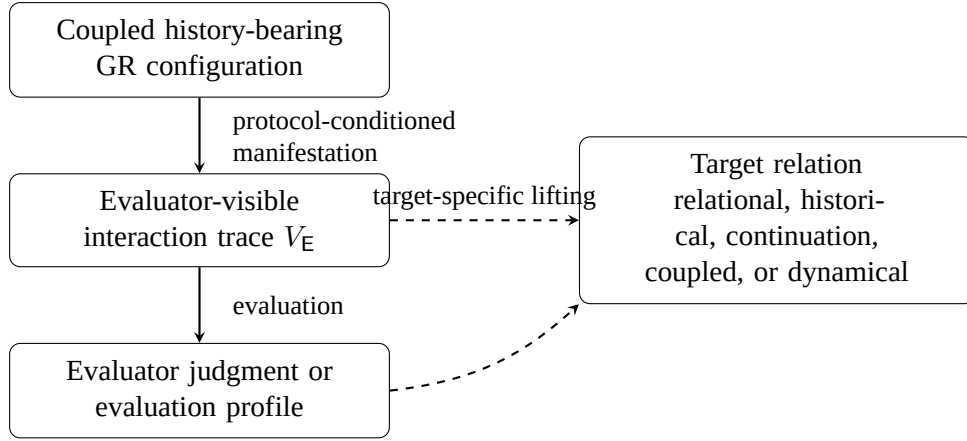


Figure 9: Taxonomic placement of Turing-style evidence. The vertical solid path is the direct evaluation architecture: a coupled relational configuration is manifested through a declared interaction protocol and is then mapped to an evaluator judgment or profile. The target relation is a separate comparison object. Dashed arrows mark inferences from visible evidence to that target; each such inference requires the lifting, separation, historical, continuation, coupling, or diagnostic-sufficiency conditions appropriate to the declared relation.

Table 10: Taxonomic placement and inferential scope of Turing-style evidential objects. Stronger conclusions become available only after the additional conditions indicated in the final column have been established.

Evaluation object	Direct taxonomic location	Additional requirement for a stronger claim
Evaluator-visible interaction trace	Manifestation trajectory under a declared protocol and context family	Separation/lifting conditions for a target relational or historical relation
Evaluator judgment	Task- or judgment-induced equivalence on visible traces	Evidence that the judgment kernel separates the target equivalence classes
Repeated-evaluation profile	Exact diagnostic/profile equivalence when profiles are equal	Sufficiency of the profile family for the target relation; tolerance alone supports approximate similarity
Relational equivalence claim	Relational structure retained by κ	Common carrier plus trace/judgment separation relative to $\simeq_\kappa$
Historical equivalence claim	Ordered state and transition structure	Historical tracking, order preservation, and transition-sensitive evidence
Continuation claim	Branching admissible continuation structure	Continuation isomorphism, simulation, or bisimulation witness according to the claim type
Coupled replacement claim	Coupled subject–environment trajectory	Comparison of the coupled configuration and interface, not only the individual subject
Dynamical-regime claim	Attractor, response, algebraic, topological, or other dynamical structure	Structural correspondence or a diagnostic family proven sufficient for the target equivalence

14.5 Inferential Scope of Turing-Style Results

A Turing-style result is therefore best reported together with the target relation it is intended to support. Let E^* be any declared target equivalence relation on a common evaluation anchor domain D , and let R_E denote the exact relation induced by the chosen evaluation result, such as $\sim_E^{\text{tr}}$, $\sim_E^J$, or $\sim_E^Q$. The relation

R_E is sufficient for positive inference to E^* precisely when the inclusion in Equation 127 holds.

$$R_E \subseteq E^*. \quad (127)$$

If the target relation should also guarantee the same evaluation result, the complementary invariance condition is $E^* \subseteq R_E$. Equality gives exact recovery on D .

Definition 14.4 (Turing-style adequacy relative to a target relation). A Turing-style evaluation relation R_E is positively adequate for a target relation E^* on a declared domain D when Equation 127 holds on D . It is exactly adequate when $R_E = E^*$ on D .

Definition 14.4 makes adequacy target-relative. The same evaluation can be exactly adequate for a narrow task-equivalence relation, positively adequate but non-exact for another retained relation, and inadequate for a richer historical or continuation relation whose classes are not separated by the interface.

The definition also avoids a categorical claim that Turing-style evaluation is either sufficient or insufficient in the abstract. Sufficiency depends on the relation being inferred. For example, an evaluation architecture can be designed so that its judgment classes coincide with a narrowly specified task relation. A broader claim concerning provenance-sensitive relational interchangeability, transition-preserving history, continuation bisimulation, or coupled replacement requires the additional structure associated with that target.

Caution 14.5 (Evaluation success and ontological equivalence). A positive Turing-style result does not remove the ontological suspension maintained by the present GR framework. Even exact recovery of a declared relational or historical equivalence establishes equivalence only relative to that modeled structure. It does not establish ultimate ontological identity, consciousness, sentience, personhood, or normative status.

Research Obligation 14.6 (Reporting Turing-style equivalence evidence). An application using Turing-style evidence should state the evaluation carrier, interaction protocol, manifestation context or context family, visible-trace representation, evaluator or statistical decision rule, relation type induced by the reported result, representational resolution, historical scope, and the target equivalence relation for any stronger inference. When the target lies outside the direct evaluation relation, the application should also state the lifting, separation, transport, continuation, coupling, or diagnostic-sufficiency evidence that licenses the inference.

Research Obligation 14.6 prepares the constructive step taken in the next section. The limitation identified here is architectural rather than specific to one historical test: an evaluation centered on a bounded manifestation interface interrogates only the structures made available through that interface and the judgments derived from them. Section 15 develops a Generative Relational Equivalence Investigation that deliberately varies contexts, retained relational content, historical scope, continuation probes, coupling structures, and dynamical diagnostics so that the scope of an equivalence claim can be investigated rather than left implicit.

15 Generative Relational Equivalence Investigation

Section 14 located Turing-style evaluation within the manifestation and task branches of the equivalence taxonomy and showed that stronger conclusions require target-relative lifting conditions. The present section develops the constructive counterpart. Its role is to specify a Generative Relational Equivalence Investigation (GREI): a structured research procedure for examining whether a comparison survives changes in manifestation context, retained relational content, historical scope, continuation structure, coupling, generative organization, and dynamical diagnostics. The objective is methodological and formal. GREI does not introduce one universal test of whether two entities are “the same.” It organizes a family of already typed comparisons and records where equivalence, non-equivalence, simulation, diagnostic agreement, or unresolved comparison occurs.

The construction proceeds in seven stages. Subsection 15.1 specifies the anchor carrier, target relation, and family of taxonomic comparisons. Subsection 15.2 designs manifestation-context families and records distinctions exposed by contextual enrichment. Subsection 15.3 investigates survival of equivalence under refinement of retained relational content. Subsection 15.4 adds realized-history, transition, and continuation probes. Subsection 15.5 treats coupled replacement and generative probes. Subsection 15.6 derives a typed equivalence profile without aggregating heterogeneous relations into one scalar verdict. Subsection 15.7 separates formal verification, finite computational checking, and empirical validation. Throughout, a positive result is indexed by the structure actually investigated; ontological equivalence and normative status remain outside the procedure unless separately supplied.

15.1 Investigation Object and Declared Scope

A GR equivalence investigation begins from a common anchor rather than from an unrestricted pair of names. Let D_G be the declared anchor carrier and let $x, y \in D_G$ be the relata retained for comparison. The anchor can contain individual history-bearing entities, coupled subject–environment configurations, historical evolution records, or another carrier already typed by the taxonomy. Its resolution, entity boundaries, and historical scope are therefore part of the investigation rather than background assumptions.

Let $\mathfrak{T}^*$ denote the target taxonomic specification in the sense of Section 4, and let $E^* \subseteq D_G^2$ denote the corresponding target relation after any required transport to the anchor. GREI need not begin with a claim that E^* is identifiable from currently available evidence. The target records what the investigation is intended to assess, while the subsequent probe family records which comparisons can actually be constructed.

Instead of introducing one additional universal equivalence relation, GREI uses a declared family of typed comparison structures. Let A be an index set and let

$$\mathbb{I} := \{(\mathfrak{T}_a, f_a, R_a)\}_{a \in A}, \quad (128)$$

where $\mathfrak{T}_a$ is a taxonomic specification, $f_a : D_G \rightarrow D_a$ is the transport, readout, restriction, or probe map required to reach the carrier on which the comparison is defined, and $R_a \subseteq D_a^2$ is the declared relation type at that location. Equation 128 is a bookkeeping family. Its members can include equivalence relations, directional simulations, isomorphism relations, approximate similarities, or diagnostic-agreement relations, provided that the relation type is reported explicitly.

Each member of $\mathbb{I}$ induces a relation on the anchor by pullback. The anchor-level relation associated with probe a is stated in Equation 129.

$$\widehat{R}_a := f_a^* R_a = \{(u, v) \in \text{Dom}(f_a)^2 \mid (f_a(u), f_a(v)) \in R_a\}. \quad (129)$$

Equation 129 reuses the pullback discipline of Section 9. It makes heterogeneous comparison structures jointly reportable without identifying them. Two probe relations can disagree because they retain different content, operate at different historical extents, or use different relation types. Such disagreement is part of the result.

Definition 15.1 (Generative Relational Equivalence Investigation). A Generative Relational Equivalence Investigation for a pair (x, y) consists of a declared anchor carrier D_G , a target taxonomic specification $\mathfrak{T}^*$ with target relation E^* when one is retained, and a family $\mathbb{I}$ of typed comparison structures of the form in Equation 128. The investigation reports the induced anchor relations, their domains, relation types, historical scopes, resolutions, and any established implication or lifting relations among them. It does not infer an unrestricted equivalence beyond those declared structures.

Definition 15.1 makes the procedure extensible. An application can begin with a small family of manifestation and task comparisons and add richer probes when the research question requires them. The definition also permits the investigation to terminate with an unresolved target relation when the available probes do not separate the relevant classes.

15.2 Context-Family Design

The first enrichment dimension concerns manifestation. Section 5 showed that enlarging a context family refines exact multi-context manifestation equivalence on a common domain. GREI uses this result as an investigative device while retaining the target-relative caution of Section 6: additional contexts can reveal relational distinctions, but they can also introduce manifestation distinctions that the target relational criterion intentionally ignores.

Let $\mathcal{C}_1 \subseteq \mathcal{C}_2$ be two context families with a common comparison domain D_{12} . Their induced manifestation-equivalence relations satisfy

$$\sim_{\mathcal{C}_2} \upharpoonright_{D_{12}} \subseteq \sim_{\mathcal{C}_1} \upharpoonright_{D_{12}}. \quad (130)$$

The pairs separated by the enrichment are collected in the context-distinction gain set defined in Equation 131.

$$\text{Gain}(\mathcal{C}_1 \rightarrow \mathcal{C}_2) := (\sim_{\mathcal{C}_1} \setminus \sim_{\mathcal{C}_2}) \cap D_{12}^2. \quad (131)$$

Equation 131 records pairs that were indistinguishable under the smaller context family and become distinguishable after enrichment. It is a set-valued diagnostic of newly exposed distinction, not a scalar information measure.

Context design should be tied to the target relation. Let E^* be defined on the common anchor domain. A useful context family for positive identification should be examined for target separation, $\sim_{\mathcal{C}} \subseteq E^*$, while a context family intended to preserve target-equivalent pairs should be examined for target invariance, $E^* \subseteq \sim_{\mathcal{C}}$. The exact-recovery case requires both inclusions. The mismatch sets developed in Section 6 therefore remain available as diagnostics of collapsed target distinctions and target-irrelevant manifestation distinctions.

Remark 15.2 (Purpose-sensitive context enrichment). Context enrichment is monotone with respect to the exact manifestation relation under the common-domain assumptions of Equation 130. Its adequacy for a target relation is not automatically monotone. A newly added context can reduce target-relevant collapse while simultaneously distinguishing pairs that the target relation treats as equivalent. GREI therefore records contextual refinement and target adequacy separately.

A context family can also be adaptive. If an initial family fails to separate target classes that matter to the inquiry, later contexts can be selected to interrogate those unresolved distinctions. The procedure remains conditional on the available contexts: failure to find a separating family does not establish that no separating family exists.

15.3 Relational-Content Refinement

Manifestation enrichment changes the epistemic interface. Relational-content refinement changes what the target comparison itself retains. Section 9 introduced a partially ordered family $(\mathcal{K}, \preceq_{\mathcal{K}})$ and the refinement-survival set $\text{Surv}_{\mathcal{K}}(x, y)$. GREI uses that construction to locate the boundary at which a coarse equivalence ceases to survive a declared enrichment of relational content.

For a declared refinement family $\mathcal{K}^* \subseteq \mathcal{K}$, the investigation records the membership pattern

$$\kappa \mapsto \mathbf{1}_{\simeq_{\kappa}}(x, y), \quad \kappa \in \mathcal{K}^*, \quad (132)$$

whenever (x, y) lies in the comparison domain of $\simeq_{\kappa}$. Equation 132 is a family of exact membership statements rather than a numerical equivalence strength.

When maximal elements exist, a useful derived object is the refinement frontier defined in Equation 133.

$$\text{Fr}_{\mathcal{K}}(x, y) := \text{Max}(\text{Surv}_{\mathcal{K}}(x, y) \cap \mathcal{K}^*). \quad (133)$$

The frontier contains the finest retained-content specifications, within the declared investigation family, under which the pair remains equivalent. It can contain several incomparable maximal elements or be empty when the relevant poset has no maximal survivor. Equation 133 therefore does not impose a one-dimensional scale.

Definition 15.3 (Refinement witness). Let $\kappa_1 \preceq_{\mathcal{K}} \kappa_2$ with $x \simeq_{\kappa_1} y$ and $x \not\simeq_{\kappa_2} y$. A refinement witness is an explicitly identified part of the additional retained structure represented by κ_2 whose comparison differentiates x and y under the construction of $\simeq_{\kappa_2}$. The witness is structural evidence for the refinement result; it is not by itself a causal explanation of why the underlying entities differ.

A refinement witness can concern provenance, responsibility, dependency, historical transition structure, operation domains, institutional embedding, coupling, continuation structure, or another declared feature. Definition 15.3 prevents a refined non-equivalence from being reported merely as a change in notation. The additional retained distinction should be identifiable within the model.

Context enrichment and relational refinement can be coordinated. For each retained $\kappa \in \mathcal{K}^*$, the investigation can ask whether the current context family $\mathcal{C}$ is κ -invariant, κ -separating, exactly recovering, or underdetermining. Refining κ therefore creates a new identification problem even when the manifestation interface itself is unchanged.

15.4 Historical and Continuation Probes

Static and cross-level comparisons can leave the historical organization of the relata unresolved. GREI therefore adds historical probes only after a tracking convention and historical carrier have been supplied. A historical probe can enlarge the retained window, expose transition structure, or compare alternative continuation branches. The procedure preserves the distinction among realized history, admissible continuation, and weighted continuation developed in Section 7.

Let $\mathcal{H}^*$ be a declared family of historical scopes equipped with whatever restriction or inclusion relations the model uses to compare them. For each $H \in \mathcal{H}^*$, let E_H denote the target historical relation actually tested at that scope. GREI records

$$H \mapsto \mathbf{1}_{E_H}(x, y), \quad (134)$$

only where both the historical tracking and the relation are defined. Equation 134 does not assume that enlarging historical scope must monotonically refine the relation. Such monotonicity requires a compatible family in which the richer scope preserves all distinctions retained by the narrower scope.

Transition-sensitive investigation requires more than matching stagewise states. When historical records Γ_x and Γ_y are available, GREI tests the order and transition-preservation conditions of Section 7. A statewise equivalence that fails transition commutation is reported as statewise historical agreement with transition divergence, rather than promoted to trajectory equivalence.

Continuation probes move from realized history to branching possibility structure. Let $\mathcal{U}_x$ and $\mathcal{U}_y$ be declared subfamilies of the admissible extension families associated with K_x^- and K_y^- , respectively. A directional continuation probe seeks a simulation witness from $\mathcal{U}_x$ to $\mathcal{U}_y$; a reciprocal probe seeks the back-and-forth witness required by continuation bisimulation. If the investigation retains continuation weights, the weight-preservation condition of Section 7 is tested separately.

Caution 15.4 (Finite continuation coverage). A continuation probe normally examines a declared subfamily of admissible extensions. Successful simulation or bisimulation on that subfamily licenses only the corresponding restricted claim unless the subfamily has independently been shown to be exhaustive for the target continuation relation.

Caution 15.4 is especially important in empirical and computational applications. Finite branch sampling can reveal a counterexample to a proposed equivalence, while absence of a counterexample in the sampled family does not by itself establish global continuation equivalence.

15.5 Coupling and Generative Probes

A replacement claim can fail because the relation between an entity and its environment changes even when selected properties of the entity remain equivalent. GREI therefore permits probes whose carrier is a coupled configuration. Let $C[x; \mathcal{E}]$ and $C[y; \mathcal{E}]$ denote comparable coupled configurations in which

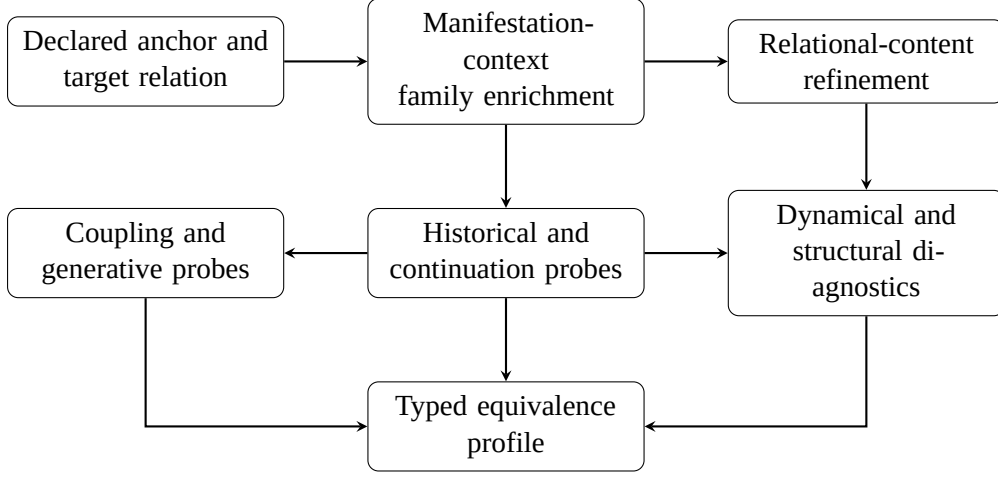


Figure 10: Architecture of a Generative Relational Equivalence Investigation. The procedure begins from a declared anchor and target relation, enriches manifestation contexts and retained relational content, and can add historical, continuation, coupling, generative, and dynamical probes. The resulting object is a typed profile of comparison outcomes; the diagram does not impose a universal ordering among the probe families.

x or y occupies a declared role relative to retained environment $\mathcal{E}$. The notation records a comparison construction; it does not imply that the empirical system can literally replace one entity while holding every other relation fixed.

A coupling probe compares the retained interface and subsequent coupled trajectory. If the target relation on coupled configurations is E^{cpl} , the direct replacement claim is stated in Equation 135.

$$(C[x; \mathcal{E}], C[y; \mathcal{E}]) \in E^{\text{cpl}}. \quad (135)$$

Equation 135 is meaningful only after E^{cpl} has been defined. The relation can retain interface structure, reciprocal operation domains, historical obligations, environmental responses, or continuation organization according to the application.

Generative probing adds a difference-making comparison. Let p be a declared operation type that is defined on both coupled configurations, with state-dependent realizations $F_p(C[x; \mathcal{E}])$ and $F_p(C[y; \mathcal{E}])$. Let G_λ be a declared extraction of later relational conditions relevant to retained generative content λ . The generated-condition comparison is stated in Equation 136.

$$G_\lambda(F_p(C[x; \mathcal{E}])) \quad \text{and} \quad G_\lambda(F_p(C[y; \mathcal{E}])). \quad (136)$$

Equation 136 does not introduce a new equivalence relation by itself. The application must specify the relation used to compare the two extracted generated-condition objects. The probe asks whether corresponding present configurations produce relationally corresponding later conditions under the declared operation.

Dynamical probes can be added at the same stage when the research question concerns attractor–basin organization, susceptibility, algebraic composition, topology, or another structure from Section 8. Their outputs remain diagnostic agreements unless the retained diagnostic family has been shown sufficient for the target dynamical equivalence.

Figure 10 summarizes the investigative architecture. Context and relational-content enrichment interrogate static and cross-level structure; historical and continuation probes interrogate realized and possible becoming; coupling and generative probes test relational embedding and later production; dynamical diagnostics supply further structured comparisons. The arrows represent an investigative sequence rather than an implication hierarchy.

15.6 Typed Equivalence Profiles and Reporting

The result of GREI is a structured profile rather than one unrestricted pass/fail label. Let $\hat{R}_a$ be the anchor-level relation induced by probe a as in Equation 129. For the ordered pair (x, y) , define the three-status comparison indicator in Equation 137.

$$s_a(x, y) := \begin{cases} 1, & (x, y) \in \hat{R}_a, \\ 0, & x, y \in \text{Dom}(f_a) \text{ and } (x, y) \notin \hat{R}_a, \\ \perp, & \text{the comparison is undefined on } (x, y). \end{cases} \quad (137)$$

The value $\perp$ preserves the distinction between undefined comparison and established non-equivalence. For directional relations, $s_a(x, y)$ and $s_a(y, x)$ are recorded separately.

Let $\text{rtype}(R_a)$ denote the declared relation-type label carried by $\mathfrak{T}_a$, such as equivalence relation, isomorphism, directional simulation, bisimulation, approximate similarity, or diagnostic agreement. The typed equivalence profile is defined in Equation 138.

$$\mathcal{P}_{\mathbb{I}}(x, y) := \{(a, \mathfrak{T}_a, \text{rtype}(R_a), s_a(x, y))\}_{a \in A}. \quad (138)$$

Equation 138 preserves the taxonomic location and relation type of every result. It is not assumed to be a vector, metric, ranking, or scalar score. Two investigations can report different profiles because they use different context families, refinements, historical scopes, probes, or resolutions.

Definition 15.5 (Target-resolved GREI result). Let E^* be a declared target relation on the anchor domain. A GREI result is *positively target-resolving* for probe a when $\hat{R}_a \subseteq E^*$ on the relevant common domain and $s_a(x, y) = 1$. It is *negatively target-resolving* when $E^* \subseteq \hat{R}_a$ on that domain and $s_a(x, y) = 0$. It is *exactly target-resolving* when $\hat{R}_a = E^*$. When none of these conditions is established, the probe result remains target-underdetermined even if its own local comparison is exact.

Definition 15.5 reuses the one-sided lifting logic of Sections 9 and 14. It also shows why a profile cannot be combined mechanically. One probe may be sufficient for positive inference, another may support only exclusion, and a third may remain incomparable with the target relation.

Table 11 summarizes the principal probe families and the direct objects they contribute to the profile.

A report should therefore expose the profile rather than summarize it as “equivalent” without qualification. For a Human–AI application, for example, one profile can contain task equivalence, manifestation divergence under a provenance context, simulation-like operational substitutability in one direction, unresolved continuation bisimulation, and non-equivalence under a responsibility-sensitive κ . GREI treats these entries as jointly informative rather than contradictory because they answer different typed questions.

15.7 Formal and Empirical Validation

GREI separates three validation layers. Formal verification concerns whether the investigation is internally well typed and whether the claimed relation-theoretic consequences follow from the stated assumptions. Computational checking concerns finite instances in which relation tables, quotient fibers, historical correspondences, simulation conditions, diagnostic profiles, or refinement frontiers can be calculated. Empirical validation concerns whether the formal carriers, maps, contexts, histories, and probe results adequately represent a substantive case.

Formal verification should include at least domain and codomain checks, relation-type checks, required equivalence axioms, historical transport compatibility, commutative-diagram conditions, relation inclusions used for lifting, and the exact scope of every undefined comparison. A claimed equivalence relation that has not been shown reflexive, symmetric, and transitive on its declared domain should retain a weaker relation label.

Table 11: Principal probe families in a Generative Relational Equivalence Investigation. Each row reports the direct comparison object; stronger conclusions require the target-relative lifting or sufficiency conditions developed earlier in the paper.

Probe family	Direct comparison object	Principal inferential boundary
Context-family probe	Exact manifestation equivalence under one or several declared contexts	Additional contexts refine visible distinctions; relational recovery requires target separation and invariance
Relational-refinement probe	Membership across a declared family of κ -relative relations	Survival is family-relative; refined non-equivalence should identify the retained distinction responsible
Historical probe	Statewise, order-preserving, or transition-preserving historical relation	Longer or richer history requires tracking and does not follow from endpoint agreement
Continuation probe	Continuation isomorphism, simulation, bisimulation, or weighted correspondence on retained branches	Finite branch coverage does not establish an unrestricted continuation relation
Coupling probe	Relation between coupled subject–environment configurations or trajectories	Componentwise equivalence does not determine equivalence of the coupling
Generative probe	Comparison of later relational conditions generated by corresponding operations	Endpoint agreement does not establish equivalence of the generating organization
Dynamical/structural probe	Attractor, response, algebraic, spectral, topological, or related diagnostic/structural comparison	Diagnostic agreement becomes structural equivalence only after sufficiency is established

Finite computational checking can be useful even when the substantive system is not finite. A finite model can enumerate equivalence classes, test whether context enrichment refines a kernel relation, locate refinement frontiers, search for failed simulation clauses, compare continuation trees, or evaluate whether selected diagnostics separate candidate structural classes. Such calculations verify consequences of the chosen representation; they do not establish empirical adequacy by themselves.

Empirical validation adds identification burdens. An application must justify the entity or coupled-system boundary, manifestation contexts, coding of visible traces, retained relational content, historical tracking, intervention or replacement construction, continuation family, and any dynamical diagnostic. Uncertainty can enter each mapping. Competing representations should therefore be compared where plausible, especially when a claimed equivalence disappears or appears under modest changes of granularity.

Research Obligation 15.6 (Minimum report for a GR equivalence investigation). A substantive GREI application should report: (i) the anchor carrier and entity or coupled-system boundaries; (ii) the target taxonomic specification, when a target relation is claimed; (iii) the investigation family $\mathbb{I}$ with the domain, codomain, relation type, resolution, and historical scope of each member; (iv) context and relational-refinement choices; (v) historical, continuation, coupling, generative, or dynamical probes used; (vi) the typed profile in Equation 138; (vii) every relation inclusion, transport, simulation, separation, or diagnostic-sufficiency result used for stronger inference; and (viii) the formal, computational, and empirical limitations that remain unresolved.

Research Obligation 15.6 defines the constructive endpoint of the paper’s evaluation programme. A GR equivalence investigation does not replace specialized behavioral tests, historical research, institutional analysis, dynamical modeling, or empirical experimentation. It supplies a common architecture in which the relation tested by each method, the structures omitted by that method, and the conditions

required to connect one result to another remain explicit. The following section states the epistemic, normative, and empirical boundaries that continue to constrain this architecture.

16 Epistemic, Normative, and Empirical Boundaries

Section 15 assembled the preceding taxonomy into a structured investigation architecture. The present section states the boundaries that continue to constrain that architecture. Its role is to prevent formal equivalence, empirical identification, ontological interpretation, and normative judgment from being merged into one conclusion. The objective is therefore delimitative and constructive: each boundary identifies both a restriction on inference and the additional structure that would be required to proceed further. The discussion addresses representation dependence, empirical identification, ontological suspension, normative evaluation, and open formal problems. These boundaries apply to the taxonomy as a whole and to Human–AI, AI–AI, Human–Human, institutional, and other future applications.

16.1 Model Dependence and Granularity

Every equivalence claim in the present paper is made relative to a declared representation. Entity boundaries, historical extent, manifestation contexts, retained relational content, admissible operations, continuation families, dynamical diagnostics, and resolution all influence which distinctions are available to the comparison. Sections 5–8 therefore treat representation as part of the taxonomic specification rather than as a transparent window onto an independently fixed object.

Model dependence has two analytically different forms. The first concerns *suppressed structure*. A coarse representation can identify configurations that a finer representation distinguishes, as formalized by the coarse-graining collapse constructions in Sections 8.7 and 13.4. The second concerns *introduced structure*. A highly articulated representation can retain distinctions that are irrelevant to the local research question and thereby separate relata whose equivalence was meaningful relative to a coarser retained content. Finer resolution therefore supplies additional distinctions without automatically supplying a superior answer to every equivalence question.

The appropriate robustness question is consequently family-relative. Let $\mathcal{R}$ be a declared family of admissible representation specifications for the same substantive inquiry. For each $r \in \mathcal{R}$, let $\mathcal{P}_{\mathbb{I}}^{(r)}(x, y)$ denote the typed equivalence profile obtained from the corresponding instantiated investigation family. A profile component indexed by a is representation-stable over $\mathcal{R}$ when its relation type remains well typed and its status is constant across the retained family, as stated in Definition 16.1.

Definition 16.1 (Representation-stable profile component). Fix an investigation index a and a family $\mathcal{R}$ of admissible representation specifications for which the corresponding profile component is defined. The component is *representation-stable over $\mathcal{R}$* for (x, y) when the instantiated relation type is preserved under the declared representation correspondences and there exists $s \in \{1, 0, \perp\}$ such that

$$s_a^{(r)}(x, y) = s \quad \text{for every } r \in \mathcal{R}. \quad (139)$$

Equation 139 is intentionally exact. Applications that require approximate robustness should supply a metric or discrepancy on the relevant output space rather than reinterpret unstable categorical statuses as numerical noise. Stability also remains relative to the declared family $\mathcal{R}$. A result can be stable under plausible changes of manifestation context while remaining sensitive to entity-boundary refinement, or stable under historical-window changes while depending strongly on a selected task criterion.

Remark 16.2 (Representation stability and truth). Representation stability supports robustness of a formal conclusion across a declared family of models. It does not by itself establish that the retained representation is empirically adequate, complete, or ontologically privileged. A systematically misspecified family can yield a stable result.

Table 12: Empirical identification burdens across principal GR equivalence objects.

Formal object	Identification burden	Typical unresolved issue
Manifestation relation	Specify the context, readout, coding, and comparison domain that generate the manifested distinction	Observed similarity can depend on prompt, interface, evaluator access, or measurement design
Retained relational content κ	Identify the relations and properties whose preservation is substantively relevant	Omitted history, responsibility, dependency, embodiment, or institutional position can alter the relation
Historical correspondence	Establish continuity, transport, or tracking across relational slices or event domains	Repeated labels can conceal discontinuity, merger, branching, or re-identification
Continuation structure	Justify admissible unrealized extensions and the operations generating them	Counterfactual branches can be model dependent and incompletely observed
Coupled configuration	Identify the focal relata together with interfaces, environment, institutions, and shared history	Bilateral evidence can omit structures that generate the apparent relation
Dynamical or invariant diagnostic	Connect observations to the state, perturbation, transition, algebraic, spectral, or topological object being calculated	Diagnostic equality can reflect misspecification or incomplete invariants rather than structural equivalence

Remark 16.2 separates robustness from adequacy. The distinction is especially important for equivalence because coarse models can reproduce the same equivalence class across many nearby parameter or coding choices while omitting a consequential relation that none of those choices represents.

16.2 Empirical Identification Burdens

The formal relations developed in this paper become empirically informative only after substantive evidence has been mapped to their carriers, operations, histories, and comparison structures. This mapping is an identification problem. Formal consistency can show that a relation is an equivalence relation, that one relation refines another, or that a lifting implication follows under stated assumptions. It cannot establish that the selected formal objects correctly identify the entities, histories, contexts, or mechanisms present in an empirical case.

The identification burden differs by taxonomic location. Manifestation equivalence requires evidence that the selected readout and context represent the presentation actually relevant to the inquiry. Relational structural equivalence requires evidence for the relations, dependencies, permissions, responsibilities, capacities, or other retained content encoded by κ . Historical equivalence requires defensible tracking across slices or event domains. Continuation relations require an admissibility model for unrealized branches. Coupled-system equivalence requires evidence about the interface and environment rather than only the two focal relata. Dynamical-regime comparison requires a justified mapping from evidence to the relevant state, perturbation, transition, or invariant structure.

Table 12 summarizes these burdens. The table distinguishes the formal object from the evidential task required before that object can support a substantive application.

Empirical uncertainty should remain visible at the mapping stage. An uncertain entity boundary, ambiguous historical match, partially observed operation domain, or estimated continuation weight can propagate into the reported equivalence profile. The present paper does not supply one statistical framework for these uncertainties. A domain-specific application can use confidence regions, posterior distri-

butions, sensitivity sets, interval-valued diagnostics, qualitative uncertainty classes, or another justified method. The requirement is narrower: uncertainty in the empirical-to-formal mapping should not disappear merely because the resulting formal relation is exact.

A second burden concerns alternative explanations. A detected difference between two continuation structures can arise from the retained relata, from a surrounding institution omitted from one model, from different data availability, or from the comparison rule itself. Conversely, an observed equivalence can arise because the chosen interface suppresses the relation that would differentiate the cases. Empirical applications therefore require process evidence, intervention evidence, comparative reconstruction, or another domain-appropriate basis for attributing the formal distinction to the substantive structure under investigation.

Research Obligation 16.3 (Empirical identification of an equivalence claim). An empirical application should state the evidence used to identify each carrier, relation, context, historical correspondence, operation family, continuation structure, and diagnostic that materially supports the reported equivalence profile. The application should distinguish directly observed material from reconstructed or counterfactual structure, report relevant uncertainty, and identify plausible alternative representations when those alternatives can change the result.

Research Obligation 16.3 complements the formal reporting obligation in Section 15.7. A formally complete GREI report can remain empirically weak when the mapping from evidence to the formal objects is unsupported.

16.3 Ontological Suspension

The taxonomy uses relational, manifestation, historical, dynamical, algebraic, categorical, and topological representations without identifying any of them with ultimate ontology. This constraint applies with particular force to equivalence. A formally established equivalence at the manifestation or relational level does not establish that the relata are ultimately the same kind of being, possess the same consciousness, share one metaphysical identity, or instantiate one exhaustive ontological structure. Likewise, a formally established non-equivalence does not establish an ultimate ontological difference.

Section 5.1 already treated ontological equivalence as suspended unless a conditional ontology is introduced. The present boundary extends that discipline across the later formalism. A successful lift from manifestation equivalence to a target relational equivalence terminates at the target relation actually specified. A successful trajectory bisimulation terminates at the declared continuation structure. A topological equivalence of model carriers terminates at the modeled topology. No additional ontological conclusion follows merely from the mathematical strength of the representation.

Conditional ontological inquiry remains possible. An application can introduce an explicit ontological model $\mathfrak{O}$, specify its admissible relata and equivalence criterion, and then ask whether evidence or lower-level structure identifies classes in that model. Such an inquiry adds assumptions beyond the present GR taxonomy. The resulting conclusion should therefore be reported as conditional on $\mathfrak{O}$ rather than as a consequence of GR's relational or manifestation formalism alone.

Caution 16.4 (Formal completeness and ontological reach). Increasing the completeness of a formal equivalence characterization within one declared model does not, by itself, increase its ontological reach. Completeness relative to a carrier, retained structure, and relation type is different from completeness about ultimate reality.

Caution 16.4 is relevant to both Human–AI and same-type comparisons. The paper can establish that a Human and an AI system occupy the same equivalence class under a task relation, or that two AI systems fail a history-sensitive relational equivalence, without using either result to settle personhood, consciousness, species identity, or another ontological classification.

16.4 Normative Independence of Structural Equivalence

Structural equivalence and normative evaluation answer different questions. The taxonomy asks which distinctions are preserved under a declared relation. Normative analysis asks which distinctions ought to matter, which substitutions are permissible, which responsibilities attach to an action, which burdens are justifiable, or which forms of treatment are legitimate. A structural result can inform such analysis while leaving the normative criterion undetermined.

This separation can be represented without introducing a universal moral metric. Let D_N be a declared normative-decision carrier obtained from the relevant relational and empirical material, and let $N \subseteq D_N \times D_N$ be an application-specified normative relation, such as permissible substitution, equal entitlement, or another explicitly interpreted evaluative relation. An application can alternatively generate judgments through a map into a judgment space, but the formal connection between that judgment structure and a GR equivalence relation must still be supplied separately. The normative carrier can retain rights, duties, consent, harm, responsibility, distributive effects, institutional legitimacy, or other evaluative material justified by the application.

Three inferential separations follow. First, xEy does not by itself imply that replacing x by y is permissible. A task-equivalent substitute can carry different obligations, consent conditions, provenance, or institutional responsibilities. Second, $x \not Ey$ does not imply that unequal moral standing or unequal treatment is justified; structural difference and normative hierarchy are distinct conclusions. Third, equivalence of a procedure's outputs does not establish equivalence of the procedure's legitimacy, accountability, or distribution of burdens.

Proposition 16.5 (Normative conclusions require additional premises). *Let E be any equivalence relation established by the present taxonomy on a carrier D , and let $N \subseteq D_N \times D_N$ be an application-specified normative relation on a declared normative carrier. Unless the application supplies the required domain correspondence and a rule connecting E to N , neither $xEy \Rightarrow N(x, y)$ nor $N(x, y) \Rightarrow xEy$ follows from the definition of E alone.*

Proof. The definition of E specifies preservation of a declared structural content. The definition of N introduces additional evaluative content. Without a rule connecting their domains and predicates, the truth conditions of one relation impose no logical condition on the other. Therefore either implication requires an additional premise. $\square$

Proposition 16.5 blocks two opposite errors. Equivalence does not automatically license substitution, and difference does not automatically license discrimination. The proposition also clarifies the role of responsibility-sensitive or provenance-sensitive κ introduced in Section 12. Retaining such structure can make a relational equivalence more relevant to a normative inquiry, while the normative conclusion still requires a criterion explaining why that retained structure carries evaluative significance.

16.5 Open Formal Problems

The taxonomy is preliminary because several formal interfaces remain incomplete. The open problems concern the generation of equivalence criteria, evolving carrier types, approximate and probabilistic relations, multi-scale consistency, continuation identification, higher-order coupling, and the computational tractability of rich GREI profiles. These problems delimit future formal work without invalidating the conditional results established in the present manuscript.

The first problem concerns the *generation of retained structure*. The present paper usually treats κ , manifestation contexts, task criteria, and diagnostic families as declared objects whose provenance must be stated. GR also permits these structures to evolve historically. A fuller theory should therefore analyze how an equivalence criterion itself forms, acquires persistence and efficacy, becomes relationally real, and later refines, coarsens, or reorganizes. Section 7.9 formalizes the comparison of changing equivalence structures after a tracking relation is supplied; it does not provide a general dynamics of their generation.

The second open problem concerns changes in carrier type. Current trajectory constructions assume enough continuity to compare stages through historical transport, partial maps, or typed correspondences. Some artificial systems can modify their own architecture, while institutional transformation, biological development, and historical reconfiguration can alter the kinds of objects or operations represented by the model. A stronger GR formalism may therefore require typed or indexed categories, fibrational structures, higher categories, or another language capable of representing changes in the comparison carrier without forcing all stages into one fixed ambient type.

The third problem concerns *approximate equivalence*. Many empirical applications require tolerances, noisy measurements, graded similarities, or probabilistic evidence. Section 5.4 deliberately separates exact kernel equivalence from tolerance-based similarity because a threshold relation need not be transitive. Future work can investigate pseudometrics, uniform structures, probabilistic bisimulation, approximate categorical equivalence, or application-specific quotient constructions that recover useful equivalence classes without hiding the assumptions required for transitivity.

The fourth problem concerns *multi-scale compatibility*. A relation can be stable at one resolution and fail after refinement, while another relation may become meaningful only after coarse-graining suppresses irrelevant microscopic variation. The present paper records resolution as a taxonomic coordinate and supplies compatibility conditions. A fuller theory would characterize families of equivalence relations across scales, identify renormalization-like or functorial consistency conditions where appropriate, and distinguish scale-stable structure from scale-specific analytical convenience.

The fifth problem concerns *continuation identification*. Simulation and bisimulation over admissible extensions are formally strong but empirically demanding because many branches are unrealized. Finite models can enumerate continuation structures, whereas open social, institutional, biological, or technical systems can leave the admissible future family incompletely specified. Future work should connect continuation relations to intervention models, causal generative models, planning structures, formal verification, and uncertainty-aware historical reconstruction without treating unobserved branches as directly observed facts.

The sixth problem concerns *higher-order coupling*. Section 7.8 shows that equivalence of individual trajectories does not imply equivalence of their coupling. Systems with many interacting entities can contain higher-order relations that are not recoverable from pairwise projections. A general treatment may therefore require hypergraph, simplicial, 2-complex, categorical, or field-theoretic structures whose equivalence notions preserve higher-order composition and shared relational history.

The seventh problem concerns *computational tractability*. A rich GREI can combine many contexts, refinements, histories, continuation branches, coupled structures, and diagnostic families. Exhaustive comparison can become infeasible even in finite models. Future computational work should investigate abstraction, compositional verification, counterexample-guided refinement, sparse witness search, and adaptive probe selection while preserving the taxonomic distinction between untested structure and established equivalence.

Table 13 summarizes these research interfaces and the formal caution associated with each one.

The open problems in Table 13 also clarify the status of the present contribution. The paper supplies a taxonomy, several exact formal relations, conditional implication results, and a structured investigation architecture. It does not supply a complete ontology of equivalence, a universal empirical measurement protocol, a normative theory of substitution, or a general dynamics of equivalence formation. The final section draws together the claims that remain supported within these boundaries.

Table 13: Open formal problems in the GR equivalence programme.

Research interface	Required extension	Present caution
Generation of equivalence criteria	Dynamics of κ , contexts, comparison rules, and their relationalization	Declared criteria should not be treated as timeless primitives
Changing carrier types	Formalism for histories whose object and operation types change	Fixed-carrier trajectory results apply only where the required transport remains defined
Approximate equivalence	Typed tolerance, metric, probabilistic, or approximate-structural relations	Similarity should not be promoted to equivalence without the needed axioms
Multi-scale consistency	Cross-resolution family of relations and compatibility conditions	Finer resolution is not automatically the uniquely adequate comparison scale
Continuation identification	Intervention- or generative-model interface for unrealized branches	Bisimulation claims exceed evidence from one realized trajectory
Higher-order coupling	Equivalence of multi-entity and higher-order relational structures	Pairwise equivalence can omit system-level relational differences
Computational tractability	Abstraction and adaptive verification methods	Failure to test a branch is different from evidence of equivalence

17 Conclusion

This paper has developed a preliminary Generative Relational taxonomy of equivalence for comparisons whose meaning depends on reality level, retained relational structure, historical extent, representational resolution, and the type of comparison relation being used. Its central objective has been to replace unrestricted statements of the form “ X is equivalent to Y ” with typed claims whose carrier, domain, preservation criterion, historical scope, and inferential limits are explicit. The resulting framework remains provisional. It does not supply one universal equivalence relation, one privileged resolution, or one final ontology of the entities being compared. It provides an architecture for determining what kind of equivalence has been established and which additional structures would be required for a stronger claim.

The first contribution is the organization of equivalence around GR’s three-level architecture while preserving the asymmetry among those levels. Section 5 distinguished manifestation equivalence induced by declared readouts from relational structural equivalence indexed by retained content κ , and from symmetry-relative interchangeability of relational positions. Ontological equivalence remained suspended unless an additional conditional ontology is supplied. This separation permits exact equivalence relations to be constructed where their axioms are earned while keeping similarity, tolerance, substitutability, simulation, bisimulation, isomorphism, and diagnostic agreement in their appropriate mathematical roles. The distinction matters because a relation that is useful for one research purpose can remain too weak, too directional, or too representation-dependent to support another.

The second contribution is the extension from equivalence within one level to equivalence of structures connecting levels. Section 6 treated a manifestation map as part of the comparison object rather than as a transparent channel. Relational and manifestation components can each admit an appropriate equivalence while the coupling between them remains different. Conversely, a coarse manifestation can identify relationally distinct structures. The factorization, invariance, and separation results therefore make cross-level recovery conditional: manifestation equivalence supports a retained relational equivalence only when the declared readout is sufficiently compatible with, and sufficiently discriminating for,

the target relation. Equality of the induced partitions represents a particularly strong case rather than the default interpretation of observational similarity.

The third contribution is the historical extension of the taxonomy. Section 7 distinguished endpoint equivalence from statewise, order-preserving, transition-preserving, operational, generative, continuation, and weighted-continuation relations. A trajectory was represented through ordered historical carriers and explicit transports rather than through an assumed universal clock. This construction permits a pair to agree at a selected endpoint while differing in the histories that produced that endpoint, the transformations available along those histories, or the branches that remain admissible afterward. It also permits equivalence structures themselves to evolve through preservation, refinement, coarsening, or non-nested reorganization after an appropriate historical transport has been supplied. Equivalence can therefore be treated as historically situated structure rather than as a timeless label attached to isolated relata.

The fourth contribution concerns dynamical and structural characteristics. Section 8 separated agreement of selected diagnostics from equivalence of the structures those diagnostics describe. Attractor and basin organization, metastable and near-critical regimes, response and susceptibility structure, algebraic and compositional organization, and topological or homological characteristics provide useful comparison resources only at the level justified by the retained invariants. A changed genuine invariant can establish a corresponding structural difference when that invariant is preserved by the target relation. Agreement of a finite diagnostic family supports the stronger conclusion only when the family has independently been shown to separate the target classes. This one-sided discipline prevents calculable summaries from acquiring an unearned status as complete descriptions of dynamical organization.

The fifth contribution is the synthesis of these categories through explicit lifting, refinement, and identifiability conditions. Section 9 required relations defined on different carriers to be transported to a common anchor before implication is assessed. Relation inclusion then supplies a precise language for one-sided evidential claims: one inclusion can express preservation or invariance, the opposite inclusion can express positive identification, and equality can express exact recovery on the declared domain. The refinement-survival construction similarly permits a pair to remain equivalent under some retained contents and become differentiated under finer ones without assigning the pair a universal scalar degree of equivalence. These results turn cross-taxonomy movement into a stated formal problem rather than an implicit change of vocabulary.

The Human–AI analysis in Section 12 illustrates the practical consequence of this architecture. Manifestation similarity, task equivalence, relational structural equivalence, position interchangeability, operational substitution, historical equivalence, continuation simulation, and coupled-trajectory equivalence are different claims. Provenance, responsibility, institutional attachment, embodiment, memory, technical architecture, and other distinctions enter only when a declared comparison retains them. The comparative stress tests in Section 13 show that the same discipline applies to AI–AI and Human–Human cases. Shared nominal type does not guarantee equivalence across richer relational or historical structures, while cross-type relata can satisfy a local equivalence when the retained structure warrants that result. The formalism therefore locates substantive differences in the structures actually represented rather than in an unrestricted type label.

Section 14 then located Turing-style evaluation inside this taxonomy. An evaluator-visible interaction protocol can establish exact or approximate relations over manifestations, traces, judgments, or repeated profiles. Such results can be fully adequate for a target relation defined at that level. A broader relational, historical, continuation, coupled, or dynamical conclusion requires the additional lifting conditions appropriate to that target. The limitation is consequently structural rather than dismissive: an evaluation architecture can be rigorous and informative while remaining insufficient for a claim whose target retains information that the evaluation interface does not separate.

The constructive response developed in Section 15 is the Generative Relational Equivalence Investigation. GREI organizes a family of typed probes over a declared anchor carrier and reports their results as a structured equivalence profile. Context enrichment, relational-content refinement, historical and continuation probes, coupled replacement analysis, generative comparison, and dynamical diagnostics

can therefore coexist without being aggregated into one universal score. A pair may be equivalent under one task criterion, differentiated under a provenance-sensitive relational criterion, simulation-related in one operational direction, unresolved under a continuation probe, and diagnostically similar in a selected dynamical regime. Preserving these heterogeneous outcomes is a feature of the method because the disagreement can identify which structure produces the change in classification.

The boundary conditions in Section 16 remain part of the contribution. Formal equivalence does not by itself establish empirical adequacy, ontological identity, permissible replacement, moral standing, legal status, or justice. Empirical applications require defensible mappings from observations to the formal carriers and relations. Representation-stability must be assessed over declared families of plausible models rather than inferred from one convenient encoding. Normative conclusions require normative premises in addition to structural results. These separations preserve the possibility that equivalence analysis can inform consequential decisions without silently making those decisions on the user's or institution's behalf.

Several research directions follow directly from the present construction. A fuller GR theory of equivalence would need stronger accounts of how equivalence criteria themselves are generated, how comparison proceeds when carrier types change through history, how approximate relations can be organized without losing their mathematical type, how multi-scale equivalence behaves under refinement and coarse-graining, and how continuation or higher-order coupled structures can be identified empirically. Computational work can examine finite instances of the taxonomy, while empirical work can test whether declared contexts, histories, and continuation structures actually distinguish the relational differences attributed to them. These developments can extend the present architecture without requiring a single final notion of sameness.

The main methodological conclusion is therefore limited but general. Within GR, equivalence is best treated as a typed, historically situated, and structure-relative relation whose status depends on the transformations and distinctions that the analysis preserves. Equivalence can be defined on a manifestation, on a relational structure, on the coupling among levels, on an ordered historical evolution, on a family of admissible continuations, or on a dynamical regime. It can also change as those structures change. A disciplined analysis consequently asks not only whether two relata are equivalent, but which structure makes them equivalent, which distinctions remain outside the comparison, how stable the relation is under enrichment, and what additional evidence would be required for the next inferential step. That framework provides the basis on which future GR analyses of artificial systems, institutions, normative orders, knowledge structures, and other history-bearing relational entities can develop more specific theories of equivalence and interchangeability.

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